GRADE 12 EXAMINATION
NOVEMBER 2016

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

Time: 2 hours 200 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 7 pages and an Information Booklet of 2 pages (i–ii). Please check that your question paper is complete.
2. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.
3. All necessary calculations must be clearly shown and writing should be legible.
4. Diagrams have not been drawn to scale.
5. Trigonometric calculations should be done using radians and answers should be given in radians.
6. Round off your answers to two decimal digits, unless otherwise indicated.

QUESTION 1

1.1 Solve for $x \in R$, without the use of a calculator:

(a) $|x+3| + 2x = 4$ (7)

(b) $\cos^{-1}\left(\frac{x^2}{8}\right) = \frac{\pi}{3}$ (4)

(c) $lnx^2 - 3log_x e = 1$ (8)

1.2 Determine the domain and range of the graph of:

$y = ln(e^2 - x^2)$ (8)

[27]

QUESTION 2

2.1 Simplify: $\sqrt{i^4}$ (2)

2.2 Find the real values of $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $(3+2i)(a+3i) = bi$ (7)

2.3 One of the solutions to the equation $x^2 - 2x + p = 0$ is $x = q + \sqrt{3}i$.

Find the rational values of $\boldsymbol{p}$ and $\boldsymbol{q}$. (7)

[16]

QUESTION 3

3.1 State whether each of the following statements is TRUE or FALSE:

(a) If a function is differentiable at a point, then the limit of the function must exist at that point.

(b) If a function is continuous at a point, then it must also be differentiable at that point.

(c) If the limit of the function does not exist at a point, then the graph will have an asymptote at that point.

(d) If the second derivative of a function at a point is equal to zero, then there will be a point of inflection on the graph at that point.

(e) A function can exist at a point, whether or not the limit of the function exists at that point.

(f) At a local maximum, the ***gradient*** of the graph is decreasing. (12)

3.2 A function is defined as follows:

$$f(x) = \begin{cases} p - x^2 & if \quad x \le 2 \\ qx + 10 & if \quad x > 2 \end{cases}$$

Calculate the value(s) of p and q if f is differentiable at $x = 2$. (8)

3.3 It is given that $f(x) = x^2 - 6x + 5$ and $g(x) = |x|$.

Sketch on separate sets of axes:

(a) $y = g(f(x))$ (5)

(b) $y = f(g(x))$ (8)

[33]

QUESTION 4

Alisha wants to prove by induction that $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Her teacher has taught her the following procedure:

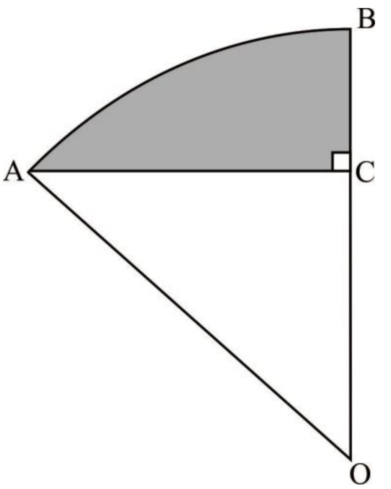
Step 1: Prove true for $n = 1$
Step 2: Assume true for $n = k$
Step 3: Prove true for $n = k + 1$
Step 4: Conclude the proof.

Show Alisha's working for Step 3.

[10]

QUESTION 5

The diagram shows an arc, AB, of a circle with centre O and radius r. The line AC is drawn perpendicular to the line OCB. The region bounded by AC, BC and arc AB is shaded. $A\hat{O}B = \frac{\pi}{6}$ radians.



5.1 Find the area of ΔAOC, in terms of r. (6)

5.2 Find the area of the sector OAB, in terms of r. (4)

5.3 If the shaded area is $\frac{2\pi - 3\sqrt{3}}{6}$ cm^2, calculate the value of r. (6)

[16]

QUESTION 6

Match the following rational functions to the appropriate graphs, A–F, below:

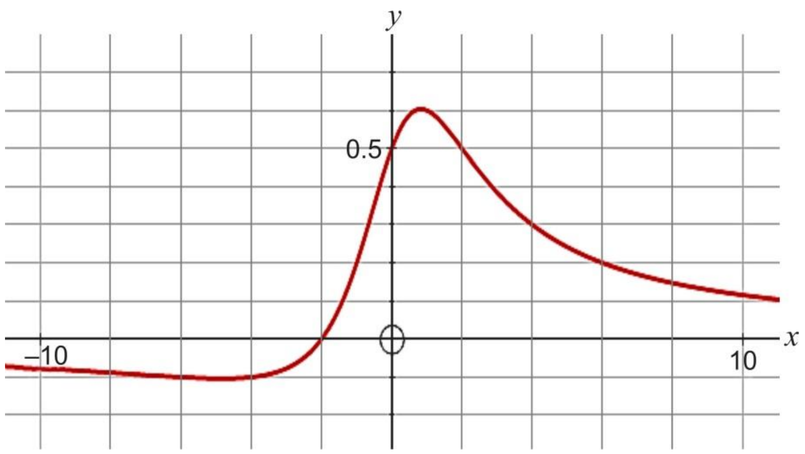
6.1 $f(x) = \frac{x^2-4}{x+2}$

6.2 $f(x) = \frac{x-2}{x^2-x-6}$

6.3 $f(x) = \frac{x^2+x-6}{x^2-6x+9}$

6.4 $f(x) = \frac{x^2-4x+4}{x-3}$

A



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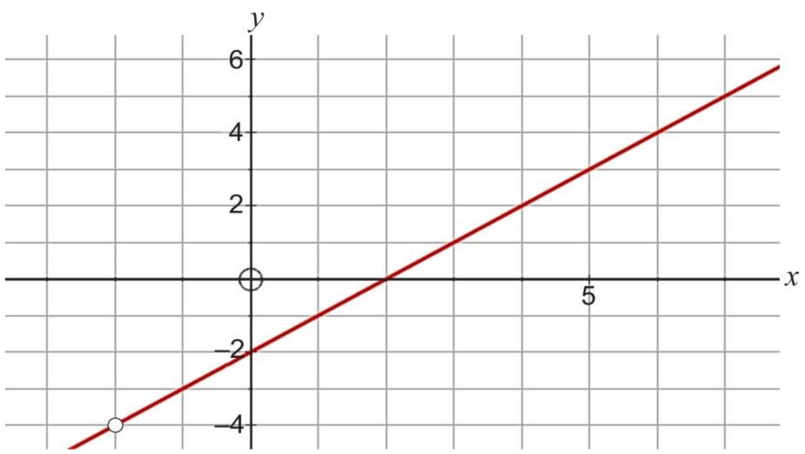

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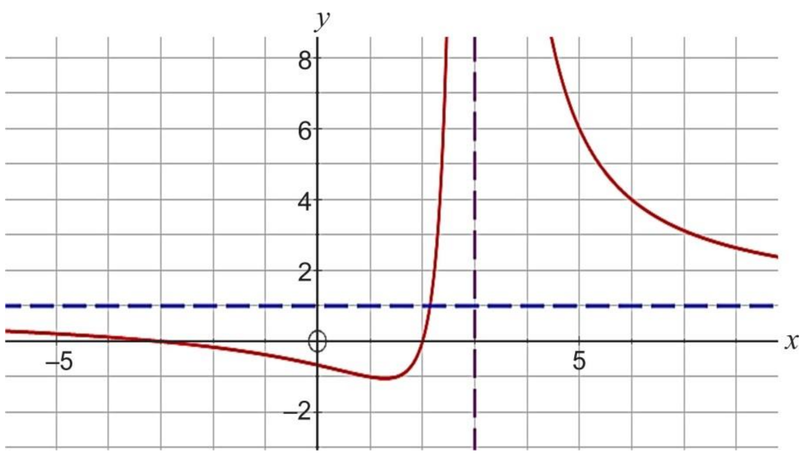

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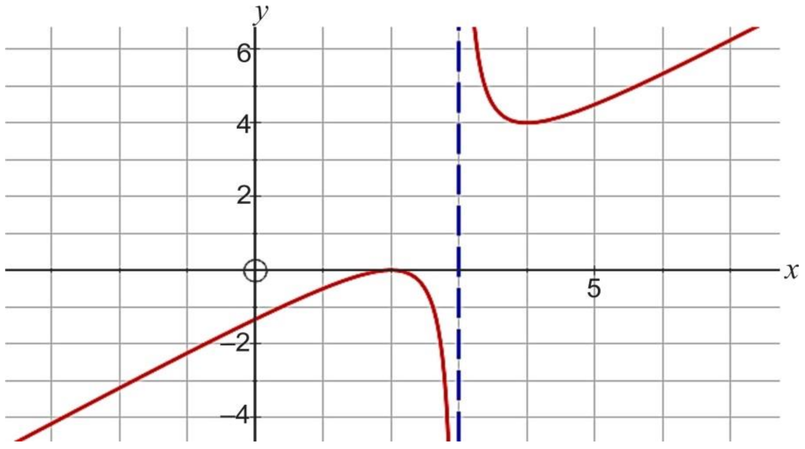

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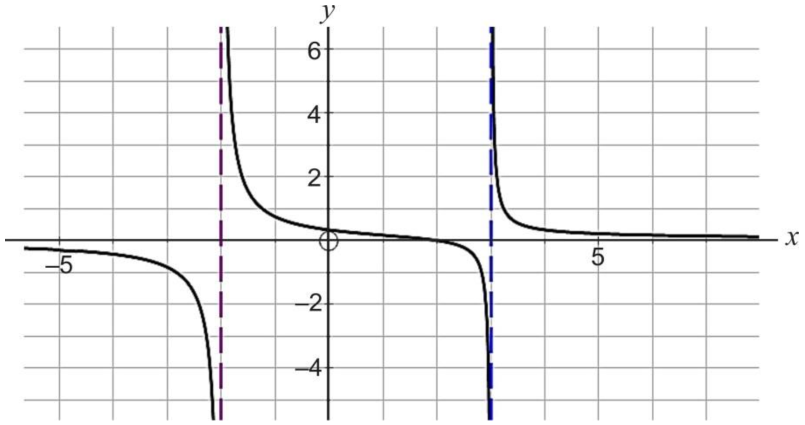

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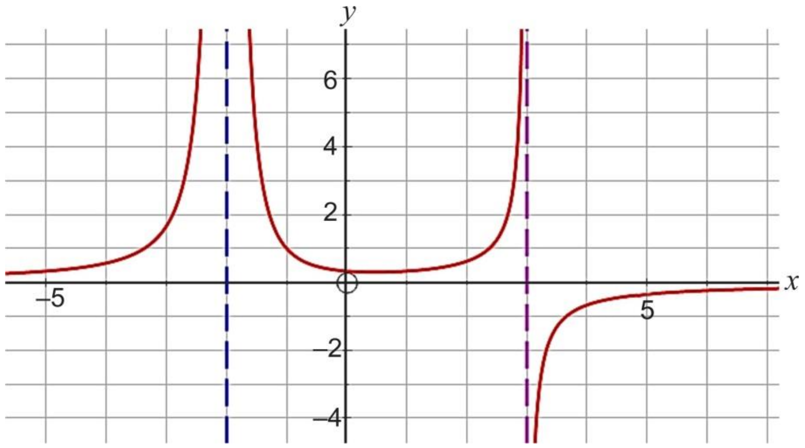


[12]

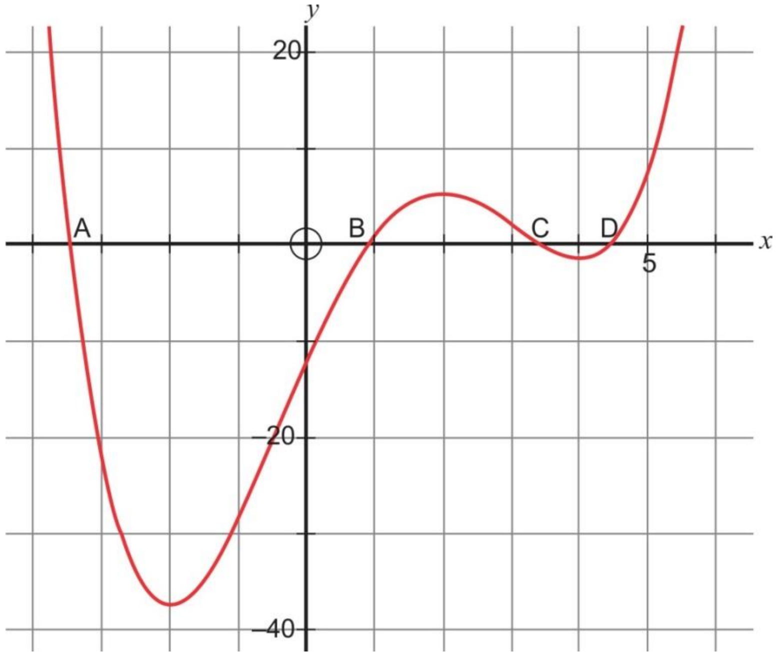
QUESTION 7

Find the equation of the tangent to the curve $x^3 - 2y^2 = 14 - 4x$ at the point (2; 1).

[11]

QUESTION 8

The graph of $f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 - 2x^2 + 16x - 12$ is shown, with stationary points at $x = -2$, $x = 2$ and $x = 4$. The graph has x-intercepts at the points indicated by A, B, C and D.



8.1 Without first solving the equation, state with clear justification which of the intercepts, A, B, C or D, will be found using Newton's method with an initial approximation of $x_0 = 2{,}1$. (3)

8.2 State any restrictions on the initial approximation of x. (2)

8.3 Given $x_0 = 3$, determine the x-intercept at C, correct to 6 decimal places. (8)

[13]

QUESTION 9

A function is given as $f(x) = x + 4(x + 1)^{-2}$

9.1 Determine the coordinates of the stationary point and prove that this is a local minimum. (10)

9.2 Write down the equation of the oblique asymptote. (2)

9.3 Determine the area between the graph and the x-axis on the interval $-p \leq x \leq p$, with $| p | < 1$, giving your answer in terms of p, in its simplest form. (6)

[18]

QUESTION 10

10.1 Integrate the following:

(a) $\int \left(\sqrt{x} + x^{-1}\right)^2 dx$ (7)

(b) $\int \tan^5 2x \cdot \sec^2 2x \, dx$ (8)

(c) $\int \frac{x}{\sqrt{2-x}} dx$ (9)

10.2 The integral of a function f is found using a Riemann sum, such that:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{3}{n} \sum_{r=1}^{n} \left[2\left(-1 + \frac{3r}{n}\right)^2 + 1 \right]$$

Deduce, from this statement, the values of a and b, and state the function, f.
(You are not required to evaluate the Riemann sum.) (7)

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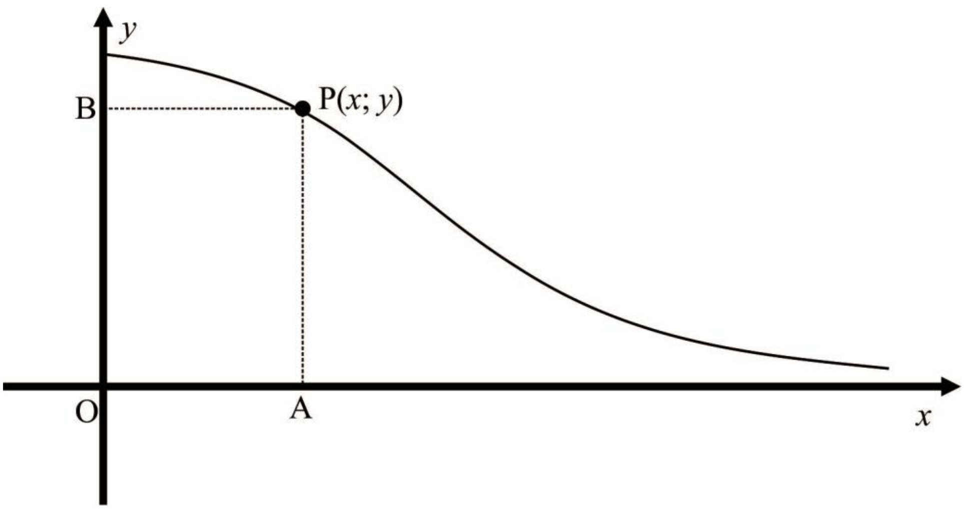
QUESTION 11

Refer to the diagram below.

A point P(x; y) moves along the curve defined by $y = \frac{1}{x^2+4}$ and $x > 0$. Point A lies on the x-axis and point B on the y-axis such that OAPB is a rectangle.



Find, by using calculus methods, the maximum area of rectangle OAPB.

[13]

Total: 200 marks