Dynamic Asset Allocation

Using Stochastic Programming and Stochastic Dynamic Programming Techniques

Gerd Infanger

Stanford University

Outline

- Motivation
- Background and Concepts
- Risk Aversion
- Applying Stochastic Dynamic Programming
  - Superiority of Dynamic Strategies
- Applying Multi-Stage Stochastic Programming
- Summary

Asset Allocation

- Allocation between asset classes accounts for the major part of return and risk of a portfolio
  - Equity investments
  - Interest-bearing investments
- Selection of individual instruments is a lower-level decision with much smaller influence on portfolio performance
- Asset Allocation should consider all financial aspects
  - Current and future wealth, income, and financial needs
  - Financial goals
  - Liquidity (plan for the unexpected)
- Financial industry suggests that investors need customized investment strategies

Typical Financial Advice

- Questionnaires to assess the risk aversion of an investor
  - (E*Trade, Charles Schwab, Fidelity, Financial Engines, ASI, etc…)
  - Taking measure, often with great sophistication and much detail
  - => risk aversion of the investor (typically assuming constant relative risk aversion, CRRA)
- Choose from standardized portfolios:
  - Conservative, e.g., 20% stocks
  - Dynamic, e.g., 40% stocks
  - Progressive, e.g., 60% stocks
- Is that a customized portfolio?

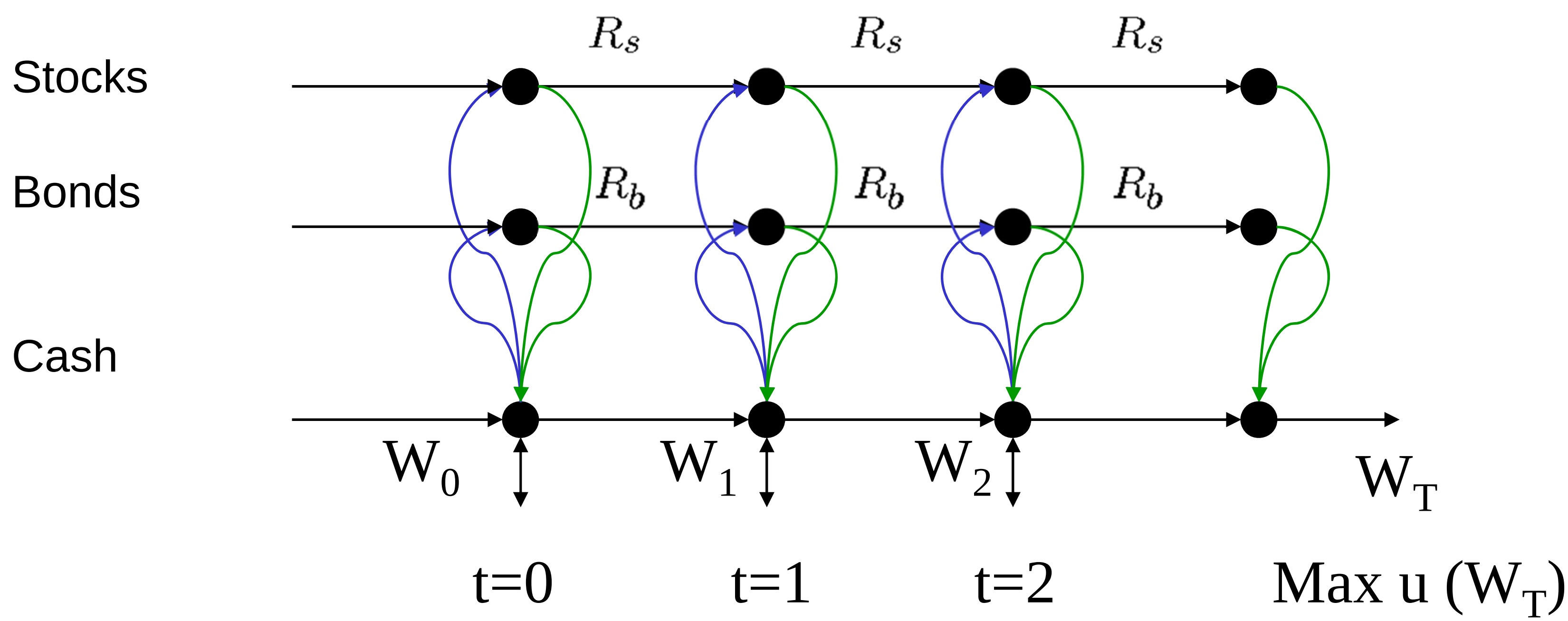
Financial Advice (cont.)

- More recently life-cycle funds have emerged
  - E.g., Fidelity Freedom 2020
  - Asset allocation is purely time-dependent
- Often practiced rule of thumb: % stocks = 100 – age
- But these strategies do not depend on wealth, expected performance, cash flow, etc.

Multi-Period Investment

Asset classes

Returns process



(Network formulation based on John Mulvey (1989))

Utility

- A utility function is an integrating measure, assigning a value (utility) to each point (possible outcome) of the distribution of returns or wealth.
- Maximizing expected utility is equivalent to choosing a certain distribution (of return or wealth) from all possible obtainable distributions.
- Risk measures, like mean, standard deviation, Sharpe ratio, downside risk, value at risk, etc., are all quantities describing various aspects of a distribution.

Dynamic Asset Allocation

- In real life investors change their asset allocation as time goes on and new information becomes available.
- In theory investors value wealth at the end of the planning horizon (and along the way) using a specific utility function and maximize expected utility.
- Fixed-mix strategies are optimal only under certain conditions.
- In general and in most practical cases the optimal investment strategy is dynamic and reflects real-life behavior.

Utility Functions

$\max E\ u(W)$ maximizing expected utility,
$(u'(W) > 0,\ u''(W) < 0)$

Risk Aversion (Arrow (1971) – Pratt (1964)),
and Risk Tolerance

Absolute
$$\text{ARA}(W) = -\frac{u''(W)}{u'(W)}, \qquad \text{ART}(W) = 1/\text{ARA}(W)$$

Relative
$$\text{RRA}(W) = -W\frac{u''(W)}{u'(W)}, \quad \text{RRT}(W) = 1/\text{RRA}(W)$$

Utility Functions

HARA utility functions: ART$(W) = a + bW$, a, b constants

Type	Function	ARA	RRA
Exponential (CARA)	$u(W) = -\exp(-\lambda W)$	λ	λW
Power (CRRA)	$u(W) = \frac{W^{1-\gamma}-1}{1-\gamma}, \gamma > 1$	$\frac{\gamma}{W}$	γ
Generalized Log	$u(W) = \log(\alpha + W)$	$\frac{1}{\alpha+W}$	$\frac{W}{\alpha+W}$

Bell (1988)

$$u(W) = -b_1 e^{-c_1 W} - b_2 e^{-c_2 W}$$

Musumeci and Musumeci (1999)

$$u(W) = -W^{-1} - be^{-\alpha W}$$

Utility Functions (cont.)

Trade-off expected return versus downside risk

Lower partial moment

$u(W) = W - \lambda\{(W_d - W)^+\}^2$

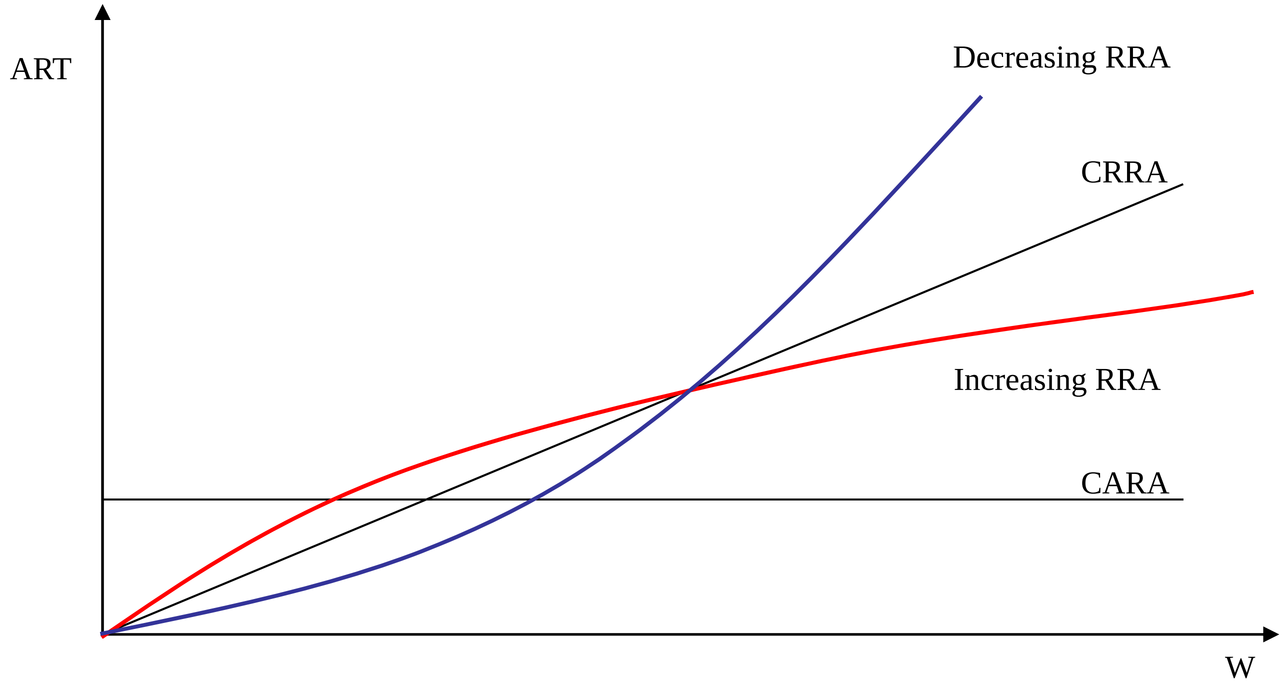
$\max E\ u(W) = E\ W - \lambda E\ (W_d - W)^{+2}$

"Put-call efficient frontier"

$u(W) = W - \lambda(W_d - W)^+$

(λ risk aversion parameter, W_d target)

Increasing and Decreasing Relative Risk Aversion



Represented as piecewise CARA approximation (Infanger, 2006)

Increasing and Decreasing Relative Risk Aversion (cont.)

- After a stock market crash (with significant losses in the stock portion of the portfolio) an investor would:
  - Rebalance back to the original allocation
    (=> constant RRA)
  - Buy more stocks and assume a larger stock allocation than in the original portfolio
    (=> increasing RRA)
  - Do nothing and keep the new stock allocation or sell stocks to assume a smaller stock allocation than in the original portfolio
    (=> decreasing RRA)
- Quantities to be assessed by additional questions

Samuelson (1969) and Merton (1969, 1990)

- The optimal investment strategy is independent of wealth and constant over time if
  - Asset return distribution is iid
  - Utility function is CRRA
  - Only investment income is considered
  - No transaction costs
  - (If the utility function is logarithmic, non-iid asset returns result in a constant strategy as well)
- Dynamic strategies are optimal if any of the above conditions is violated

Partial Myopia, Mossin (1968), Hakanson (1971)

- Invest in each period as if you were to invest in all future periods only in the risk-free asset if:
  - Asset return distribution is iid
  - Utility function is HARA (power, exponential and generalized logarithmic)
  - No transaction costs
  - Absence of any borrowing and short sales constraints
- More recently, analytical solutions have been obtained also for HARA utility functions with borrowing and short sale constraints

Approaches for Dynamic Asset Allocation

- Stochastic Programming
  - Can efficiently solve the most general model. Successfully used for asset allocation and asset liability management (ALM)
- Dynamic Programming (Stochastic Control)
  - When the state space is small, say, up to 3 or 4 state variables, “value function approximation” methods show promise
- Analytical Solutions
  - Myopic portfolio strategies, discrete and continuous time analysis, e.g., Cox and Huang (1999), Campbell and Viceira (2002)
- Fixed-mix strategies

Stochastic Programming

- Monte Carlo Sampling within decomposition
  - Multi-stage dual decomposition with sampling and application of variance reduction techniques, Infanger (1994). Best suited for dynamic asset allocation for many stages, serially independent returns processes, and transaction costs, Dantzig and Infanger (1991)
- Monte Carlo Pre-Sampling
  - Generating a multi-stage stochastic program using sampling and solving it. General model, with serial dependency, e.g., vector auto-regressive returns processes, and transaction costs, Collomb and Infanger (2005)
  - Bond Portfolio Optimization, Diaco and Infanger (2008)
  - Funding mortgage pools using portfolios of bonds, Infanger (1999, 2007)
- Stochastic programming solver DECIS, Infanger (1997)

Stochastic Dynamic Programming

- Numerical approaches with one risky asset and one risk-free asset have been discussed in the literature, e.g.,
  - Musumeci and Musumeci (1999) condition in each period on the amount invested in the risky asset.
  - Jeff Adachi (1996 Ph.D. thesis) conditions in each period on wealth but reports on results for two assets classes only.
- The following novel approach is efficient for multiple asset classes and multiple time periods, all reasonable types of utility functions and distributions
  - Normal, lognormal, and distributions with fat tails using bootstrapping from historical observations
  - Extensions to a restricted class of autoregressive processes

Dynamic (Stochastic) Programming Recursion

$$
\begin{aligned}
v_t(W_t) = \max E\ [\delta v_{t+1}(W_{t+1}) + u_{t+1}(W_{t+1})] \\
e^T x_t \ &= \ 1 \\
A x_t \ &= \ b,\ l \le x_t \le u,
\end{aligned}
$$

where

$v_T(W_T) = U(W_T) + u_T(W_T)$,

$W_{t+1} = (W_t + s_t) R_t x_t$, W_0 given.

R_t random rate of return, independent over time or $R_t | R_{t-1}$

x_t portfolio holdings, W_t wealth, s_t cash flow,

$Ax_t = b$ linear constraints, l, u bounds on holdings

$U(W_T)$ terminal utility, $u_t(W_t)$ on the way utility

In Practice, in-sample/out-of-sample approach

Optimization:

$$\widehat{\widehat{v}}_t(W_t) = \max \frac{1}{|S_t^i|} \sum_{\omega \in S_t^i} [\delta \widehat{v}_{t+1}(W_{t+1}^{\omega_t}) + u_{t+1}(W_{t+1}^{\omega_t})]$$

$$\begin{aligned} e^T x_t &= 1 \\ A x_t &= b, \ l \le x_t \le u \end{aligned}$$

Out-of-sample evaluation:

$$\widehat{v}_t(W_t) = \frac{1}{|S_t^o|} \sum_{\omega \in S_t^o} [\delta \widehat{v}_{t+1}(W_{t+1}^{\omega_t} + u_{t+1}(W_{t+1}^{\omega_t})]$$

where

$\widehat{v}_T(W_T) = U(W_T) + u_T(W_T)$,

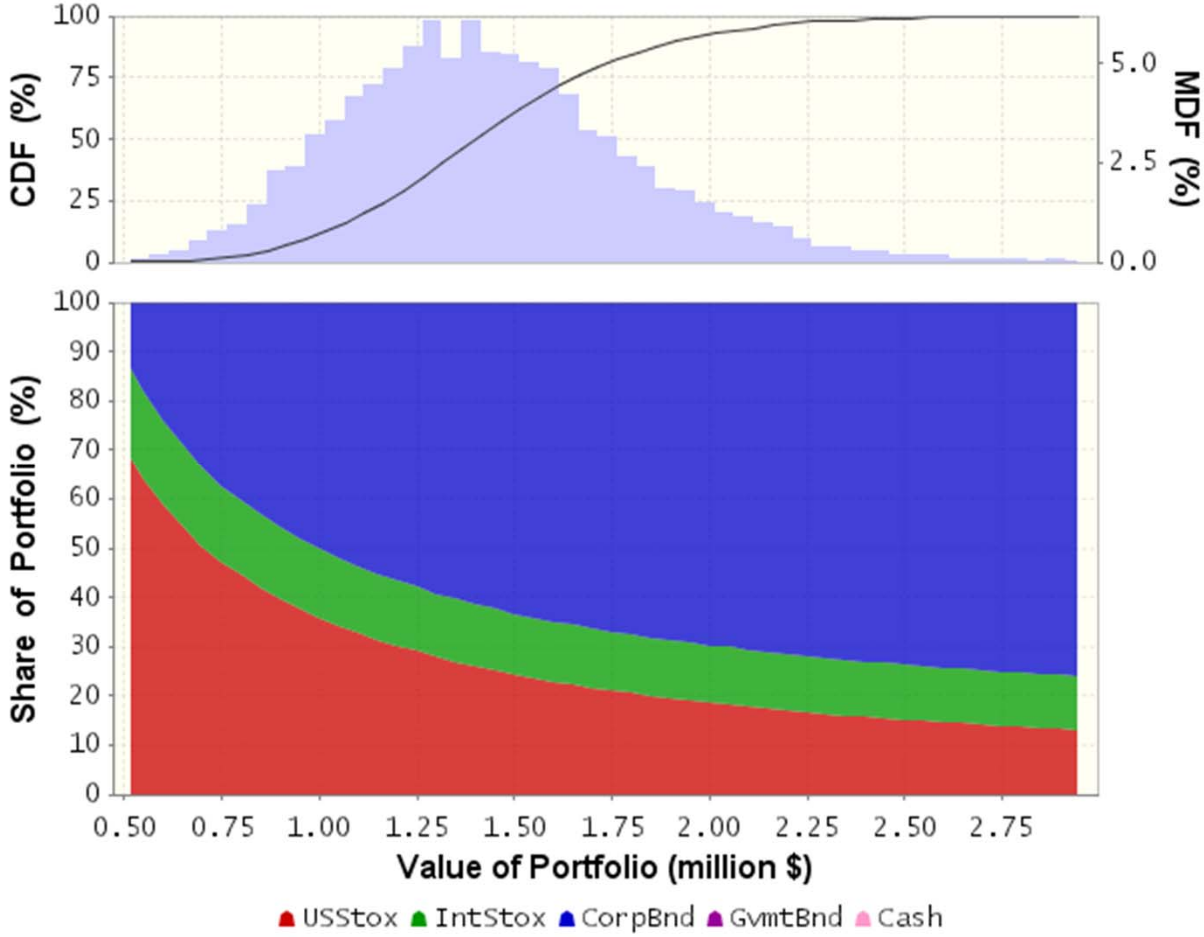
$W_{t+1}^{\omega_t} = (W_t + s_t) R_t^{\omega_t} x_t$, W_0 given.

S_t^i, S_t^o are independent return samples

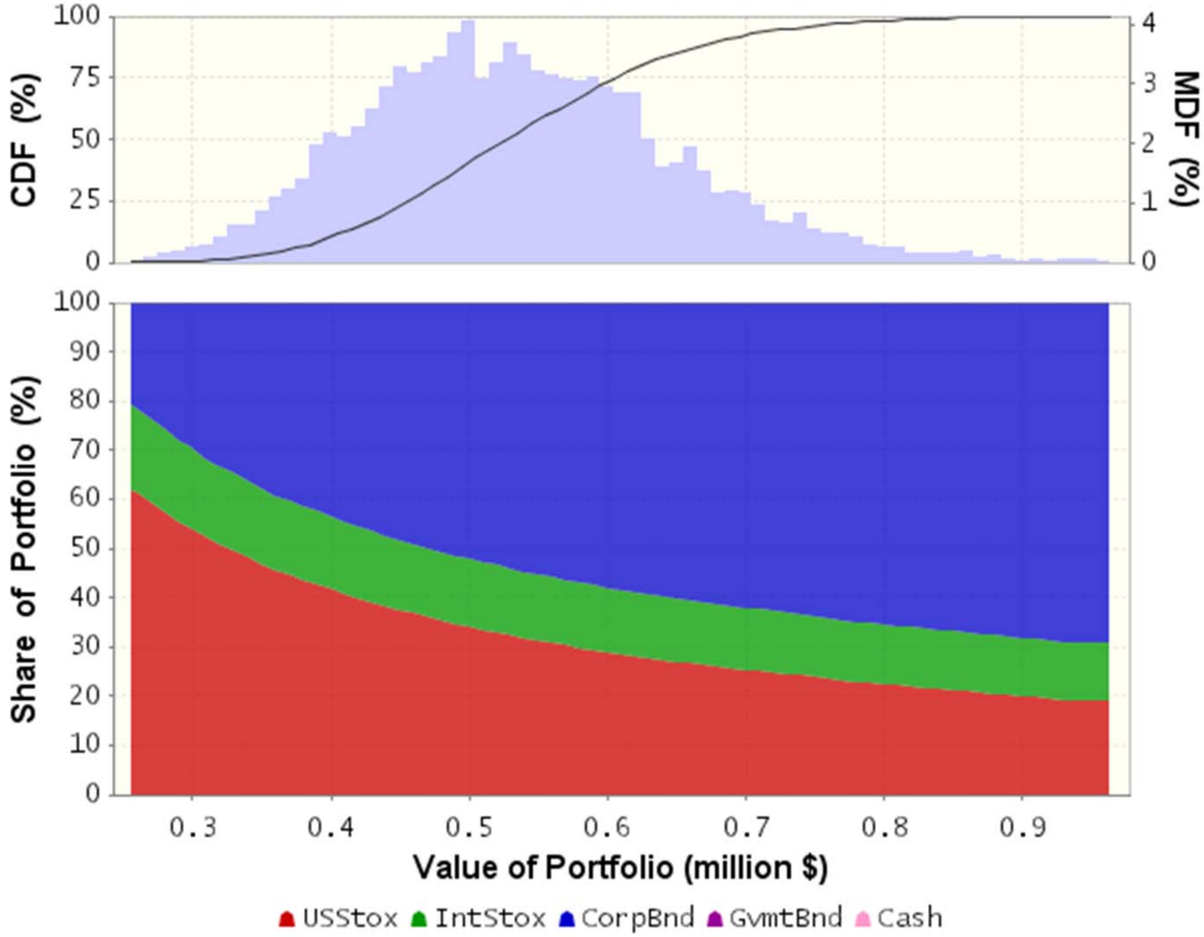
An Investment Example

- Current wealth \$100k,
- Cash contributions (savings) of \$0.015M per year
- Investment horizon 20 years
- Example of terminal utility function:
  - Exponential, ARA = 2 (A)
  - Increasing relative risk aversion and decreasing absolute risk aversion (B)
    - 2.0 at wealth of \$0.25M and below, increasing to 3.5 at wealth of \$3.5M and above
  - Decreasing relative risk aversion and decreasing absolute risk aversion (C)
    - 8.0 at wealth of \$1.0M and below, decreasing to 1.01 at wealth of \$1.5M and above
  - Quadratic (downside) (D)
    - Quadratic and linear penalty of 1000 for underperforming \$1.0M
- US Stocks, Intern. Stocks, Corp. Bonds, Gvmt. Bonds, and Cash

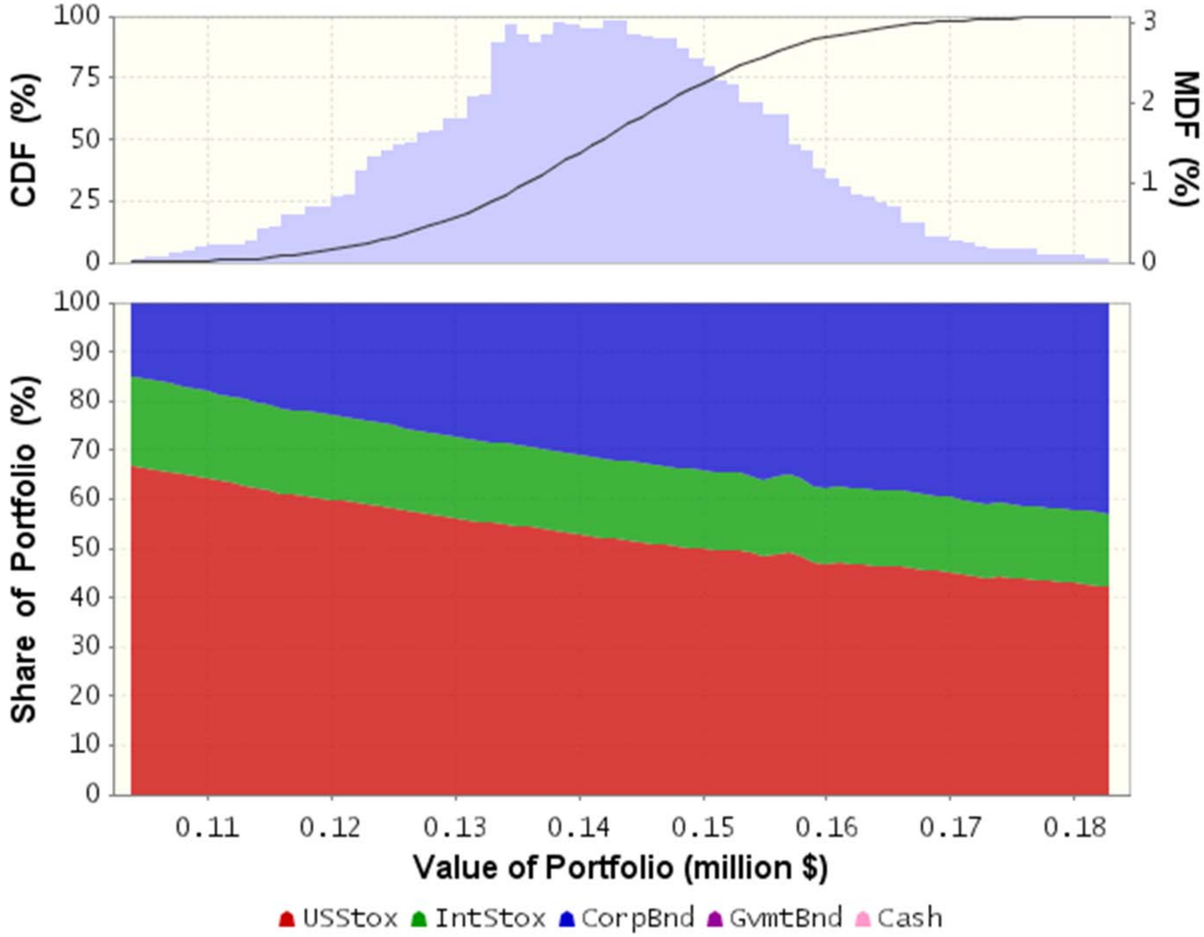
Dynamic Strategy (A) (1 year to go)



Dynamic Strategy (A) (10 years to go)

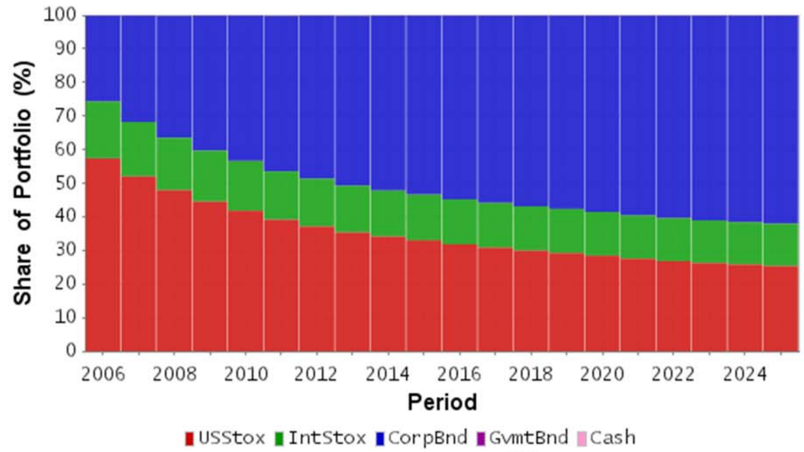


Dynamic Strategy (A) (19 years to go)

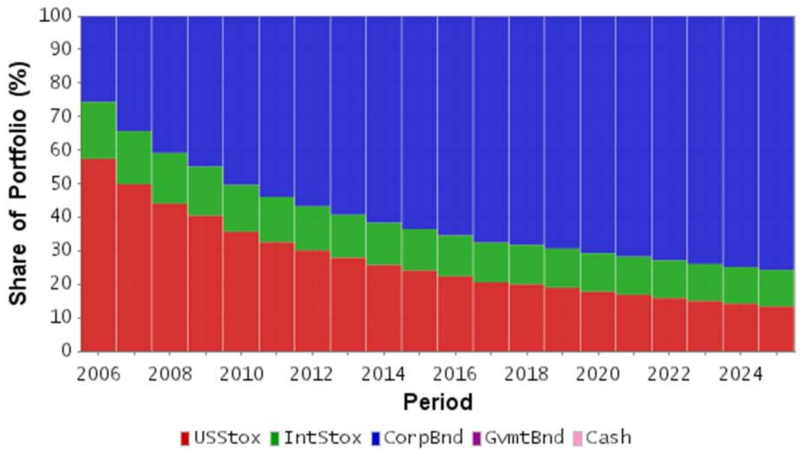


Dynamic Strategy over Time (A)

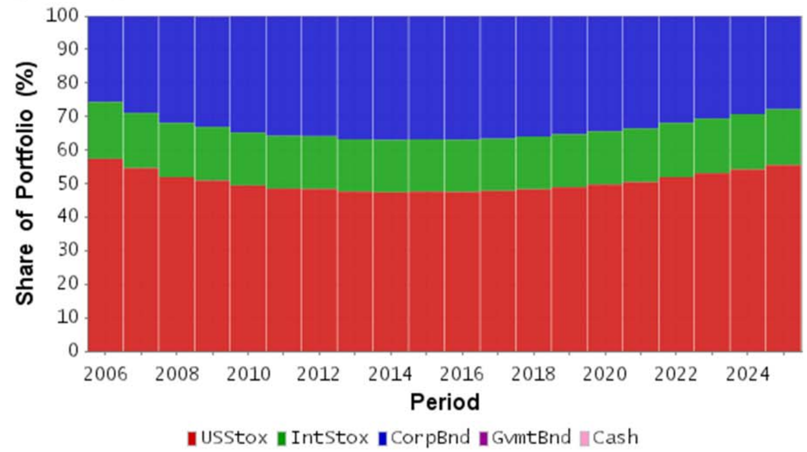
Expected



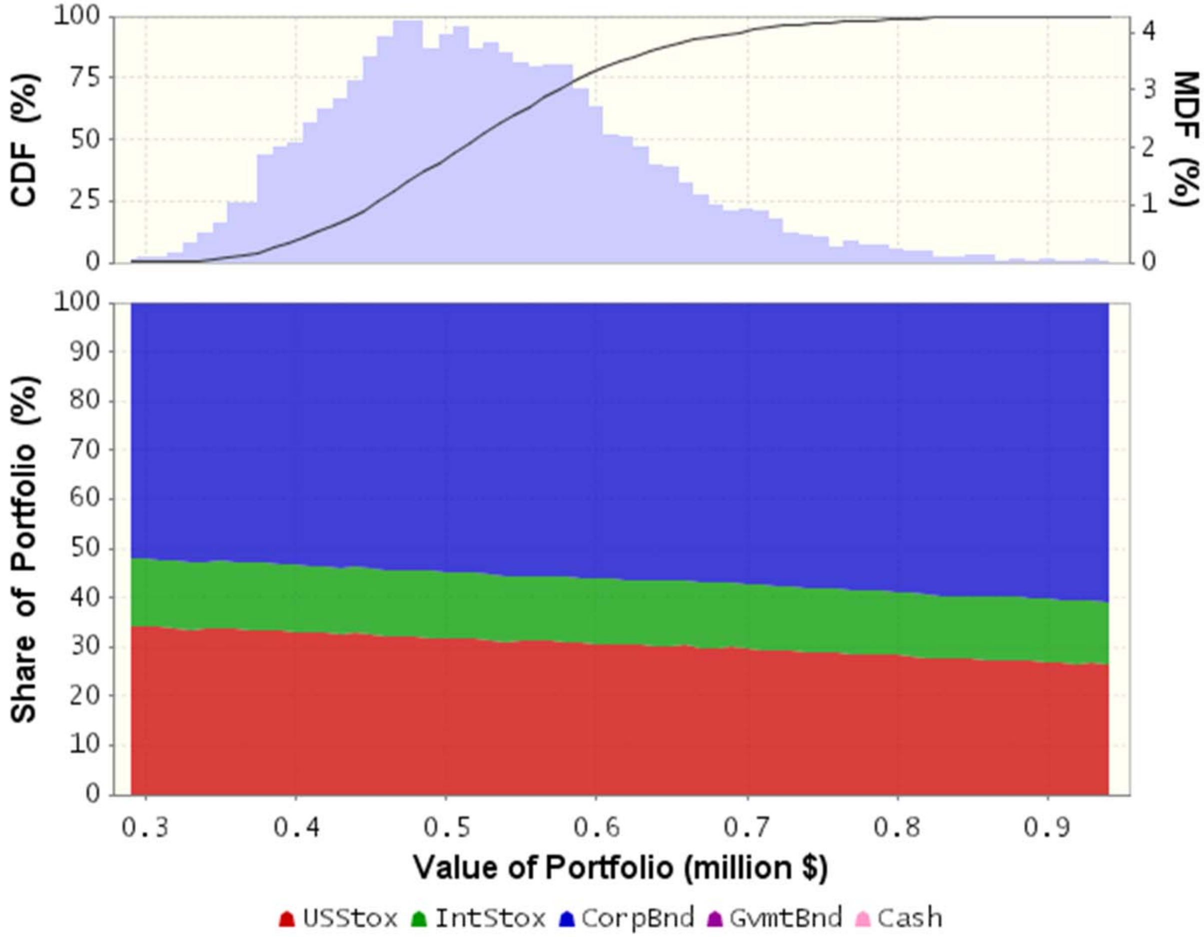
Favorable



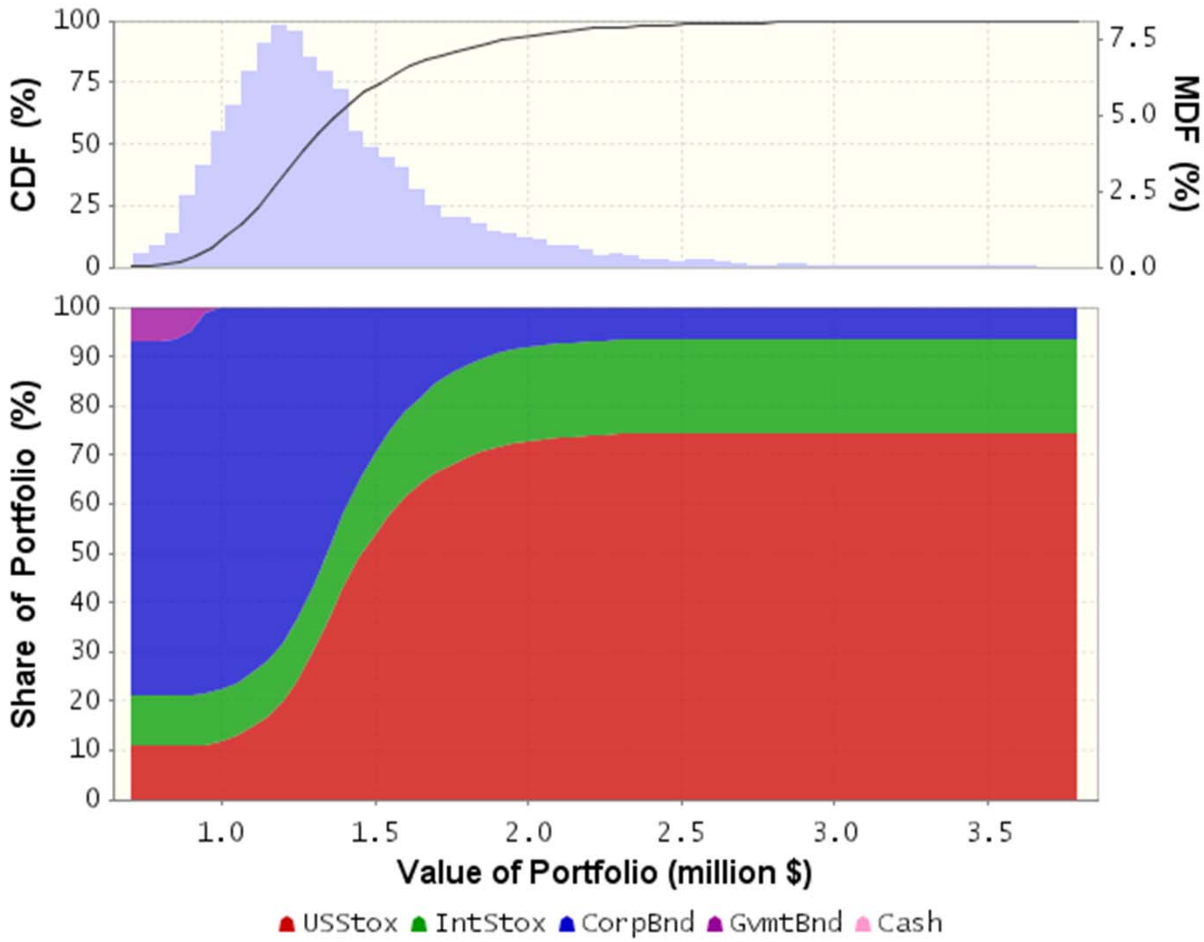
Unfavorable



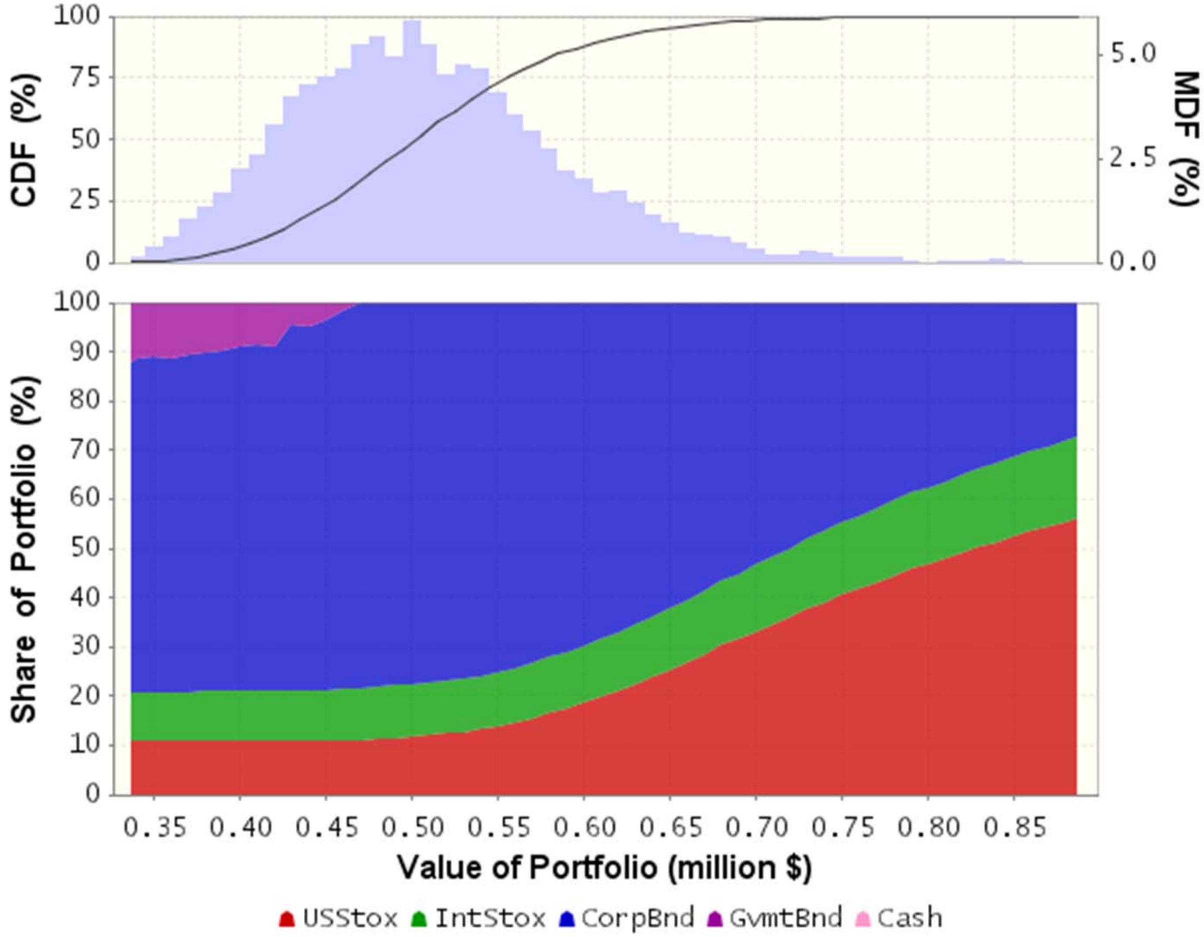
Dynamic Strategy (B) (10 years to go)



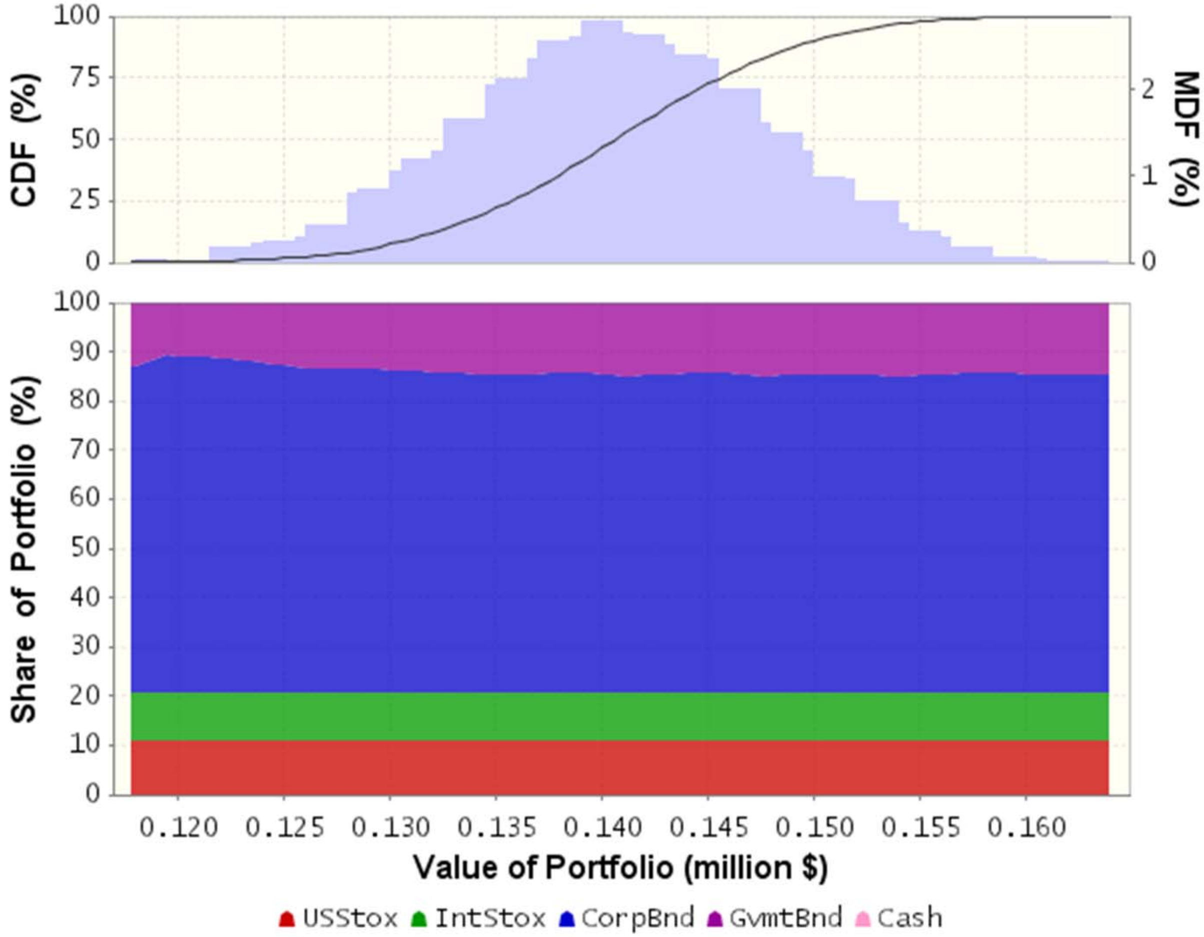
Dynamic Strategy (C) (1 year to go)



Dynamic Strategy (C) (10 years to go)

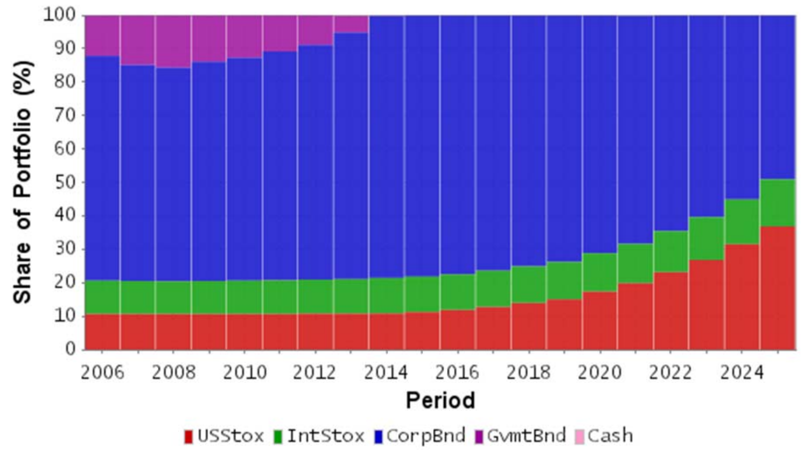


Dynamic Strategy (C) (19 years to go)

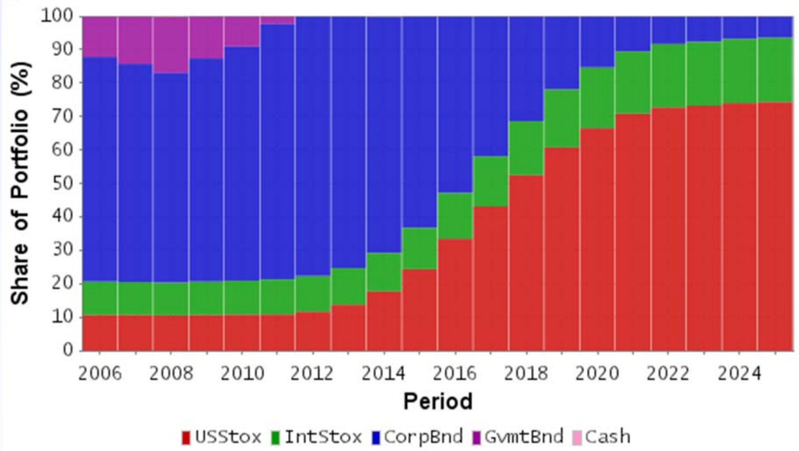


Dynamic Strategy over Time (C)

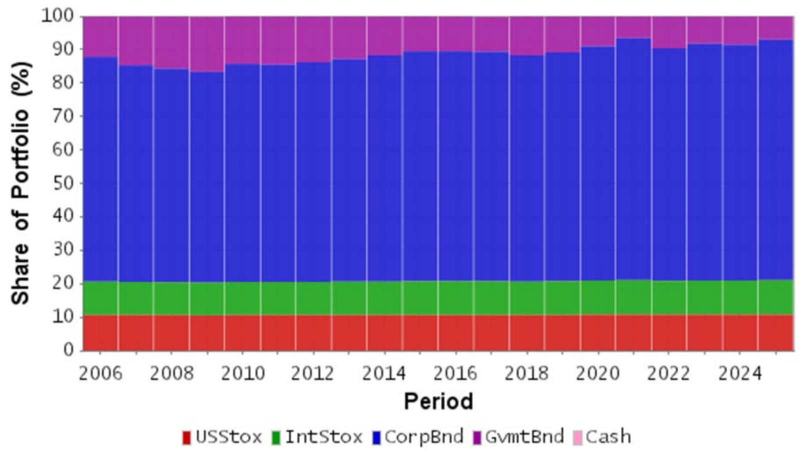
Expected



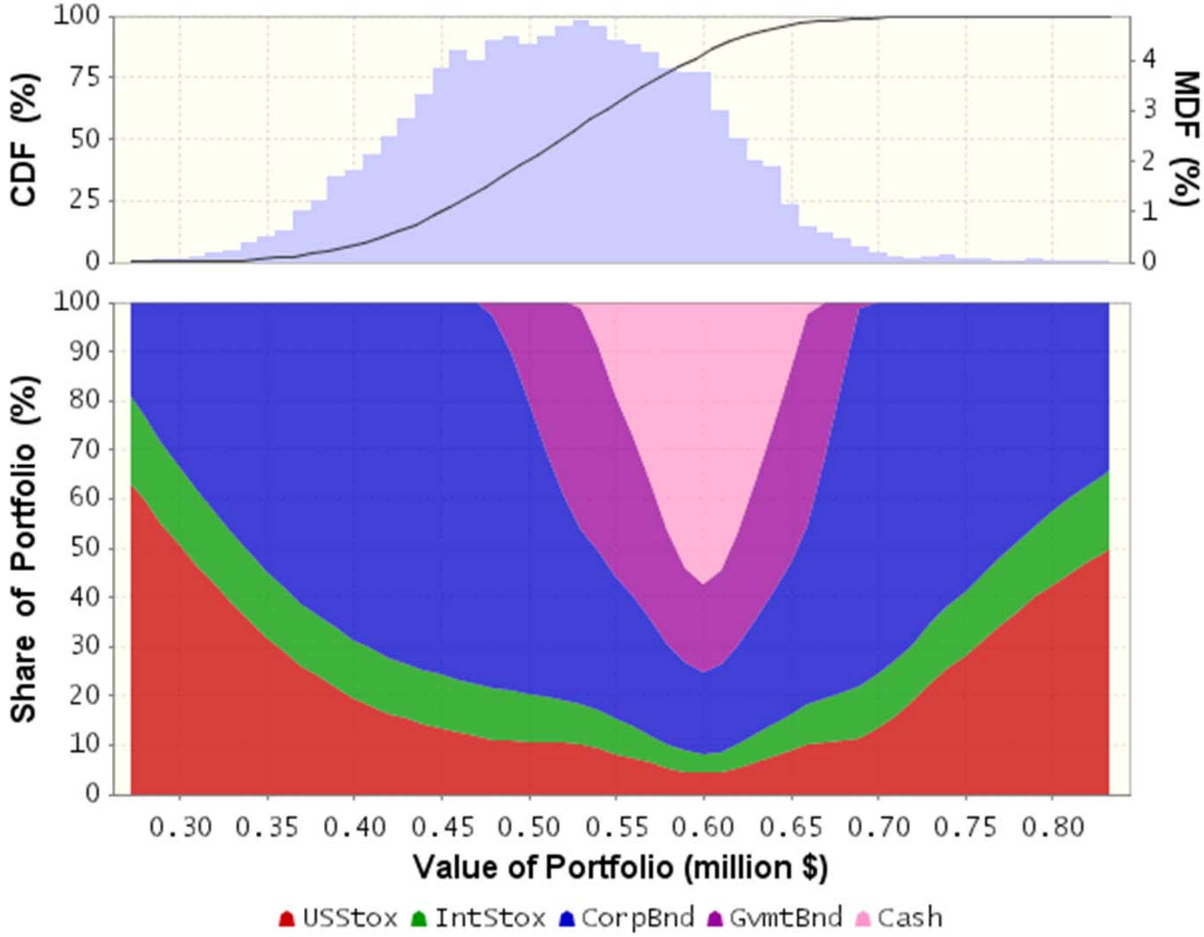
Favorable



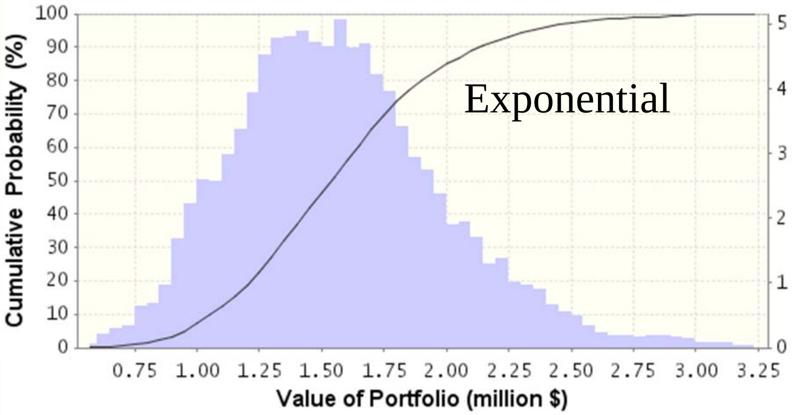
Unfavorable

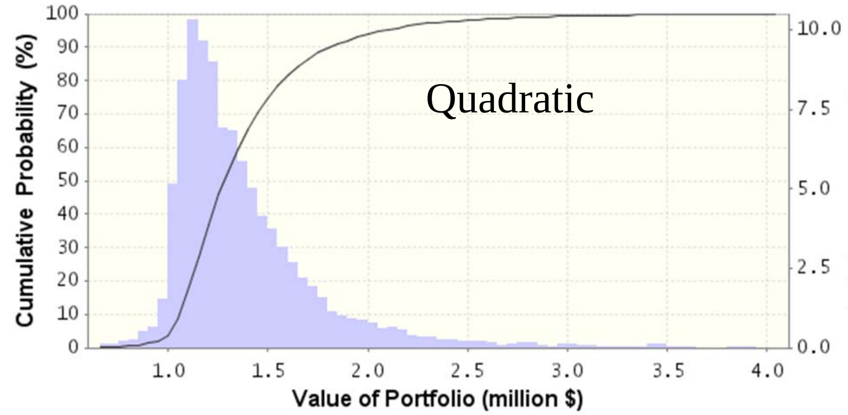


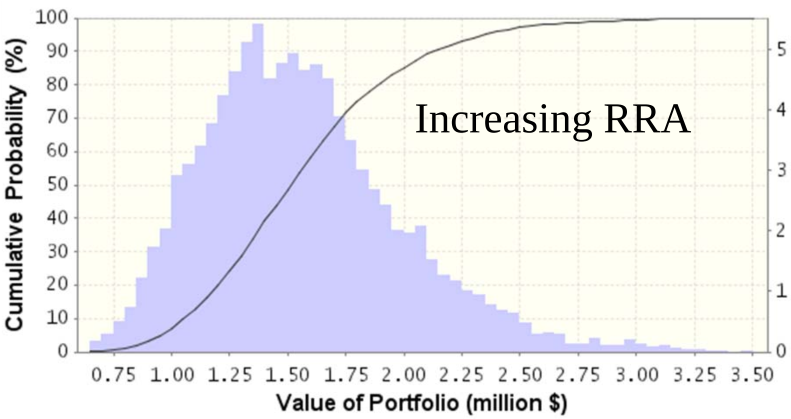
Dynamic Strategy (D) (10 years to go)

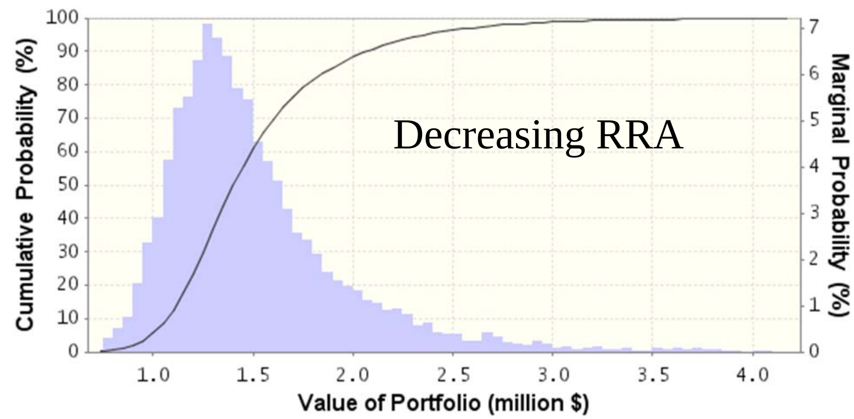


Distribution of Terminal Wealth



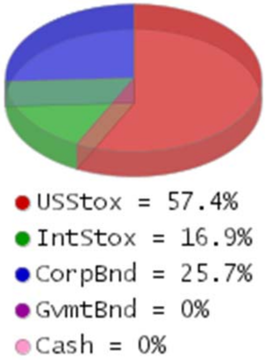




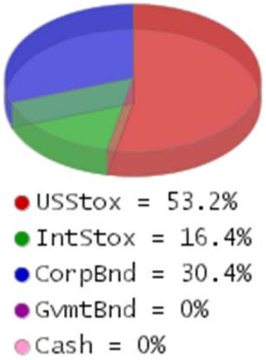


Initial Optimal Portfolios

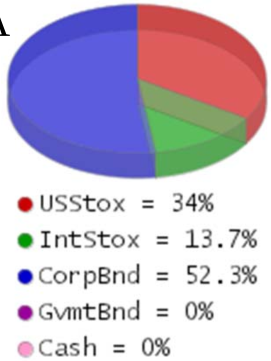
Exponential



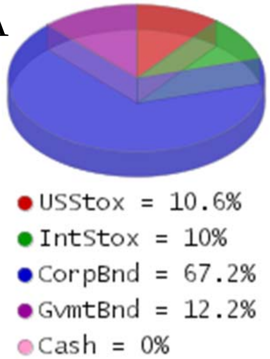
Quadratic



Increasing RRA



Decreasing RRA



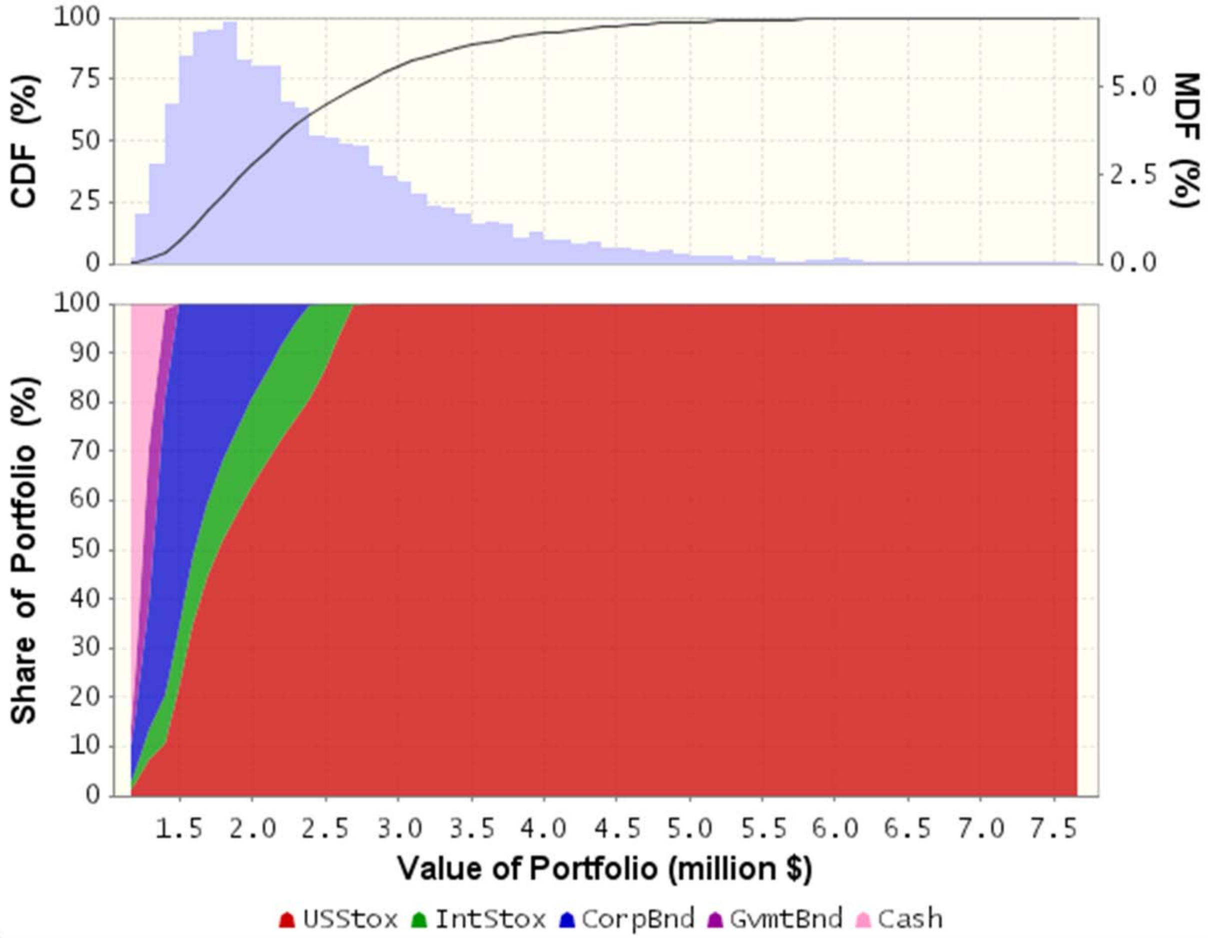
Performance (measured via Certainty Equivalent Wealth)

	Exponential RA=2		Increasing RRA		Decreasing RRA		Quadratic	
	CEW	Impr.	CEW	Impr.	CEW	Impr.	CEW	Impr.
Dynamic	1.41193		1.44004		1.35163		0.98204	
US Stocks	1.28811	9.61%	1.34365	7.17%	0.68918	96.12%	0.87639	12.06%
Cash	0.86732	62.79%	0.86728	66.04%	0.86596	56.08%	0.86628	13.36%
Equally weighted	1.27082	11.10%	1.28235	12.30%	1.17981	14.56%	0.96254	2.03%
Averse	1.08667	29.93%	1.08747	32.42%	1.0605	27.45%	0.97207	1.03%
Medium	1.40415	0.55%	1.4292	0.76%	1.34326	0.62%	0.97051	1.19%
Prone	1.38933	1.63%	1.43378	0.44%	1.0925	23.72%	0.93699	4.81%

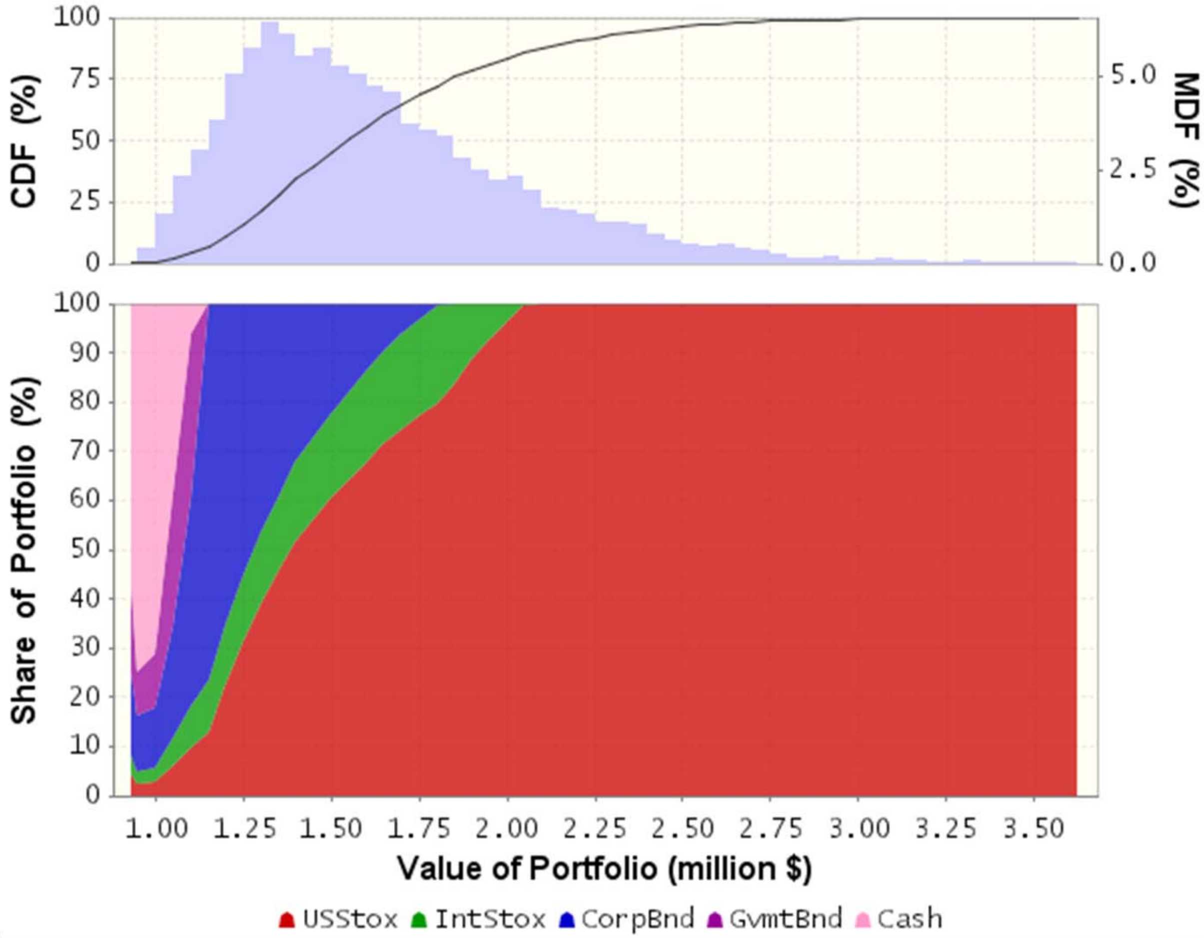
Hedging Downside Risk

- Current wealth $1M
- No cash contributions
- Investment horizon 10 years
- Desired minimum return of 2% per year

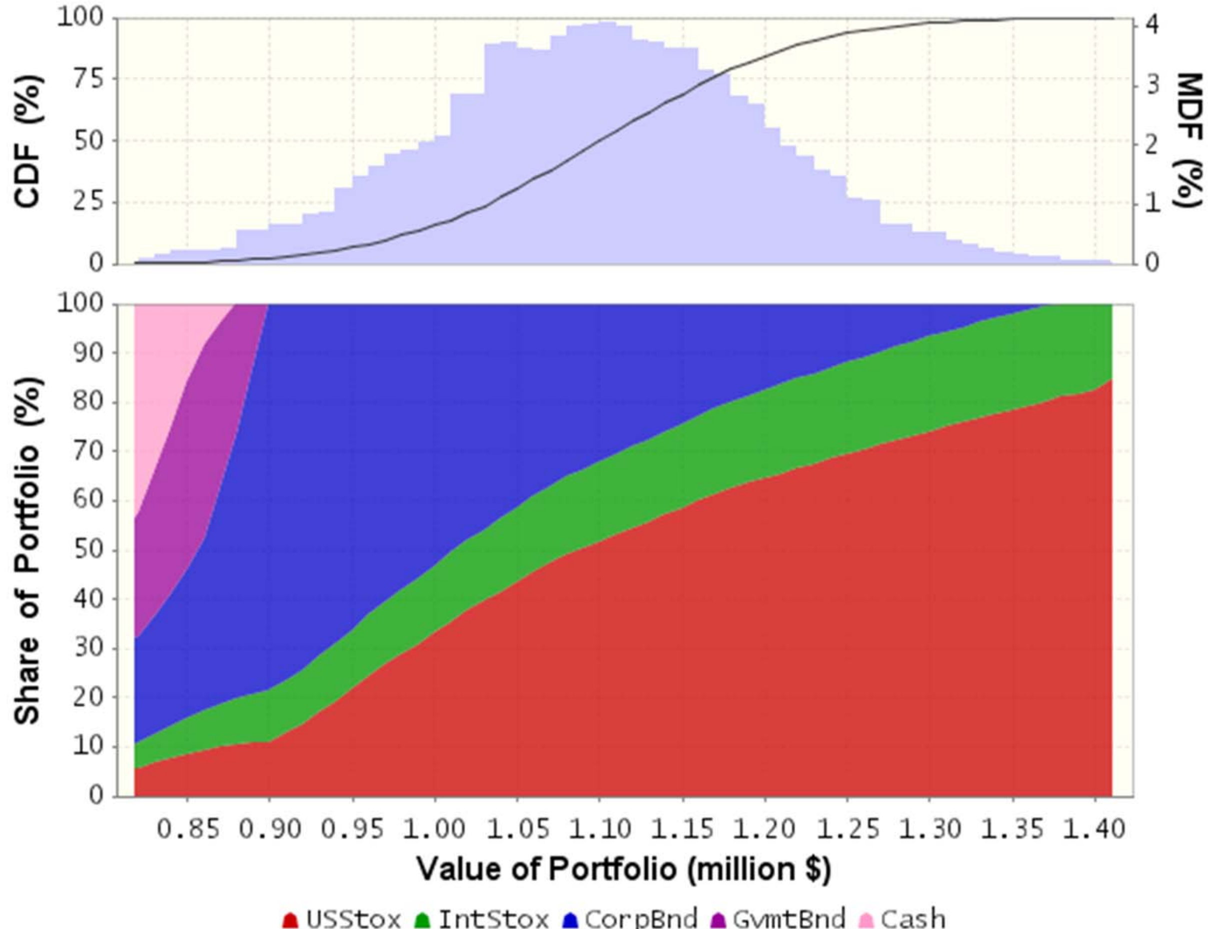
Dynamic Strategy (Q) (1 year to go)



Dynamic Strategy (Q) (5 years to go)

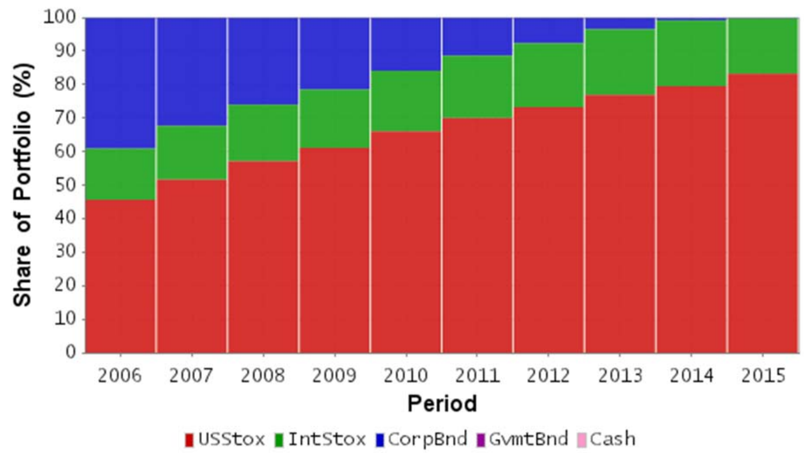


Dynamic Strategy (Q) (9 years to go)

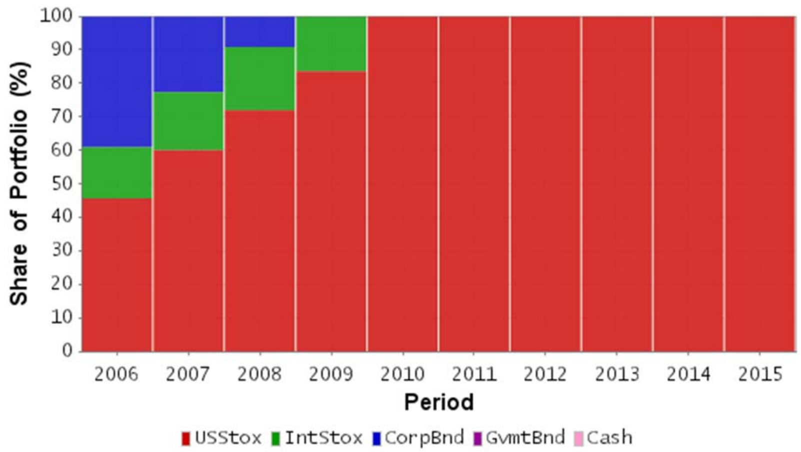


Dynamic Strategy over Time (Q)

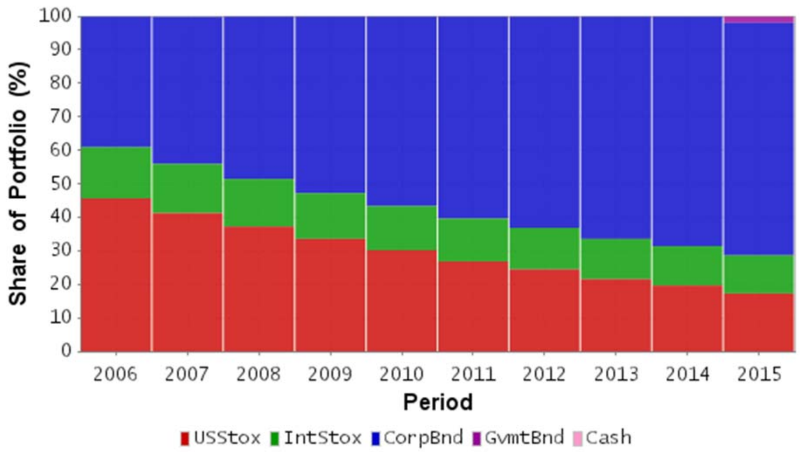
Expected



Favorable



Unfavorable



Quadratic Downside Risk at Terminal Wealth

Quadratic dynamic downside risk						
Period	Exp Wealth	99% Wealth	Min Wealth	Exp Ret.	99% Ret.	Min Ret.
1	1.10008	0.8814	0.8	10.01%	-11.86%	-20.00%
2	1.21161	0.89267	0.825	10.07%	-5.52%	-9.17%
3	1.33354	0.92828	0.86	10.07%	-2.45%	-4.90%
4	1.47004	0.97231	0.9	10.11%	-0.70%	-2.60%
5	1.62245	1.01714	0.9616	10.16%	0.34%	-0.78%
6	1.79528	1.07636	1	10.24%	1.23%	0.00%
7	1.98402	1.1296	1.05333	10.28%	1.76%	0.74%
8	2.19368	1.18286	1.12	10.32%	2.12%	1.43%
9	2.42128	1.25091	1.16071	10.32%	2.52%	1.67%
10	2.67628	1.31957	1.22778	10.35%	2.81%	2.07%

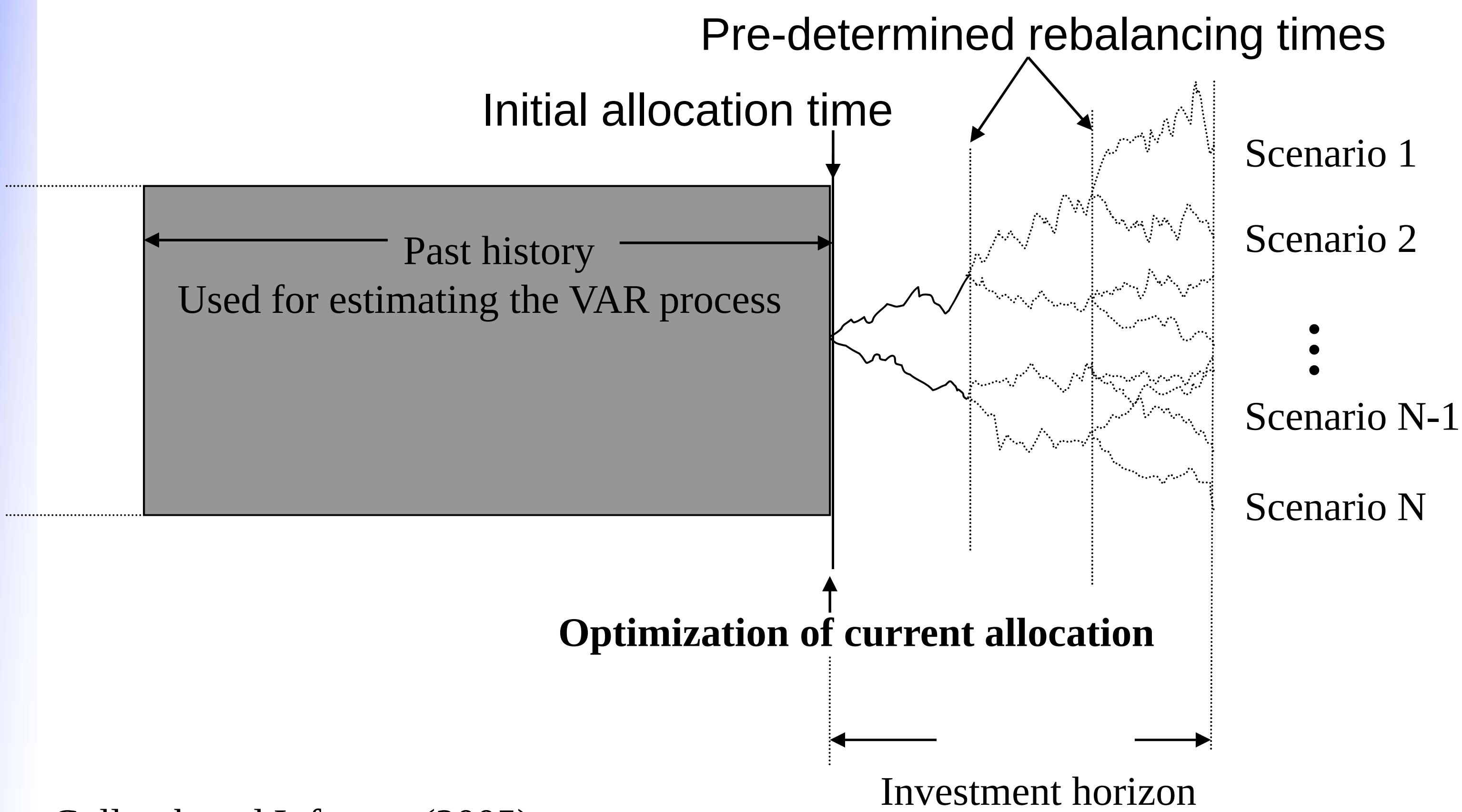
Quadratic Downside Risk at Every Period

Quadratic dynamic downside risk						
Period	Exp Wealth	99% Wealth	Min Wealth	Exp Ret.	99% Ret.	Min Ret.
1	1.06575	1.02225	1.00533	6.58%	2.23%	0.53%
2	1.14561	1.06108	1.032	7.03%	3.01%	1.59%
3	1.24271	1.09137	1.06222	7.51%	2.96%	2.03%
4	1.35621	1.13641	1.095	7.91%	3.25%	2.29%
5	1.48549	1.176	1.12667	8.24%	3.30%	2.41%
6	1.63401	1.22638	1.152	8.53%	3.46%	2.39%
7	1.79975	1.27385	1.2	8.76%	3.52%	2.64%
8	1.9845	1.32121	1.23	8.94%	3.54%	2.62%
9	2.18661	1.37778	1.26	9.08%	3.62%	2.60%
10	2.41527	1.43966	1.3125	9.22%	3.71%	2.76%

Computation Times (Elapsed CPU Seconds)

Rebalancing	Horizon	Periods	Utility	
			Exponential	Increasing RRA
yearly	20	20	10.53	10.92
			Quadratic	Decreasing RRA
yearly	20	20	10.48	11.81
			Quadratic	Decreasing RRA
yearly	10	10	6.58	7.79
quarterly	10	40	15.85	17.06
monthly	10	120	41.35	41.35

Multi-Stage Stochastic Program



Collomb and Infanger (2005)

Solving Multi-Stage Programs with DECIS

	NUMBER OF SCENARIOS				
	40,000	60,000	80,000	100,000	120,000
GAMS/DECIS+ CPLEX Solve time (sec.)	234.6	386.8	472.9	706.3	1,659.6
GAMS/CPLEX Solve time (sec.)	574.8	1,046.3	2,013.0	3,493.9	6,555.5

In comparison, an optimization over 20 periods using the dynamic program requires about 10 seconds.

Summary

- Dynamic asset allocation yields superior strategies.
- For serially independent asset returns, general utility, and no transaction costs, a stochastic dynamic programming recursion is effective and efficient.
  - Provides insight into dynamic strategies
- For general serially dependent asset returns and/or consideration of transaction costs, a multi-stage stochastic programming approach may be needed.
  - Solved efficiently using decomposition and sampling