Problem Set 1: Sketch of Solutions

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Problem 1.

Consider the following "portfolio choice" problem. The investor has initial wealth w and utility $u(x) = \ln(x)$. There is a *safe asset* (such as a US government bond) that has net real return of zero. There is also a *risky asset* with a random net return that has only two possible returns, R_1 with probability q and R_0 with probability $1-q$. Let A be the amount invested in the risky asset, so that $w - A$ is invested in the safe asset.

1. Find A as a function of w. Does the investor put more or less of his portfolio into the risky asset as his wealth increases?
2. Another investor has the utility function $u(x) = -e^{-x}$. How does her investment in the risky asset change with wealth?
3. Find the coefficients of absolute risk aversion $r(x) = -\frac{u''(x)}{u'(x)}$ for the two investors. How do they depend on wealth? How does this account for the qualitative difference in the answers you obtain in parts (1) and (2)?

Solution of Problem 1:

Firstly, let's set up the problem:

$$\max_{A\in[0,w]} \{q\,u\left((1+R_1)A + w - A\right) + (1-q)\,u\left((1+R_0)A + w - A\right)\}$$

Part 1: When we have a specific utility function $u(x) = \ln(x)$, we can get the first order condition as follows:

$$\begin{aligned} q\frac{R_1}{R_1A+w} + (1-q)\frac{R_0}{R_0A+w} &= 0 \\ \Longrightarrow A &= -w\frac{qR_1 + (1-q)R_0}{R_0R_1} \end{aligned}$$

Since, her utility function is concave, basically we can say, she is risk averse. So, we can argue that $qR_1 + (1-q)R_0 > 0 = r$. Otherwise, the investor will not invest in the risky asset at all. WLOG, we assume $R_1 < 0$, $R_0 > 0$. Otherwise, the investor will not invest in the risky asset or will invest all her wealth in the risky asset. Therefore, we can observe $\frac{dA}{dw} > 0$. That is, the investor will put more of her portfolio into the risky asset when she gets wealthier.

Part 2: From the first order condition,

$$\begin{aligned} R_1 q e^{-(R_1 A + w)} + R_0(1-q)e^{-(R_0 A + w)} &= 0 \\ \implies A &= \frac{1}{R_0 - R_1} \ln\left[-\frac{R_0(1-q)}{R_1 q}\right] \end{aligned}$$

Observe that $\frac{dA}{dw} = 0$; that is, her investment in the risky asset doesn't change with wealth.

Part 3: For $u(x) = \ln(x)$, we have $u'(x) = \frac{1}{x}$ and $u''(x) = -\frac{1}{x^2}$. So $r(x) = \frac{1}{x}$; i.e., as x gets bigger, $r(x)$ gets smaller, and so the wealthier the investor is, the less risk averse she is. Therefore, she will put more wealth into the risky asset.

For $u(x) = -e^{-x}$, we have $u'(x) = e^{-x}$ and $u''(x) = -e^{-x}$. So $r(x) = 1$. Therefore, the amount that the investor allocates to the risky asset is independent of her wealth.

Problem 2.

You have an opportunity to place a bet on the outcome of an upcoming race involving a certain female horse named Bayes: if you bet x dollars and Bayes wins, you will have $w_0 + x$, while if she loses you will have $w_0 - x$, where w_0 is your initial wealth.

1. Suppose that you believe the horse will win with probability p and that your utility for wealth w is $\ln(w)$. Find your optimal bet as a function of p and w_0.
2. You know little about horse racing, only that racehorses are either winners or average, that winners win 90% of their races, and that average horses win only 10% of their races. After all the buzz you've been hearing, you are 90% sure that Bayes is a winner. What fraction of your wealth do you plan to bet?
3. As you approach the betting window at the track, you happen to run into your uncle. He knows rather a lot about horse racing: he correctly identifies a horse's true quality 95% of the time. You relay your excitement about Bayes. "Don't believe the hype," he states. "That Bayes mare is only an average horse." What do you bet now (assume that the rules of the track permit you to receive money only if the horse wins)?

Solution of Problem 2:

Part 1: The expected utility from betting x is:

$$EU(x) = p\ln(w_0 + x) + (1-p)\ln(w_0 - x)$$

Your objective is to choose x to maximize your expected utility. The first order condition w.r.t x is

$$\begin{aligned} \frac{p}{w_0 + x} &= \frac{1-p}{w_0 - x} \\ x^* &= w_0(2p - 1) \end{aligned}$$

Part 2: Your probability that Bayes will win can be determined as follows:

$$p = 0.9 \times 0.9 + 0.1 \times 0.1 = 0.82$$

Therefore, using the formula from part 1, we obtain $x^* = w_0(2 \times 0.82 - 1) = 0.64w_0$

Part 3: Let θ denote the true type of Bayes. "$\theta = 1$" means Bayes is a winner, "$\theta = 0$" means Bayes is average. Let s denote the signal from your uncle. "$s = 1$" means uncle asserts Bayes is a winner, and "$s = 0$" means uncle asserts Bayes is average. The uncle's signal is accurate 95% of the time, *i.e.*,

$$\Pr(s = 1|\theta = 1) = \Pr(s = 0|\theta = 0) = 0.95$$

Therefore, the updated belief is

$$\begin{aligned}
\Pr(\theta = 1|s = 0) &= \frac{\Pr(\theta = 1,\, s = 0)}{\Pr(s = 0)} \\
&= \frac{\Pr(s = 0|\theta = 1) \times \Pr(\theta = 1)}{\Pr(s = 0,\, \theta = 1) + \Pr(s = 0,\, \theta = 0)} \\
&= \frac{\Pr(s = 0|\theta = 1) \times \Pr(\theta = 1)}{\Pr(s = 0|\theta = 1) \times \Pr(\theta = 1) + \Pr(s = 0|\theta = 0) \times \Pr(\theta = 0)} \\
&= \frac{0.05 \times 0.9}{0.05 \times 0.9 + 0.95 \times 0.1} = 0.32
\end{aligned}$$

Using the formula from part 1 again, we obtain $x^* = -0.357w_0 < 0$. You would like to bet against Bayes, but this is not allowed, so the optimal choice is to bet nothing.

Problem 3.

If an individual devotes a units of effort in preventative care, then the probability of an accident is $1 - a$ (thus, effort can only assume values in $[0, 1]$). Each individual is an expected utility maximizer with utility function $p \ln(x) + (1-p) \ln(y) - a^2$, where p is the probability of an accident, x is wealth if there is an accident, and y is wealth if there is no accident. If there is no insurance, then $x = 50$, while $y = 150$.

1. Suppose first there is no market for insurance. What level of a would the typical individual choose? What would her expected utility be?

2. Assume that a is verifiable. What relationship do you expect to prevail between x and y in a competitive insurance market? What relationship do you then expect to prevail between x and a?

3. Derive the value of a, x and y that maximize the typical customer's expected utility. What is the value of this maximized expected utility?

4. Suppose that a is not verifiable. What would happen (*i.e.*, what would the level of a and expected utility be) if the same contract (*i.e.*, same x and y values) as in (3) were offered by competitive firms? Do you expect this would be an equilibrium?

5. Under the non-verifiability assumption, what relationship must prevail between x, y, and a? Use this relationship along with the assumption of perfect competition to derive a relationship between x and

a that contracts offered by insurers must have. Finally, find the level of a that maximizes the expected utility of the typical consumer, and find that level of expected utility.

6. Summarize your answers by ranking the levels of a and the expected utilities for each of the cases in (1), (3), (4) and (5). What do you notice?

Solution of Problem 3:

Part 1: The consumer solves

$$\max_a \left\{(1-a)\ln(50) + a\ln(150) - a^2\right\}$$

The first order condition w.r.t a is $-\ln(50) + \ln(150) = 2a$, which in turn implies that $a = \frac{\ln(3)}{2} \simeq 0.55$.

Part 2: Without moral hazard, the consumer will be fully insured, so $x = y$. Perfect competition implies that firms will make 0 profits, so $x = y = (1-a)50 + a150 = 50 + 100a$.

Part 3: Using the answer from part 2, we solve

$$\max_a \left\{\ln(50+100a) - a^2\right\}$$

It follows from the first order condition that $a = \frac{1}{2}$, and so $x = 50 + 100a = 100$

Part 4: if $x = y = 100$ and a is not verfiable, then the consumer will set $a = 0$. Since $x = 100$ from above, firms lose money (they get 50 and pay out 100 always), so this cannot be an equilibrium.

Part 5: Each firm solves

$$\begin{aligned}
\max_{a,x,y} \quad & \left\{(1-a)\ln(x) + a\ln(y) - a^2\right\} \\
\text{s.t.} \quad & (1-a)x + ay = (1-a)50 + a150 \qquad \text{(ZP)} \\
& a \in \arg\max \left\{(1-a)\ln(x) + a\ln(y) - a^2\right\} \qquad \text{(IC)}
\end{aligned}$$

The first order condition for (IC) is $\ln\left(\frac{y}{x}\right) = 2a$, so $y = e^{2a}x$. Plugging this into (ZP) yields $x = \frac{50+100a}{1+a(e^{2a}-1)}$, and so the problem simplifies to

$$\max_a \left\{a^2 + \ln\left(\frac{50+100a}{1+a\left(e^{2a}-1\right)}\right)\right\}$$

After some algebra, the first order condition simplifies to $2(2a^3 - a^2 - a) = \frac{e^{2a}-3}{e^{2a}-1}$. Using a software package such as Matlab, one obtains $a \simeq 0.37$, and plugging back we get the expected utility 4.26.

Part 6: The first best in (c) gives the consumer the highest expected utility with $a = 0.5$, the second best in (e) yields the second highest expected utility with $a = 0.37$, and no insurance has the lowest expected utility with $a = 0.55$. Hence, we notice that, consumers will have a lower expected utility without insurance, even though they have devoted a higher effort a to prevent accidence.

Problem 4.

An agent can work for a principal. The agent's effort, a affects current profits, $q_1 = a + \varepsilon_{q_1}$, and future profits, $q_2 = a + \varepsilon_{q_2}$, where ε_{q_t} are random shocks, and they are i.i.d with normal distribution $N(0, \sigma_q^2)$. The agent retires at the end of the first period, and his compensation cannot be based on q_2. However, his compensation can depend on the stock price $P = 2a + \varepsilon_P$, where $\varepsilon_P \sim N(0, \sigma_P^2)$. The agent's utility function is exponential and equal to

$$-e^{-\eta\left[t - c\frac{a^2}{2}\right]}$$

where t is the agent's income, while his reservation utility is $\bar{t}$.[1] The principal chooses the agent's compensation contract $t = w + fq_1 + sP$ to maximize her expected profit, while accounting for the agent's IR and IC constraints.

1. Derive the optimal compensation contract $t = w + fq_1 + sP$.
2. Discuss how it depends on σ_P^2 and on its relation with σ_q^2. Offer some intuition?

Solution of Problem 4:

The program for this problem is the following,

$$\max_{a,w,f,s} E\left(q_1 + q_2 - t\right)$$

subject to

$$\begin{aligned} E\left(-e^{-\eta\left[t - c\frac{a^2}{2}\right]}\right) &\geq -e^{-\eta\bar{t}} \quad (1) \\ a &\in \arg\max_{\widehat{a}} E\left(-e^{-\eta\left[t - c\frac{a^2}{2}\right]}\right) \quad (2) \\ t &= w + fq_1 + sP \quad (3) \end{aligned}$$

$\Leftrightarrow$

$$\max_{a,w,f,s} 2a - (w + fa + 2sa)$$

subject to

$$w + fa + 2sa - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{c}{2}a^2 \geq \bar{t}$$
$$a \in \arg\max_{\widehat{a}} w + f\widehat{a} + 2s\widehat{a} - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{c}{2}\widehat{a}^2.$$

We can apply the first-order approach and substitute the first order condition

$$a = \frac{f + 2s}{c}$$

for the incentive compatibility constraint. Introducing this efficient level of effort and substituting the out-

[1]Reservation utility $\bar{t}$ means that the agent's IR constraint requires that $\mathbb{E} - e^{-\eta\left[t - c\frac{a^2}{2})\right]} \geq -e^{-\eta\bar{t}}$.

side opportunity level plus the risk premium plus the cost of effort for the wage, we obtain

$$\max_{f,s} 2\frac{f+2s}{c} - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{(f+2s)^2}{2c} - \bar{t}.$$

The first order conditions with respect to f and s are respectively

$$\begin{aligned} \frac{2}{c} - \eta\left(f^*\sigma_q^2 + s^*\sigma_{qP}\right) - \frac{f^*+2s^*}{c} &= 0 \\ \frac{4}{c} - \eta\left(s^*\sigma_P^2 + f^*\sigma_{qP}\right) - 2\frac{f^*+2s^*}{c} &= 0. \end{aligned}$$

After some rewriting, the equations become

$$\begin{aligned} f^* &= \frac{2 - s^*(2+\eta c\sigma_{qP})}{1+\eta c\sigma_q^2} \\ f^* &= \frac{2 - s^*(2+\frac{\eta c\sigma_P^2}{2})}{1+\frac{\eta c\sigma_{qP}}{2}}, \end{aligned}$$

and we finally find

$$\begin{aligned} f^* &= \frac{\sigma_P^2 - 2\sigma_{qP}}{2\sigma_q^2 + \frac{\sigma_P^2}{2} - 2\sigma_{qP} + \frac{\eta c}{2}(\sigma_P^2\sigma_q^2 - \sigma_{qP}^2)} \\ s^* &= \frac{2\sigma_q^2 - \sigma_{qP}}{2\sigma_q^2 + \frac{\sigma_P^2}{2} - 2\sigma_{qP} + \frac{\eta c}{2}(\sigma_P^2\sigma_q^2 - \sigma_{qP}^2)}. \end{aligned}$$

Problem 5.

Two agents can work for a principal. The output of agent i ($i = 1, 2$), is $q_i = a_i + \varepsilon_i$, where a_i is agent i's effort level and ε_i is a random shock. The ε_i's are independent of each other and normally distributed with mean 0 and variance σ^2. In addition to choosing a_2, agent 2 can engage in a second activity b_2. This activity does not affect output directly, but rather reduces the effort cost of agent 1. The interpretation is that agent 2 can *help* agent 1 (but not the other way around). The effort cost functions of the agents are

$$\psi_1(a_1, b_2) = \frac{1}{2}(a_1 - b_2)^2$$

and

$$\psi_2(a_2, b_2) = \frac{1}{2}a_2^2 + b_2^2.$$

Agent 1 chooses her effort level a_1 only after she has observed the level of help b_2. Agent i's utility function is exponential and equal to

$$-e^{[-\eta(w_i - \psi_i(a_i, b_2))]}$$

where w_i is the agent's income. The agent's reservation utility is -1, which corresponds to a reservation wage of 0. The principal is risk neutral and is restricted to linear incentive schemes. The incentive scheme for agent i is

$$w_i = z_i + v_i q_i + u_i q_j$$

1. Assume that a_1, a_2, and b_2 are observable. Solve the principal's problem by maximizing the total expected surplus with respect to a_1, a_2, and b_2. Explain intuitively why $a_1 > a_2$.
2. Assume from now on that a_1, a_2, and b_2 are not observable. Solve again the principal's problem. Explain intuitively why $u_1 = 0$.
3. Assume that the principal cannot distinguish whether a unit of output was produced by agent 1 or agent 2. The agents can thus engage in *arbitrage*, claiming that all output was produced by one of them. Assume that they will do so whenever it increases the sum of their wages. Explain why the incentive scheme in part 2 above leads to arbitrage. What additional constraint does arbitrage impose on the principal's problem? Solve this problem, and explain intuitively why $u_1 > 0$.

Solution of Problem 5:

Part 1: First-Best Outcome The principal can observe a_1, a_2, and b_2 and maximizes the total expected surplus

$$\max_{a_1,a_2,b_2} \mathbb{E}S = a_1 + a_2 - \frac{1}{2}\left[(a_1 - b_2)^2 + a_2^2 + 2b_2^2 + \eta\mathbb{V}(w)\right].$$

The first order conditions yield

$$\begin{aligned} 1-(a_1-b_2) &= 0 \\ 1-a_2 &= 0 \\ a_1-b_2-2b_2 &= 0. \end{aligned}$$

Solving these equations we obtain

$$\begin{aligned} a_1 &= \frac{3}{2} \\ a_2 &= 1 \\ b_2 &= \frac{1}{2}. \end{aligned}$$

Note that $a_1 > a_2$, that is, agent 1 exerts more effort in activity a_1 than agent 2 in a_2 since at an interior solution where agent 2 exerts positive effort in activity b_2 agent 1's marginal cost is lower.

Part 2: Unobservable Effort and Linear Contracts The principal's problem is to solve

$$\max \mathbb{E}\pi = a_1(1 - v_1 - u_2) + a_2(1 - v_2 - u_1) - z_1 - z_2$$

subject to

$$z_1 + v_1a_1 + u_1a_2 - \frac{\eta}{2}\sigma^2(v_1^2 + u_1^2) - \frac{1}{2}(a_1 - b_2)^2 \geq 0 \tag{4}$$

$$z_2 + v_2a_2 + u_2a_1 - \frac{\eta}{2}\sigma^2(v_2^2 + u_2^2) - \frac{1}{2}a_2^2 - b_2^2 \geq 0 \tag{5}$$

and

$$
\begin{aligned}
a_1 &\in \arg\max\left\{z_1 + v_1 a_1 + u_1 a_2 - \frac{\eta}{2}\sigma^2(v_1^2 + u_1^2) - \frac{1}{2}(a_1 - b_2)^2\right\} && (6) \\
a_2, b_2 &\in \arg\max\left\{z_2 + v_2 a_2 + u_2 a_1 - \frac{\eta}{2}\sigma^2(v_2^2 + u_2^2) - \frac{1}{2}a_2^2 - b_2^2\right\} && (7)
\end{aligned}
$$

From the incentive compatibility constraints we can obtain the best-response functions of the two agents. Note that agent 1 chooses his effort level a_1 after he has observed the level of help b_2. Agent 1's first order condition for a_1 yields

$$a_1 = v_1 + b_2. \tag{8}$$

Hence we can rewrite the maximization problem for agent 2 as

$$\max_{a_2, b_2} z_2 + v_2 a_2 + u_2(v_1 + b_2) - \frac{\eta}{2}\sigma^2(v_2^2 + u_2^2) - \frac{1}{2}a_2^2 - b_2^2.$$

Solving the resulting system of equations we obtain

$$
\begin{aligned}
a_1 &= v_1 + \frac{u_2}{2} \\
a_2 &= v_2 \\
b_2 &= \frac{u_2}{2}.
\end{aligned}
$$

Substituting these equations as well as the binding participation constraints into the principal's problem we are left with the following unconstrained maximization problem

$$\max_{v_1, v_2, u_1, u_2}\left\{v_1 + \frac{u_2}{2} + v_2 - \frac{\eta}{2}\sigma^2(v_1^2 + u_1^2 + v_2^2 + u_2^2) - \frac{1}{2}\left(v_1^2 + v_2^2 + \frac{u_2^2}{2}\right)\right\}.$$

The first order conditions are

$$
\begin{aligned}
1 - \eta\sigma^2 v_1 - v_1 &= 0 \\
1 - \eta\sigma^2 v_2 - v_2 &= 0 \\
-\eta\sigma^2 u_1 &= 0 \\
\frac{1}{2} - \eta\sigma^2 u_2 - \frac{u_2}{2} &= 0
\end{aligned}
$$

which yield

$$
\begin{aligned}
v_1 &= \frac{1}{1 + \eta\sigma^2} \\
v_2 &= \frac{1}{1 + \eta\sigma^2} \\
u_1 &= 0 \\
u_2 &= \frac{1}{1 + 2\eta\sigma^2}.
\end{aligned}
$$

Note that even though $v_1 = v_2$ we have $a_1 > a_2$ as before in the first-best solution. This is the result of a positive u_2. Agent 2 has to receive a share of agent 1's output in order to incentivize him to help agent 2. On the other hand, agent 1 does not receive a share of agent 2's output ($u_1 = 0$) since he cannot help agent 1 and making his wage contingent on agent 1's output only introduces costly variance into his wage payment.

Part 3: Arbitrage The optimal payment scheme proposed above is not robust to arbitrage since $v_1 = v_2$, yet $u_2 > u_1$. As a result in order to increase the sum of their wages the agents will always claim that all output was produced by agent 1 (since agent 2 receives a share of his output). In order to prevent arbitrage the principal has to ensure that for the sum of wages the marginal return to both outputs is the same so that the agents are indifferent between claiming that outputs are produced by one or the other:

$$v_1 + u_2 = v_2 + u_1.$$

The principal now maximizes

$$\max_{v_1,v_2,u_1,u_2} v_1 + \frac{u_2}{2} + v_2 - \frac{\eta}{2}\sigma^2(v_1^2 + u_1^2 + v_2^2 + u_2^2) - \frac{1}{2}\left(v_1^2 + v_2^2 + \frac{u_2^2}{2}\right)$$

subject to the new arbitrage constraint. Eliminating u_1 from the maximand using the above constraint, results in the first order conditions for v_1, v_2 and u_2 being

$$\begin{aligned}
(1 + 2\eta\sigma^2)v_1 - \eta\sigma^2 v_2 + \eta\sigma^2 u_2 &= 1 \\
-\eta\sigma^2 v_1 + (1 + 2\eta\sigma^2)v_2 - \eta\sigma^2 u_2 &= 1 \\
2\eta\sigma^2 v_1 - 2\eta\sigma^2 v_2 + (1 + 4\eta\sigma^2)u_2 &= 1
\end{aligned}$$

which can be solved to obtain

$$\begin{aligned}
v_1 &= \frac{7\eta^2\sigma^4 + 6\eta\sigma^2 + 1}{(8\eta^2\sigma^4 + 7\eta\sigma^2 + 1)(\eta\sigma^2 + 1)} \\
v_2 &= \frac{9\eta^2\sigma^4 + 8\eta\sigma^2 + 1}{(8\eta^2\sigma^4 + 7\eta\sigma^2 + 1)(\eta\sigma^2 + 1)} \\
u_1 &= \frac{\eta\sigma^2 + 1}{8\eta^2\sigma^4 + 7\eta\sigma^2 + 1} \\
u_2 &= \frac{3\eta\sigma^2 + 1}{8\eta^2\sigma^4 + 7\eta\sigma^2 + 1}.
\end{aligned}$$

First note that $v_1 < v_2$ and $u_1 < u_2$, so agent 1 receives a lower wage payment scheme in equilibrium. However, in contrast to the previous case where we ruled out arbitrage, we have $u_1 > 0$. In order to satisfy the arbitrage constraint the principal has to raise v_2 and u_1 even though the latter only introduces additional noise into the wage payment to the agents.