Composition Functions

Composition functions are functions that combine to make a new function. We use the notation $\circ$ to denote a composition.

$f \circ g$ is the composition function that has f composed with g. Be aware though, $f \circ g$ is not the same as $g \circ f$. (This means that composition is not commutative).

$f \circ g \circ h$ is the composition that composes f with g with h.
Since when we combine functions in composition to make a new function, sometimes we define a function to be the composition of two smaller function. For instance,

$$h = f \circ g \tag{1}$$

h is the function that is made from f composed with g.

For regular functions such as, say:

$$f(x) = 3x^2 + 2x + 1 \tag{2}$$

What do we end up doing with this function? All we do is plug in various values of x into the function because that's what the function accepts as inputs. So we would have different outputs for each input:

$$f(-2) = 3(-2)^2 + 2(-2) + 1 = 12 - 4 + 1 = 9 \tag{3}$$

$$f(0) = 3(0)^2 + 2(0) + 1 = 1 \tag{4}$$

$$f(2) = 3(2)^2 + 2(2) + 1 = 12 + 4 + 1 = 17 \tag{5}$$

When composing functions we do the same thing but instead of plugging in numbers we are plugging in whole functions. For example let's look at the following problems below:

Examples

- Find $(f \circ g)(x)$ for f and g below.

$$f(x) = 3x + 4 \tag{6}$$

$$g(x) = x^2 + \frac{1}{x} \tag{7}$$

When composing functions we always read from right to left. So, first, we will plug x into g (which is already done) and then g into f. What this means, is that wherever we see an x in f we will plug in g. That is, g acts as our new variable and we have $f(g(x))$.

$$g(x) = x^2 + \frac{1}{x} \tag{8}$$
$$f(x) = 3x + 4 \tag{9}$$
$$f(\quad) = 3(\quad) + 4 \tag{10}$$
$$f(g(x)) = 3(g(x)) + 4 \tag{11}$$
$$f(x^2 + \frac{1}{x}) = 3(x^2 + \frac{1}{x}) + 4 \tag{12}$$
$$f(x^2 + \frac{1}{x}) = 3x^2 + \frac{3}{x} + 4 \tag{13}$$

Thus, $(f \circ g)(x) = f(g(x)) = 3x^2 + \frac{3}{x} + 4$.

Let's try one more composition but this time with 3 functions. It'll be exactly the same but with one extra step.

- Find $(f \circ g \circ h)(x)$ given f, g, and h below.

$$f(x) = 2x \tag{14}$$
$$g(x) = x^2 + 2x \tag{15}$$
$$h(x) = 2x \tag{16}$$
$$\tag{17}$$

We wish to find $f(g(h(x)))$. We will first find $g(h(x))$.

$$h(x) = 2x \tag{18}$$
$$g(\quad) = (\quad)^2 + 2(\quad) \tag{19}$$
$$g(h(x)) = (h(x))^2 + 2(h(x)) \tag{20}$$
$$g(2x) = (2x)^2 + 2(2x) \tag{21}$$
$$g(2x) = 4x^2 + 4x \tag{22}$$

Thus $g(h(x)) = 4x^2 + 4x$. We now wish to find $f(g(h(x)))$.

$$g(h(x)) = 4x^2 + 4x \tag{23}$$
$$f(\quad) = 2(\quad) \tag{24}$$
$$f(g(h(x))) = 2(g(h(x))) \tag{25}$$
$$f(4x^2 + 4x) = 2(4x^2 + 4x) \tag{26}$$
$$f(4x^2 + 4x) = 8x^2 + 8x \tag{27}$$
$$\tag{28}$$

Thus $(f \circ g \circ h)(x) = f(g(h(x))) = 8x^2 + 8x$.

Here are some example problems for you to work out on your own with their respective answers at the bottom:
Find $(s \circ p)(x)$ for f and g below.

$$s(x) = 4x^2 + 8x + 8 \tag{29}$$
$$p(x) = x + 4 \tag{30}$$

Find $(g \circ f \circ q)(t)$ for g, f, and q below.

$$q(t) = \sqrt{x} \tag{31}$$
$$f(t) = x^2 \tag{32}$$
$$g(t) = 5x^9 \tag{33}$$

Find $(f \circ g \circ h \circ j)(x)$ for the functions below. HINT: Look at f and think about what will happen to it no matter what we plug into f.

$$j(x) = 4x^9 + 3sin(x) \tag{34}$$
$$h(x) = ln(x) \tag{35}$$
$$g(x) = 4x \tag{36}$$
$$f(x) = 1 \tag{37}$$

answers in order: $4x^2 + 40x + 104$, $5t^9$, 1