5.5 Exponential Bases other than *e*

TOOTLIFTST:

- Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions.

We have already seen how *e*asy it is to work with the exponential and logarithmic bases *e*, but as you can imagine, not all the problems we will come in contact with will have this nice base. So the question naturally arises: how do we deal with bases other than *e*? That is what this lesson is about.

The good news is that we treat them almost the same, there is only a slight modification, and we can always make a problem that has a base other than *e* in terms of *e*. I know it sounds too good to be true, but it is. Here's how

Definition of Exponential Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined as

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

The reason the above equation works is due to the properties of natural logs from the last section. Here they are again in case you need them.

$\ln e^x = e^{\ln x} = x$ and $\ln a^x = x(\ln a)$

Here's the above definition's counterpart:

Definition of Logarithmic Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base *a*** is denoted by $\log_a x = \frac{1}{\ln a}\ln x = \frac{\ln x}{\ln a}$

This is nothing more than the change of base formula from your Algebra II and Precalculus Class. All the rules of natural logs (expanding, condensing, etc.) also hold for logs of any other base a.

Now from the definition of exponential and logarithmic functions to the base *a*, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other. We summarize this in the familiar theorem below that allows us to convert between the two notations.

$y = a^x$ if and only if $x = \log_a y$
$a^{\log_a x} = x$ for $x > 0$
$\log_a a^x = x$ for all x

Also, remember the common base 10 is notated $\log_{10} x$ or it is understood to be 10 without it written, as in $\log x$.

Let's sharpen up our exponential equation solving ability before we get to the calculus part.

Solve for *x* in the following examples.

a. $3^x = \dfrac{1}{27}$

Recall the first method you hope to employ is what I call the GTBTS or Getting The Bases The Same method. With a little rewriting, if you can get a base to the "somethingth" power equal to the same base to a "somethingth" power, you can set the "somethings" equal to each other.

$3^x = 3^{-3}$
$x = -3$

b. $\log_2 x = -4$

To solve this one, it is more helpful to rewrite it in exponential format, by using the above theorem.

$= 2^{-4} = x = \dfrac{1}{2^4} = \dfrac{1}{16}$

c. $3^{x+1} = 8$

Notice on this one, we cannot get the bases the same. In times like these, we can always take the natural log (or log of any base) of both sides.

$$\ln 3^{x+1} = \ln 8$$
$$(x+1)\ln 3 = \ln 8$$
$$x+1 = \frac{\ln 8}{\ln 3}$$
$$x = \frac{\ln 8}{\ln 3} - 1 \approx 0.893$$

Now for the moment you all have been waiting for: the application to Calculus.

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options:

1. use the definitions of a^x and $\log_a x$ from above, and differentiate using the rules for the natural exponential and logarithmic functions.
2. Use Logarithmic Differentiation (LOG DIFF—Remember this one?!)
3. use the following differentiation rules for bases other than *e*.

Derivatives for Bases other than *e*

Let *a* be a positive real number $(a \neq 1)$ and let *u* be a differentiable function of *x*.

1. $\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$
2. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \underbrace{\left(\frac{du}{dx}\right)}_{\text{chain rule}}$
3. $\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}\left[\log_a u\right] = \frac{1}{(\ln a)u} \underbrace{\left(\frac{du}{dx}\right)}_{\text{chain rule}}$

Notice the rules are almost identical for those of the natural base, except for the additional factor of $\ln a$, which is either multiplied or divided by.

Now these rules are not that difficult to memorize, but you already have the first two methods from above memorized, so it is entirely up to YOU which way you want to pursue these problems. I will work some examples showing all three ways, and from then on, I will use the new rules predominately, or LOG DIFF.

Find the derivative of the following:

a. $y = 2^x$

i. Method 1: Rewriting this as $\log_2 y = x$ will not make this problem any easier.

ii. Method 2: LOG DIFF

$\ln y = x\ln 2$

$\frac{d}{dx}[\ln y = x\ln 2]$

$\Rightarrow \frac{y'}{y} = \ln 2$

$\Rightarrow y' = y\ln 2 = (\ln 2)2^x$

iii. By the rule, the answer comes immediately.

b. $y = 2^{3x}$

$$y' = \underbrace{(2^{3x})}_{itself}\underbrace{(3)}_{chain\,rule}\underbrace{(\ln 2)}_{rule} = 3\ln 2(2^{3x})$$

this final expression can be condensed to $\ln 2^3(2^{3x}) = \ln 8(2^{3x})$ although either answer is correct. If you rewrote the original expression as $y = (2^3)^x = 8^x$, you get an equivalent answer.

c. $y = \log_{10}\cos x$

$$y' = \underbrace{\left(\frac{-\sin x}{\cos x}\right)}_{\text{deriv over itself}}\underbrace{\left(\frac{1}{\ln 10}\right)}_{\text{rule}} = \frac{-\tan x}{\ln 10}$$

Notice in b. and c., you can use the exact same rules you did for natural logs, you just cannot forget to either multiply or divide by $\ln a$, whatever a is, appropriately.

Now for Integration:

If we were staring at $\int a^x dx$, just as if a were an e, we would anticipate the integral to be a^x, but we now know that if this were our answer, upon checking by differentiating, we would generate $(\ln a)a^x$, leaving us with an unwanted $\ln a$, which remember is just a number, a scalar multiple. Our guess then would have to have a correction of $\frac{1}{\ln a}$ in it. This gives us a new rule:

$$\int a^x dx = \left(\frac{1}{\ln a}\right)a^x + C \quad \text{or} \quad \frac{a^x}{\ln a} + C$$

Let's take it out for a spin!

Evaluate $\int 2^x \, dx$

Using the rule, we get $= \dfrac{2^x}{\ln 2} + C$

Not so bad . . . if you know the rule, and can recognize it.

The following are some interesting examples.

a. $\dfrac{d}{dx} e^e = 0$, Remember that e^e is just a constant!

b. $\dfrac{d}{dx} x^e = ex^{e-1}$, Remember that e is just a constant. See letter a.

c. $\dfrac{d}{dx}\left[y = x^x\right]$, we need to use LOG DIFF here since we have not encountered a rule for this yet. It is NOT $y' = x\left(x^{x-1}\right)$.

$\Rightarrow \dfrac{d}{dx}\left[\ln y = x \ln x\right]$

$\Rightarrow \dfrac{y'}{y} = \underbrace{(1)(\ln x) + (x)\left(\dfrac{1}{x}\right)}_{\text{product rule}} = \ln x + 1$

$\Rightarrow y' = x^x\left(\ln x + 1\right)$

THE END