Chapter 1

Linear Models and Systems of Linear Equations

Contents

1.1 Mathematical Models

Augustin Cournot, 1801-1877

The first significant work dealing with the application of mathematics to economics was Cournot's *Researches into the Mathematical Principles of the Theory of Wealth*, published in 1836. It was Cournot who originated the supply and demand curves that are discussed in this section. Irving Fisher, a prominent economics professor at Yale University and one of the first exponents of mathematical economics in the United States, wrote that Cournot's book "seemed a failure when first published. It was far in advance of the times. Its methods were too strange, its reasoning too intricate for the crude and confident notions of political economy then current."

Application: Cost, Revenue, and Profit Models

A firm has weekly fixed costs of $80,000 associated with the manufacture of dresses that cost $25 per dress to produce. The firm sells all the dresses it produces at $75 per dress. Find the cost, revenue, and profit equations if x is the number of dresses produced per week. See Example 3 for the answer.

We will first review some basic material on functions. An introduction to the mathematical theory of the business firm with some necessary economics background is provided. We study mathematical business models of cost, revenue, profit, and depreciation, and mathematical economic models of demand and supply. We will only consider *linear* relationships, so you may wish to review material located in the Algebra Review chapter on straight lines.

1.1.1 Functions

Mathematical modeling is an attempt to describe some part of the real world in mathematical terms. Our models will be **functions** that show the relationship between two or more variables. These variables will represent quantities that we wish to understand or describe. Examples include the price of gasoline, the cost of producing cereal or the number of video games sold. The idea of representing these quantities as variables in a function is central to our goal of creating models to describe their behavior. We will begin by reviewing the concept of functions. In short, we call any rule that assigns or corresponds to each element in one set precisely one element in another set a **function**.

For example, suppose you are going a steady speed of 40 miles per hour in a car. In one hour you will travel 40 miles; in two hours you will travel 80 miles; and so on. The distance you travel depends on (corresponds to) the time. Indeed, the equation relating the variables distance (d), velocity (v), and time (t), is $d = v \cdot t$. In our example, we have a constant velocity of $v = 40$, so $d = 40 \cdot t$. We can view this as a correspondence or rule: Given the time t in hours, the rule gives a distance d in miles according to $d = 40 \cdot t$. Thus, given $t = 3$, $d = 40 \cdot 3 = 120$. Notice carefully how this rule is *unambiguous*. That is, given any time t, the rule specifies one and only one distance d. This rule is therefore a function; the correspondence is between time and distance.

Often the letter f is used to denote a function. Thus, using the previous example, we can write $d = f(t) = 40 \cdot t$. The symbol $f(t)$ is read "f of t." One can think of the variable t as the "input" and the value of the variable $d = f(t)$ as the "output." For example, an input of $t = 4$ results in an output of $d = f(4) = 40 \cdot 4 = 160$ miles.

The following gives a general definition of a function.

Definition of a Function

A **function** f from D to R is a rule that assigns to each element x in D one and only one element $y = f(x)$ in R. See Figure 1.1.1.
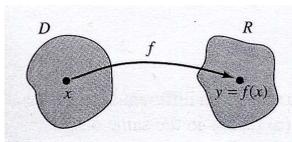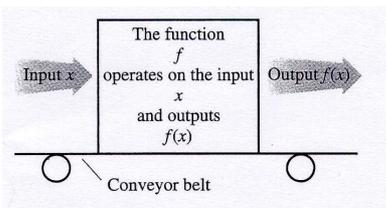

Figure 1.1.1

The caption is here, if needed

The set D in the definition is called the **domain** of f. We might think of the domain as the set of inputs. We then can think of the values $f(x)$ as outputs. The set of outputs, R is called the **range** of f.

Another helpful way to think of a function is shown in Figure 1.1.2. Here the function f accepts the input x from the conveyor belt, operates on x, and outputs (assigns) the new value $f(x)$.

Figure 1.1.2

The letter representing elements in the domain is called the **independent variable**, and the letter representing the elements in the range is called the **dependent variable**. Thus, if $y = f(x)$, x is the independent variable, and y is the dependent variable, since the value of y *depends* on x. In the equation $d = 40t$, we can write $d = f(t) = 40t$ with t as the independent variable. The dependent variable is d, since the distance *depends* on the spent time t traveling. We are free to set the independent variable t equal to any number of values in the domain. The domain for this function is $t \geq 0$ since only nonnegative time is allowed.

Note that the domain in an application problem will always be those values that are allowed for the independent variable in the particular application. This often means that we are restricted to non-negative values or perhaps we will be limited to the case of whole numbers only, as in the next example:

Example 1   **Steak Specials** A restaurant serves a steak special for $12. Write a function that models the amount of revenue made from selling these specials. How much revenue will 10 steak specials earn?

Solution: We first need to decide if the independent variable is the price of the steak specials, the number of specials sold, or the amount of revenue earned. Since the price is fixed at $12 per special and revenue depends on the number of specials sold, we choose the independent variable, x, to be the number of specials sold and the dependent variable, $R = f(x)$ to be the amount of revenue. Our rule will be $R = f(x) = 12x$ where x is the number of steak specials sold and R is the revenue from selling these specials in dollars. Note that x must be a whole number, so the domain is $x = 0, 1, 2, 3, \ldots$. To determine the revenue made on selling 10 steak specials, plug $x = 10$ into the model:

$$R = f(10) = 12(10) = 120$$

So the revenue is $120. $\checkmark$

$\textcircled{T}$ **Technology Option.** You may wish to see Technology Note 1 for the solution to the question using the graphing calculator.

Recall (see Appendix A) that lines satisfy the equation $y = mx + b$. Actually, we can view this as a *function*. We can set $y = f(x) = mx + b$. Given any number x, $f(x)$ is obtained by multiplying x by m and adding b. More specifically, we call the function $y = f(x) = mx + b$ a **linear function**.

Definition of Linear Function

A **linear function** f is any function of the form

$$y = f(x) = mx + b$$

where m and b are constants.

Example 2   **Linear Functions** Which of the following functions are linear?
   a. $y = -0.5x + 12$
   b. $5y - 2x = 10$
   c. $y = 1/x + 2$
   d. $y = x^2$

Solution: a. This is a linear function. The slope is $m = -0.5$ and the y-intercept is $b = 12$.
   b. Rewrite this function first as,

$$\begin{aligned}
5y - 2x &= 10 \\
5y &= 2x + 10 \\
y &= (2/5)x + 2
\end{aligned}$$

Now we see it is a linear function with $m = 2/5$ and $b = 2$.

c. This is not a linear function. Rewrite $1/x$ as x^{-1} and this shows that we do not have a term mx and so this is not a linear function.
d.   Here x is raised to the second power and so this is not a linear function. $\sqrt{}$

1.1.2   Mathematical Modeling

When we use mathematical modeling we are attempting to describe some part of the real world in mathematical terms, just as we have done for the distance traveled and the revenue from selling meals. There are three steps in mathematical modeling: formulation, mathematical manipulation, and evaluation.

Formulation

First, on the basis of observations, we must state a question or formulate a hypothesis. If the question or hypothesis is too vague, we need to make it precise. If it is too ambitious, we need to restrict it or subdivide it into manageable parts. Second, we need to identify important factors. We must decide which quantities and relationships are important to answer the question and which can be ignored. We then need to formulate a *mathematical* description. For example, each important quantity should be represented by a variable. Each relationship should be represented by an equation, inequality, or other mathematical construct. If we obtain a function, say, $y = f(x)$, we must carefully identify the input variable x and the output variable y and the units for each. We should also indicate the interval of values of the input variable for which the model is justified.

Mathematical Manipulation

After the mathematical formulation, we then need to do some mathematical manipulation to obtain the answer to our original question. We might need to do a calculation, solve an equation, or prove a theorem. Sometimes the mathematical formulation gives us a mathematical problem that is impossible to solve. In such a case, we will need to reformulate the question in a less ambitious manner.

Evaluation

Naturally, we need to check the answers given by the model with real data. We normally expect the mathematical model to describe only a very limited aspect of the world and to give only approximate answers. If the answers are wrong or not accurate enough for our purposes, then we will need to identify the sources of the model's shortcomings. Perhaps we need to change the model entirely, or perhaps we need to just make some refinements. In any case, this requires a new mathematical manipulation and evaluation. Thus, modeling often involves repeating the three steps of formulation, mathematical manipulation, and evaluation.

We will next create linear mathematical models by find equations that relate cost, revenue, and profits of a manufacturing firm to the number of units produced and sold.

1.1.3   Cost, Revenue, and Profits

Any manufacturing firm has two types of costs: fixed and variable. **Fixed costs** are those that do not depend on the amount of production. These costs include real estate taxes, interest on loans, some management salaries, certain minimal maintenance, and protection of plant and equipment. **Variable costs** depend on the amount of production. They include the cost of material and labor. Total cost, or simply **cost**, is the sum of fixed and variable costs:

$$\text{cost} = (\text{variable cost}) + (\text{fixed cost}).$$

Let x denote the number of units of a given product or commodity produced by a firm. (Notice that we must have $x \geq 0$.) The units could be bales of cotton, tons of fertilizer, or number of automobiles. In the **linear cost model** we assume that the cost m of manufacturing one unit is the same no matter how many units are produced. Thus, the variable cost is the number of units produced times the cost of each unit:

$$
\begin{aligned}
\text{variable cost} \quad &= \quad (\text{cost per unit}) \times (\text{number of units produced}) \\
&= \quad mx
\end{aligned}
$$

If b is the fixed cost and $C(x)$ is the cost, then we have the following:

$$
\begin{aligned}
C(x) \quad &= \quad \text{cost} \\
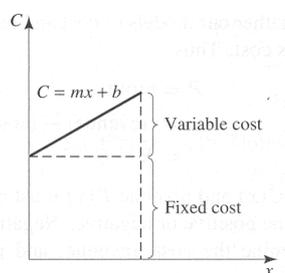&= \quad (\text{variable cost}) + (\text{fixed cost}) \\
&= \quad mx + b
\end{aligned}
$$

Notice that we must have $C(x) \geq 0$. In the graph shown in Figure 1.1.3, we see that the y-intercept is the fixed cost and the slope is the cost per item.

Figure 1.1.3

CONNECTION[*]
What Are Costs? Isn't it obvious what the costs to a firm are? Apparently not. On July 15, 2002, Coca-Cola Company announced that it would begin treating stock-option compensation as a cost, thereby lowering earnings. If all companies in the Standard and Poors 500 stock index were to do the same, the earnings for this index would drop by 23%.
[*]*The Wall Street Journal, July 16, 2002.*

In the **linear revenue model** we assume that the price p of a unit sold by a firm is the same no matter how many units are sold. (This is a reasonable assumption if the number of units sold by the

firm is small in comparison to the total number sold by the entire industry.) Revenue is always the price per unit times the number of units sold. Let x be the number of units sold. (For convenience, we always assume that *the number of units sold equals the number of units produced.*) Then, if we denote the revenue by $R(x)$,
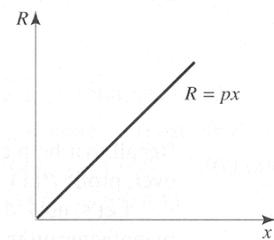
$$
\begin{aligned}
R(x) &= \text{revenue} \\
&= (\text{price per unit}) \times (\text{number sold}) \\
&= px
\end{aligned}
$$

Since $p > 0$, we must have $R(x) \geq 0$. Notice in Figure 1.1.4. that the straight line goes through $(0,0)$ because nothing sold results in no revenue. The slope is the price per unit.

Figure 1.1.4

Connection: What Are Revenues?

The accounting practices of many telecommunications companies such as Cisco and Lucent, have been criticized for what the companies consider revenues. In particular, these companies have loaned money to other companies, which then use the proceeds of the loan to buy telecommunications equipment from Cisco and Lucent. Cisco and Lucent then book these sales as "revenue." But is this revenue?

Regardless of whether our models of cost and revenue are linear or not, **profit** P is always revenue less cost. Thus

$$
\begin{aligned}
P &= \text{profit} \\
&= (\text{revenue}) - (\text{cost}) \\
&= R - C
\end{aligned}
$$

Recall that both cost $C(x)$ and revenue $R(x)$ must be nonnegative functions. However, the profit $P(x)$ can be positive or negative. Negative profits are called *losses*.

Let's now determine the cost, revenue, and profit equations for a dress-manufacturing firm.

Example 3   **Cost, Revenue, and Profit Equations** A firm has weekly fixed costs of $80,000 associated with the manufacture of dresses that cost $25 per dress to produce. The firm sells all the dresses it produces at $75 per dress.
a.   Find the cost, revenue, and profit equations if x is the number of dresses produced per week.
b.   Make a table of values for cost, revenue, and profit for production levels of 1000, 1500 and 2000 dresses and discuss what is the table of numbers telling you.

Solution:

a.   The fixed cost is $80,000 and the variable cost is $25x$. So

$$\begin{aligned} C &= \text{(variable cost)} + \text{(fixed cost)} \\ &= mx + b \\ &= 25x + 80,000 \end{aligned}$$

See Figure 1.1.5a. Notice that $x \geq 0$ and $C(x) \geq 0$.
The revenue is just the price \$75 that each dress is sold multiplied by the number x of dresses sold. So

$$\begin{aligned} R &= \text{(price per dress)} \times \text{(number sold)} \\ &= px \\ &= 75x \end{aligned}$$

See Figure 1.1.5b. Notice that $x \geq 0$ and $R(x) \geq 0$. Also notice that if there are no sales, then there is no revenue, that is, $R(0) = 0$.

Profit is always revenue less cost. So

$$\begin{aligned} P &= \text{(revenue)} - \text{(cost)} \\ &= R - C \\ &= (75x) - (25x + 80,000) \\ &= 50x - 80,000 \end{aligned}$$

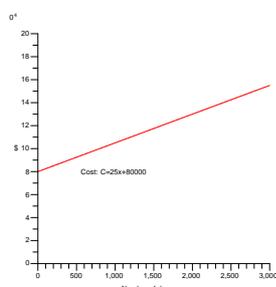See Figure 1.1.5c. Notice in Figure 1.1.5c that profits can be negative.
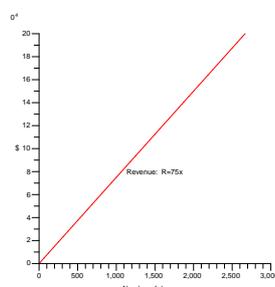

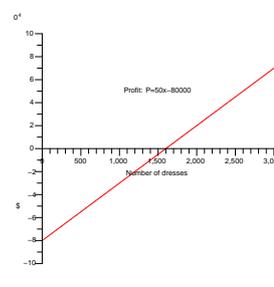Figure 1.1.5a   Figure 1.1.5b   Figure 1.1.5c

b.   To be specific, suppose 1000 dresses are produced and sold. Then $x = 1000$ and

$$\begin{aligned} C(1000) &= 25(1000) + 80,000 = 130,000 \\ R(1000) &= 75(1000) = 75,000 \\ P(1000) &= 75,000 - 130,000 = -55,000 \end{aligned}$$

Thus, if 1000 dresses are produced and sold, the cost is \$130,000, the revenue is \$75,000, and there is a negative profit or *loss* of \$55,000.

Doing the same for 1500 and 200 dresses, we have the results shown in Table 1.1.

Number of Dresses Made and Sold	1000	1500	2000
Cost in dollars	130,000	117,500	130,000
Revenue in dollars	75,000	112,500	150,000
Profit (or loss) in dollars	-55,000	-5,000	20,000
Table 1.1			

We can see in Figure 1.1.5c or in Table 1.1, that for smaller values of x, $P(x)$ is *negative*; that is, the firm has losses as their costs are greater than their revenue. For larger values of x, $P(x)$ turns positive and the firm has (positive) profits. $\sqrt{}$

1.1.4   Supply and Demand

In the previous discussion we assumed that the number of units produced and sold by the given firm was small in comparison to the number sold by the industry. Under this assumption it was reasonable to conclude that the price, p, was constant and did not vary with the number x sold. But if the number of units sold by the firm represented a *large* percentage of the number sold by the entire industry, then trying to sell significantly more units could only be accomplished by *lowering* the price of each unit. Since we just stated that the price effects the number sold, you would expect the price to be the independent variable and thus graphed on the horizontal axis. However, by custom, the price is graphed on the vertical axis and the quantity x on the horizontal axis. This convention was started by English economist Alfred Marshall (1842 - 1924) in his important book, *Principles of Economics*. We will abide by this custom in this text.
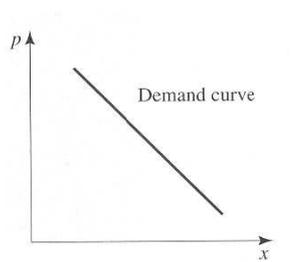
For most items the relationship between quantity and price is a decreasing function (there are some exceptions to this rule, such as certain luxury good, medical care and higher eduction, to name a few). That is, for the number of items to be sold to increase, the price must decrease. We assume now for mathematical convenience that this relationship is linear. Then the graph of this equation is a straight line that slopes downward as shown in Figure 1.1.6.

Figure 1.1.6

We assume that x is the number of units produced and sold by the entire industry during a given time period and that $p = D(x) = -cx + d$, $c > 0$, is the price of one unit if x units are sold; that is, $p = -cx + d$ is the price of the xth unit sold. We call $p = D(x)$ the **demand equation** and the graph the **demand curve**.

Estimating the demand equation is a fundamental problem for the management of any company or business. In the next example we consider the situation when just two data points are available and the demand equation is assumed to be linear.

Example 5   **Finding the Demand Equation** Timmins estimated the municipal water demand in Delano, California. He estimated the demand x, measured in acre-feet (the volume of water needed to cover one acre of ground at a depth of one foot), with price p per acre-foot.
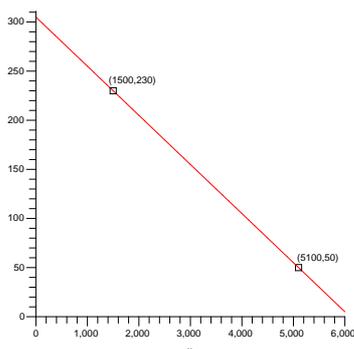
Figure 1.1.7

He indicated two points on the demand curve, $(x, p) = (1500, 230)$ and $(x, p) = (5100, 50)$. Use this data to estimate the demand curve using a linear model. Estimate the price when the demand is 3000 acre-feet.

Figure 1.1.7 shows the two points $(x, p) = (1500, 230)$ and $(x, p) = (5100, 50)$ that lie on the demand curve. We are assuming that the demand curve is a straight line. The slope of the line is

$$m = \frac{50 - 230}{5100 - 1500} = -0.05$$

Now using the point-slope equation for a line with $(1500, 230)$ as the point on the line, we have

$$
\begin{aligned}
p - 230 &= m(x - 1500) \\
&= -0.05(x - 1500) \\
p &= -0.05x + 75 + 230 \\
&= -0.05x + 305
\end{aligned}
$$

When demand is 3000 acre-feet, then $x = 3000$, and

$$p = -0.05(3000) + 305 = 155$$

or \$155 per acre-foot. Thus, according to this model, if 3000 acre-feet is demanded, the price of each acre-foot will be \$155. $\checkmark$

CONNECTION
Demand for Apartments The figure below shows that during the minor recession of 2001, vacancy rates for apartments rose, that is, the demand for apartments decreased. Also notice from the figure that as demand for apartments *decreased*, rents also *decreased*. For example, in South Francisco's South Beach area, a two-bedroom apartment that had rented for \$3000 a month two years before saw the rent drop to \$2100 a month.
*Source: Wall Street Journal, 4-11-02*
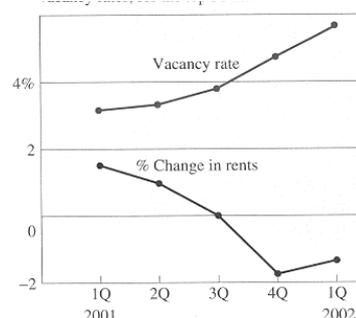
CONNECTION[*]
Demand for Television Sets As sleek flat-panel and high-definition television sets became more affordable, sales soared during the holidays. Sales of ultra-thin, wall-mountable LCD TVs rose over 100% in 2005 to about 20 million sets while plasma-TV sales rose at a similar pace, to about 5 million sets. Normally set makers and retailers lower their prices after the holidays, but since there was strong demand and production shortages for these sets, prices were kept high.
[*]*http://biz.yahoo.com/weekend/tvbargain_1.html 1-21-2006*
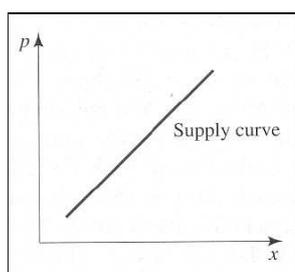
The **supply equation** $p = S(x)$ gives the price p necessary for suppliers to make available x units to the market. The graph of this equation is called the **supply curve**. A reasonable supply curve rises, moving from left to right, because the suppliers of any product naturally want to sell more if the price is higher. (See Shea[6] who looked at a large number of industries and determined that the supply curve does indeed slope upward.) If the supply curve is linear, then as shown in Figure 1.1.8, the graph is a line sloping upward. Note the positive y-intercept. The y-intercept represents the *choke point* or lowest price a supplier is willing to accept.

Figure 1.1.8   **Example 6**   **Finding the Supply Equation** Antle and Capalbo estimated a spring wheat supply curve. Use a mathematical model to determine a linear curve using their estimates that the supply of spring wheat of 50 million bushels at a price of $2.90 per bushel and 100 million bushels at a price of $4.00 per bushel. Estimate the price when 80 million bushels is supplied.

Solution: Let x be in millions of bushels of wheat. We are then given two points on the linear supply curve, $(x, p) = (50, 2.9)$ and $(x, p) = (100, 4)$. The slope is

$$m = \frac{4 - 2.9}{100 - 50} = 0.022$$

The equation is then given by

$$p - 2.9 = 0.022(x - 50)$$

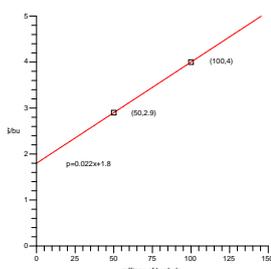or $p = 0.022x + 1.8$. See Figure 1.1.9 and note that the line rises.

Figure 1.1.9   When supply is 80 million bushels, $x = 80$, and we have

$$p = 0.022(80) + 1.8 = 3.56$$

This gives a price of $3.56 per bushel. $\sqrt{}$

CONNECTION[*]
Supply of Cotton On May 2, 2002, the U.S. House of Representatives passed a farm bill that promises billions of dollars in subsidies to cotton farmers. With the prospect of a greater supply of cotton, cotton prices dropped 1.36 cents to 33.76 cents per pound.
[*]*The Wall Street Journal, May 3, 2002.*

1.1.5   Straight-Line Depreciation.

Many assets, such as machines or buildings, have a *finite* useful life and furthermore *depreciate* in value from year to year. For purposes of determining profits and taxes, various methods of depreciation can be used. In **straight-line depreciation** we assume that the value

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[6]John Shea. 1993. Do supply curves slope up? *Quart. J. Econ.* cviii: 1-32.

V of the asset is given by a *linear* equation in time t, say, $V = mt + b$. The slope m must be *negative* since the value of the asset *decreases* over time. The y-intercept is the initial value of the item and the slope gives the rate of depreciation (how much the item decreases in value per time period).

Example 7   **Straight-Line Depreciation**   A company has purchased a new grinding machine for \$100,000 with a useful life of 10 years, after which it is assumed that the scrap value of the machine is \$5000. Use straight-line depreciation to write an equation for the value V of the machine where t is measured in years. What will be the value of the machine after the first year? After the second year? After the ninth year? What is the rate of depreciation?
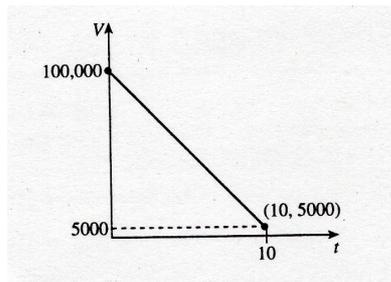
Figure 1.1.10

We assume that $V = mt + b$, where m is the slope and b is the V-intercept. We then must find both m and b. We are told that the machine is initially worth \$100,000, that is, when $t = 0$, $V = 100,000$. Thus, the point $(0,\ 100,000)$ is on the line, and 100,000 is the V-intercept, b. (see Figure 1.1.10). Note the domain of t is $0 \leq t \leq 10$.

Since the value of the machine in 10 years will be \$5000, this means that when $t = 10$, $V = 5000$. Thus, $(10, 5000)$ is also on the line. From Figure 1.1.10, the slope can then be calculated since we now know that the two points $(0,\ 100,000)$ and $(10, 5000)$ are on the line. Then

$$m = \frac{5000 - 100,000}{10 - 0} = -9500$$

The rate of depreciation is therefor -9500\$/year. Then, using the point-slope form of a line,

$$V = -9500t + 100,000$$

Where the time t is in years since the machine was purchased and V is the value in dollars. Now we can find the value at different time periods,

$$V(1) = -9500(1) + 100,000 = 90,500 \text{ or } \$90,500.$$
$$V(2) = -9500(2) + 100,000 = 81,000 \text{ or } \$81,000.$$
$$V(9) = -9500(9) + 100,000 = 14,500 \text{ or } \$14,500. \qquad \checkmark$$

(T) **Technology Option.** You may wish to see Technology Note 2 for the solution to this example using the graphing calculator.
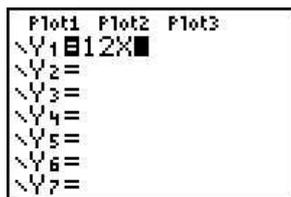
⌈ **Technology Corner** ⌋

Technology Note 1 - Example 1 on a Graphing Calculator. Begin by pressing the $\boxed{\text{Y=}}$ button on the top row of your calculator. Enter 12 from the keypad and the variable X using the $\boxed{\text{X,T,}\theta\text{,n}}$ button. Next choose the viewing window by pressing the $\boxed{\text{WINDOW}}$ button along the top row of buttons. Since the smallest value for x is 0
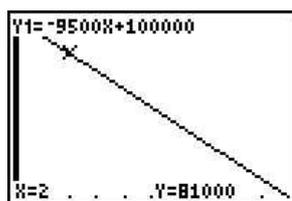
Screen 1.1.1

Screen 1.1.2

(no steak specials sold), enter 0 for `Xmin`. We want to evaluate the function for 12 steak specials, or $x = 12$, so choose an `Xmax` that is greater than 12. The graph below used `Xmin = 20` and `Xscl=5` (that is, a tick mark is placed every 5 units). The range of values for y must be large enough to view the function. The window below was `Ymin=0`, `Ymax=200` and `Yscl=10`. The `Xres` setting can be left at 1 as we want the full resolution on the screen. Press the $\boxed{\text{GRAPH}}$ button to see the function displayed.

To find the value of our function at a particular x-value, choose the `CALC` menu (above the $\boxed{\text{TRACE}}$ button) as shown in Screen 1.1.2. The trace function should avoided as it will not go to an exact x-value. Choose the first option, `1:value` and then enter the value 10. Pressing enter again to evaluate, we see in Screen 1.1.3 the value of the function at $x = 10$ is 120.

Technology Note 2 - Example 7 on a Graphing Calculator

The depreciation function can be graphed as done in the Technology Note 1 above. Screen 1.1.4 shows the result of graphing `Y`$_1$`=-9500X+100000` and finding the value at `X=2`. The window was chosen by entering `Xmin=0` and `Xmax=10`, the known domain of this function, and then pressing $\boxed{\text{ZOOM}}$ and scrolling down to `0:ZoomFit` and enter. This useful feature will evaluate the functions to be graphed from `Xmin` to `Xmax` and choose the values for `Ymin` and `Ymax` to allow the functions to be seen.

We were asked to find the value of the grinding machine at several different times, the table function can be used to simplify this task. Once a function is entered, go to the `TBLSET` feature by pressing $\boxed{\text{2ND}}$ and then $\boxed{\text{WINDOW}}$, see Screen 1.1.5. We want to start at `X=0` and count by 1's, so set `TblStart = 0` and Δ`Tbl=1`. To see the table, press $\boxed{\text{2ND}}$ and then $\boxed{\text{GRAPH}}$, see Screen 1.1.6.

Screen 1.1.4   **Screen 1.1.5**   **Screen 1.1.6**

1.1 SELF HELP EXERCISES

1. Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical medium-sized plant they estimated fixed costs at \$400,000 and estimated the cost of each ton of fertilizer was \$200 to produce. The plant sells its fertilizer output at \$250 per ton.

(a) Find and graph the cost, revenue, and profit equations.

(b) Determine the cost, revenue, and profits when the number of tons produced and sold is 5000, 7000, and 9000 tons.

2. The excess supply and demand curves for wheat worldwide were estimated by Schmitz and coworkers to be

Supply: $p = 7x - 400$
Demand: $p = 510 - 3.5x$

where p is price per metric ton and x is in millions of metric tons. Excess demand refers to the excess of wheat that producer countries have over their own consumption. Graph these two functions. Find the prices for the supply and demand models when x is 70 million metric tons. Is the price for supply or demand larger? Repeat these questions when x is 100 million metric tons.

EXERCISE SET   1.1

In Exercises 1 and 2 you are given the cost per item and the fixed costs. Assuming a linear cost model, find the cost equation, where C is cost and x is the number produced.

1. Cost per item = \$3, fixed cost = \$10,000
2. Cost per item = \$6, fixed cost = \$14,000

In Exercises 3 and 4 you are given the price of each item, which is assumed to be constant. Find the revenue equation, where R is revenue and x is the number sold.

3. Price per item=\$5   4.Price per item = \$0.1

5. Using the cost equation found in Exercise 1 and the revenue equation found in Exercise 3, find the profit equation for P, assuming that the number produced equals the number sold.

6. Using the cost equation found in Exercise 2 and the revenue equation found in Exercise 4, find the profit equation for P, assuming that the number produced equals the number sold.

In questions 7 to 10, find the demand equation using the given information.

7. A company finds it can sell 10 items at a price of \$8.00 each and sell 15 items at a price of \$6.00 each.

8. A company finds it can sell 40 items at a price of \$60.00 each and sell 60 items at a price of \$50.00 each.

9. A company finds that at a price of \$35, a total of 100 items will be sold. If the price is lowered by \$5, then 20 additional items will be sold.

10. A company finds that at a price of \$200, a total of 30 items will be sold. If the price is raised \$50, then 10 fewer items will be sold.

In Exercises 11 to 14, find the supply equation using the given information.

11. A supplier will supply 50 items to the market if the price is \$95 per item and supply 100 items if the price is \$175 per item.

12. A supplier will supply 1000 items to the market if the price is \$3.00 per item and supply 2000 items if the price is \$4.00 per item.

13. At a price of \$60 per item, a supplier will supply 10 of these items. If the price increases by \$20, then 4 additional items will be supplied.

14. At a price of \$800 per item, a supplier will supply 90 items. If the price decreases by \$50, then the supplier will supply 20 fewer items.

In Exercises 15 to 18, find the depreciation equation and corresponding domain using the given information.

15. A calculator is purchased for \$130 and the value decreases by \$15 per year for 7 years.

16. A violin bow is purchased for \$50 and the value decreases by \$5 per year for 6 years.

17. A car is purchased for \$15,000 and is sold for \$6000 six years later.

18. A car is purchased for $32,000 and is sold for $23,200 eight years later.

APPLICATIONS

19. **Revenue for red wine grapes in Napa Valley.** Brown and colleagues report that the price of red varieties of grapes in Napa Valley was $2274 per ton. Determine a revenue function and clearly indicate the independent and dependent variables.

20. **Revenue for wine grapes in Napa Valley.** Brown and colleagues report that the price of wine grapes in Napa Valley was $617 per ton. They estimated that 6 tons per acre was yielded. Determine a revenue function using the independent variable as the number of acres.

21. **Ecotourism Revenue.** Velazquez and colleagues studied the economics of ecotourism. A grant of $100,000 was given to a certain locality to use to develop an ecotourism alternative to destroying forest and the consequent biodiversity. The community found that each visitor spent $40 on average. If x is the number of visitors, find a revenue function. How many visitors are needed to reach the initial $100,000 invested? (This community was experiencing about 2500 visits per year.)

22. **Heinz Ketchup Revenue.** Besanko and colleagues reported that a Heinz ketchup 32 oz size yielded a price of $0.043 per ounce. Write an equation for revenue as a function of the number of 32 oz bottles of Heintz ketchup.

23. **Fishery Revenue.** Grafton created a mathematical model for revenue for the northern cod fishery. We can see from this model that when 150,000 kilograms of cod were caught, $105,600 of revenue were yielded. Using this information and assuming a linear revenue model, find a revenue function R in units of 1000 dollars where x is given in units of 1000 kilograms.

24. **Fishery Cost Function.** The cost function for wild crayfish was estimated by Bell to be a function $C(x)$, where x is the number of millions of pounds of crayfish caught and C is the cost in millions of dollars. From

this function we can see two points that are on the graph: $(x, C) = (8, 0.157)$ and $(x, C) = (10, 0.190)$. Using this information and assuming a linear model, determine a cost function.

25. **Wood Chipper Cost Function.** A contractor needs to rent a wood chipper for a day for $150 plus $10 per hour. Find the cost function.

26. **Rental Cost Function.** A builder needs to rent a dump truck for a day for $75 plus $0.40 per mile. Find the cost function.

27. **Machine Cost Function.** A shirt manufacturer is considering purchasing a sewing machine for $91,000 and for which will cost $2 to sew each of their standard shirts. Find the cost function.

28. **Copying Cost Function.** At Lincoln Library there are two ways to pay for copying. You can pay 5 cents a copy, or you can buy a plastic card for $5 and then pay 3 cents a copy. Let x be the number of copies you make. Write an equation for your costs for each way of paying.

29. **Cost Function for the Cotton Ginning Industry.** Misra and colleagues estimated the cost function for the ginning industry in the Southern High Plains of Texas. They give a (total) cost function C by

$$C(x) = 21x + 674,000$$

where C is in dollars and x is the number of bales of cotton. Find the fixed and variable costs.

30. **The Costs Associated with Raising a Steer.** Kaitibie and colleagues estimated the costs of raising a young steer purchased for $428 and the variable food cost per day for $0.67. Determine the cost function based on the number of days this steer is grown.

31. **Costs of Manufacturing Fenders.** Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using five different materials: steel, aluminum, and three injection-molded polymer blends: rubber-modified polypropylene (RMP), nylon-polyphenyle-neoxide (NPN),

and polycarbonate-polybutylene terephthalate (PPT). The following table gives the fixed and variable costs of manufacturing each pair of fenders.

Variable and Fixed Costs of Pairs of Fenders				
Costs	Steel	Aluminum	RMP	NPN
Variable	$5.26	$12.67	$13.19	$9.53
Fixed	$260,000	$385,000	$95,000	$95,000

Write down the cost function associated with each of the materials.

32. 32. NEW DEPR. QUESTION

33. Cost, Revenue, and Profit in Rice Production.
Kekhora and McCann estimated a cost function for the rice production function in Thailand. They gave the fixed costs per hectare of $75 and the variable costs per hectare of $371. The revenue per hectare was given as $573.
a. Determine the total cost for one hectare.
b. Determine the profit for one hectare.

34. Cost, Revenue, and Profit in Shrimp Production.
Kekhora and McCann estimated a cost function for a shrimp production function in Thailand. They gave the fixed costs per hectare of $1838 and the variable costs per hectare of $14,183. The revenue per hectare was given as $26,022
a. Determine the total cost for one hectare.
b. Determine the profit for one hectare.

35. Cost, Revenue, and Profit Equations.
In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical large-sized plant they estimated fixed costs at $447,917 and estimated that it cost $209.03 to produce each ton of fertilizer. The plant sells its fertilizer output at $266.67 per ton. Find the cost,revenue, and profit equations.

36. Cost, Revenue, and Profit Equations.
In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical small-sized plant they estimated fixed costs at $235,487 and estimated that it cost $206.68 to produce each ton of fertilizer. The plant sells its fertilizer output at $266.67 per ton. Find the cost,revenue, and profit equations.

37. NEW DEMAND QUESTION

38. Profit Function.
Roberts formulated a mathematical model of corn yield response to nitrogen fertilizer in low-yield response land given by a equation $Y(N)$, where Y is bushels of corn per acre and N is pounds of nitrogen per acre. They estimated that the farmer obtains $2.42 for a bushel of corn and pays $0.22 a pound for nitrogen fertilizer. For this model they assume that the only cost to the farmer is the cost of nitrogen fertilizer.
a. We are given that $Y(20) = 24.36$ and $Y(120) = 51.96$. Let the revenue be given by $R(N)$. Then find $R(20)$ and $R(120)$. Determine a revenue function using this information and assuming a linear model.
b. Determine a cost function $C(N)$ assuming a linear model.
c. Now calculate a profit function $P(N)$.

39. NEW DEMAND QUESTION

40. NEW DEMAND QUESTION

41. Demand for Recreation.
Shafer and others estimated a demand curve for recreational power boating in a number of bodies of water in Pennsylvania. They estimated the price p of a power boat trip including rental cost of boat, cost of fuel, and rental cost of equipment. For the Three Rivers Area they collected data indicating that for a price (cost) of $99, individuals made 10 trips, and for a price of $43, individuals made 20 trips. Assuming a linear model determine the demand curve. For 15 trips, what was the cost?

42. Demand for Recreation.
Shafer and others estimated a demand curve for recreational power boating in a number of bodies of water in Pennsylvania. They estimated the price p of a power boat trip including rental cost of boat, cost of fuel, and rental cost of equipment. For the Lake Erie/Presque Isle Bay Area they collected data indicating that for a price (cost) of $144, individuals made 10 trips, and for a price of $50, individuals made 20 trips. Assuming a linear model determine the demand curve. For 15 trips, what was the cost?

43. Demand for Rice.
Suzuki and Kaiser estimated the demand equation for rice in Japan

to be $p = 1,195,789 - 0.1084753x$, where x is in tons of rice and p is in yen per ton. Graph this equation. In 1995, the quantity of rice consumed in Japan was 8,258,000 tons.
a. According to the demand equation, what was the price in yen per ton?
b. What happens to the price of a ton of rice when the demand increases by 1 ton. What has this number to do with the demand equation?

44.   **Fishery Demand** Grafton created a mathematical model for demand for the northern cod fishery. We can see from this model that when 100,000 kilograms of cod were caught the price was $0.81 per kilogram and when 200,000 kilograms of cod were caught the price was $0.63 per kilogram. Using this information and assuming a linear demand model, find a demand function.

45. **Supply.** Blau and Mocan gathered data over a number of states and estimated a supply curve that related quality of child care with price. For quality q of child care they developed an index of quality and for price p they used their own units. In their graph they gave $q = S(p)$, that is, the price was the independent variable. On this graph we see the following points: $(p, q) = (1, 2.6)$ and $(p, q) = (3, 5.5)$. Use this information and assuming a linear model, determine the supply curve.

46. **Supply.** Suppose that 8000 units of a certain item are sold per day by the entire industry at a price of $150 per item and that 10,000 units can be sold per day by the same industry at a price of $200 per item. Find the demand equation for p, assuming the demand curve to be a straight line.

47. **Straight-Line Depreciation.** Consider a new machine that costs $50,000 and has a useful life of nine years and a scrap value of $5000. Using straight-line depreciation, find the equation for the value V in terms of t, where t is in years. Find the value after one year and after five years.

48.   **Straight-Line Depreciation.**   A new building that costs $1,100,000 has a useful life of 50 years and a scrap value of $100,000. Using straight-line depreciation (refer to Exercise 27), find the equation for the value V in terms of t,

where t is in years. Find the value after one year, after two years, and after 40 years.

49. **Oil Production Technology.** D'Unger and coworkers studied the economics of conversion to saltwater injection for inactive wells in Texas. (By injecting saltwater into the wells, pressure is applied to the oil field, and oil and gas are forced out to be recovered.) The expense of a typical well conversion was estimated to be $31,750. The monthly revenue as a result of the conversion was estimated to be $2700. If x is the number of months the well operates after conversion, determine a revenue function as a function of x. How many months of operation would it take to recover the initial cost of conversion?

50. **Rail Freight.** In a report of the Federal Trade Commission (FTC) an example is given in which the Portland, Oregon, mill price of 50,000 board square feet of plywood is $3525 and the rail freight is $0.3056 per mile.
a. If a customer is located x rail miles from this mill, write an equation that gives the total freight f charged to this customer in terms of x for delivery of 50,000 board square feet of plywood.
b. Write a (linear) equation that gives the total c charged to a customer x rail miles from the mill for delivery of 50,000 board square feet of plywood. Graph this equation.
c. In the FTC report, a delivery of 50,000 board square feet of plywood from this mill is made to New Orleans, Louisiana, 2500 miles from the mill. What is the total charge?

51. Assume that the linear cost model applies and fixed costs are $1000. If the *total* cost of producing 800 items is $5000, find the cost equation.

52. Assume that the linear revenue model applies. If the *total* revenue from producing 1000 items is $8000, find the revenue equation.

53. Assume that the linear cost model applies. If the *total* cost of producing 1000 items at $3 each is $5000, find the cost equation.

54. Assume that the linear cost and revenue models applies. An item that costs $3 to make sells for $6. If profits of $5000 are made when

1000 items are made and sold, find the cost equation.

55. Assume that the linear cost and revenue models applies. An item costs $3 to make. If fixed costs are $1000 and profits are $7000 when 1000 items are made and sold, find the revenue equation.

56. Assume that the linear cost and revenue model applies. An item sells for $10. If fixed costs are $2000 and profits are $9000 when 1000 items are made and sold, find the cost equation.

57. When 50 silver beads are ordered they cost $1.25 each. If 100 silver beads are ordered, they cost $1.00 each. How much will each silver bead cost if 150 are ordered?

58. You find that when you order 75 magnets, the average cost per magnet is $0.90 and when you order 200 magnets, the average cost per magnet is $0.80. What is the cost equation for these custom magnets?

EXTENSIONS

59. **Revenue Equation.** Assuming a linear revenue model, explain in a complete sentence where you expect the y-intercept to be. Give a reason for your answer.

60. **Cost and Profit Equations.** Assuming a linear cost and revenue model, explain in complete sentences where you expect the y-intercepts to be for the cost and profit equations. Give reasons for your answers.

61. **Demand Curve.** In the figure we see a demand curve with a point (x_0, p_0) on it. We also see a rectangle with a corner on this point. What do you think the area of this rectangle represents?
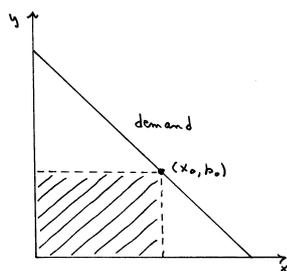
62. **Demand Curves.** Price and Connor studied the difference between demand curves between loyal customers and nonloyal customers in ready-to-eat cereal. The figure shows two such as demand curves. (Note that the independent variable is the quantity.) Discuss the differences and the possible reasons. For example, why do you think that the p-intercept for the loyal demand curve is higher than the other? Why do you think the loyal demand is above the other? What do you think the producers should do to make their customers more loyal?

63. **Cost of Irrigation Water.** Using an argument that is too complex to give here, Tolley and Hastings argued that if c is the cost in 1960 dollars per acre-foot of water in the area of Nebraska and x is the acre-feet of water available, then $c = 12$ when $x = 0$. They also noted that farms used about 2 acre-feet of water in the Ainsworth area when this water was free. If we assume (as they did) that the relationship between c and x is linear, then find the equation that c and x must satisfy.

64. **Kinked and Spiked Demand and Profit Curves.** Stiving determined demand curves. A figure is shown. Note that a manufacturer can decide to produce a durable good with a varying quality.

a. The figure shows a demand curve for which the quality of an item depends on the price. Explain if this demand curve seems reasonable.

b. Notice that the demand curve is kinked and spiked at prices at which the price ends in the digit, such at $39.99. Explain why you think this could happen.

Exer 1.1.47
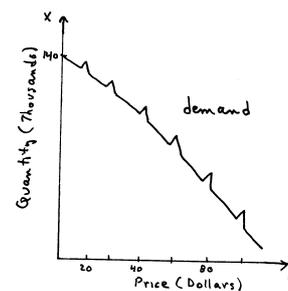
Exercise kinked

65.   Costs, Revenues, and Profits on Kansas Beef Cow Farms. Featherstone and coauthors studied 195 Kansas beef cow farms. The average fixed and variable costs are found in the following table.

Variable and Fixed Costs	
Costs per cow	
Feed costs	$261
Labor costs	$82
Utilities and fuel costs	$19
Veterinary expenses costs	$13
Miscellaneous costs	$18
Total variable costs	$393
Total fixed costs	$13,386

The farm can sell each cow for $470. Find the cost, revenue, and profit functions for an average farm. The average farm had 97 cows. What was the profit for 97 cows? Can you give a possible explanation for your answer?

66.   Profit Function. Roberts formulated a mathematical model of corn yield response to nitrogen fertilizer in high-yield response land given by a equation $Y(N)$, where Y is bushels of corn per acre and N is pounds of nitrogen per acre. They estimated that the farmer obtains $2.42 for a bushel of corn and pays $0.22 a pound for nitrogen fertilizer. For this model they assume that the only cost to the farmer is the cost of nitrogen fertilizer.

a.   We are given that $Y(20) = 47.8$ and $Y(120) = 125.8$. Let the revenue be given by $R(N)$. Then find $R(20)$ and $R(120)$. Determine a revenue function using this information and assuming a linear model.

b. Determine a cost function $C(N)$ assuming a linear model.

c. Now calculate a profit function $P(N)$.

67. Cost, Revenue, and Profit Equations in the Cereal Manufacturing Industry Cotterill estimated the the costs and prices in the cereal-manufacturing industry. The table summarizes the costs in both pounds and tons in the manufacture of a typical cereal

Item	$/lb	$/ton
Manufacturing cost:		
Grain	0.16	320
Other ingredients	0.20	400
Packaging	0.28	560
Labor	0.15	300
Plant costs	0.23	460
Total manufacturing costs	1.02	2040
Marketing expenses:		
Advertising	0.31	620
Consumer promo (mfr. coupons)	0.35	700
Trade promo (retail in-store)	0.24	480
Total marketing costs	0.90	1800
Total variable costs	1.92	3840
Table		

The manufacturer obtained a price of $2.40 a pound, or $4800 a ton. Let x be the number of tons of cereal manufactured and sold and let p be the price of a ton sold. Nero estimated fixed costs for a typical plant to be $300 million. Let the cost, revenue, and profits be given in thousands of dollars. Find the cost, revenue and profit equations. Also make a table of values for cost, revenue, and profit for production levels of 200,000, 300,000 and 400,000 tons and discuss what is the table of numbers telling you.

1.1 Solutions to Self-Help Exercises

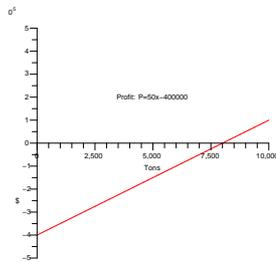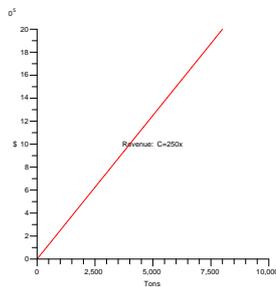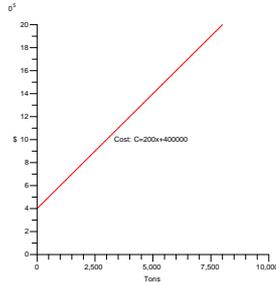1.   Let x be the number of tons of fertilizer produced and sold.

a.   Then the cost, revenue, and profit equations are

$$
\begin{aligned}
C(x) &= (\text{variable cost}) + (\text{fixed cost}) \\
&= mx + b \\
&= 200x + 400,000 \\
R(x) &= (\text{price per ton}) \times (\text{number tons sold}) \\
&= px \\
&= 250x \\
P(x) &= (\text{revenue}) - (\text{cost}) = R - C \\
&= (250x) - (200x + 400,000) \\
&= 50x - 400,000
\end{aligned}
$$

The cost, revenue, and profit equations are graphed in the figures below.

b.   If $x = 5000$, then

$$C(5000) \;=\; 200(5000) + 400,000$$
$$R(5000) \;=\; 250(5000) = 1,250,00$$
$$P(5000) \;=\; 1,250,000 - 1,400,00$$

Thus, if 5000 tons are produced an cost is \$1,400,000, the revenue is and there is a *loss* of \$150,000.

   Doing the same for some other we have the results shown in the fo ble.

Number Made and Sold	5000	
Cost	1,400,00	1
Revenue	1,250,000	1
Profit (or loss)	-150,000	

2. The graphs are shown in the fig $x = 70$, we have
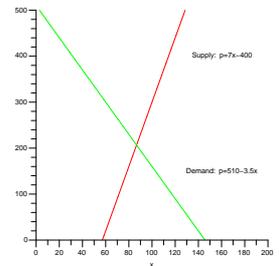   supply: $p = 7(70) - 400 = 49$
   demand: $p = 510 - 3.5(70) = 26$
Demand is larger.
   When $x = 100$, we have
   supply: $p = 7(100) - 400 = 300$
   demand: $p = 510 - 3.5(100) = 1$
Supply is larger.

1.2 Systems of Linear Equations

- Two Linear Equations in Two Unknowns

- Decision Analysis

- Supply and Demand Equilibrium

- Decision Analysis Complications [Optional]

- Technology Corner

Adam Smith, 1723-1790

Adam Smith was a Scottish political economist. His *Inquiry into the Nature and Causes of the Wealth of Nations* was one of the earliest attempts to study the development of industry and commerce in Europe. That work helped to create the modern academic discipline of economics. In the Western world, it is arguably the most influential book on the subject ever published.

One of the main points of *The Wealth of Nations* is that the free market, while appearing chaotic and unrestrained, is actually guided to produce the right amount and variety of goods. If a product shortage occurs, for instance, its price rises, creating a profit margin that creates an incentive for others to enter production, eventually cutting the shortage. If too many producers enter the market, the increased competition among manufactures and increased supply would lower the price of the product toward to its production cost. Smith believed that while human motives are often selfish and greedy, the competition in the free market would tend to benefit society as a whole by keeping prices low, while still building in an incentive for a wide variety of goods and services.

Application: Cost, Revenue, and Profit Models

In Example 3 in the last section we found the cost and revenue equations in the dress manufacturing industry. Let x be the number of dresses made and sold. Recall the cost and revenue functions were found to be $C(x) = 25x + 80,000$ and $R(x) = 75x$. Find the point at which the profit is zero. See Example 2 for the answer.

We now begin to look at systems of linear equations in many unknowns. In this section we first consider systems of two linear equations in two unknowns. We will see that solutions of such a system have a variety of applications.

1.2.1   Two Linear Equations in Two Unknowns

In this section we will encounter applications that have a unique solution to a system of two linear equations in two unknowns. For example, consider two lines,

$$L_1 : y = m_1 x + b_1$$
$$L_2 : y = m_2 x + b_2$$

If these two linear equations are not parallel ($m_1 \neq m_2$), then the lines must intersect at a unique point, say (x_0, y_0). See Figure 1.2.1.
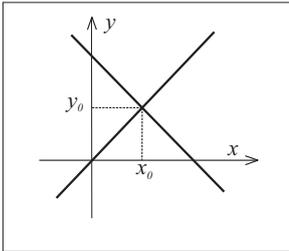
This means that (x_0, y_0) is a *solution* to the two linear equations and must satisfy both of the equations

$$y_0 = m_1 x_0 + b_1$$
$$y_0 = m_2 x_0 + b_2$$

Figure 1.2.1

Example 1   **Intersection of Two Lines** Find the solution (intersection) of the two lines.

$$L_1 : y = 7x - 3$$
$$L_2 : y = -4x + 9$$

Solution: To find the solution, set the two lines equal to each other, $L_1 = L_2$,

$$
\begin{aligned}
y_0 &= y_0 \\
7x_0 - 3 &= 4x_0 + 9 \\
11x_0 &= 12 \\
x_0 &= \frac{12}{11}
\end{aligned}
$$

To find the value of y_0, substitute the x_0 value into either equation,

$$
\begin{aligned}
y_0 &= 7\left(\frac{12}{11}\right) - 3 = \frac{51}{11} \\
y_0 &= 4\left(\frac{12}{11}\right) + 9 = \frac{51}{11}
\end{aligned}
$$

So, the solution to this system is the intersection point, $\left(\frac{12}{11}, \frac{51}{11}\right)$.
$\sqrt{}$

Ⓣ **Technology Option.** You may wish to see Technology Note 1 for the solution to Example 1 using a graphing calculator.

1.2.2   Decision Analysis

In the last section we considered linear mathematical models of cost, revenue, and profit for a firm. In Figure 1.2.2 we see the graphs of two typical cost and revenue functions. We can see in this figure that for smaller values of x, the cost line is *above* the revenue line and therefore the profit P is *negative*. Thus the firm has losses. As x becomes larger, the revenue line becomes *above* the cost line and therefore the profit becomes *positive*. The value of x at which the profit is zero is called the **break-even quantity**. Geometrically, this is the point of intersection of the cost line and the revenue line. Mathematically, this requires us to solve the equations $y = C(x)$ and $y = R(x)$ simultaneously.

Example 2   **Finding the Break-Even Quantity** In Example 3 in the last section we found the cost and revenue equations in a dressing-manufacturing firm. Let x be the number of dresses manufactured and sold and let the cost and revenue be given in dollars. Then recall that the cost and revenue equations were found to be $C(x) = 25x + 80,000$ and $R(x) = 75x$. Find the break-even quantity.
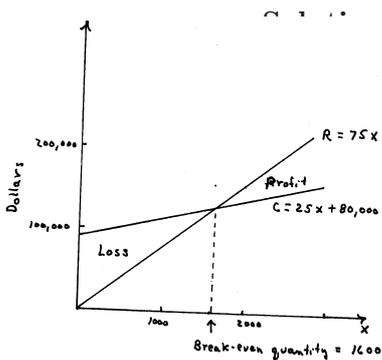
Solution : To find the break-even quantity, we need to solve the equations $y = C(x)$ and $y = R(x)$ simultaneously. To do this we set $R(x) = C(x)$. Doing this we have

$$
\begin{aligned}
R(x) &= C(x) \\
75x &= 25x + 80,000 \\
50x &= 80,000 \\
x &= 1600
\end{aligned}
$$

Figure 1.2.2

Thus, the firm needs to produce and sell 1600 dresses to break even (i.e., for profits to be zero). See Figure 1.2.2. $\checkmark$

$\textcircled{T}$ **Technology Option.** You may wish to see Technology Note 2 to see this example using a graphing calculator.

REMARK: Notice that $R(1600) = 120,000 = C(1600)$ so it costs the company \$120,000 to make the dresses and they bring in \$120,000 in revenue when the dresses are all sold.

In the following example we consider the total energy consumed by automobile fenders using two different materials. We need to decide how many miles carrying the fenders result in the same energy consumption and which type of fender will consume the least amount of energy for large numbers of miles.

Example 3   **Break-Even Analysis.**   Saur and colleagues did a careful study of the amount of energy consumed by each type of automobile fender using various materials. The total energy was the sum of the energy needed for production plus the energy consumed by the vehicle used in carrying the fenders. If x is the miles traveled, then the total energy consumption equations for steel and rubber-modified polypropylene (RMP) were as follows:

Steel: $E = 225 + 0.012x$

RPM: $E = 285 + 0.007x$

Graph these equations, and find the number of miles for which the total energy consumed is the same for both fenders. Which material uses the least energy for 15,000 miles?
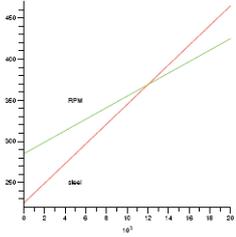

Figure 1.2.3

ion: The total energy using steel is $E_1(x) = 225 + 0.012x$ and for RMP is $E_2(x) = 285 + 0.007x$. The graphs of these two linear energy functions are shown in Figure 1.2.3. We note that the graphs intersect.

To find this intersection we set $E_1(x) = E_2(x)$ and obtain

$$
\begin{aligned}
E_1(x) &= E_2(x) \\
225 + 0.012x &= 285 + 0.007x \\
0.005x &= 60 \\
x &= 12,000
\end{aligned}
$$

So, 12,000 miles results in the total energy used by either material being the same.

Setting $x = 0$ gives the energy used in production, and we note that steel uses less energy to produce these fenders than does RPM. However, since steel is heavier than RPM, we suspect that carrying steel fenders might require more total energy when the number of pair of fenders is large. Indeed, we see in Figure 1.2.3 that the graph corresponding to steel is above that of RPM when $x > 12,000$. Checking this for $x = 15,000$, we have

$$
\begin{aligned}
\text{steel}: \quad E_1(x) &= 225 + 0.012x \\
E_1(15,000) &= 225 + 0.012(15,000) \\
&= 405 \\
\text{RPM}: \quad E_2(x) &= 285 + 0.007x \\
E_2(15,000) &= 285 + 0.007(15,000) \\
&= 390
\end{aligned}
$$

So for traveling 15,000 miles, the total energy used by RPM is less than that for steel. $\checkmark$
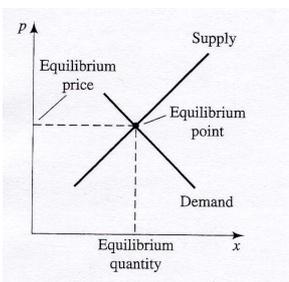
1.2.3   Supply and Demand Equilibrium

The best-known law of economics is the law of supply and demand. Figure 1.2.4 shows a demand equation and a supply equation that intersect. The point of intersection, or the point at which supply equals demand, is called the **equilibrium point**. The x-coordinate of the equilibrium point is called the **equilibrium quantity**, x_0, and the p-coordinate is called the **equilibrium price**, p_0. In other words, at a price p_0, the consumer is willing to buy x_0 items and the producer is willing to supply x_0 items.

Figure 1.2.4

Example 4 **Finding the Equilibrium Point** Tauer determined demand and supply curves for milk in this country. If x is billions of pounds of milk and p is in dollars per hundred pounds, he found that the demand function for milk was $p = D(x) = 56 - 0.3x$ and the supply function was $p = S(x) = 0.1x$. Graph the demand and supply equations. Find the equilibrium point.
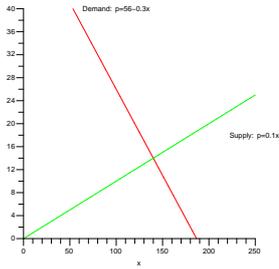

Figure 1.2.5

ion: The demand equation $p = D(x) = 56 - 0.3x$ is a line with negative slope -0.3 and y-intercept 56 and is graphed in Figure 1.2.4. The supply equation $p = S(x) = 0.1x$ is a line with positive slope 0.1 with y-intercept 0. This is also graphed in Figure 1.2.5.

To find the point of intersection of the demand curve and the supply curve, set $S(x) = D(x)$ and solve:

$$
\begin{aligned}
S(x) &= D(x) \\
0.1x &= 56 - 0.3x \\
0.4x &= 56 \\
x &= 140
\end{aligned}
$$

Then since $p(x) = 0.1x$,

$$p(140) = 0.1(140) = 14$$

We then see that the equilibrium point is $(x, p) = (140, 14)$. That is, 140 billions pounds of milk at \$14 per hundred pounds of milk. ✓
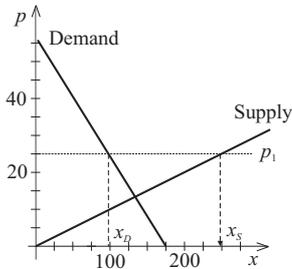

Figure 1.2.6a

Example 5 **Supply and Demand** Refer to Example 4. What will consumers and suppliers do if the price is $p_1 = 25$ shown in Figure 1.2.6a? What if the price is $p_2 = 5$ as shown in Figure 1.2.6b?
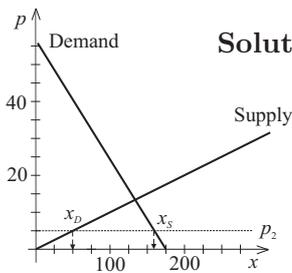

Figure 1.2.6b

Solution: If the price is at $p_1 = 25$ shown in Figure 1.2.6a, then let the supply of milk be denoted by x_S. Let us find x_S.

$$
\begin{aligned}
p &= S(x_S) \\
25 &= 0.1x_S \\
x_S &= 250
\end{aligned}
$$

That is, 250 billion pounds of milk will be supplied. Keeping the same price of $p_1 = 25$ shown in Figure 1.2.6a, then let the demand of milk be denoted by x_D. Let us find x_D. Then

$$
\begin{aligned}
p &= D(x_D) \\
25 &= 56 - 0.3x_D \\
x_D &\approx 103
\end{aligned}
$$

So only 103 billions of pounds of milk are demanded by consumers. There will be a surplus of $250 - 103 = 147$ billions of pounds of milk. To work off the surplus, the price should fall toward the equilibrium price of $p_0 = 14$.

If the price is at $p_2 = 5$ shown in Figure 1.2.6b, then let the supply of milk be denoted by x_S. Let us find x_S. Then

$$
\begin{aligned}
p &= S(x_S) \\
5 &= 0.1x_S \\
x_S &= 50
\end{aligned}
$$

That is, 50 billion pounds of milk will be supplied. Keeping the same price of $p_2 = 5$ shown in Figure 1.2.6b, then let the demand of milk be denoted by x_D. Let us find x_D. Then

$$
\begin{aligned}
p &= D(x_D) \\
5 &= 56 - 0.3x_D \\
x_D &\approx 170
\end{aligned}
$$

So 170 billions of pounds of milk are demanded by consumers. There will be a shortage of $170 - 50 = 220$ billions of pounds of milk, and the price should rise toward the equilibrium price.

CONNECTION[*]
Demand for Steel Outpaces Supply In early 2002 President George W. Bush imposed steep tariffs on imported steel to protect domestic steel producers. As a result millions of tons of imported steel were locked out of the country. Domestic steelmakers announced on March 27, 2002, that they had been forced to ration steel to their customers and boost prices because demand has outpaced supply.

[*] *The Wall Street Journal March 28, 2002.*

1.2.4   Enrichment: Decision Analysis Complications

In the following example we look at the cost of manufacturing automobile fenders using two different materials. We determine the number of pairs of fenders that will be produced by using the same cost. However, we must keep in mind that we do not produce *fractional* numbers of fenders, but rather only *whole* numbers. For example, we can produce one or two pairs of fenders, but not 1.43 pairs.

Example 6   **Decision Analysis for Manufacturing Fenders.**   Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using two different materials: steel and a rubber-modified polypropylene blend (RMP). The following table gives the fixed and variable costs of manufacturing each pair of fenders.

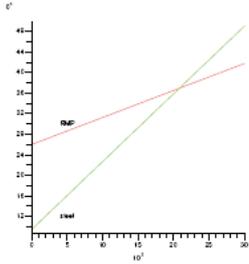| Variable and Fixed Costs of Pairs of Fenders		
Costs	Steel	RMP
Variable	$5.26	$13.19
Fixed	$260,000	$95,000

Figure 1.2.7

Graph the cost function for each material. Find the number of fenders for which the cost of each materials is the same. Which material will result in the lowest cost if a large number of fenders are manufactured?

ion: The cost function for steel is $C_1(x) = 5.26x + 260,000$ and for RMP is $C_2(x) = 13.19x + 95,000$. The graphs of these two cost functions are shown in Figure 1.2.7.   For a small number of fenders, we see from the graph that the cost for steel is greater than that for RMP. However, for a large number of fenders the cost for steel is less. To find the number of pairs that yield the same cost for each material, we need to solve $C_2(x) = C_1(x)$.

$$
\begin{aligned}
C_2(x) &= C_1(x) \\
13.19x + 95,000 &= 5.26x + 260,000 \\
7.93x &= 165,000 \\
x &= 20,807.062
\end{aligned}
$$

This is a real application, so only an *integer* number of fenders can be manufactured. We need to round off the answer given above and obtain 20,807 pair of fenders. $\checkmark$

Remark. Note that $C_2(20,807) = 369,444.44$ and $C_1(20,807) = 369,444.82$. The two values are not *exactly* equal.

⌈ **Technology Corner** ⌉

Technology Note 1 - Finding the Intersection Graphically

Begin by entering the two lines as Y_1 and Y_2. Choose a window where the intersection point is visible. The screens below used Xmin=0, Xmax=4, Ymin=-5, and Ymax=20. To find the exact value of the intersection, go to CALC (via 2nd   TRACE) and choose 5:intersect.   You will be prompted to select the lines.   Press ENTER for "First curve?", "Second curve?", and "Guess?". The intersection point will be displayed as in Screen 1.2.1.   To avoid rounding errors, the intersection point must be converted to a fraction. To do this, QUIT to the home screen using 2ND   and MODE . Then press X,T,θ,n , then the MATH   button, as shown in Screen 1.2.2. Choose 1:▷Frac and then ENTER   to convert the x-value of the intersection to a fraction, see Screen 1.2.3. To convert the y-value to a fraction, press ALPHA   then 1   to get the variable Y. Next the MATH   and 1:▷Frac to see Y as a fraction.
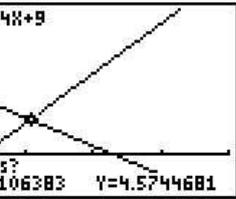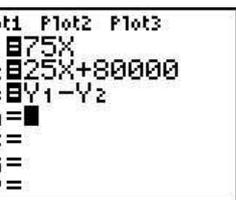
Screen 1.2.1

Screen 1.2.2
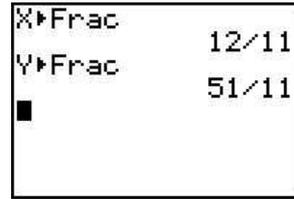
Screen 1.2.3

**Technology Note 2 - Finding the Break-Even Quantity Graphically**

You can find the break-even quantity in Example 2 on your graphing calculator by finding where $P = 0$ or by finding where $C = R$. Begin by entering the revenue and cost equations into Y_1 and Y_2. You can subtract these two on paper to find the profit equation or have the calculator find the difference, as shown in Screen 1.2.4. To access the names Y_1 and Y_2, press the $\boxed{\text{VARS}}$, then right arrow to Y-VARS and $\boxed{\text{ENTER}}$ to select 1:Function then choose Y_1 or Y_2, as needed.

Pick an appropriate window to view the intersection of revenue and cost equations. Screen 1.2.5 used Xmin=0, Xmax=3000, Ymin=-100000, and Ymax=250000. The intersection can be found in the same manner as Technology Note 1.

To find where the profit is zero, return to the CALC menu and choose 2:zero. Note you will initially be on the line Y_1. Use the down arrow twice to be on Y_3. Then use the left or right arrows to move to the left side of the zero of Y_3 and hit $\boxed{\text{ENTER}}$ to answer the question "Left Bound"'". Right arrow over to the right side of the place where Y_3 crosses the x-axis and hit $\boxed{\text{ENTER}}$ to answer the question "Right Bound"'". Place your cursor between these two spots and press $\boxed{\text{ENTER}}$ to answer the last question, "Guess?". The result is shown in Screen 1.2.6.

Screen 1.2.4
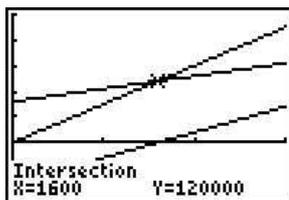
Screen 1.2.5
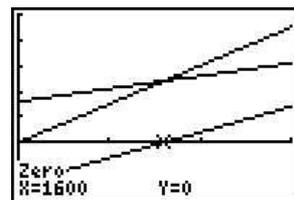
Screen 1.2.6

1.2 SELF HELP EXERCISES

1. Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical medium-sized plant they estimated fixed costs at $400,000 and estimated that it cost $200 to produce each ton of fertilizer. The plant sells its fertilizer output at $250 per ton. Find the break-even point. (Refer to Self-Help Exercise 1 in Section 1.1.)

2. The excess supply and demand curves for wheat worldwide were estimated by Schmitz and coworkers to be

   Supply: $p = S(x) = 7x - 400$
   Demand: $p = D(x) = 510 - 3.5x$

where p is price per metric ton and x is in millions of metric tons. Excess demand refers to the excess of wheat that producer countries have over their own consumption. Graph and find the equilibrium price and equilibrium quantity. (Note that the independent variable is the price p.)

EXERCISE SET 1.2

Exercises 1 through 4 show linear cost and revenue equations. Find the break-even quantity.

1. $C = 2x + 4$, $R = 4x$ 2. $C = 3x + 10$, $R = 6x$ 3. $C = 0.1x + 2$, $R = 0.2x$ 4. $C = 0.03x + 1$, $R = 0.04x$

In Exercises 5 through 8 you are given a demand equation and a supply equation. Sketch the demand and supply curves, and find the equilibrium point.

5. Demand: $p = -x + 6$, supply: $p = x + 3$ 6. Demand: $p = -3x + 12$, supply: $p = 2x + 5$ 7. Demand: $p = -10x + 25$, supply: $p = 5x + 10$ 8. Demand: $p = -0.1x + 2$, supply: $p = 0.2x + 1$

APPLICATIONS

9. **Break-Even Quantity.** A firm has weekly fixed costs of $40,000 associated with the manufacture of purses that cost $15 per purse to produce. The firm sells all the purses it produces at $35 per purse. Find the cost, revenue, and profit equations. Find the break-even quantity.

10. **Break-Even Quantity.** A firm has fixed costs of $1,000,000 associated with the manufacture of lawn mowers that cost $200 per mower to produce. The firm sells all the mowers it produces at $300 each. Find the cost, revenue, and profit equations. Find the break-even quantity.

11. **Break-even Quantity in Rice Production.** Kekhora and McCann estimated a cost function for the rice production function in Thailand. They gave the fixed costs per hectare of $75 and the variable costs per hectare of $371. The revenue per hectare was given as $573. Suppose the price for rice went down. What would be the minimum price to charge per hectacre to determine the break-even quantity.

12. **Break-even Quantity in Shrimp Production.** Kekhora and McCann estimated a cost function for a shrimp production function in Thailand. They gave the fixed costs per hectare of $1838 and the variable costs per hectare of $14,183. The revenue per hectare was given as $26,022. Suppose the price for shrimp went down. What would be the revenue to determine the break-even quantity.

13. **Break-even Quantity.** In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical small-sized plant they estimated fixed costs at $235,487 and estimated that it cost $206.68 to produce each ton of fertilizer. The plant sells its fertilizer output at $266.67 per ton. Find the break-even quantity.

14. **Break-even Quantity.** In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical large-sized plant they estimated fixed costs at $447,917 and estimated that it cost $209.03 to produce each ton of fertilizer. The plant sells its fertilizer output at $266.67 per ton. Find the break-even quantity.

15. **Break-even Quantity on Kansas Beef Cow Farms.** Featherstone and coauthors studied 195 Kansas beef cow farms. The average fixed and variable costs are found in the following table.

Variable and Fixed Costs	
Costs per cow	
Feed costs	$261
Labor costs	$82
Utilities and fuel costs	$19
Veterinary expenses costs	$13
Miscellaneous costs	$18
Total variable costs	$393
Total fixed costs	$13,386

The farm can sell each cow for $470. Find the break-even quantity.

16. Decision Analysis. At Lincoln Library there are two ways to pay for copying. You can pay 5 cents a copy, or you can buy a plastic card for $5 and then pay 3 cents a copy. Let x be the number of copies you make. Write an equation for your costs for each way of paying.
(a) How many copies do you need to make for buying the plastic card is the same as cash?
(b) If you wish to make 300 copies, which way of paying has the least cost.

17. Rent or Buy Decision Analysis. A forester has the need to cut many trees and to chip the branches. On the one hand he could, when needed, rent a large wood chipper to chip branches and logs up to 12 inches in diameter for $320 a day. He estimates that his crew would use the chipper exactly 8 hours each day of rental use. Since he has a large amount of work to do, he is considering purchasing a new 12-inch wood chipper for $28,000. He estimates that he will need to spend $40 on maintenance per every 8 hours of use.
(a) Let x be the number of hours he will use a wood chipper. Write a formula that gives him the total cost of renting for x hours.
(b) Write a formula that gives him the total cost of buying and maintaining the wood chipper for x days of use.
(c) If the forester estimates he will need to use the chipper for 1000 hours, should he buy or rent?
(d) Determine the number of hours of use before the forester can save as much money by buying the chipper as oppose to renting.

18. Compensation Decision Analysis. A salesman for carpets has been offered two possible compensation plans. The first offers him an monthly salary of $2000 plus a royalty of 10% of the total dollar amount of sales he makes. The second offers him an monthly salary of $1000 plus a royalty of 20% of the total dollar amount of sales he makes.
(a) Write a formula that gives each compensation packages as a function of the dollar amount x of sales he makes.
(b) Suppose he believes he can sell $15,000 of carpeting each month. which compensation package should he choose?
(c) How much carpeting will he sale each month if he earns the same amount of money with either compensation package.

19.   Energy Decision Analysis. Many homes and businesses in northern Ohio can successfully drill for natural gas on their property. They are faced with the choice of obtaining natural gas free from their own gas well or from buying the gas from a utility company. A garden center determines that they will need to buy $4000 worth of gas each year from the local utility company to heat their greenhouses. They determine that the cost of drilling a small commercial gas well for the garden center will be $40,000 and they assume that their well will need $1000 of maintenance each year.
(a) Write a formula that gives the cost of the natural gas bought from the utility for x years.
(b) Write a formula that gives the cost of obtaining the natural gas from their well over x years.
(c) How many years will it take for the garden center to have the same cost of gas from the utility or from their own well?

> Connection: We know an individual living in a private home in northern Ohio who has a gas and oil well drilled some years ago. The well yields both natural gas and oil. Both products go into a splitter that separates the natural gas and the oil. The oil goes into a large tank and is sold to a local utility. The natural gas is used to heat the home and the excess is fed into the utility company pipes, where it is measured and purchased by the utility.

20. Rental Decision Analysis. A contractor wants to rent a wood chipper from Acme Rental for a day for $150 plus $10 per hour or from Bell Rental for a day for $165 plus $7 per

hour. Find a cost function for using each rental firm.

(a) Find the number of hours for which each cost function will give the same cost.

(b) If the contractor wants to rent the chipper for 8 hours, which rental place will cost less?

21. **Decision Analysis.** A builder needs to rent a dump truck from Acme Rental for a day for $75 plus $0.40 per mile and the same one from Bell Rental for $105 plus $0.25 per mile. Find a cost function for using each rental firm.

(a) Find the number of hours for which each cost function will give the same cost.

(b) If the builder wants to rent a dump truck for 150 days, which rental place will cost less?

22. **Decision Analysis.** A shirt manufacturer is considering purchasing a standard sewing machine for $91,000 and for which will cost $2 to sew each of their standard shirts. They are also considering purchasing a more efficient sewing machine for $100,000 and for which will cost $1.25 to sew each of their standard shirts. Find a cost function for purchasing and using each machine.

(a) Find the number of hours for which each cost function will give the same cost.

(b) If the manufacturer wishes to sew 10,000 shirts, which machine should they purchase?

Decision Analysis. In Exercises 23 and 24 use the following information. In the Saur study of fenders mentioned in Example 2, the amount of energy consumed by each type of fender was also analyzed. The total energy was the sum of the energy needed for production plus the energy consumed by the vehicle used in carrying the fenders. If x is the miles traveled, then the total energy consumption equations for steel, aluminum, and NPN were as follows:

　　Steel: $E = 225 + 0.012x$
　　Al: $E = 550 + 0.007x$
　　NPN: $E = 565 + 0.007x$

23. Find the number of miles traveled for which the total energy consumed is the same for steel and aluminum fenders. If 5000 miles is traveled, which material would use the least energy?

24. Find the number of miles traveled for which the total energy consumed is the same for steel and NPN fenders. If 6000 miles is traveled, which material would use the least energy?

Decision Analysis. For Exercises 25 and 26 refer to the following information. In the Saur study of fenders mentioned in Example 2, the amount of CO_2 emissions in kg per 2 fenders of the production and utilization into the air of each type of fender was also analyzed. The total CO_2 emissions was the sum of production plus the use phase of the vehicle used in carrying the fenders. If x is the miles traveled, then the total CO_2 emission equations for steel, aluminum, and NPN were as follows:

　　Steel: $CO = 21 + 0.00085x$
　　Aluminum: $CO = 43 + 0.00045x$
　　NPN: $CO = 23 + 0.00080x$

25. Find the number of pairs of fenders for which the total CO_2 emissions is the same for both steel and NPN fenders. If 30,000 miles are traveled, which material would yield the least CO_2?

26. Find the number of pairs of fenders for which the total CO_2 emissions is the same for both steel and aluminum fenders. If 60,000 miles are traveled, which material would yield the least CO_2?

27. **Make or Buy Decision.** A company includes a manual with each piece of software it sells and is trying to decide whether to contract with an outside supplier to produce or to produce in house. The lowest bid of any outside supplier is $0.75 per manual. The company estimates that producing the manuals in-house will require fixed costs of $10,000 and variable costs of $0.50 per manual. Find the number of manuals resulting in same cost for contracting with the outside supplier or to produce in house. If 50,000 manuals are needed, should the company go with outside supplier or go in-house?

28. EQUIL QUESTION HERE

29. EQUIL QUESTION HERE

30. **Supply and Demand.** Demand and supply equations for milk were given by Tauer In

this paper he estimated demand and supply equations for bovine somatotropin-produced milk. The demand equation is $p = 55.9867 - 0.3249x$, and the supply equation is $p = 0.07958$, where again p is the price in dollars per hundred pounds and x is the amount of milk measured in billions of pounds. Find the equilibrium point.

31. **Facility Location.** A company is trying to decide whether to locate a new plant in Houston or Boston. Information on the two possible locations is given in the following table. The initial investment is in land, buildings, and equipment.

	Houston	Boston
Variable cost	$.25 per item	$.22 per item
Annual fixed costs	$4,000,000	$4,210,000
Initial investment	$16,000,000	$20,000,000

(a) Suppose 10,000,000 items are produced each year. Find which city has the lower annual total costs, not counting the initial investment.

(b) Find the number of items yield the same cost for each city.

32. **Facility Location.** Use the information found in the previous exercise.

(a) Determine which city has the lower total cost over five years, counting the initial investment if 10,000,000 items are produced each year?

(b) Find the number of items yield the same cost for each city counting the initial investment.

EXTENSIONS

Decisions Analysis. For Exercises 33 through 36 consider the following study. As mentioned in Example 2 Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using five different materials: steel, aluminum, and three injection-molded polymer blends: rubber-modified polypropylene (RMP), nylon-polyphenyle-neoxide (NPN), and polycarbonate-polybutylene terephthalate (PPT). The following table gives the fixed and variable costs of manufacturing each pair of fenders. Keep in mind that only an *integer* number of pair of fenders can be counted. Your answer must reflect this.

Variable and Fixed Costs of Pairs of Fenders					
Costs	Steel	Aluminum	RMP	NPN	PPT
Variable	$5.26	$12.67	$13.19	$9.53	$12.55
Fixed	$260,000	$385,000	$95,000	$95,000	$95,000

33. How many pairs of fenders are required for the cost of the aluminum ones to equal the cost of the RMP ones?

34. How many pairs of fenders are required for the cost of the steel ones to equal the cost of the NPN ones?

35. How many pairs of fenders are required for the cost of the steel ones to equal the cost of the PPT ones?

36. How many pairs of fenders are required for the cost of the aluminum ones to equal the cost of the RMP ones?

37. **Process Selection and Capacity.** A machine shop needs to drill holes in a certain plate. An inexpensive manual drill press could be purchased that will require large labor costs to operate, or an expensive automatic press can be purchased that will require small labor costs to operate. The following table summarizes the options.

Machine	Annual Fixed Costs	Variable Labor Costs	Production Rate
Manual	$1000	$16.00/hour	10 plates/hour
Automatic	$8000	$2.00/hour	100 plates/hour

Suppose these are the only fixed and variable costs.

(a) If x is the number of plates produced per hour, find the total cost using the manual drill press per hour and the cost function using the automatic drill press.

(b) Find the number of plates produced per hour for which the manual and automatic drill presses will cost the same.

38. **Decision Analysis.** Roberts formulated a mathematical model of corn yield response to nitrogen fertilizer in high-yield response land

and low-yield response land. They estimated a profit equation $P = f(N)$ that depended only on the amount of pounds of nitrogen fertilizer per acre used. For the high-yield response land they estimated that $P = H(N) = 0.17N + 96.6$ and for the low-yield response land they estimated that $P = L(N) = 0.48N + 26.0$. A farmer has both types of land in tw[...] fields but does not have the time t[...] fields. How much nitrogen will res[...] response land to yield the same pr[...] field should be selected if 250 poun[...] gen is used?

1.2 Solutions to Self-Help Exercises

1. Let x be the number of tons of fertilizer produced and sold. Then the cost and revenue equations are

(a) $C(x) = \text{(variable cost)} + \text{(fixed cost)}$
$$= mx + b$$
$$= 200x + 400,000$$

(b) $R(x) = \text{(price per ton)} \times \text{(number tons sold)}$
$$= px$$
$$= 250x$$

The cost and revenue equations are graphed in the figure.

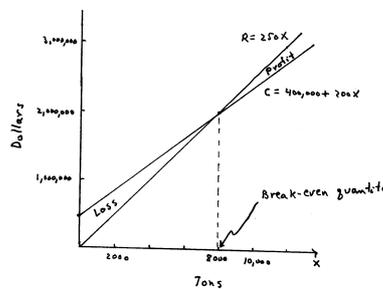To find the break-even quantity set $C(x) = R(x)$ and solve for x:

$$
\begin{aligned}
R(x) &= C(x) \\
250x &= 200x + 400,000 \\
50x &= 400,000 \\
x &= 8000
\end{aligned}
$$

rounded to the nearest ton. Thus, the plant needs to produce and sell 8000 tons of fertilizer to break-even (i.e., for profits to be zero).

2. The graphs are shown in the figu[...] the equilibrium price, set $D(p) = S$[...] tain

$$
\begin{aligned}
D(p) &= S(p) \\
510 - 3.5x &= 7x - 400 \\
10.5x &= 910 \\
x &\approx 86.7
\end{aligned}
$$

With $x = 86.7$, $p = D(86.7)$[...] $3.5(86.7) \approx 207$. The equilibrium p[...] per metric ton, and the equilibrium [...] 86.7 million metric tons.