Proof by Contradiction

MAT231

Transition to Higher Mathematics

Fall 2014

Outline

Motivating Example

Proposition

For all integers n, if $n^3 + 5$ is odd then n is even.

How should we proceed to prove this statement?

- A direct proof would require that we begin with $n^3 + 5$ being odd and conclude that n is even.
- A contrapositive proof seems more reasonable: assume n is odd and show that $n^3 + 5$ is even.

The second approach works well for this problem. However, today we want try another approach that works well here and in other important cases where a contrapositive proof may not.

Motivating Example

Proposition

For all integers n, if $n^3 + 5$ is odd then n is even.

Motivating Example

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Proof.

Let n be any integer and suppose, for the sake of contradiction, that $n^3 + 5$ and n are both odd. In this case integers j and k exist such that $n^3 + 5 = 2k + 1$ and $n = 2j + 1$. Substituting for n we have

$$\begin{aligned} 2k + 1 &= n^3 + 5 \\ 2k + 1 &= (2j + 1)^3 + 5 \\ 2k + 1 &= 8j^3 + 3(2j)^2(1) + 3(2j)(1)^2 + 1^3 + 5 \\ 2k &= 8j^3 + 12j^2 + 6j + 5. \end{aligned}$$

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Motivating Example

Proof.

(Continued from previous slide) We found

$$2k = 8j^3 + 12j^2 + 6j + 5.$$

Dividing by 2 and rearranging we have

$$k - 4j^3 - 6j^2 - 3j = \frac{5}{2}.$$

This, however, is impossible: $5/2$ is a non-integer rational number, while $k - 4j^3 - 6j^2 - 3j$ is an integer by the closure properties for integers. Therefore, it must be the case that our assumption that when $n^3 + 5$ is odd then n is odd is false, so n must be even. □

Proof by Contradiction

This is an example of **proof by contradiction**. To prove a statement P is true, we begin by assuming P false and show that this leads to a contradiction; something that always false.

Many of the statements we prove have the form $P \Rightarrow Q$ which, when negated, has the form $P \Rightarrow \sim Q$. Often proof by contradiction has the form

Proposition

$P \Rightarrow Q$.

Proof.

Assume, for the sake of contradiction P is true but Q is false.

$$\ldots$$

Since we have a contradiction, it must be that Q is true. □

Proof: $\sqrt{2}$ is irrational

Proof.

Suppose $\sqrt{2}$ is rational. Then integers a and b exist so that $\sqrt{2} = a/b$. Without loss of generality we can assume that a and b have no factors in common (i.e., the fraction is in simplest form). Multiplying both sides by b and squaring, we have

$$2b^2 = a^2$$

so we see that a^2 is even. This means that a is even (how would you prove this?) so $a = 2m$ for some $m \in \mathbb{Z}$. Then

$$2b^2 = a^2 = (2m)^2 = 4m^2$$

which, after dividing by 2, gives $b^2 = 2m^2$ so b^2 is even. This means $b = 2n$ for some $n \in \mathbb{Z}$.

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Proof: $\sqrt{2}$ is irrational

Proof.

(Continued from previous slide)

We've seen that if $\sqrt{2} = a/b$ then both a and b must be even and so are both multiples of 2.

This contradicts the fact that we know a and b can be chosen to have no common factors. Thus, $\sqrt{2}$ must not be rational, so $\sqrt{2}$ is irrational. $\square$

Quantifications and Contradiction

Sometimes we need prove statements of the form

$$\forall x, P(x).$$

These are often particularly well suited to proof by contradiction as the negation of the statement is

$$\exists x, \sim P(x)$$

so all that is necessary to complete the proof is to assume there is an x that makes $\sim P(x)$ true and see that it leads to a contradiction.

Quantifications and Contradiction

Proposition

There exist no integers a and b for which $18a + 6b = 1$.

This could be written as "$\forall a, b \in \mathbb{Z}, 18a + 6b \neq 1$." Negating this yields "$\exists a, b \in \mathbb{Z}, 18a + 6b = 1$."

Proof.

Assume, for the sake of contradiction, that integers a and b can be found for which $18a + 6b = 1$. Dividing by 6 we obtain

$$3a + b = \frac{1}{6}.$$

This is a contradiction, since by the closure properties $3a + b$ is an integer but $1/6$ is not. Therefore, it must be that no integers a and b exist for which $18a + 6b = 1$. □

Practice

Use contradiction to prove each of the following propositions.

Proposition

The sum of a rational number and an irrational number is irrational.

Proposition

Suppose a*,* b*, and* c *are positive real numbers. If* $ab = c$ *then* $a \leq \sqrt{c}$ *or* $b \leq \sqrt{c}$*.*

Practice

Use a direct proof, a contrapositive proof, or a proof by contradiction to prove each of the following propositions.

Proposition

Suppose $a, b \in \mathbb{Z}$*. If* $a + b \geq 19$*, then* $a \geq 10$ *or* $b \geq 10$*.*

Proposition

Suppose $a, b, c, d \in \mathbb{Z}$ *and* $n \in \mathbb{N}$*. If* $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$*, then* $a + c \equiv b + d \pmod{n}$*.*

Proposition

Suppose n is a composite integer. Then n has a prime divisor less than or equal to $\sqrt{n}$*.*