SAMPLE SECTIONS

A-Level Mathematics

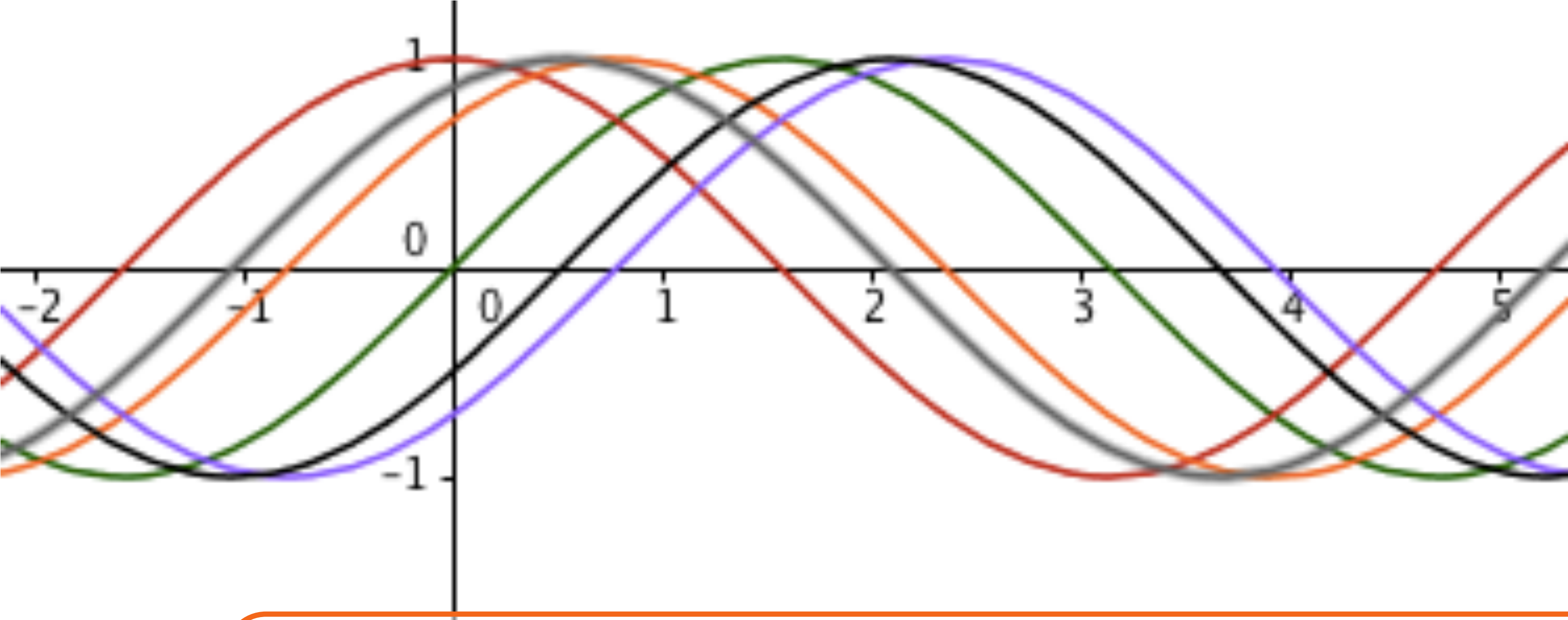
A Comprehensive and Supportive Companion to the Unified Curriculum

TEACHER BOOK

Tarquin

YEAR ONE

Edited by Tom Bennison and Edward Hall



1. The Small Angle Approximations

Before the advent of calculators, evaluation of the trigonometric ratios was complicated, and for small angles (less than 15° say) the so called *small angle approximations* proved sufficiently accurate for most tasks.
As shown in Figure 1.1, the functions x, $\sin(x)$ and $\tan(x)$ agree very closely between $x = 0$ and $x \approx 0.25$ radians. This figure leads us to consider the use of $y = x$ to approximate both $y = \sin(x)$ and $y = \tan(x)$. Of course, a more mathematical justification of these approximations (and of a similar approximation for $y = \cos(x)$) is desirable and is the objective of this chapter.

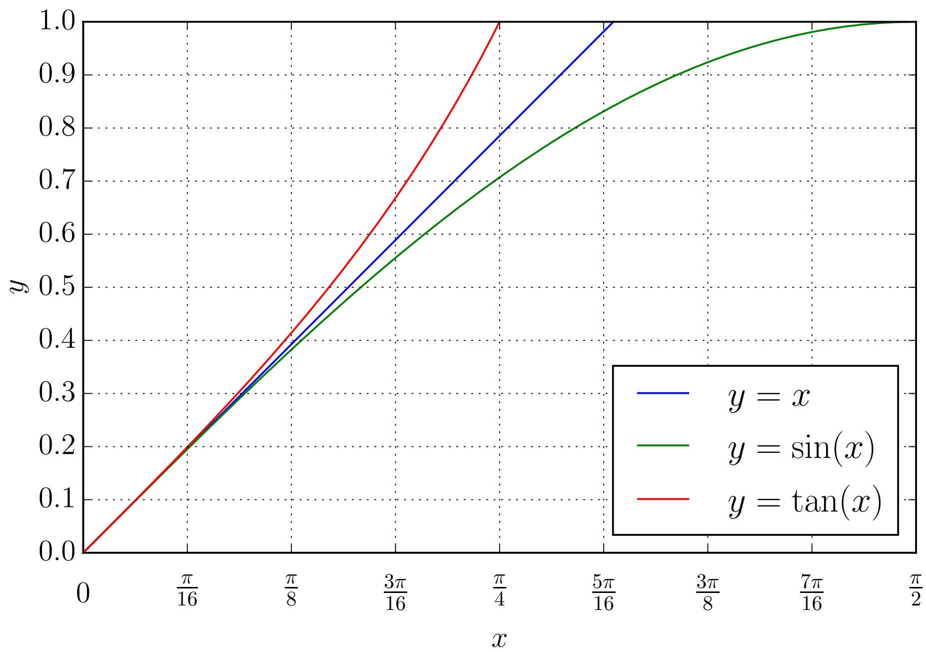


Figure 1.1: The functions $y = x$, $y = \sin(x)$ and $y = \tan(x)$ plotted over the interval $[0, \frac{\pi}{2}]$.

1.1 A Geometrical Derivation of the Small Angle Approximations

Using the definition of the trigonometric ratios for a right angle triangle we can geometrically derive the small angle approximations.

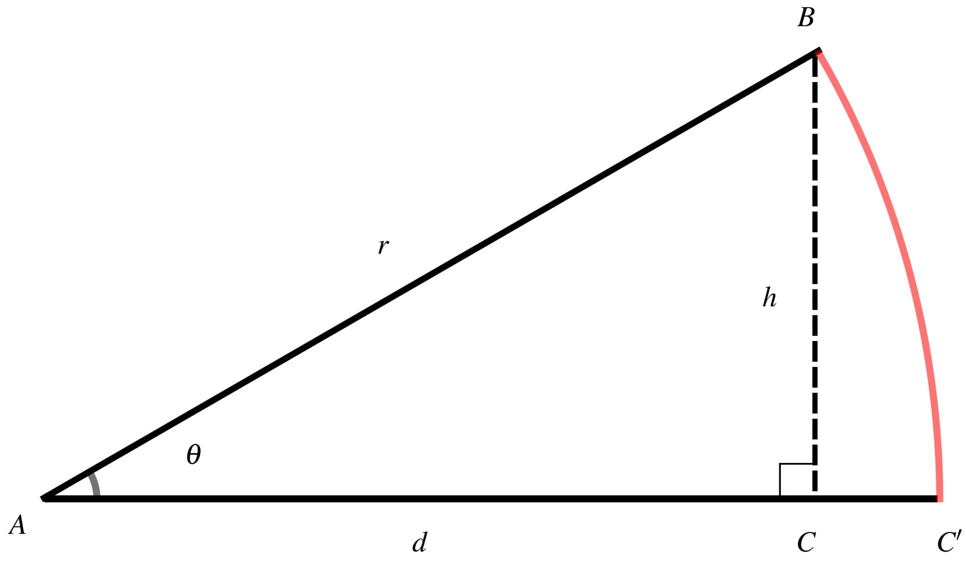


Figure 1.2: Derivation of the small angle approximations.

Consider the right angled triangle ABC shown in Figure 1.2, then, by trigonometry, the perpendicular height, h, can be calculated in the following two ways:

$$h = d\tan(\theta) \quad \text{and} \quad h = r\sin(\theta).$$

As the angle θ becomes close to zero then $r \approx d$. In addition, the height h becomes close to the length of the circular arc joining B to C', which can be calculated as $r\theta$ (provided the angle is given in radians). Hence, we have

$$h = r\sin(\theta) \approx r\theta \approx r\tan(\theta),$$

which lead to the approximations

$$\sin(\theta) \approx \tan(\theta) \approx \theta.$$

To obtain an approximation for $y = \cos(\theta)$ we make use of the double angle formula

$$\cos(2x) = 1 - 2\sin^2(x),$$

with $x = \frac{\theta}{2}$ and apply the small angle approximation for $\sin(x)$. Hence,

$$\begin{aligned}\cos(\theta) &= 1 - 2\sin^2\left(\frac{\theta}{2}\right)\\ &\approx 1 - 2\left(\frac{\theta}{2}\right)^2\\ &= 1 - \frac{\theta^2}{2}.\end{aligned}$$

More formally, the trigonometric functions can be expressed using their Taylor Series approximations (Taylor Series are part of the Further Mathematics A-Level course). These are infinite power series which get increasingly close to the value of the underlying function as more terms are included. For the three common trigonometric ratios, their Taylor Series expansions about the point $x = 0$ are

$$\begin{aligned}
\sin(\theta) &\approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots, \\
\tan(\theta) &\approx \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \cdots, \\
\cos(\theta) &\approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots.
\end{aligned}$$

Teaching Comment

The Taylor Series of a function, expanded about zero, is given by,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + +\frac{x^3}{3!}f^{(3)} \cdots + \frac{x^r}{r!}f^{(r)}(0) + \cdots. \tag{T1.1}$$

As an example, consider the derivation of the Taylor series for $f(x) = \sin(x)$. Differentiating, we have,

$$\begin{aligned}
f'(x) &= \cos(x), \\
f''(x) &= -\sin(x), \\
f^{(3)}(x) &= -\cos(x), \\
f^{(4)}(x) &= \sin(x), \\
f^{(5)}(x) &= \cos(x).
\end{aligned}$$

From the above we can evaluate the successive derivatives at zero, namely; $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0$ and $f^{(5)}(0) = 1$. Using these values in (T1.1) we obtain the Taylor Series expansion for $f(x) = \sin(x)$ centered at 0.

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots \tag{T1.2}$$

Note that the form for a general term in the expansion can be deduced by observing the pattern shown in the low order terms and the cyclic properties of the derivatives of $\sin(x)$ and $\cos(x)$. A Taylor Series which has been expanded about zero is commonly known as a Maclaurin Series.

Neglecting any terms of order 3 or greater in the above expansions also lead to the small-angle approximations.

Formulae 1.1 — Small Angle Approximations.

For θ close to zero and measured in radians, the small angle approximations are

$$\begin{aligned}
\sin(\theta) &= \theta, \\
\cos(\theta) &= 1 - \frac{\theta^2}{2}, \\
\tan(\theta) &= \theta.
\end{aligned}$$

Example 1.2
Approximate $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$ for $\theta = 15^\circ$ using the small angle approximations.

When applying the small angle approximations we must first ensure that we are working in radians, and so

$$\sin(15^\circ) = \sin\frac{\pi}{12} \approx \frac{\pi}{12} \approx 0.2618,$$
$$\cos(15^\circ) = \cos\frac{\pi}{12} \approx 1 - \frac{(\frac{\pi}{12})^2}{2} \approx 0.9657,$$
$$\tan(15^\circ) = \tan\frac{\pi}{12} \approx \frac{\pi}{12} \approx 0.2618.$$

Exercise 1.1

Q1. Approximate $\sin(\theta), \cos(\theta)$ and $\tan(\theta)$ for
 (a) $\theta = 0$;
 (b) $\theta = \frac{\pi}{48}$;
 (c) $\theta = \frac{\pi}{24}$;
 (d) $\theta = \frac{\pi}{15}$;
 (e) $\theta = \frac{\pi}{12}$;
 (f) $\theta = \frac{\pi}{6}$.

Q2. (a) What is the percentage error made when approximating $\sin(18^\circ)$? Take a calculator's value as the exact answer.
 (b) What is the percentage error made when approximating $\cos(18^\circ)$? Take a calculator's value as the exact answer.
 (c) What is the percentage error made when approximating $\tan(18^\circ)$? Take a calculator's value as the exact answer.

1.2 Applications of the Small Angle Approximations

The small angle approximations can be used to express a trigonometric function in terms of a polynomial which is valid for small arguments. Calculating powers of a number is quicker than computing values of trigonometric functions and so, for small arguments, this is sometimes preferred.

Example 1.3
Show that, for small angles, the function $f(x) = \sin^2(x)\cos(x)$ can be approximated by a function of the form $h(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$ and use this approximation to evaluate $\sin^2(\frac{\pi}{24})\cos(\frac{\pi}{24})$.

Solution:

$$\begin{aligned}\sin^2(x)\cos(x) &\approx (x)^2\left(1 - \frac{x^2}{2}\right)\\ &= x^2 - \frac{x^4}{2},\end{aligned}$$

which is of the desired form with $A = B = D = 0$, $C = 1$ and $E = \frac{1}{2}$.
Using this approximation,

$$\sin^2\left(\frac{\pi}{24}\right)\cos\left(\frac{\pi}{24}\right) \approx \left(\frac{\pi}{24}\right)^2 - \frac{\left(\frac{\pi}{24}\right)^4}{2}$$
$$\approx 0.01699.$$

Teaching Comment
In your digital textbook there is a card sort that could be printed and used in the classroom to let students practice applying the small angle approximations. This activity could be made harder by removing one of the original expressions and asking "How many different expressions can you make which have the same small angle approximation?".

Since we used a geometric argument to derive the small angle approximations, we can also use the small angle approximations to find the derivatives of trigonometric functions from first principles, as shown in the next example.

Example 1.4
Show, from first principles, that the derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$.

$$f'(x) = \lim_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin(x)}{\delta x}.$$

Applying the addition formula for $\sin(A+B)$ with $A = x$ and $B = \delta x$ gives:

$$f'(x) = \lim_{\delta x \to 0} \frac{\sin(x)\cos(\delta x) + \cos(x)\sin(\delta x) - \sin(x)}{\delta x},$$

Then, using the small angle approximations, we obtain

$$\begin{aligned}
f'(x) &= \lim_{\delta x \to 0} \frac{\sin(x)\left(1 - \frac{1}{2}(\delta x)^2\right) + \cos(x)(\delta x) - \sin(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\sin(x) - \frac{1}{2}(\delta x)^2\sin(x) + \delta x\cos(x) - \sin(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\delta x\cos(x) - \frac{1}{2}(\delta x)^2\sin(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \cos(x) - \frac{1}{2}\delta x\sin(x).
\end{aligned}$$

As $\delta x \to 0$, $\cos(x) - \frac{1}{2}\delta x\sin(x) \to \cos(x)$ and so $f'(x) = \cos(x)$.

Activity 1.1
Why is it valid to use the small angle approximations for functions of δx here?

Teaching Comment
This question will let you test students' understanding of both small angle approximations and differentiation from first principles.

Exercise 1.2

Q1. Find a polynomial approximation of the function $f(\theta) = \cos^2(\theta)\tan(\theta)$.

Q2. Find a polynomial approximation of the function $\frac{\sin^2(\theta)+1}{\cos(\theta)}$.

Q3. Find a polynomial approximation of the function $\sec^2(x)$.

Q4. Find a polynomial approximation of the function $\sin^2(x)\cos(x)+\tan(x)$.

Q5. For the function $f(x) = \sec(x)(\sin(x)+\cos(x))$:

(a) Find a polynomial approximation of $f(x)$ using the standard small angle approximations.

(b) Find a polynomial approximation of $f(x)$ using $\sin(x) \approx x$ and the lower order approximation for $\cos(x) \approx 1$.

(c) Find the difference in percentage error when approximating $f(x)$ using the two approximations derived in parts (a) and (b).

Q6. Show, from first principles, that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.

Q7. Find, from first principles, the derivative of $f(x) = \tan(x)$.

1.3 The Accuracy of the Small Angle Approximations

Teaching Comment
The interactive activity contained in your digital textbook can be used by students to explore how the accuracy of the small angle approximations change as the angle varies.

The accuracy of the small angle approximations can be assessed in terms of the percentage error in the approximation when compared with the exact (up to machine accuracy) value. Tables1.1 show the percentage accuracy when approximating $\sin(x)$ for angles between 5° and 45°.

Angle, x	$\sin(x)$	**Approximation**	**% Error**
5°	0.087156	0.087266	0.13%
10°	0.173648	0.174533	0.51%
15°	0.258819	0.261799	1.15%
20°	0.342020	0.349065	2.06%
30°	0.500000	0.523599	4.72%
40°	0.642788	0.698132	8.61%
45°	0.707107	0.785398	11.07%

Table 1.1: The error in approximating $\sin(x)$ using the small angle approximations.

This information is displayed graphically in Figure 1.3, along with the percentage errors for $\cos(x)$ and $\tan(x)$.

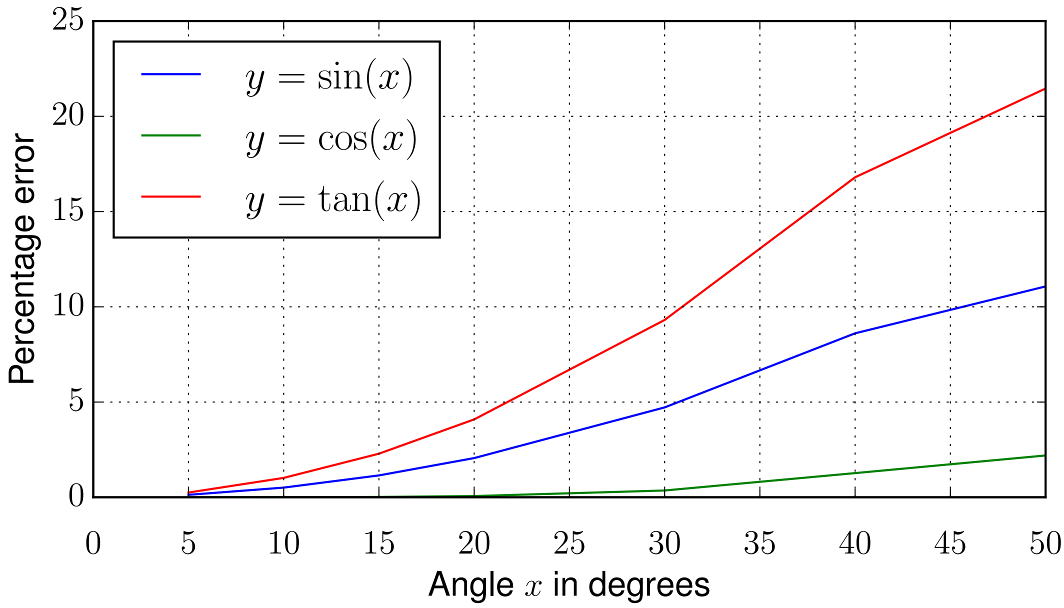
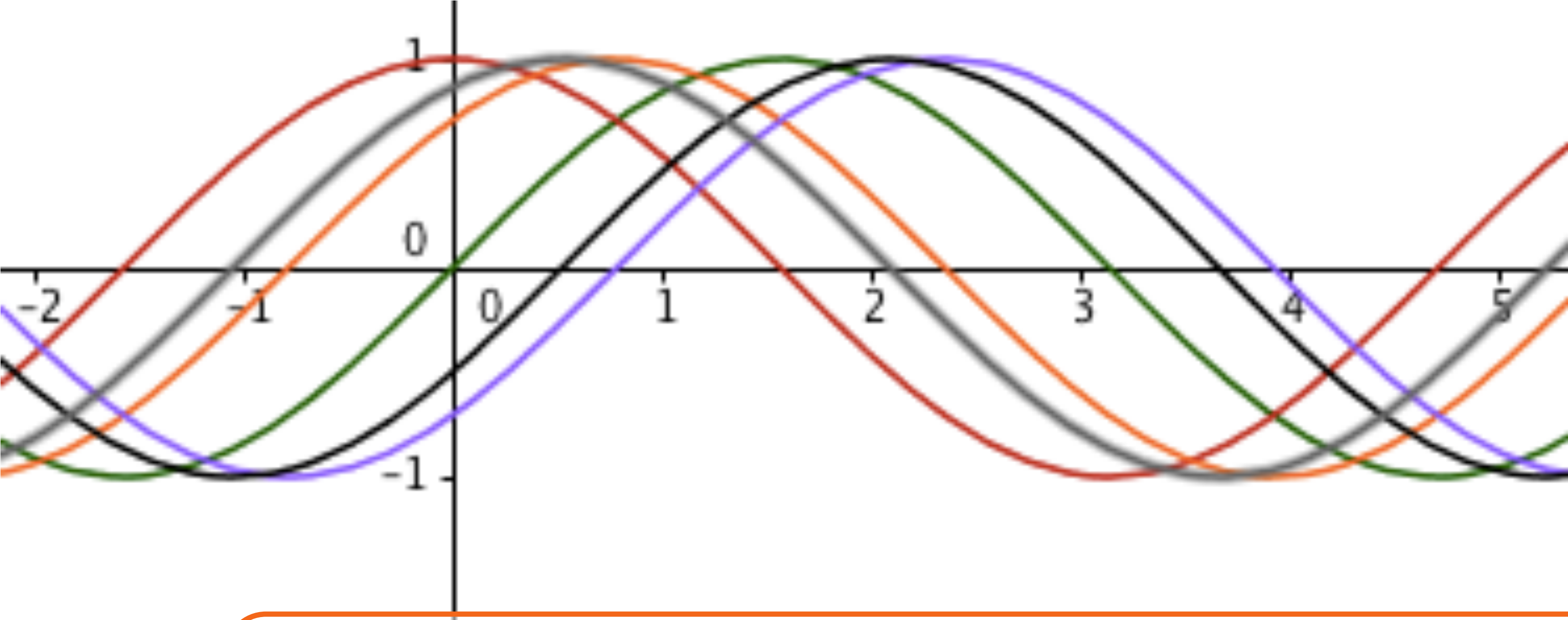


Figure 1.3: The percentage error made when using the small angle approximations.

As can be seen, the approximation for $y = \tan(x)$ performs poorly as the angle size increases. By contrast, the approximation for $y = \cos(x)$ performs remarkably well; with an error of less than 5% even when the angle is $50°$.

Activity 1.2
Why are the approximation for the tangent and sine functions significantly worse than that for cosine?

Teaching Comment
This activity is designed to get students thinking about order of approximations and how the error grows in magnitude as the angle increases. To help students who are struggling you could highlight the order of the polynomial approximations used in each case. They should also consider why the approximation to $y = \tan(x)$ is worse than that for $y = \sin(x)$.



2. Exploring Quadratic Functions

We study quadratic functions in depth for two important reasons. Firstly, quadratic functions often arise when we model physical situations mathematically, for example, if we consider a projectile moving through the air (assuming no air-resistance) then a quadratic function can be used to model its path. Secondly, quadratic functions are the lowest order polynomials which exhibit interesting behaviour and so, from a theoretical perspective, they warrant study.
A quadratic function is a polynomial which maps a set of numbers (the domain) to another set of numbers (the range). For example, the set $\{-2,-1,0,1,2,3\}$ is mapped to the set $\{0,0,2,6,20\}$ by the function $f(x) = x^2+3x+2$. More formally, we can define a quadratic function in the following way.

Definition 2.1
The function $f(x)$ is a function of x and is said to be a quadratic function if $f(x) = ax^2+bx+c$, where $a,b,c \in \mathbb{R}$ and $a \neq 0$.

Teaching Comment
For A-Level, the coefficients are real numbers but, depending on the group, it would be nice to introduce quadratics with complex valued coefficients, especially if the group is studying further mathematics.

Teaching Comment
The interactive activity available in your digital textbook can be used to explore the general quadratic. When initially introducing quadratics, we would expect that students may not be able to articulate the effect of a, b and c on the curve. The effect of a is the easiest to describe, as it clearly changes the orientation of the curve as a passes through zero. Encourage students to explicitly explain what happens when $a = 0$, even it seems obvious. The value of c being the y-intercept is also easy to describe, but emphasise the need to use precise mathematical language.

Students will find describing the influence of b harder, encourage them to fix the values of a and c and vary b and then do the same for a few different values of a and c.

Exercise 2.1

Q1. Which of the following are quadratic functions? Please explain your answer.

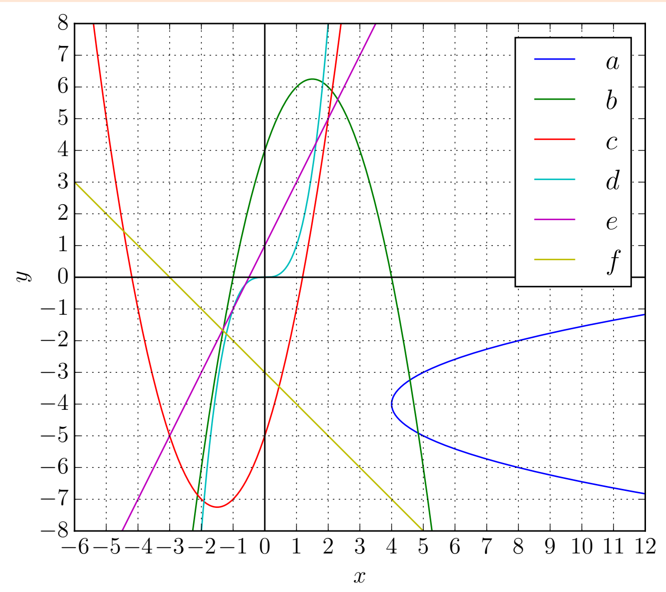
a) $f(x) = x^2 + 3x + 2$.
b) $y = 3x + 2$.
c) $x = y^2 + 2y - 15$.
d) $f(x) = (x+3)^2$.
e) $f(x) = 5x^2 + 2x$.
f) $f(x) = (2x-3)^2 - 4x^2$.

2.1 Plotting Quadratic Functions

We begin the exploration of quadratic functions by considering how to plot them. Plotting a function is distinct from sketching a function and we shall briefly discuss sketching quadratics towards the end of this chapter.

Exercise 2.2

Q1. Which of the functions in the figure below are quadratic functions?



When plotting a quadratic equation function it is helpful, at least initially, to set your work out in a table as shown in the following example.

Example 2.2
To plot the graph of $y = x^2 - 4x + 3$ we can construct Table 2.1.

x	x^2	$-4x$	$+3$	y
-1	1	$+4$	$+3$	8
0	0	0	$+3$	$+3$
1	1	-4	$+3$	0
2	4	-8	$+3$	-1
3	9	-12	$+3$	0
4	16	-16	$+3$	$+3$
5	25	-20	$+3$	8

Table 2.1: Table to calculate plotting points for Example 2.2.

Plotting these points on a pair of coordinate axes and joining them up with a smooth curve leads to the plot shown in Figure 2.1.

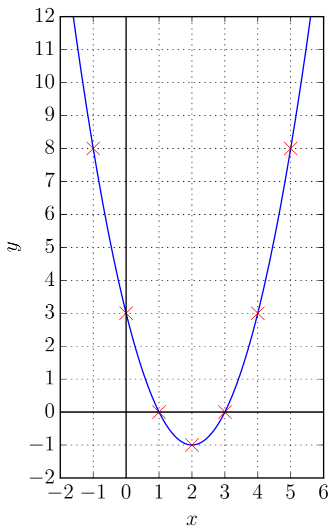


Figure 2.1: The graph of $y = x^2 - 4x + 3$.

The shape of this quadratic (and indeed the graph of any quadratic function) is known as a parabola. The parabola is one of a family of curves that are collectively known as conic sections - these are studied in more detail in the Further Mathematics course.
Notice that this curve is symmetrical about the line $x = 2$ and that this line of symmetry passes through the vertex $(2, -1)$ of the parabola, which in this case is a minimum.

Teaching Comment
Plotting such a detailed table will not be necessary for a lot of students, but we would still encourage a table of the form

x	-1	0	1	2	3	4	5
y	8	3	0	-1	0	3	8

Exercise 2.3

Q1. Plot the graphs of the following quadratics between $x = -3$ and $x = 3$. State the line of symmetry of the curve and the vertex of the parabola.

(a) $y = x^2 + 1$.
(b) $y = x^2 + 2x$.
(c) $y = x^2 + 2x + 1$.
(d) $y = x^2 + 2x + 5$.
(e) $y = (x+2)^2$.
(f) $y = -x^2 + 4x + 7$.
(g) $y = -x^2 + 4x - 4$.
(h) $x = y^2 + y + 2$.

Q2. Plot on the same pair of axes the functions $f(x) = x^2 - 4x + 3$ and $f(x) = -x^2 + 4x - 3$. What can be said about these curves?

2.2 The Role of the Discriminant

From the previous exercise it should be apparent that the graphs of all quadratic functions have a similar 'look'. Hence, we should be able to categorise quadratic functions in some way. To do this, it helps if we make the following definition.

Definition 2.3
The discriminant of the quadratic function $f(x) = ax^2 + bx + c$ is defined to be $\Delta = b^2 - 4ac$.

This definition allows us to compute a numerical quantity associated with every possible quadratic.

Example 2.4
We compute the discriminant of the quadratic $y = 3x^2 + 4x - 2$ in the following way, noting that in this instance, $a = 3$, $b = 4$ and $c = -2$.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4 \times 3 \times (-2) \\ &= 40.\end{aligned}$$

Teaching Comment
Contained within your digital textbook is an electronic card sorting activity allowing students to practice calculating the determinant. The cards will only land over a target if they match.

For the general quadratic $f(x) = ax^2 + bx + c$ we can use the discriminant together with the sign of the coefficient a to determine the shape of the graph of $f(x)$. Table 2.2 illustrates the six possibilities. From Table 2.2 we can also classify the algebraic properties of the roots of the quadratic equation $ax^2 + bx + c = 0$ using the discriminant.

- $b^2 - 4ac > 0$ implies that the quadratic equation $ax^2 + bx + c = 0$ has two real and distinct roots.

- $b^2 - 4ac = 0$ implies that the quadratic equation $ax^2 + bx + c = 0$ has one repeated root.
- $b^2 - 4ac < 0$ implies that the quadratic equation $ax^2 + bx + c = 0$ has no real roots.

If we are only interested in classifying the roots of a quadratic equation and not finding them, then it suffices to just investigate the discriminant.

Activity 2.1
What can be said if the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is greater than zero and a whole number? What can be said if the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is greater than zero and a perfect square?

Teaching Comment
If the discriminant of a quadratic equation is a perfect square then the equation can be factorised with two rational roots. Many students will incorrectly say the same thing for the weaker condition that the discriminant is a positive whole number. This could also be a good time to ask students "What is the same and what is different about the quadratics $y = \left(x - \frac{5}{3}\right)\left(x - \frac{7}{2}\right)$ and $y = (3x - 5)(2x - 7)$?". A common mistake is to convert the first quadratic into the second by "multiplying up" the denominator inside each bracket, which of course does not work. They are distinct quadratics which just happen to share the same roots.

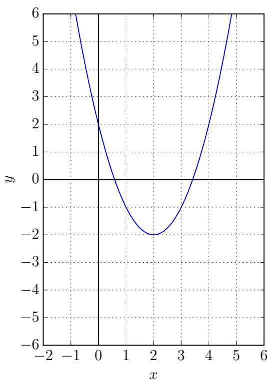
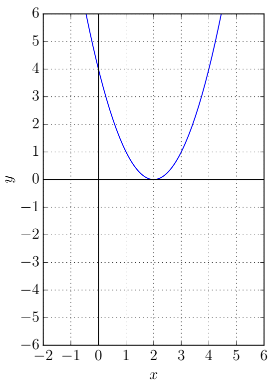
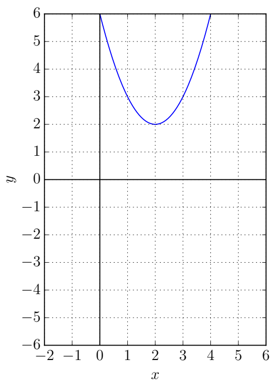
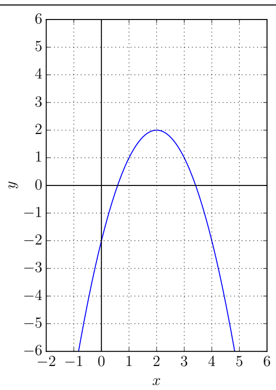
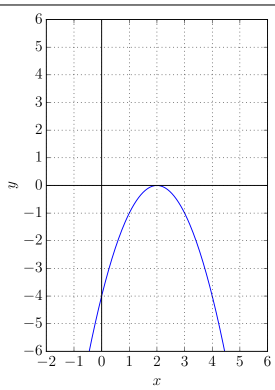
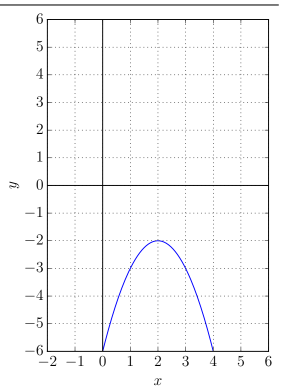
	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
$a > 0$			
$a < 0$			

Table 2.2: Graphical meaning of the discriminant.

Example 2.5
Consider the quadratic equation $x^2 + 2x - 2 = 0$, then the discriminant is given by:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 2^2 - 4 \times 1 \times (-2) \\ &= 4 - (-8) \\ &= 1,\end{aligned}$$

therefore, this quadratic equation has two distinct real roots.

Teaching Comment
The use of the discriminant for a quadratic is well known, however the formula for the discriminant of a cubic equation is less well known.
For the cubic polynomial $ax^3 + bx^2 + cx + d$ the discriminant can be calculated using

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

This can be used in a similar way to the discriminant of a quadratic equation to classify the roots of a cubic equation:
- $\Delta > 0$: the equation has 3 distinct real roots.
- $\Delta = 0$: at least two roots coincide and they are all real.
- $\Delta < 0$: the equation has 1 real root and two complex roots.

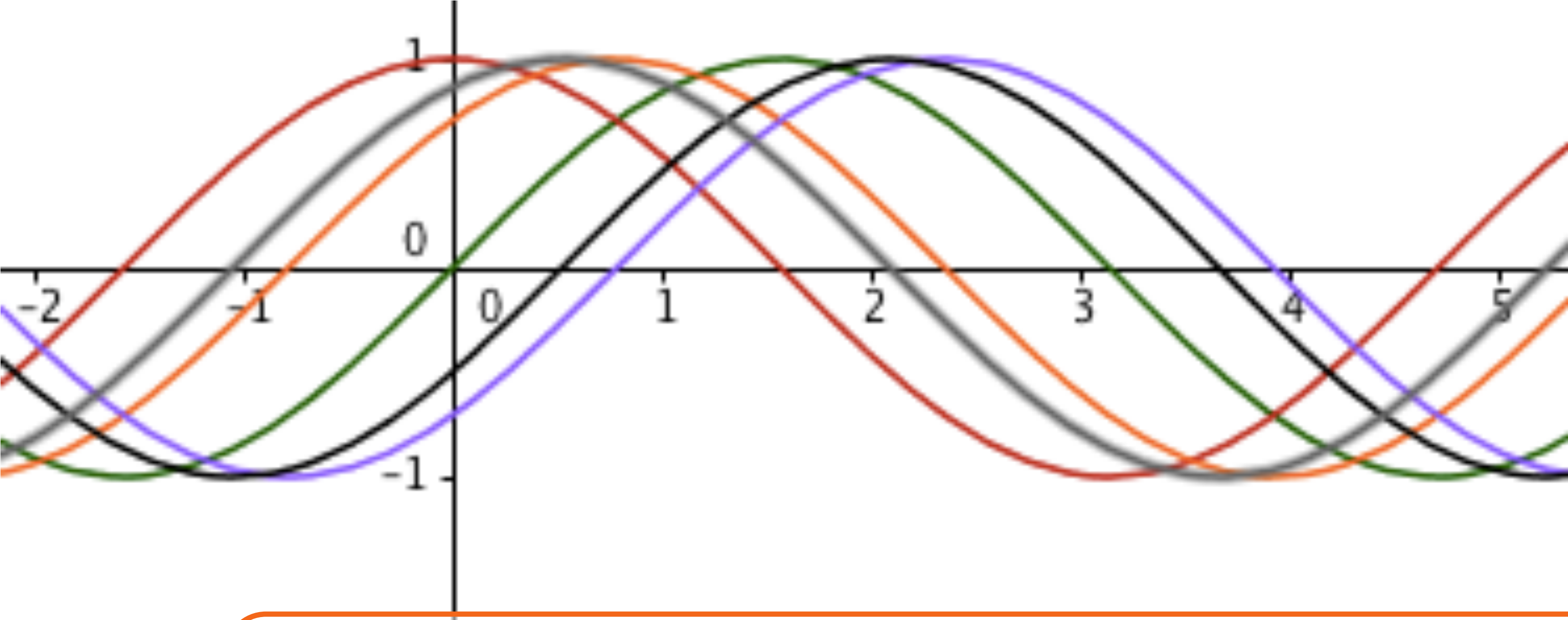
Exercise 2.4
Q1. Find the discriminant of the quadratic functions below.

(a) $y = x^2 + 3x - 2$.
(b) $y = x^2 + x + 5$.
(c) $y = 2x^2 + 2x - 1$.
(d) $y = x^2 - 2x + 6$.
(e) $y = -x^2 - 2x + 5$.
(f) $y = -x^2 - 6x + 1$.
(g) $y = x^2 + 8x + 4$.
(h) $y = x^2 + 3x + 4$.
(i) $y = 2x^2 + 4x + 2$.
(j) $y = x^2 + 2x - 5$.

Q2. Classify the roots of the following quadratic equations.

(a) $x^2 + 5x + 6 = 0$.
(b) $x^2 + 5x + 1 = 0$.
(c) $x^2 + 1$.
(d) $3x^2 + 6x + 3$.
(e) $x^2 - 3x + 2$.
(f) $x^2 - 2x - 1$.
(g) $x^2 - 2x + 8$.
(h) $5x^2 + 10x + 5$.

Q3. Show that, for all $k \in \mathbb{R}$, the quadratic $2x^2 + 4kx - 2$ will have two real and distinct roots.
Q4. Find the p such that $x^2 + 3x + p = 0$ has a repeated root.
Q5. Find the range of k for which $x^2 + 2kx - k = 0$ has real roots.



3. Numerical Integration

Definite integrals have many important uses throughout science and engineering. For example, the lift and drag produced by an aircraft wing can be found via definite integrals and the noise pollution from a jet engine can be approximated using a definite integral. Similarly, determining the safe operating conditions of nuclear reactors relies heavily on the computation of definite integrals. Being able to compute definite integrals is also essential in finding the probability that an event will occur.

Unfortunately, most functions do not have known anti-derivatives, or the functions are not known exactly, but are sampled at specific points. In these cases numerical methods must be used to find approximations to the integrals. In this section we explore some methods of numerical integration (*quadrature*) for carrying out these approximations.

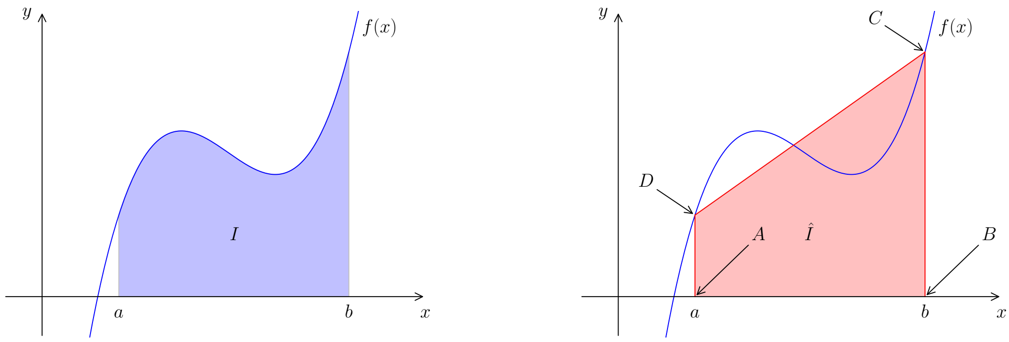
3.1 Trapezium Rule

Suppose we wish to find the definite integral

$$\int_a^b f(x)\mathrm{d}x.$$

One of the simplest numerical methods for approximating a definite integral is the trapezium rule. As the name suggests, the area under the curve is approximated by the area of a trapezium, which is very simple to compute.

Figure 3.1 shows how this trapezium is constructed, with its vertices having the coordinates $A = (a,0)$, $B = (b,0)$, $C = (b, f(b))$ and $D = (a, f(a))$. A general trapezium with parallel sides of length c and d and perpendicular height h has area $A = \frac{c+d}{2}h$. For the trapezium in Figure 3.1 the parallel sides are BC and DA, which have length $f(b)$ and $f(a)$, respectively. The perpendicular height is simply given by $b-a$. Hence, trapezium $ABCD$ has area $\frac{f(a)+f(b)}{2}(b-a)$.



(a) (b)

Figure 3.1: (a) The definite integral $I = \int_a^b f(x)\mathrm{d}x$ and (b) its trapezium rule approximation $\hat{I}$.

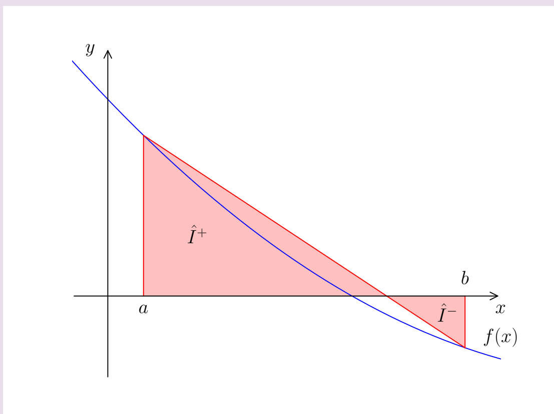
Explicity, we have the following formula.

Formulae 3.1 — Trapezium Rule.
The trapezium rule to approximate a definite integral of $f(x)$ has the formula

$$\int_a^b f(x)\mathrm{d}x \approx \frac{f(a)+f(b)}{2}(b-a). \tag{3.1}$$

Activity 3.1
Show that the same formula (3.1) is correct when $f(a)$ and $f(b)$ have different signs.

Teaching Comment
We first use a geometric argument to find the formulae. Without loss of generality, we assume that $f(a) > 0$ and $f(b) < 0$. In this situation we have a positive contribution to the integral from the left, and a negative contribution from the right, as can be seen in the figure below.



The line joining $f(a)$ and $f(b)$ crosses the x-axis at

$$x = -\frac{f(a)(b-a)}{f(b)-f(a)} + a = b - \frac{f(b)(b-a)}{f(b)-f(a)}.$$

Hence, using the formula for the area of a triangle, the positive area $\hat{I}^+$ is given by

$$\hat{I}^+ = -\frac{f(a)}{2}\frac{f(a)(b-a)}{f(b)-f(a)}$$

and the negative area $\hat{I}^-$ is given by

$$\hat{I}^- = \frac{f(b)}{2}\frac{f(b)(b-a)}{f(b)-f(a)}.$$

Summing these together gives

$$\begin{aligned}
\hat{I} = \hat{I}^+ + \hat{I}^- &= \frac{f(b)}{2}\frac{f(b)(b-a)}{f(b)-f(a)} - \frac{f(a)}{2}\frac{f(a)(b-a)}{f(b)-f(a)} \\
&= \frac{b-a}{2}\frac{(f(b)-f(a))(f(a)+f(b))}{f(b)-f(a)} \qquad \text{(Difference of two squares)} \\
&= \frac{f(a)+f(b)}{2}(b-a),
\end{aligned}$$

which is the same formula as in (3.1).
Perhaps even simpler than this is direct integration of the linear function that is being used to approximate $f(x)$. The equation of the linear function can be shown to be

$$y = \frac{f(b)-f(a)}{b-a}(x-b) + f(b).$$

By integration, we obtain

$$\begin{aligned}
\hat{I} &= \int_a^b \frac{f(b)-f(a)}{b-a}(x-b) + f(b)\mathrm{d}x \\
&= \left[\frac{f(b)-f(a)}{b-a}\frac{(x-b)^2}{2} + f(b)x\right]_a^b \\
&= f(b)b + \frac{f(b)-f(a)}{2}(b-a) - f(b)a \\
&= \frac{f(a)+f(b)}{2}(b-a),
\end{aligned}$$

as required. We note that this approach works for $f(a)$ and $f(b)$ of any sign.

Example 3.2
Use the trapezium rule to find an approximation of the definite integral

$$\int_1^2 (3x^2 - x^3)\mathrm{d}x.$$

Solution: In this example we have $f(x) = 3x^2 - x^3$, $a = 1$ and $b = 2$. We evaluate the function at these values:

$$f(a) = 2, \quad f(b) = 4.$$

Therefore,

$$\begin{aligned}\int_1^2 f(x)\mathrm{d}x &\approx \frac{f(a)+f(b)}{2}(b-a) \\ &= \frac{f(a)+f(b)}{2} \\ &= 3.\end{aligned}$$

In fact, the exact integral is

$$\begin{aligned}\int_1^2 (3x^2 - x^3)\mathrm{d}x &= \left[x^3 - \frac{x^4}{4}\right]_1^2 \\ &= 8 - 4 - \left(1 - \frac{1}{4}\right) \\ &= 3\frac{1}{4}.\end{aligned}$$

In this case, the approximation given by the trapezium rule is accurate up to an error of $\frac{1}{4}$.

In our next example we show that the trapezium rule does not always give an accurate answer.

Example 3.3
Use the trapezium rule to find an approximation to the definite integral

$$\int_0^3 \mathrm{e}^{2x}\mathrm{d}x.$$

Solution: In this case $f(x) = \mathrm{e}^{2x}$, $a = 0$, $b = 3$ and the approximation is given by

$$\int_0^3 f(x)\mathrm{d}x \approx \frac{3}{2}\left(\mathrm{e}^0 + \mathrm{e}^6\right) = 606.6 \text{ to 1 d.p.}$$

In fact, the true integral is given by

$$\int_0^3 f(x)\mathrm{d}x = \left[\frac{\mathrm{e}^{2x}}{2}\right]_0^3 = 200.4 \text{ to 1 d.p.}$$

In this case, the approximation is incorrect by a factor of over 3.

Exercise 3.1

Q1. Use the trapezium rule to find approximations to the following integrals and evaluate the errors made by these approximations.

(a) $\int_2^4 \frac{1}{2x}\mathrm{d}x$;

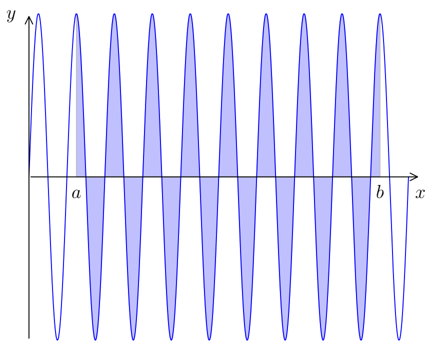
(b) $\int_1^2 \ln(x)\mathrm{d}x$;

(c) $\int_0^{\frac{\pi}{4}} \sin(x)\mathrm{d}x$;

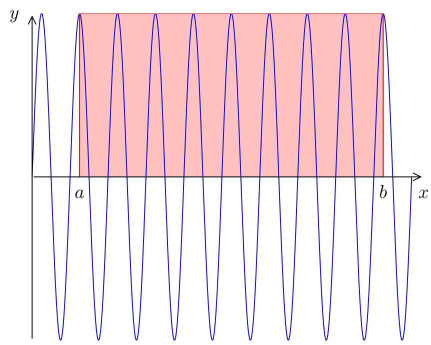
(d) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(10x)\mathrm{d}x$.

3.2 Composite Trapezium Rule

We saw in Example 3.3 that the trapezium rule can give very inaccurate results. In fact, for most integrals, the trapezium rule will not perform well. If the solution rapidly varies between a and b, as in shown in the example in Figure 3.2, then the function cannot be well approximated by a straight line and the trapezium rule will give poor results.



(a)



(b)

Figure 3.2: The trapezium rule can give very inaccurate results when the function $f(x)$ is rapidly changing. The approximate integral (b) is much larger than the actual integral, which is 0 (a).

Rather than using just one trapezium, we could approximate the area under the curve by two trapeziums of equal width. Let us divide the interval into two subintervals, $[x_0, x_1]$ and $[x_1, x_2]$, where $x_0 = a$, $x_2 = b$, $x_1 = \frac{a+b}{2}$ and let $h = x_2 - x_1 = x_1 - x_0 = \frac{b-a}{2}$, see Figure 3.3.

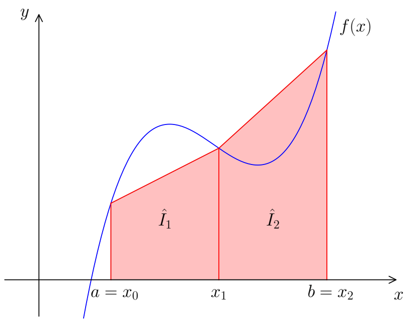


Figure 3.3: Approximating $\int_a^b f(x)\mathrm{d}x$ using two trapeziums.

We have the following approximation:

$$\int_a^b f(x)\mathrm{d}x \approx \underbrace{\frac{f(x_0)+f(x_1)}{2}h}_{\hat{I}_1} + \underbrace{\frac{f(x_1)+f(x_2)}{2}h}_{\hat{I}_2}$$
$$= h\left[\frac{f(x_0)}{2} + f(x_1) + \frac{f(x_2)}{2}\right].$$

We can continue increasing the number of subintervals and can expect that the approximation will improve. Suppose we have N subintervals of equal width $h = \frac{b-a}{N}$ and let $x_k = a + kh$, for $k = 0, 1, \ldots, N$. The area under the curve $f(x)$ can be approximated by the area under the *piecewise linear* curve joining the points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_N, f(x_N))$. Two examples of this are shown in Figures 3.4(a) and 3.4(b), where $N = 4$ and $N = 10$, respectively.

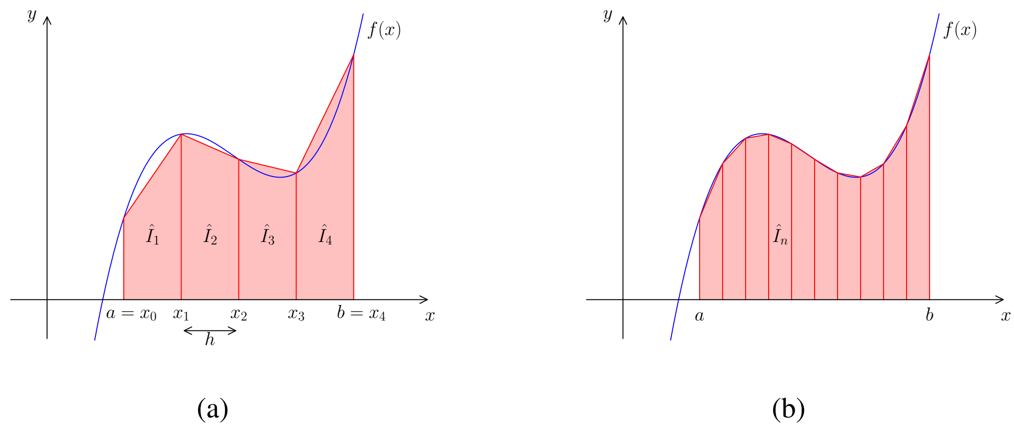


Figure 3.4: Approximating $\int_a^b f(x)\mathrm{d}x$ using four trapeziums (a) and ten trapeziums (b).

Over each subinterval the area of the approximation is once again computed by a trapezium. In fact, the area $\hat{I}_n$ is given by

$$\hat{I}_n = \frac{f(x_{n-1})+f(x_n)}{2}h.$$

The area from each subinterval can then be summed so that

$$\int_a^b f(x)\mathrm{d}x \approx \hat{I}_1 + \hat{I}_2 + \ldots + \hat{I}_{N-1} + \hat{I}_N$$
$$= \frac{f(x_0)+f(x_1)}{2}h + \frac{f(x_1)+f(x_2)}{2}h + \ldots + \frac{f(x_{N-2})+f(x_{N-1})}{2}h$$
$$+ \frac{f(x_{N-1})+f(x_N)}{2}h$$
$$= h\left[\frac{f(x_0)}{2} + \frac{f(x_1)}{2} + \frac{f(x_1)}{2} + \frac{f(x_2)}{2} + \ldots + \frac{f(x_{N-2})}{2} + \frac{f(x_{N-1})}{2}\right.$$
$$\left. + \frac{f(x_{N-1})}{2} + \frac{f(x_N)}{2}\right]$$
$$= h\left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \ldots + f(x_{N-2}) + f(x_{N-1}) + \frac{f(x_N)}{2}\right]. \tag{3.2}$$

This numerical integration method is called the *composite trapezium rule*. If we rewrite (3.2) using summation notation (see Chapter 5), then we obtain the following formula.

Formulae 3.4 — Composite Trapezium Rule.
The composite trapezium rule with N subintervals of equal width $h = \frac{b-a}{N}$ to approximate $\int_a^b f(x)\mathrm{d}x$ has the formula

$$\int_a^b f(x)\mathrm{d}x \approx h\left[\frac{f(x_0)}{2} + \sum_{k=1}^{N-1} f(x_k) + \frac{f(x_N)}{2}\right], \tag{3.3}$$

where $x_k = a + kh$ for $k = 0, \ldots, N$.

Remark
A *piecewise linear function* is simply a function that is composed of straight-line sections or 'pieces'. In the composite trapezium rule, $f(x)$ is approximated by a continuous piecewise linear function, which is known as the *linear interpolant*. The linear interpolant passes through the graph of $f(x)$ at the points$(x_0, f(x_0))$, $(x_1, f(x_1)$, $\ldots$, $(x_N, f(x_N))$ and it enables an approximation to $f(x)$ to be found for any x in the interval $[x_0, x_N]$ needing only the values $f(x_k)$, $k = 0, \ldots, N$ to be known. It is also possibly to approximate $f(x)$ with higher order polynomials, which are name *the quadratic interpolant*, *the cubic interpolant* and so on.

Tip
Some exam boards call the composite trapezium rule just *the trapezium rule*. Be careful of the definition they use when sitting the exam.

Example 3.5
Use the composite trapezium rule with 4 subintervals to approximate the definite integral

$$\int_0^\pi \sin(\theta)\mathrm{d}\theta.$$

Note, here θ is measured in radians and this will always be the case when integrating trigonometric functions.
Solution:
We first set up the points at which $\sin(\theta)$ will be evaluated. To do this, we find h: the number of intervals $N = 4$, so

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

The points are then given by

$$\theta_0 = 0, \quad \theta_1 = \frac{\pi}{4}, \quad \theta_2 = \frac{\pi}{2}, \quad \theta_3 = \frac{3\pi}{4}, \quad \theta_4 = \pi.$$

Evaluate $\sin(\theta)$ at these points:

$$\sin(\theta_0) = 0, \quad \sin(\theta_1) = \frac{1}{\sqrt{2}}, \quad \sin(\theta_2) = 1, \quad \sin(\theta_3) = \frac{1}{\sqrt{2}}, \quad \sin(\theta_4) = 0.$$

Hence,

$$\int_0^{\pi} \sin(\theta)\mathrm{d}\theta \approx \frac{\pi}{4}\left[\frac{0}{2}+\frac{1}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}+\frac{0}{2}\right]$$
$$= \frac{\pi}{4}(1+\sqrt{2}) = 1.896 \text{ to 3 d.p.}$$

The actual value of the integral is 2, so the trapezium rule has underestimated this value. sin is a concave function over the interval $[0,\pi]$ and thus, on every subinterval, the area is underestimated. No matter how many intervals are chosen, the trapezium rule will always give an underestimate of this particular integral.

Activity 3.2
Investigate how well the integral

$$\int_0^{2\pi} \sin(\theta)\mathrm{d}\theta$$

is approximated using the composite trapezium rule.

Teaching Comment
For any value of N, students should see that the composite trapezium rule gives a value of 0, which is exact for this problem. In the interval $\left[0,\frac{\pi}{2}\right]$, the trapezium rule underestimates the area, as seen in Example 3.2, but in the interval $\left[\frac{\pi}{2},\pi\right]$ the rule overestimates the negative integral. These over- and underestimations combine to give a zero error. In general, the trapezium rule performs particularly well when integrating periodic functions.

Exercise 3.2

Q1. Use the composite trapezium rule with 4 subintervals to approximate the following integrals:
  (a) $\int_{-1}^{1}(x^3+2x^2)\mathrm{d}x$;
  (b) $\int_0^{\pi/4}\tan(x)\mathrm{d}x$;
  (c) $\int_1^2 \mathrm{e}^{-2x}\mathrm{d}x$.

Q2. Suppose the composite trapezium rule with N intervals has been used to approximate $I=\int_a^b f(x)\mathrm{d}x$ and that the resultant approximation $\hat{I}$ is an overestimate of I. In the following questions, assume that the trapezium rule has again been used with N intervals.
  (a) What is the value of the approximation of $\int_a^b (f(x)+12)\mathrm{d}x$?
  (b) What is the value of the approximation of $\int_a^b (f(x)+3x)\mathrm{d}x$?
  (c) What is the value of the approximation of $\int_a^b (f(x)+cx)\mathrm{d}x$, where c is any real number?
  (d) What can be said about the approximation of $\int_a^b 2f(x))\mathrm{d}x$?
  (e) What can be said about the approximation of $\int_a^b 5f(x-3))\mathrm{d}x$?
  (f) What can be said about the approximation of $\int_a^b (2-f(x))\mathrm{d}x$?

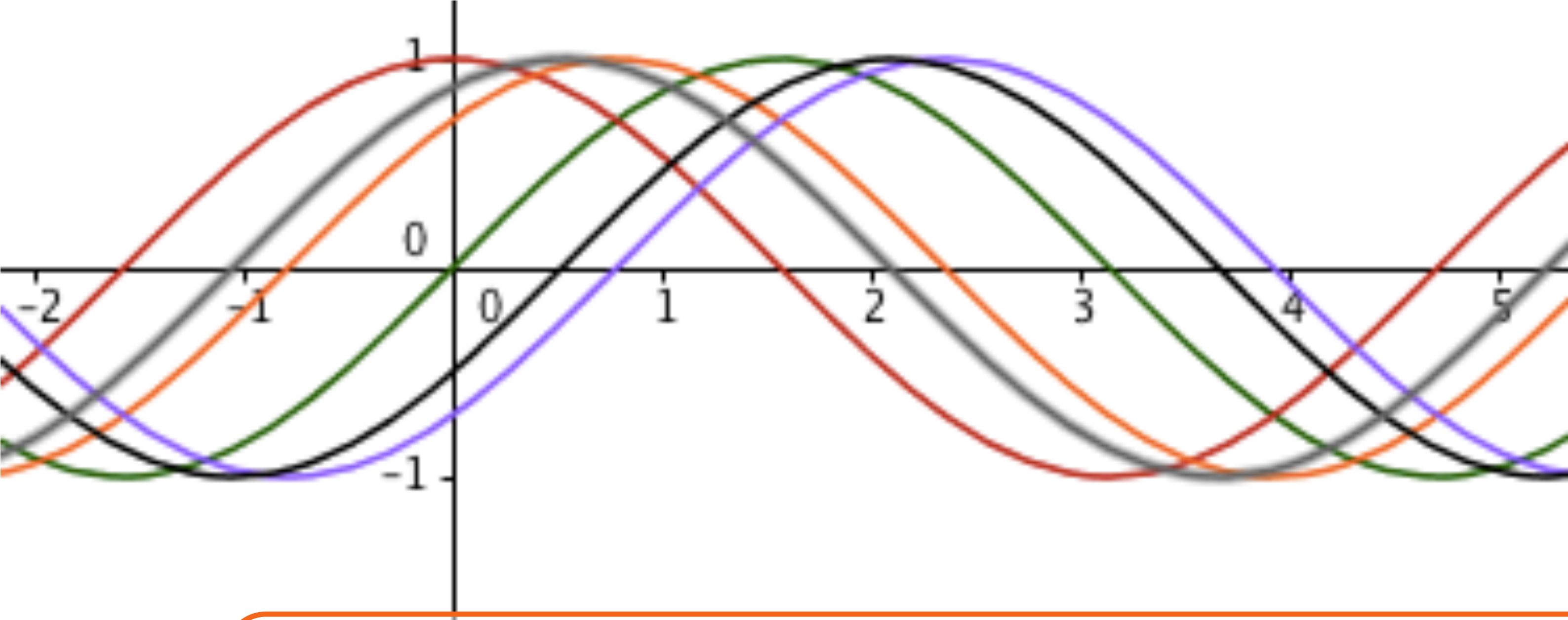
Q3. Consider the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Use the trapezium rule with $N = 5$ to approximate the definite integral

$$\int_{-\infty}^{z} f(x)\mathrm{d}x,$$

when
(a) $z = 0.1$;
(b) $z = 0.5$;
(c) $z = 1.0$;
(d) $z = 2.0$,
assuming $\int_{-\infty}^{0} f(x)\mathrm{d}x = 0.5$. How do these results compare with the tabulated cumulative probabilities for the normal distribution, with $z = 0.1$, $z = 0.5$, $z = 1.0$ and $z = 2.0$?

Q4. Consider the integral $I = \int_0^2 3^x \mathrm{d}x$.
(a) Write down the trapezium rule approximation of I in summation notation, when N subintervals are used.
(b) By noticing that the terms in the sum form a geometric progression, find an expression for the trapezium rule approximation that no longer requires a sum.
(c) Use the expression from above to calculate approximations to I when
i. $N = 10$;
ii. $N = 100$;
iii. $N = 1000$.
(d) How do the values found in part (c) compare with the exact values of I?

Q5. On a motorway, a car is recorded travelling at speeds of $30\mathrm{ms}^{-1}$, $20\mathrm{ms}^{-1}$, $16\mathrm{ms}^{-1}$, $32\mathrm{ms}^{-1}$, $9\mathrm{ms}^{-1}$ and $25\mathrm{ms}^{-1}$ at times $t = 0\mathrm{s}$, $t = 17\mathrm{s}$, $t = 58\mathrm{s}$, $t = 78\mathrm{s}$, $t = 118\mathrm{s}$ and $t = 143\mathrm{s}$, respectively, by speed cameras equally spaced along the road. Use the trapezium rule to obtain an estimate for the total distance travelled by the car and hence an approximation to the distance between speed cameras. (Hint: The composite trapezium rule will need some modification before it can be used for this problem.)



4. Worked Solutions and Numerical Answers

Teachers will have access to full worked solutions for all exercises. A sample for one of the exercises is shown below.

Worked Solutions — Exercise 1.1.

Q1. Given in the order $\sin(\theta)$,$\cos(\theta)$, $\tan(\theta)$ and all to 4 decimal places.

(a) $0, 1, 0$

(b) $0.0654, 0.9979, 0.0654$

(c) $0.1309, 0.9914, 0.1309$

(d) $0.2094, 0.9781, 0.2094$

(e) $0.2618, 0.9657, 0.2618$

(f) $0.5235, 0.8629, 0.5236$

Q2. First we need to convert the angle into radians, $18^\circ = \pi/10^c$, and so $\sin(18^\circ) \approx 0.3141$.

(a) For the percentage error we calculate:

$$\frac{\left|\sin\left(\frac{\pi}{10}\right) - \frac{\pi}{10}\right|}{\sin\left(\frac{\pi}{10}\right)} \times 100 \approx 1.66\%.$$

(b) For the percentage error we calculate:

$$\frac{\left|\cos\left(\frac{\pi}{10}\right) - \left(1 - \frac{1}{2}\left(\frac{\pi}{10}\right)^2\right)\right|}{\cos\left(\frac{\pi}{10}\right)} \times 100 \approx 0.04\%.$$

(c) For the percentage error we calculate:

$$\frac{\left|\tan\left(\frac{\pi}{10}\right) - \frac{\pi}{10}\right|}{\tan\left(\frac{\pi}{10}\right)} \times 100 \approx 3.31\%.$$