Introduction to Complex Fourier Series

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Fourier series come in two flavors. What we have studied so far are called *real Fourier series*: these decompose a given periodic function into terms of the form $\sin(nx)$ and $\cos(nx)$. This document describes an alternative, where a function is instead decomposed into terms of the form e^{inx}. These series are called *complex Fourier series*, since they make use of complex numbers.

In practice, it is easier to work with the complex Fourier series for most of a calculation, and then convert it to a real Fourier series only at the end. This pattern is very typical of many of the situations where complex numbers are useful.

Throughout this document, I will make two significant simplifications, in order to focus on the conceptual points and avoid technical baggage.

- I will focus on **finite series**, i.e. finite sums of terms. This are often called "trigonometric polynomials" in other contexts.
- I will consider only functions with **period** 2π.

1 Complex Fourier coefficients

Recall that we begun discussing Fourier series by attempting to write a given 2π-periodic function $f(x)$ in the following form (notation differs from author to author; I am following Stewart's notation here).

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

In words, the goal was to break $f(x)$ into its constituent frequencies. The miracle of Fourier series is that as long as $f(x)$ is continuous (or even piecewise-continuous, with some caveats discussed in the Stewart text), such a decomposition is always possible. The functions $\sin(nx)$ and $\cos(nx)$ form a sort of periodic table: they are the atoms that all other waves are built out of.

By contrast, a complex Fourier series aims instead to write $f(x)$ in a series of the following form.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

There are some technical points about what this notation means (namely what I mean by a sum that starts at $-\infty$), but I am going to gloss over them since we will focus on the case of finite series anyway.

Here the numbers c_n are *complex* constants. They are called the *complex Fourier coefficients* of $f(x)$.

Example 1.1. Consider the following function.

$$f(x) = 2e^{-2ix} + (1+i)e^{-ix} + 5 + (1-i)e^{ix} + 2e^{2ix}$$

The complex Fourier coefficients of this function are just the constants in front of these terms. The ones that aren't zero are as follows.

$$
\begin{aligned}
c_{-2} &= 2 \\
c_{-1} &= 1+i \\
c_0 &= 5 \\
c_1 &= 1-i \\
c_2 &= 2
\end{aligned}
$$

The other Fourier coefficients (c_n for all other values of n) are all 0. $\triangleleft$

There are two primary ways to identify the complex Fourier coefficients.

1. By computing an integral similar to the integrals used to find real Fourier coefficients.
2. By first finding the real Fourier coefficients, and converting the real Fourier series into a complex Fourier series.

Since our scope is quite narrow in this course, we will focus on the second of these two options, and specifically on the case where the real Fourier series is finite.

2 Converting between real and complex Fourier series

Recall Euler's formula, which is the basic bridge that connects exponential and trigonometric functions, by way of complex numbers. It states that $e^{ix} = \cos x + i \sin x$. This formula is probably the most important equation in all of mathematics. It is often important to notice that when x is replaced with $-x$, this formula changes in a simple way. This simply reflects the facts that $\cos(-x) = \cos x$ (cosine is an even function) and $\sin(-x) = -\sin x$ (sine is an odd function).

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x && (1) \\
e^{-ix} &= \cos x - i \sin x && (2)
\end{aligned}
$$

Together, these two formulas show how a complex exponential can always be converted to trigonometric functions. The following two formulas show that it is also possible to convert the other direction.

$$
\begin{aligned}
\cos x &= \frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix} && (3) \\
\sin x &= \frac{i}{2}e^{-ix} - \frac{i}{2}e^{ix} && (4)
\end{aligned}
$$

Both of these formulas follow from the first two formulas: adding them together yields $2\cos x$ (and dividing by 2 yields $\cos x$ alone), while subtracting the first from the second yields $-2i \sin x$ (and multiplying by $\frac{i}{2}$ yields $\sin x$ alone).

2.1 Real to complex

The most straightforward way to convert a real Fourier series to a complex Fourier series is to use formulas 3 and 4. First each sine or cosine can be split into two exponential terms, and then the matching terms must be collected together.

The following examples show how to do this with a finite real Fourier series (often called a trigonometric polynomial).

Example 2.1. Convert the (finite) real Fourier series

$$5\cos x + 12\sin x$$

to a (finite) complex Fourier series. What are the complex Fourier coefficients c_n?

Solution. Use formulas 3 and 4 as follows.

$$\begin{aligned}
5\cos x + 12\sin x &= 5\left(\frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix}\right) + 12\left(\frac{i}{2}e^{-ix} - \frac{i}{2}e^{ix}\right) \\
&= \frac{5}{2}e^{-ix} + \frac{5}{2}e^{ix} + 6ie^{-ix} - 6ie^{-ix} \\
&= \left(\frac{5}{2} + 6i\right)e^{-ix} + \left(\frac{5}{2} - 6i\right)e^{ix}
\end{aligned}$$

This last line is the complex Fourier series. From it we can directly read off the complex Fourier coefficients:

$$\begin{aligned}
c_{-1} &= \frac{5}{2} + 6i \\
c_1 &= \frac{5}{2} - 6i \\
c_n &= 0 \text{ for all other } n.
\end{aligned}$$

$\triangleleft$

Example 2.2. Convert the (finite) real Fourier series

$$7 + 4\cos x + 6\sin x - 8\sin(2x) + 10\cos(24x)$$

to a (finite) complex Fourier series.

Solution. Use formulas 3 and 4 as follows.

$$\begin{aligned}
7 + 4\cos x + 6\sin x - 8\sin(2x) + 10\cos(24x) &= 7 + 4\left(\frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix}\right) + 6\left(\frac{i}{2}e^{-ix} - \frac{i}{2}e^{ix}\right) \\
&\quad -8\left(\frac{i}{2}e^{-2ix} - \frac{i}{2}e^{2ix}\right) + 10\left(\frac{1}{2}e^{-24ix} + \frac{1}{2}e^{24ix}\right) \\
&= 7 + 2e^{-ix} + 2e^{ix} + 3ie^{-ix} - 3ie^{ix} \\
&\quad -4ie^{-2ix} + 4ie^{2ix} + 5e^{-24ix} + 5e^{24ix} \\
&= 5e^{-24ix} - 4ie^{-2ix} + (2+3i)e^{-ix} \\
&\quad +7 + (2-3i)e^{ix} + 4ie^{2ix} + 5e^{24ix}
\end{aligned}$$

$\triangleleft$

2.2 Complex to real: first method

To convert the other direction, from a complex Fourier series to a real Fourier series, you can use Euler's formula (equations 1 and 2). Similar to before, each exponential term first splits into two trigonometric terms, and then like terms must be collected. The following two examples show how this works.

Example 2.3. Convert the (finite) complex Fourier series

$$(3+4i)e^{-2ix} + (3-4i)e^{2ix}$$

to a (finite) real Fourier series.

Solution. Using formulas 1 and 2 and collecting like terms:

$$\begin{aligned}
(3+4i)e^{-2ix}+(3-4i)e^{2ix} &= (3+4i)\left[\cos(2x)-i\sin(2x)\right]+(3-4i)\left[\cos(2x)+i\sin(2x)\right] \\
&= (3+4i)\cos(2x)+(4-3i)\sin(2x)+(3-4i)\cos(2x)+(4+3i)\sin(2x) \\
&= \left[(3+4i)+(3-4i)\right]\cos(2x)+\left[(4-3i)+(4+3i)\right]\sin(2x) \\
&= 6\cos(2x)+8\sin(2x)
\end{aligned}$$

Note that in the second line I have used the fact that $(3+4i)i=-4+3i$ and $(3-4i)i=4+3i$ in finding the coefficients in front of the $\sin(2x)$ terms. $\triangleleft$

Example 2.4. Convert the (finite) complex Fourier series

$$2e^{-2ix}+(1+i)e^{-ix}+5+(1-i)e^{ix}+2e^{2ix}$$

to a (finite) real Fourier series.

Solution. Using formulas 1 and 2 and collecting like terms:

$$\begin{aligned}
2e^{-2ix}+(1+i)e^{-ix}+5+(1-i)e^{ix}+2e^{2ix} &= 2\left[\cos(2x)-i\sin(2x)\right]+(1+i)\left[\cos x-i\sin x\right] \\
&\quad +5+(1-i)\left[\cos x+i\sin x\right]+2\left[\cos(2x)+i\sin(2x)\right] \\
&= 2\cos(2x)-2i\sin(2x)+(1+i)\cos x+(1-i)\sin x \\
&\quad +5+(1-i)\cos x+(1+i)\sin x+2\cos(2x)+2i\sin(2x) \\
&= 5+\left[(1+i)+(1-i)\right]\cos x+\left[(1-i)+(1+i)\right]\sin x \\
&\quad +\left[2+2\right]\cos(2x)+\left[-2i+2i\right]\sin(2x) \\
&= 5+2\cos x+2\sin x+4\cos(2x)
\end{aligned}$$

Note that I have used the fact that $(1+i)i=-1+i$ and $(1-i)i=1+i$ when going from the first line to the second line. $\triangleleft$

2.3 Complex to real: another method

The process described above (splitting each e^{ins} using Euler's formula and collecting sine and cosine terms) can become tedious, especially when there are many terms. One alternative which may be less cumbersome to apply is to instead use formulas 3 and 4 after regrouping the exponential terms in a different way. The following example illustrates what I mean.

Example 2.5. Consider the same complex Fourier series as in example 2.4.

$$2e^{-2ix}+(1+i)e^{-ix}+5+(1-i)e^{ix}+2e^{2ix}$$

This time, instead of splitting the exponential terms immediately, begin by rearranging as follows.

$$\begin{aligned}
2e^{-2ix}+(1+i)e^{-ix}+5+(1-i)e^{ix}+2e^{2ix} &= 5+1\cdot\left(e^{-ix}+e^{ix}\right)+i\cdot\left(e^{-ix}-e^{ix}\right)+2\left(e^{-2ix}+e^{2ix}\right) \\
&= 5+2\cos x+2\sin x+4\cos(2x)
\end{aligned}$$

$\triangleleft$

The idea here is to find multiples either of $e^{-inx}+e^{inx}$ or $e^{-inx}-e^{inx}$.

2.4 Conversion formulas

Another way to perform the conversions above is to simply observe the following formulas relating the real Fourier coefficients to the complex Fourier coefficients. I hesitate to worsen the already dense thicket of formulas in this unit of the course, but I will produce the formulas here in case you find them useful.

To compute complex coefficients from real coefficients, the following formulas can be used.

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}(a_n - ib_n) \quad (\text{for } n \geq 1) \\ c_{-n} &= \frac{1}{2}(a_n + ib_n) \quad (\text{for } n \geq 1) \end{aligned}$$

These formulas are easy to remember as long as you know formulas 3 and 4: simply think about which of the terms $a_n \cos(nx)$ and $b_n \sin(nx)$ will end up contributing to which terms in the complex Fourier series.

The compute real coefficients from complex coefficients, the following formulas can be used.

$$\begin{aligned} a_0 &= c_0 \\ a_n &= c_n + c_{-n} \quad (\text{for } n \geq 1) \\ b_n &= i\,(c_n - c_{-n}) \end{aligned}$$