Section 6.4 Permutations and Combinations: Part 1

Permutations

1. How many ways can you arrange three people in a line?

$\underline{3} \times \underline{2} \times \underline{1} = 3! = \boxed{6}$

2. Five people are waiting to take a picture. How many ways can you arrange three of them in a line to take the picture?

$\underline{5} \times \underline{4} \times \underline{3} = 5 \times 4 \times 3 = \boxed{60}$

Reminder of Factorials

n-Factorial ($n!$) For any natural number n,

$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$

$0! = 1$

Calculator Steps: Enter the number followed by $\boxed{\text{MATH}}$, scroll right to $\boxed{\text{PRB}}$ and click $\boxed{4}$ then click $\boxed{\text{ENTER}}$.

Permutations of n Distinct Objects:

The number of *permutations* of n distinct objects taken r at a time is

$$P(n,r) = \frac{n!}{(n-r)!}$$

Calculator Steps: Enter the first number followed by $\boxed{\text{MATH}}$, scroll right to $\boxed{\text{PRB}}$ and click $\boxed{2}$ then enter the second number and click $\boxed{\text{ENTER}}$.

Note: If $n = r$, then $P(n,n) = n!$

3. Compute $P(3,3)$ and $P(5,3)$. How do these two permutations relate to the answers in examples 1 and 2?

$P(3,3) = 3! = \boxed{6}$

$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \boxed{60}$

They are the same

4. In how many ways can the names of nine candidates for political office be listed on a ballot?

$P(9,9) = 9! = \boxed{362,880}$

Note: In this class, I tend to use the Multiplication Principle interchangeably with Permutations.

5. Rework the previous example using the Multiplication Principle instead of Permutations.

$\underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 9!$

$= \boxed{362,880}$

6. A company car that has a seating capacity of eight is to be used by eight employees who have formed a car pool. If only three of these employees can drive, how many possible seating arrangements are there for the group?

$\underline{3} \,|\, \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 3 \times 7!$

$= \boxed{15,120}$

7. There are four families attending a concert together. Each family consists of 1 male and 2 females. In how many ways can they be seated in a row of twelve seats if

   (a) There are no restrictions?

   $\underline{12}\ \underline{11}\ \underline{10}\ \underline{\cdot}\ \underline{\cdot}\ \underline{\cdot}\ \underline{\cdot}\ \underline{\cdot}\ \underline{\cdot}\ \underline{3}\ \underline{2}\ \underline{1} =$

   $12! = \boxed{479,001,600}$

   (b) Each family is seated together?

   $\overbrace{\underline{3}\ \underline{2}\ \underline{1}}^{3!}$ $\overbrace{\underline{3}\ \underline{2}\ \underline{1}}^{3!}$ $\overbrace{\underline{3}\ \underline{2}\ \underline{1}}^{3!}$ $\overbrace{\underline{3}\ \underline{2}\ \underline{1}}^{3!}$

   $\underbrace{\underline{\quad}_{4}\ \underline{\quad}_{3}\ \underline{\quad}_{2}\ \underline{\quad}_{1}}_{4!}$

   $= 3! \times 3! \times 3! \times 3! \times \underline{4!} = \boxed{31,104}$

(c) The members of each gender are seated together?

$\overbrace{4\ 3\ 2\ 1}^{4!}$ $\quad$ $\overbrace{8\ 7\ 6\ \cdots\ 1}^{8!}$

$\underbrace{2 \qquad\qquad 1}_{2!}$

$= 4! \times 8! \times 2!$

$= \boxed{1,935,360}$

8. At a college library exhibition of faculty publications, two mathematics books, four social science books, and three biology books will be displayed on a shelf. (Assume that none of the books are alike.)

(a) In how many ways can the nine books be arranged on the shelf?

$9! = \boxed{362,880}$

(b) In how many ways can the nine books be arranged on the shelf if books on the same subject matter are placed together?

$\overbrace{2\ 1}^{2!}$ $\quad$ $\overbrace{4\ 3\ 2\ 1}^{4!}$ $\quad$ $\overbrace{3\ 2\ 1}^{3!}$

$\underbrace{3 \qquad\qquad 2 \qquad\qquad 1}_{3!}$

$= 2! \times 4! \times (3!)^2$

$= \boxed{1,720}$

Permutations of n Objects, Not all Distinct:

Given a set of n objects in which n_1 objects are alike and of one kind, n_2 objects alike and of another kind, . . . , and n_m objects are alike and of yet another kind, so that

$$n_1 + n_2 + \cdots + n_m = n$$

then the number of permutations of these n objects taken n at a time is given by

$$\frac{n!}{n_1!n_2!\cdots n_m!}$$

9. Find the number of distinguishable arrangements of each of the following "words."

   (a) *acdbens*

   Total # of letters = 7

   # of ways = $7! = \boxed{5,040}$

   (b) *baaaben*

   $$\text{\# of ways} = \frac{7!}{(2! \times 3!)} = \boxed{420}$$

   (c) *aaabbba*

   $$\text{\# of ways} = \frac{7!}{(4! \times 3!)} = \boxed{35}$$

10. A toy chest contains six identical blue blocks, five identical yellow blocks, and nine identical red blocks. How many distinguishable arrangements of these blocks can be made?

   # of blocks = 20

   $$\text{\# of ways} = \frac{20!}{(6! \times 5! \times 9!)} = \boxed{77,597,520}$$

11. Suppose that in the previous example, the blocks of the same color are numbered, so that the ~~yellow~~ blue blocks are numbered 1 through 6, the ~~blue~~ yellow blocks are numbered 1 through 5 and the red blocks are numbered 1 through 9. Note that this means that blocks of the same color are no longer identical.

   (a) How many distinguishable arrangements of these blocks can be made?

   $$20! = \boxed{2.432902008 \times 10^{18}}$$

   (b) How many distinguishable arrangements of these blocks can be made if blocks of the same ~~shape and~~ color should stay together?

   $$\underbrace{\underbrace{\overbrace{6\ 5\ \_\ \_\ \_\ 1}^{6!}}_{3}\quad \underbrace{\overbrace{5\ 4\ \_\ \_\ 1}^{5!}}_{2}\quad \underbrace{\overbrace{9\ 8\ \_\ \_\ \_\ \_\ \_\ \_\ 1}^{9!}}_{1}}_{3!}$$

   $$= 6! \times 5! \times 9! \times 3! = \boxed{1.88116992 \times 10^{11}}$$

Section 6.4 Permutations and Combinations Part 2

Question:

Suppose we want to choose three people from a group of four people and we do not care about the order in which we do this, that is, we will not be arranging the people we choose in any particular order. How do we do this?

Answer:

Suppose we number the people from 1 through 4 and think of the set $A = \{1, 2, 3, 4\}$. To answer this question we will count how many subsets of size 3 there are of this set...

$\{3,2,1\}, \{4,2,1\}, \{4,3,2\}, \{4,3,1\}$

Combinations of n Distinct Objects:

The number of *combinations* of n distinct objects taken r at a time is given by

$$C(n,r) = \frac{n!}{r!(n-r)!} \qquad (\text{where} \quad r \leq n)$$

Calculator Steps: Enter the first number followed by MATH, scroll right to PRB and click 3 then enter the second number and click ENTER.

1. Compute $C(4,3)$ and $C(10,5)$.

$C(4,3) = \frac{4!}{3!(4-3)!} = \frac{4!}{3!\cdot 1!} = \frac{4!}{3!} = \boxed{4}$

$C(10,5) = \frac{10!}{5!(10-5)!} = \frac{10!}{(5!\cdot 5!)} = \boxed{252}$

Language If a problem uses the word "and" ($\cap$) then you need to multiply the results. If a problem uses the word "or" ($\cup$) then you need to add the results.

2. In how many ways can a subcommittee of six be chosen from a Senate committee of six Democrats and five Republicans if

Choose 6, Total 5+6=11

   (a) All members are eligible?

   $C(11,6) = 11\ nCr\ 6 = \boxed{462}$

   (b) The subcommittee must consist of three Republicans and three Democrats?

   $(5\ nCr\ 3) \times (6\ nCr\ 3) = \boxed{200}$

   3 Reps. 3 Dems.

3. In how many different ways can a panel of 12 jurors and 2 alternates be chosen from a group of 16 prospective jurors? Total is 16

   $(16\ nCr\ 12) \times (4\ nCr\ 2) = \boxed{10{,}920}$

   12 jurors 2 alternates

4. A student planning her curriculum for the upcoming year must select one of four business courses, one of four mathematics courses, two of seven elective courses, and either one of five history courses or one of three social science courses. How many different curricula are available for her consideration?

   $(4\ nCr\ 1) \times (4\ nCr\ 1) \times (7\ nCr\ 2) \times \big((5\ nCr\ 1) + (3\ nCr\ 1)\big)$

   $= \boxed{2{,}688}$

5. From a shipment of 25 transistors, 6 of which are defective, a sample of 9 transistors is selected at random.

Total 25
6 defective
19 nondefective
Choose 9

(a) In how many different ways can the sample be selected?

$(25 \, nCr \, 9) = \boxed{2,042,975}$

(b) How many samples contain exactly 3 defective transistors?

$\underbrace{(6 \, nCr \, 3)}_{3 \text{ def.}} \cdot (19 \, nCr \, 6) = \boxed{542,640}$

(c) How many samples contain no defective transistors?

no def.

$(19 \, nCr \, 9) = \boxed{92,378}$

all are nondefec.

(d) How many samples contain at least 5 defective transistors?

At least 5 = exactly 5 "or" exactly 6

$\underbrace{(6 \, nCr \, 5) \cdot (19 \, nCr \, 4)}_{5 \text{ Def.}} + \underbrace{(6 \, nCr \, 6) \cdot (19 \, nCr \, 3)}_{6 \text{ def.}}$

$= \boxed{24,225}$

Complement Rule: Sometimes it is easier to ask how many ways there are of doing the opposite (or complement) of what you want than it is to ask how many ways there are to do what you want. So the complement rule is

$$\textbf{\# of Ways You Want = Total Ways - \# of Ways You Don't Want}$$

6. A box contains 8 red marbles, 8 green marbles, and 10 black marbles. A sample of 12 marbles is to be picked from the box.

Total 26   8R
choose 12   8G
10 B

(a) How many samples contain at least 1 red marble?

Want = Total ways − don't want

no reds

$= (26 \text{ nCr } 12) - (18 \text{ nCr } 12)$

$= \boxed{9{,}639{,}136}$

(b) How many samples contain exactly 4 red marbles and exactly 3 black marbles?

$(8 \text{ nCr } 4)\cdot(10 \text{ nCr } 3)\cdot(8 \text{ nCr } 5)$

4R   3B   5G

$= \boxed{470{,}400}$

(c) How many samples contain exactly 7 red marbles or exactly 6 green marbles?

Q: Intersection?

A: 7 + 6 = 13 > 12, No. Just add

$(8 \text{ nCr } 7)\cdot(18 \text{ nCr } 5) + (8 \text{ nCr } 6)\cdot(18 \text{ nCr } 6)$

7R   6G

$= \boxed{588{,}336}$

(d) How many samples contain exactly 5 green marbles or exactly 3 black marbles?

Q: Intersection?

A: 5 + 3 = 8 ≤ 12, yes. Subtract it off

$(8 \text{ nCr } 5)\cdot(18 \text{ nCr } 7) + (10 \text{ nCr } 3)\cdot(16 \text{ nCr } 9)$

5G 3B

$$- \overbrace{(\underbrace{8 \text{ nCr } 5}_{5G})\cdot(\underbrace{10 \text{ nCr } 3}_{3B})\cdot(\underbrace{8 \text{ nCr } 4}_{4R})}^{\text{Intersection}}$$

$= \boxed{2,684,544}$

When To Use Permutations/Multiplication Principle or Combinations?

We use permutations/multiplication principle whenever *order matters* and we use combinations whenever *order does not matter.*

- Keywords that suggest we use permutations/multiplication principle: *distinguishable arrangments, ordered list of names, distinct arrangments, arrange in a line, seated in a line or seated in any arrangment*

- Keywords that suggest we use combinations: *choose a smaller group from a larger group, select a committee or subcommittee, select a number of items, how many samples contain a number of items or people*

7. In how many ways can the names of three Republican and four Democratic candidates for political office be listed on a ballot?

(a) Should we use combinations or should we use permutations/multiplication principle?

Permutations

(b) How many ways can this be done?

$\underline{7}\ \underline{6}\ \underline{5}\ \underline{\cdot}\cdot\underline{\cdot}\ \underline{\cdot}\cdot\ \underline{1} = 7! = \boxed{5,040}$

8. A bag contains 5 red marbles and 5 blue marbles. How many ways can you select 6 marbles from the bag?

   (a) Should we use combinations or should we use permutations/multiplication principle?

   Combinations

   (b) How many ways can this be done?

   $(10 \; nCr \; 6) = \boxed{210}$

Note: Sometimes a problem requires using both permutations/multiplication principle and combinations.

9. Suppose we have 25 people on a committee. How many subcommittees contain one president, one vice president and six cabinet members?

   $\underset{\text{President}}{\underline{25}} \; \underset{\text{VP}}{\underline{24}} \; \Big| \; \underset{\text{Cabinet members}}{(23 \; nCr \; 6)} = 25 \cdot 24 \cdot (23 \; nCr \; 6)$

   $= \boxed{60,568,200}$

10. Twenty runners are competing in a half-marathon. How many ways can we award one 1*st* place prize, one 2*nd* place prize, one 3*rd* place prize, and four 4*th* place prizes?

   $\underset{\text{1st}}{\underline{20}} \; \underset{\text{2nd}}{\underline{19}} \; \underset{\text{3rd}}{\underline{18}} \cdot \underset{\text{4th}}{(17 \; nCr \; 4)}$

   $= \boxed{16,279,200}$