Cambridge

International A and AS Level Mathematics

Pure Mathematics 1

Sophie Goldie

Series Editor: Roger Porkess

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Much of the material in this book was published originally as part of the MEI Structured Mathematics series. It has been carefully adapted for the Cambridge International A & AS level Mathematics syllabus.

The original MEI author team for Pure Mathematics comprised Catherine Berry, Bob Francis, Val Hanrahan, Terry Heard, David Martin, Jean Matthews, Bernard Murphy, Roger Porkess and Peter Secker.

First published in 2012 by
Hodder Education, a Hachette UK company,
338 Euston Road
London NW1 3BH

Impression number 5 4 3 2 1
Year 2016 2015 2014 2013 2012

Cover photo by © Joy Fera / Fotolia
Illustrations by Pantek Media, Maidstone, Kent
Typeset in 10.5pt Minion by Pantek Media, Maidstone, Kent
Printed in Dubai

A catalogue record for this title is available from the British Library

ISBN 978 1444 14644 8

Contents

Key to symbols in this book

? This symbol means that you want to discuss a point with your teacher. If you are working on your own there are answers in the back of the book. It is important, however, that you have a go at answering the questions before looking up the answers if you are to understand the mathematics fully.

p This symbol invites you to join in a discussion about proof. The answers to these questions are given in the back of the book.

! This is a warning sign. It is used where a common mistake, misunderstanding or tricky point is being described.

This is the ICT icon. It indicates where you could use a graphic calculator or a computer. Graphical calculators and computers are not permitted in any of the examinations for the Cambridge International A & AS Level Mathematics 9709 syllabus, however, so these activities are optional.

b This symbol and a dotted line down the right-hand side of the page indicates material that you are likely to have met before. You need to be familiar with the material before you move on to develop it further.

e This symbol and a dotted line down the right-hand side of the page indicates material which is beyond the syllabus for the unit but which is included for completeness.

Introduction

This is the first of a series of books for the University of Cambridge International Examinations syllabus for Cambridge International A & AS Level Mathematics 9709. The eight chapters of this book cover the pure mathematics in AS level. The series also contains a more advanced book for pure mathematics and one each for mechanics and statistics.

These books are based on the highly successful series for the Mathematics in Education and Industry (MEI) syllabus in the UK but they have been redesigned for Cambridge users; where appropriate new material has been written and the exercises contain many past Cambridge examination questions. An overview of the units making up the Cambridge International A & AS Level Mathematics 9709 syllabus is given in the diagram on the next page.

Throughout the series the emphasis is on understanding the mathematics as well as routine calculations. The various exercises provide plenty of scope for practising basic techniques; they also contain many typical examination questions.

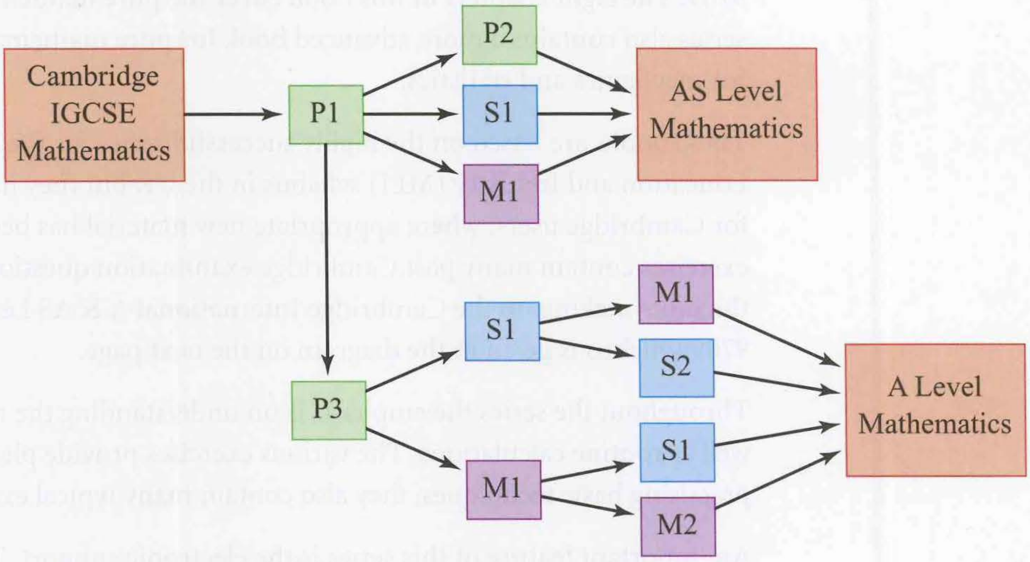
An important feature of this series is the electronic support. There is an accompanying disc containing two types of Personal Tutor presentation: examination-style questions, in which the solutions are written out, step by step, with an accompanying verbal explanation, and test yourself questions; these are multiple-choice with explanations of the mistakes that lead to the wrong answers as well as full solutions for the correct ones. In addition, extensive online support is available via the MEI website, www.mei.org.uk.

The books are written on the assumption that students have covered and understood the work in the Cambridge IGCSE syllabus. However, some of the early material is designed to provide an overlap and this is designated 'Background'. There are also places where the books show how the ideas can be taken further or where fundamental underpinning work is explored and such work is marked as 'Extension'.

The original MEI author team would like to thank Sophie Goldie who has carried out the extensive task of presenting their work in a suitable form for Cambridge International students and for her many original contributions. They would also like to thank Cambridge International Examinations for their detailed advice in preparing the books and for permission to use many past examination questions.

Roger Porkess
Series Editor

The Cambridge A & AS Level Mathematics syllabus



1 Algebra

Sherlock Holmes: 'Now the skillful workman is very careful indeed ... He will have nothing but the tools which may help him in doing his work, but of these he has a large assortment, and all in the most perfect order.'

A. Conan Doyle

Background algebra

Manipulating algebraic expressions

You will often wish to tidy up an expression, or to rearrange it so that it is easier to read its meaning. The following examples show you how to do this. You should practise the techniques for yourself on the questions in Exercise 1A.

Collecting terms

Very often you just need to collect like terms together, in this example those in x, those in y and those in z.

? What are 'like' and 'unlike' terms?

EXAMPLE 1.1

Simplify the expression $2x + 4y - 5z - 5x - 9y + 2z + 4x - 7y + 8z$.

SOLUTION

$$\begin{aligned} \text{Expression} &= 2x + 4x - 5x + 4y - 9y - 7y + 2z + 8z - 5z \quad \leftarrow \text{Collect like terms} \\ &= 6x - 5x + 4y - 16y + 10z - 5z \quad \leftarrow \text{Tidy up} \\ &= x - 12y + 5z \quad \leftarrow \text{This cannot be simplified further and so it is the answer.} \end{aligned}$$

Removing brackets

Sometimes you need to remove brackets before collecting like terms together.

EXAMPLE 1.2

Simplify the expression $3(2x-4y)-4(x-5y)$.

SOLUTION

Expression $= 6x-12y-4x+20y$ (Open the brackets) (Notice $(-4)\times(-5y)=+20y$)

$= 6x-4x+20y-12y$ (Collect like terms)

$= 2x+8y$ (Answer)

EXAMPLE 1.3

Simplify $x(x+2)-(x-4)$.

SOLUTION

Expression $= x^2+2x-x+4$ (Open the brackets)

$= x^2+x+4$ (Answer)

EXAMPLE 1.4

Simplify $a(b+c)-ac$.

SOLUTION

Expression $= ab+ac-ac$ (Open the brackets)

$= ab$ (Answer)

Factorisation

It is often possible to rewrite an expression as the product of two or more numbers or expressions, its *factors*. This usually involves using brackets and is called *factorisation*. Factorisation may make an expression easier to use and neater to write, or it may help you to interpret its meaning.

EXAMPLE 1.5

Factorise $12x-18y$.

SOLUTION

(6 is a factor of both 12 and 18.)

Expression $= 6(2x-3y)$

EXAMPLE 1.6

Factorise $x^2-2xy+3xz$.

SOLUTION

(x is a factor of all three terms.)

Expression $= x(x-2y+3z)$

Multiplication

Several of the previous examples have involved multiplication of variables: cases like

$$a \times b = ab \quad \text{and} \quad x \times x = x^2.$$

In the next example the principles are the same but the expressions are not quite so simple.

EXAMPLE 1.7

Multiply $3p^2qr \times 4pq^3 \times 5qr^2$.

SOLUTION

$$\begin{aligned} \text{Expression} &= 3 \times 4 \times 5 \times p^2 \times p \times q \times q^3 \times q \times r \times r^2 \\ &= 60 \times p^3 \times q^5 \times r^3 \\ &= 60p^3q^5r^3 \end{aligned}$$

You might well do this line in your head.

Fractions

The rules for working with fractions in algebra are exactly the same as those used in arithmetic.

EXAMPLE 1.8

Simplify $\frac{x}{2} - \frac{2y}{10} + \frac{z}{4}$.

SOLUTION

As in arithmetic you start by finding the common denominator. For 2, 10 and 4 this is 20.

Then you write each part as the equivalent fraction with 20 as its denominator, as follows.

$$\begin{aligned} \text{Expression} &= \frac{10x}{20} - \frac{4y}{20} + \frac{5z}{20} \\ &= \frac{10x - 4y + 5z}{20} \end{aligned}$$

This line would often be left out.

EXAMPLE 1.9

Simplify $\frac{x^2}{y} - \frac{y^2}{x}$.

SOLUTION

$$\begin{aligned} \text{Expression} &= \frac{x^3}{xy} - \frac{y^3}{xy} \\ &= \frac{x^3 - y^3}{xy} \end{aligned}$$

The common denominator is xy.

EXAMPLE 1.10

Simplify $\frac{3x^2}{5y} \times \frac{5yz}{6x}$.

SOLUTION

Since the two parts of the expression are multiplied, terms may be cancelled top and bottom as in arithmetic. In this case 3, 5, x and y may all be cancelled.

$$\text{Expression} = \frac{\cancel{3}x^{\cancel{2}}}{\cancel{5}\cancel{y}} \times \frac{\cancel{5}\cancel{y}z}{\cancel{6}_2\cancel{x}}$$

$$= \frac{xz}{2}$$

EXAMPLE 1.11

Simplify $\frac{(x-1)^3}{4x(x-1)}$.

SOLUTION

$(x-1)$ is a common factor of both top and bottom, so may be cancelled. However, x is not a factor of the top (the numerator), so may not be cancelled.

$$\text{Expression} = \frac{(x-1)^2}{4x}$$

EXAMPLE 1.12

Simplify $\frac{24x+6}{3(4x+1)}$.

SOLUTION

When the numerator (top) and/or the denominator (bottom) are not factorised, first factorise them as much as possible. Then you can see whether there are any common factors which can be cancelled.

$$\text{Expression} = \frac{6(4x+1)}{3(4x+1)}$$

$$= 2$$

EXERCISE 1A

1 Simplify the following expressions by collecting like terms.

(i) $8x + 3x + 4x - 6x$

(ii) $3p + 3 + 5p - 7 - 7p - 9$

(iii) $2k + 3m + 8n - 3k - 6m - 5n + 2k - m + n$

(iv) $2a + 3b - 4c + 4a - 5b - 8c - 6a + 2b + 12c$

(v) $r - 2s - t + 2r - 5t - 6r - 7t - s + 5s - 2t + 4r$

2 Factorise the following expressions.

(i) $4x + 8y$ **(ii)** $12a + 15b - 18c$

(iii) $72f - 36g - 48h$ **(iv)** $p^2 - pq + pr$

(v) $12k^2 + 144km - 72kn$

3 Simplify the following expressions, factorising the answers where possible.

(i) $8(3x + 2y) + 4(x + 3y)$

(ii) $2(3a - 4b + 5c) - 3(2a - 5b - c)$

(iii) $6(2p - 3q + 4r) - 5(2p - 6q - 3r) - 3(p - 4q + 2r)$

(iv) $4(l + w + h) + 3(2l - w - 2h) + 5w$

(v) $5u - 6(w - v) + 2(3u + 4w - v) - 11u$

4 Simplify the following expressions, factorising the answers where possible.

(i) $a(b + c) + a(b - c)$ **(ii)** $k(m + n) - m(k + n)$

(iii) $p(2q + r + 3s) - pr - s(3p + q)$ **(iv)** $x(x - 2) - x(x - 6) + 8$

(v) $x(x - 1) + 2(x - 1) - x(x + 1)$

5 Perform the following multiplications, simplifying your answers.

(i) $2xy \times 3x^2y$ **(ii)** $5a^2bc^3 \times 2ab^2 \times 3c$

(iii) $km \times mn \times nk$ **(iv)** $3pq^2r \times 6p^2qr \times 9pqr^2$

(v) $rs \times 2st \times 3tu \times 4ur$

6 Simplify the following fractions as much as possible.

(i) $\frac{ab}{ac}$ **(ii)** $\frac{2e}{4f}$ **(iii)** $\frac{x^2}{5x}$

(iv) $\frac{4a^2b}{2ab}$ **(v)** $\frac{6p^2q^3r}{3p^3q^3r^2}$

7 Simplify the following as much as possible.

(i) $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}$ **(ii)** $\frac{3x}{2y} \times \frac{8y}{3z} \times \frac{5z}{4x}$ **(iii)** $\frac{p^2}{q} \times \frac{q^2}{p}$

(iv) $\frac{2fg}{16h} \times \frac{4gh^2}{4fh} \times \frac{32fh^3}{12f^3}$ **(v)** $\frac{kmn}{3n^3} \times \frac{6k^2m^3}{2k^3m}$

8 Write the following as single fractions.

(i) $\frac{x}{2} + \frac{x}{3}$ **(ii)** $\frac{2x}{5} - \frac{x}{3} + \frac{3x}{4}$ **(iii)** $\frac{3z}{8} + \frac{2z}{12} - \frac{5z}{24}$

(iv) $\frac{2x}{3} - \frac{x}{4}$ **(v)** $\frac{y}{2} - \frac{5y}{8} + \frac{4y}{5}$

9 Write the following as single fractions.

(i) $\frac{3}{x} + \frac{5}{x}$ **(ii)** $\frac{1}{x} + \frac{1}{y}$ **(iii)** $\frac{4}{x} + \frac{x}{y}$

(iv) $\frac{p}{q} + \frac{q}{p}$ **(v)** $\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$

10 Write the following as single fractions.

(i) $\frac{x+1}{4}+\frac{x-1}{2}$ **(ii)** $\frac{2x}{3}-\frac{x-1}{5}$ **(iii)** $\frac{3x-5}{4}+\frac{x-7}{6}$

(iv) $\frac{3(2x+1)}{5}-\frac{7(x-2)}{2}$ **(v)** $\frac{4x+1}{8}+\frac{7x-3}{12}$

11 Simplify the following expressions.

(i) $\frac{x+3}{2x+6}$ **(ii)** $\frac{6(2x+1)^2}{3(2x+1)^5}$ **(iii)** $\frac{2x(y-3)^4}{8x^2(y-3)}$

(iv) $\frac{6x-12}{x-2}$ **(v)** $\frac{(3x+2)^2}{6x}\times\frac{x^4}{6x+4}$

b Linear equations

? What is a variable?

You will often need to find the value of the variable in an expression in a particular case, as in the following example.

EXAMPLE 1.13

A polygon is a closed figure whose sides are straight lines. Figure 1.1 shows a seven-sided polygon (a heptagon).

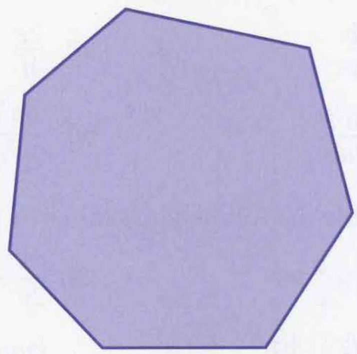


Figure 1.1

An expression for $S°$, the sum of the angles of a polygon with n sides, is

$$S = 180(n-2).$$

? How is this expression obtained?

Try dividing a polygon into triangles, starting from one vertex.

Find the number of sides in a polygon with an angle sum of **(i)** 180° **(ii)** 1080°.

SOLUTION

(i) Substituting 180 for S gives $\quad 180 = 180(n-2)$ ← This is an equation which can be solved to find n.

Dividing both sides by 180 $\Rightarrow \quad 1 = n-2$

Adding 2 to both sides $\Rightarrow \quad 3 = n$

The polygon has three sides: it is a triangle.

(ii) Substituting 1080 for S gives $\quad 1080 = 180(n-2)$

Dividing both sides by 180 $\Rightarrow \quad 6 = n-2$

Adding 2 to both sides $\Rightarrow \quad 8 = n$

The polygon has eight sides: it is an octagon.

Example 1.13 illustrates the process of solving an equation. An *equation* is formed when an expression, in this case $180(n-2)$, is set equal to a value, in this case 180 or 1080, or to another expression. *Solving* means finding the value(s) of the variable(s) in the equation.

Since both sides of an equation are equal, you may do what you wish to an equation provided that you do exactly the same thing to both sides. If there is only one variable involved (like n in the above examples), you aim to get that on one side of the equation, and everything else on the other. The two examples which follow illustrate this.

In both of these examples the working is given in full, step by step. In practice you would expect to omit some of these lines by tidying up as you went along.

Look at the statement $5(x-1) = 5x-5$.

What happens when you try to solve it as an equation?

This is an *identity* and not an equation. It is true for *all* values of x.

For example, try $x = 11$: $5(x-1) = 5 \times (11-1) = 50$; $5x-5 = 55-5 = 50$ ✓,
or try $x = 46$: $5(x-1) = 5 \times (46-1) = 225$; $5x-5 = 230-5 = 225$ ✓,
or try $x =$ anything else and it will still be true.

To distinguish an identity from an equation, the symbol $\equiv$ is sometimes used.

Thus $5(x-1) \equiv 5x-5$.

EXAMPLE 1.14 Solve the equation $5(x-3) = 2(x+6)$.

SOLUTION

Open the brackets	$\Rightarrow$	$5x-15 = 2x+12$
Subtract $2x$ from both sides	$\Rightarrow$	$5x-2x-15 = 2x-2x+12$
Tidy up	$\Rightarrow$	$3x-15 = 12$
Add 15 to both sides	$\Rightarrow$	$3x-15+15 = 12+15$
Tidy up	$\Rightarrow$	$3x = 27$
Divide both sides by 3	$\Rightarrow$	$\frac{3x}{3} = \frac{27}{3}$
	$\Rightarrow$	$x = 9$

CHECK

When the answer is substituted in the original equation both sides should come out to be equal. If they are different, you have made a mistake.

Left-hand side	**Right-hand side**
$5(x-3)$	$2(x+6)$
$5(9-3)$	$2(9+6)$
5×6	2×15
30	30 (as required).

EXAMPLE 1.15 Solve the equation $\frac{1}{2}(x+6) = x+\frac{1}{3}(2x-5)$.

SOLUTION

Start by clearing the fractions. Since the numbers 2 and 3 appear on the bottom line, multiply through by 6 which cancels both of them.

Multiply both sides by 6	$\Rightarrow$	$6\times\frac{1}{2}(x+6) = 6\times x+6\times\frac{1}{3}(2x-5)$
Tidy up	$\Rightarrow$	$3(x+6) = 6x+2(2x-5)$
Open the brackets	$\Rightarrow$	$3x+18 = 6x+4x-10$
Subtract $6x$, $4x$, and 18 from both sides	$\Rightarrow$	$3x-6x-4x = -10-18$
Tidy up	$\Rightarrow$	$-7x = -28$
Divide both sides by (-7)	$\Rightarrow$	$\frac{-7x}{-7} = \frac{-28}{-7}$
	$\Rightarrow$	$x = 4$

CHECK

Substituting $x=4$ in $\frac{1}{2}(x+6) = x+\frac{1}{3}(2x-5)$ gives:

Left-hand side	**Right-hand side**
$\frac{1}{2}(4+6)$	$4+\frac{1}{3}(8-5)$
$\frac{10}{2}$	$4+\frac{3}{3}$
5	5 (as required).

EXERCISE 1B

1 Solve the following equations.

(i) $5a - 32 = 68$

(ii) $4b - 6 = 3b + 2$

(iii) $2c + 12 = 5c + 12$

(iv) $5(2d + 8) = 2(3d + 24)$

(v) $3(2e - 1) = 6(e + 2) + 3e$

(vi) $7(2 - f) - 3(f - 4) = 10f - 4$

(vii) $5g + 2(g - 9) = 3(2g - 5) + 11$

(viii) $3(2h - 6) - 6(h + 5) = 2(4h - 4) - 10(h + 4)$

(ix) $\frac{1}{2}k + \frac{1}{4}k = 36$

(x) $\frac{1}{2}(l - 5) + l = 11$

(xi) $\frac{1}{2}(3m + 5) + 1\frac{1}{2}(2m - 1) = 5\frac{1}{2}$

(xii) $n + \frac{1}{3}(n + 1) + \frac{1}{4}(n + 2) = \frac{5}{6}$

2 The largest angle of a triangle is six times as big as the smallest. The third angle is 75°.

(i) Write this information in the form of an equation for a, the size in degrees of the smallest angle.

(ii) Solve the equation and so find the sizes of the three angles.

3 Miriam and Saloma are twins and their sister Rohana is 2 years older than them.
The total of their ages is 32 years.

(i) Write this information in the form of an equation for r, Rohana's age in years.

(ii) What are the ages of the three girls?

4 The length, d m, of a rectangular field is 40 m greater than the width.
The perimeter of the field is 400 m.

(i) Write this information in the form of an equation for d.

(ii) Solve the equation and so find the area of the field.

5 Yash can buy three pencils and have 49c change, or he can buy five pencils and have 15c change.

(i) Write this information as an equation for x, the cost in cents of one pencil.

(ii) How much money did Yash have to start with?

6 In a multiple-choice examination of 25 questions, four marks are given for each correct answer and two marks are deducted for each wrong answer. One mark is deducted for any question which is not attempted. A candidate attempts q questions and gets c correct.

(i) Write down an expression for the candidate's total mark in terms of q and c.

(ii) James attempts 22 questions and scores 55 marks. Write down and solve an equation for the number of questions which James gets right.

7 Joe buys 18 kg of potatoes. Some of these are old potatoes at 22c per kilogram, the rest are new ones at 36c per kilogram.

(i) Denoting the mass of old potatoes he buys by m kg, write down an expression for the total cost of Joe's potatoes.

(ii) Joe pays with a \$5 note and receives 20c change. What mass of new potatoes does he buy?

8 In 18 years' time Hussein will be five times as old as he was 2 years ago.

(i) Write this information in the form of an equation involving Hussein's present age, a years.

(ii) How old is Hussein now?

b Changing the subject of a formula

The area of a trapezium is given by

$$A = \tfrac{1}{2}(a + b)h$$

where a and b are the lengths of the parallel sides and h is the distance between them (see figure 1.2). An equation like this is often called a *formula*.

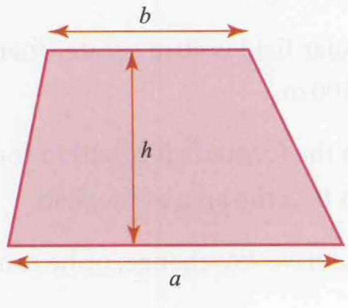


Figure 1.2

The variable A is called the subject of this formula because it only appears once on its own on the left-hand side. You often need to make one of the other variables the subject of a formula. In that case, the steps involved are just the same as those in solving an equation, as the following examples show.

EXAMPLE 1.16

Make a the subject in $A = \frac{1}{2}(a+b)h$.

SOLUTION

It is usually easiest if you start by arranging the equation so that the variable you want to be its subject is on the left-hand side.

$$\tfrac{1}{2}(a+b)h = A$$

Multiply both sides by 2 $\Rightarrow$ $(a+b)h = 2A$

Divide both sides by h $\Rightarrow$ $a+b = \dfrac{2A}{h}$

Subtract b from both sides $\Rightarrow$ $a = \dfrac{2A}{h} - b$

EXAMPLE 1.17

Make T the subject in the simple interest formula $I = \dfrac{PRT}{100}$.

SOLUTION

Arrange with T on the left-hand side $\dfrac{PRT}{100} = I$

Multiply both sides by 100 $\Rightarrow$ $PRT = 100I$

Divide both sides by P and R $\Rightarrow$ $T = \dfrac{100I}{PR}$

EXAMPLE 1.18

Make x the subject in the formula $v = \omega\sqrt{a^2 - x^2}$. (This formula gives the speed of an oscillating point.)

SOLUTION

Square both sides $\Rightarrow$ $v^2 = \omega^2(a^2 - x^2)$

Divide both sides by ω^2 $\Rightarrow$ $\dfrac{v^2}{\omega^2} = a^2 - x^2$

Add x^2 to both sides $\Rightarrow$ $\dfrac{v^2}{\omega^2} + x^2 = a^2$

Subtract $\dfrac{v^2}{\omega^2}$ from both sides $\Rightarrow$ $x^2 = a^2 - \dfrac{v^2}{\omega^2}$

Take the square root of both sides $\Rightarrow$ $x = \pm\sqrt{a^2 - \dfrac{v^2}{\omega^2}}$

EXAMPLE 1.19

Make m the subject of the formula $mv = I + mu$. (This formula gives the momentum after an impulse.)

SOLUTION

Collect terms in m on the left-hand side and terms without m on the other. $\Rightarrow$ $mv - mu = I$

Factorise the left-hand side $\Rightarrow$ $m(v-u) = I$

Divide both sides by $(v-u)$ $\Rightarrow$ $m = \dfrac{I}{v-u}$

EXERCISE 1C

1 Make **(i)** a **(ii)** t the subject in $v = u + at$.

2 Make h the subject in $V = lwh$.

3 Make r the subject in $A = \pi r^2$.

4 Make **(i)** s **(ii)** u the subject in $v^2 - u^2 = 2as$.

5 Make h the subject in $A = 2\pi rh + 2\pi r^2$.

6 Make a the subject in $s = ut + \frac{1}{2}at^2$.

7 Make b the subject in $h = \sqrt{a^2 + b^2}$.

8 Make g the subject in $T = 2\pi\sqrt{\frac{l}{g}}$.

9 Make m the subject in $E = mgh + \frac{1}{2}mv^2$.

10 Make R the subject in $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

11 Make h the subject in $bh = 2A - ah$.

12 Make u the subject in $f = \frac{uv}{u + v}$.

13 Make d the subject in $u^2 - du + fd = 0$.

14 Make V the subject in $p_1VM = mRT + p_2VM$.

? All the formulae in Exercise 1C refer to real situations. Can you recognise them?

Quadratic equations

EXAMPLE 1.20

The length of a rectangular field is 40 m greater than its width, and its area is $6000\,m^2$. Form an equation involving the length, x m, of the field.

SOLUTION

Since the length of the field is 40 m greater than the width,

the width in m must be $x - 40$

and the area in m^2 is $x(x - 40)$.

So the required equation is $x(x - 40) = 6000$

or $x^2 - 40x - 6000 = 0$.

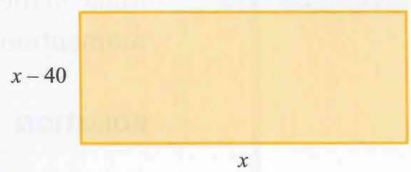


Figure 1.3

This equation, involving terms in x^2 and x as well as a constant term (i.e. a number, in this case 6000), is an example of a *quadratic equation*. This is in contrast to a linear equation. A linear equation in the variable x involves only terms in x and constant terms.

It is usual to write a quadratic equation with the right-hand side equal to zero. To solve it, you first factorise the left-hand side if possible, and this requires a particular technique.

Quadratic factorisation

EXAMPLE 1.21

Factorise $xa + xb + ya + yb$.

SOLUTION

Notice $(a + b)$ is a common factor.

$$xa + xb + ya + yb = x(a + b) + y(a + b)$$
$$= (x + y)(a + b)$$

The expression is now in the form of two factors, $(x + y)$ and $(a + b)$, so this is the answer.

You can see this result in terms of the area of the rectangle in figure 1.4. This can be written as the product of its length $(x + y)$ and its width $(a + b)$, or as the sum of the areas of the four smaller rectangles, xa, xb, ya and yb.

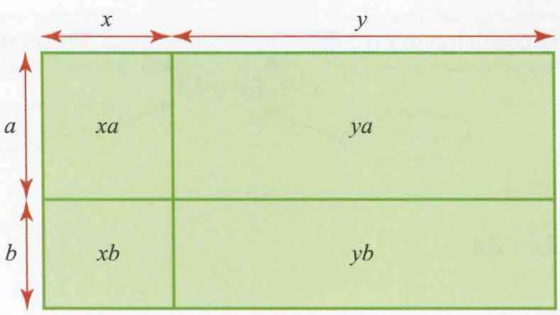


Figure 1.4

The same pattern is used for quadratic factorisation, but first you need to split the middle term into two parts. This gives you four terms, which correspond to the areas of the four regions in a diagram like figure 1.4.

P1

1

Algebra

EXAMPLE 1.22 Factorise $x^2 + 7x + 12$.

SOLUTION

Splitting the middle term, $7x$, as $4x + 3x$ you have

$$\begin{aligned} x^2 + 7x + 12 &= x^2 + 4x + 3x + 12 \\ &= x(x+4) + 3(x+4) \\ &= (x+3)(x+4). \end{aligned}$$

How do you know to split the middle term, $7x$, into $4x + 3x$, rather than say $5x + 2x$ or $9x - 2x$?

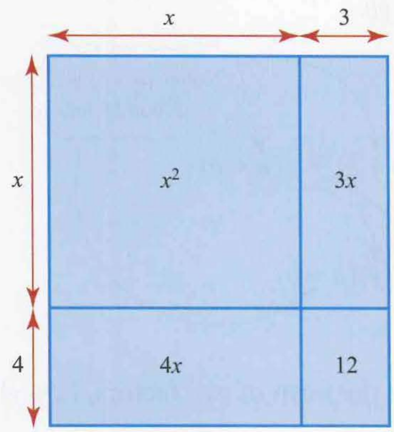
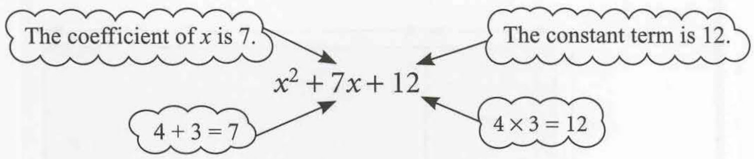


Figure 1.5

The numbers 4 and 3 can be added to give 7 (the middle coefficient) and multiplied to give 12 (the constant term), so these are the numbers chosen.



EXAMPLE 1.23 Factorise $x^2 - 2x - 24$.

SOLUTION

First you look for two numbers that can be added to give –2 and multiplied to give –24:

$$-6 + 4 = -2 \qquad -6 \times (+4) = -24.$$

The numbers are –6 and +4 and so the middle term, $-2x$, is split into $-6x + 4x$.

$$\begin{aligned} x^2 - 2x - 24 &= x^2 - 6x + 4x - 24 \\ &= x(x-6) + 4(x-6) \\ &= (x+4)(x-6). \end{aligned}$$

This example raises a number of important points.

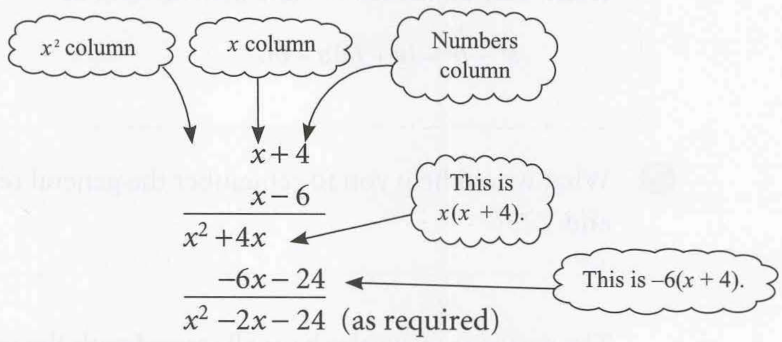
1 It makes no difference if you write $+4x - 6x$ instead of $-6x + 4x$. In that case the factorisation reads:

$$\begin{aligned} x^2 - 2x - 24 &= x^2 + 4x - 6x - 24 \\ &= x(x+4) - 6(x+4) \\ &= (x-6)(x+4) \end{aligned}$$ (clearly the same answer).

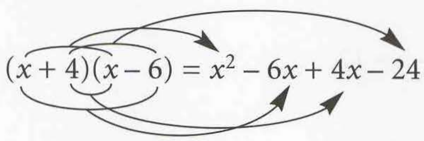
2 There are other methods of quadratic factorisation. If you have already learned another way, and consistently get your answers right, then continue to use it. This method has one major advantage: it is self-checking. In the last line but one of the solution to the example, you will see that $(x+4)$ appears twice. If at this point the contents of the two brackets are different, for example $(x+4)$ and $(x-4)$, then something is wrong. You may have chosen the wrong numbers, or made a careless mistake, or perhaps the expression cannot be factorised. There is no point in proceeding until you have sorted out why they are different.

3 You may check your final answer by multiplying it out to get back to the original expression. There are two common ways of setting this out.

(i) Long multiplication



(ii) Multiplying term by term



$= x^2 - 2x - 24$ (as required)

You would not expect to draw the lines and arrows in your answers. They have been put in to help you understand where the terms have come from.

EXAMPLE 1.24 Factorise $x^2 - 20x + 100$.

SOLUTION

$$\begin{aligned} x^2 - 20x + 100 &= x^2 - 10x - 10x + 100 \\ &= x(x-10) - 10(x-10) \\ &= (x-10)(x-10) \\ &= (x-10)^2 \end{aligned}$$

Notice:
$(-10) + (-10) = -20$
$(-10) \times (-10) = +100$

Note

The expression in Example 1.24 was a *perfect square*. It is helpful to be able to recognise the form of such expressions.

$$(x + a)^2 = x^2 + 2ax + a^2 \quad \text{(in this case } a = 10\text{)}$$
$$(x - a)^2 = x^2 - 2ax + a^2$$

EXAMPLE 1.25

Factorise $x^2 - 49$.

SOLUTION

Notice this is $x^2 - 7^2$

$x^2 - 49$ can be written as $x^2 + 0x - 49$.

$$\begin{aligned} x^2 + 0x - 49 &= x^2 - 7x + 7x - 49 \\ &= x(x - 7) + 7(x - 7) \\ &= (x + 7)(x - 7) \end{aligned}$$

$-7 + 7 = 0$
$(-7) \times 7 = -49$

Note

The expression in Example 1.25 was an example of *the difference of two squares* which may be written in more general form as

$$a^2 - b^2 = (a + b)(a - b).$$

? What would help you to remember the general results from Examples 1.24 and 1.25?

The previous examples have all started with the term x^2, that is the coefficient of x^2 has been 1. This is not the case in the next example.

EXAMPLE 1.26

Factorise $6x^2 + x - 12$.

SOLUTION

The technique for finding how to split the middle term is now adjusted. Start by multiplying the two outside numbers together:

$$6 \times (-12) = -72.$$

Now look for two numbers which add to give +1 (the coefficient of x) and multiply to give −72 (the number found above).

$$(+9) + (-8) = +1 \qquad (+9) \times (-8) = -72$$

Splitting the middle term gives

$$\begin{aligned} 6x^2 + 9x - 8x - 12 &= 3x(2x + 3) - 4(2x + 3) \\ &= (3x - 4)(2x + 3) \end{aligned}$$

$3x$ is a factor of both $6x^2$ and $9x$.

-4 is a factor of both $-8x$ and -12.

Note

The method used in the earlier examples is really the same as this. It is just that in those cases the coefficient of x^2 was 1 and so multiplying the constant term by it had no effect.

Before starting the procedure for factorising a quadratic, you should always check that the terms do not have a common factor as for example in

$$2x^2 - 8x + 6.$$

This can be written as $2(x^2 - 4x + 3)$ and factorised to give $2(x - 3)(x - 1)$.

Solving quadratic equations

It is a simple matter to solve a quadratic equation once the quadratic expression has been factorised. Since the product of the two factors is zero, it follows that one or other of them must equal zero, and this gives the solution.

EXAMPLE 1.27

Solve $x^2 - 40x - 6000 = 0$.

SOLUTION

$$\begin{aligned} x^2 - 40x - 6000 &= x^2 - 100x + 60x - 6000 \\ &= x(x - 100) + 60(x - 100) \\ &= (x + 60)(x - 100) \end{aligned}$$

$$\Rightarrow \quad (x + 60)(x - 100) = 0$$

$$\Rightarrow \quad \text{either } x + 60 = 0 \quad \Rightarrow \quad x = -60$$

$$\Rightarrow \quad \text{or } x - 100 = 0 \quad \Rightarrow \quad x = 100$$

The solution is $x = -60$ or 100.

Look back to page 12. What is the length of the field?

Note

The *solution* of the equation in the example is $x = -60$ or 100.

The *roots* of the equation are the values of x which satisfy the equation, in this case one root is $x = -60$ and the other root is $x = 100$.

Sometimes an equation can be rewritten as a quadratic and then solved.

EXAMPLE 1.28

Solve $x^4 - 13x^2 + 36 = 0$

SOLUTION

This is a quartic equation (its highest power of x is 4) and it isn't easy to factorise this directly. However, you can rewrite the equation as a quadratic in x^2.

Let $y = x^2$

$$x^4 - 13x^2 + 36 = 0$$
$$\Rightarrow (x^2)^2 - 13x^2 + 36 = 0$$
$$\Rightarrow \quad y^2 - 13y + 36 = 0$$

You can replace x^2 with y to get a quadratic equation.

Now you have a quadratic equation which you can factorise.

$$(y-4)(y-9) = 0$$

So $\quad y = 4$ or $y = 9$

Don't stop here. You are asked to find x, not y.

Since $\quad y = x^2 \quad$ then $x^2 = 4 \Rightarrow x = \pm 2$

or $x^2 = 9 \Rightarrow x = \pm 3$

Remember the negative square root.

You may have to do some work rearranging the equation before you can solve it.

EXAMPLE 1.29

Find the real roots of the equation $x^2 - 2 = \dfrac{8}{x^2}$.

SOLUTION

You need to rearrange the equation before you can solve it.

$$x^2 - 2 = \frac{8}{x^2}$$

Multiply by x^2: $\quad x^4 - 2x^2 = 8$

Rearrange: $\quad x^4 - 2x^2 - 8 = 0$

This is a quadratic in x^2. You can factorise it directly, without substituting in for x^2.

$\Rightarrow \quad (x^2 + 2)(x^2 - 4) = 0$

So $\quad x^2 = -2$ which has no real solutions.

or $\quad x^2 = 4 \Rightarrow x = \pm 2$

So this quartic equation only has two real roots. You can find out more about roots which are not real in P3.

EXERCISE 1D

1 Factorise the following expressions.

(i) $al + am + bl + bm$ **(ii)** $px + py - qx - qy$

(iii) $ur - vr + us - vs$ **(iv)** $m^2 + mn + pm + pn$

(v) $x^2 - 3x + 2x - 6$ **(vi)** $y^2 + 3y + 7y + 21$

(vii) $z^2 - 5z + 5z - 25$ **(viii)** $q^2 - 3q - 3q + 9$

(ix) $2x^2 + 2x + 3x + 3$ **(x)** $6v^2 + 3v - 20v - 10$

2 Multiply out the following expressions and collect like terms.

(i) $(a+2)(a+3)$ **(ii)** $(b+5)(b+7)$

(iii) $(c-4)(c-2)$ **(iv)** $(d-5)(d-4)$

(v) $(e+6)(e-1)$ **(vi)** $(g-3)(g+3)$

(vii) $(h+5)^2$ **(viii)** $(2i-3)^2$

(ix) $(a+b)(c+d)$ **(x)** $(x+y)(x-y)$

3 Factorise the following quadratic expressions.

(i) x^2+6x+8 **(ii)** x^2-6x+8
(iii) $y^2+9y+20$ **(iv)** $r^2+2r-15$
(v) $r^2-2r-15$ **(vi)** s^2-4s+4
(vii) x^2-5x-6 **(viii)** x^2+2x+1
(ix) a^2-9 **(x)** $(x+3)^2-9$

4 Factorise the following expressions.

(i) $2x^2+5x+2$ **(ii)** $2x^2-5x+2$
(iii) $5x^2+11x+2$ **(iv)** $5x^2-11x+2$
(v) $2x^2+14x+24$ **(vi)** $4x^2-49$
(vii) $6x^2-5x-6$ **(viii)** $9x^2-6x+1$
(ix) $t_1^2-t_2^2$ **(x)** $2x^2-11xy+5y^2$

5 Solve the following equations.

(i) $x^2-11x+24=0$ **(ii)** $x^2+11x+24=0$
(iii) $x^2-11x+18=0$ **(iv)** $x^2-6x+9=0$
(v) $x^2-64=0$

6 Solve the following equations.

(i) $3x^2-5x+2=0$ **(ii)** $3x^2+5x+2=0$
(iii) $3x^2-5x-2=0$ **(iv)** $25x^2-16=0$
(v) $9x^2-12x+4=0$

7 Solve the following equations.

(i) $x^2-x=20$ **(ii)** $\dfrac{3x^2+5x}{3}=4$
(iii) $x^2+4=4x$ **(iv)** $2x+1=\dfrac{15}{x}$
(v) $x-1=\dfrac{6}{x}$ **(vi)** $3x+\dfrac{8}{x}=14$

8 Solve the following equations.

(i) $x^4-5x^2+4=0$ **(ii)** $x^4-10x^2+9=0$
(iii) $9x^4-13x^2+4=0$ **(iv)** $4x^4-25x^2+36=0$
(v) $25x^4-4x^2=0$ **(vi)** $x-6\sqrt{x}+5=0$
(vii) $x^6-9x^3+8=0$ **(viii)** $x-\sqrt{x}-6=0$

9 Find the real roots of the following equations.

(i) $x^2+1=\dfrac{2}{x^2}$ **(ii)** $x^2=1+\dfrac{12}{x^2}$
(iii) $x^2-6=\dfrac{27}{x^2}$ **(iv)** $1+\dfrac{1}{x^2}-\dfrac{20}{x^4}=0$
(v) $\dfrac{9}{x^4}+4=\dfrac{13}{x^2}$ **(vi)** $x^3+\dfrac{2}{x^3}=3$
(vii) $\sqrt{x}+\dfrac{8}{\sqrt{x}}=6$ **(viii)** $2+\dfrac{3}{x}=\dfrac{7}{\sqrt{x}}$

10 Find the real roots of the equation $\frac{9}{x^4} + \frac{8}{x^2} = 1$.

11 The length of a rectangular field is 30 m greater than its width, w metres.

(i) Write down an expression for the area $A\,\text{m}^2$ of the field, in terms of w.

(ii) The area of the field is $8800\,\text{m}^2$. Find its width and perimeter.

12 A cylindrical tin of height h cm and radius $r$cm, has surface area, including its top and bottom, $A\,\text{cm}^2$.

(i) Write down an expression for A in terms of r, h and π.

(ii) A tin of height 6 cm has surface area $54\pi\,\text{cm}^2$. What is the radius of the tin?

(iii) Another tin has the same diameter as height. Its surface area is $150\pi\,\text{cm}^2$. What is its radius?

13 When the first n positive integers are added together, their sum is given by

$$\tfrac{1}{2}n(n+1).$$

(i) Demonstrate that this result holds for the case $n = 5$.

(ii) Find the value of n for which the sum is 105.

(iii) What is the smallest value of n for which the sum exceeds 1000?

14 The shortest side AB of a right-angled triangle is x cm long. The side BC is 1 cm longer than AB and the hypotenuse, AC, is 29 cm long.
Form an equation for x and solve it to find the lengths of the three sides of the triangle.

Equations that cannot be factorised

The method of quadratic factorisation is fine so long as the quadratic expression can be factorised, but not all of them can. In the case of $x^2 - 6x + 2$, for example, it is not possible to find two whole numbers which add to give -6 and multiply to give $+2$.

There are other techniques available for such situations, as you will see in the next few pages.

Graphical solution

If an equation has a solution, you can always find an approximate value for it by drawing a graph. In the case of

$$x^2 - 6x + 2 = 0$$

you draw the graph of

$$y = x^2 - 6x + 2$$

and find where it cuts the x axis.

x	0	1	2	3	4	5	6
x^2	0	1	4	9	16	25	36
$-6x$	0	−6	−12	−18	−24	−30	−36
$+2$	+2	+2	+2	+2	+2	+2	+2
y	+2	−3	−6	−7	−6	−3	+2

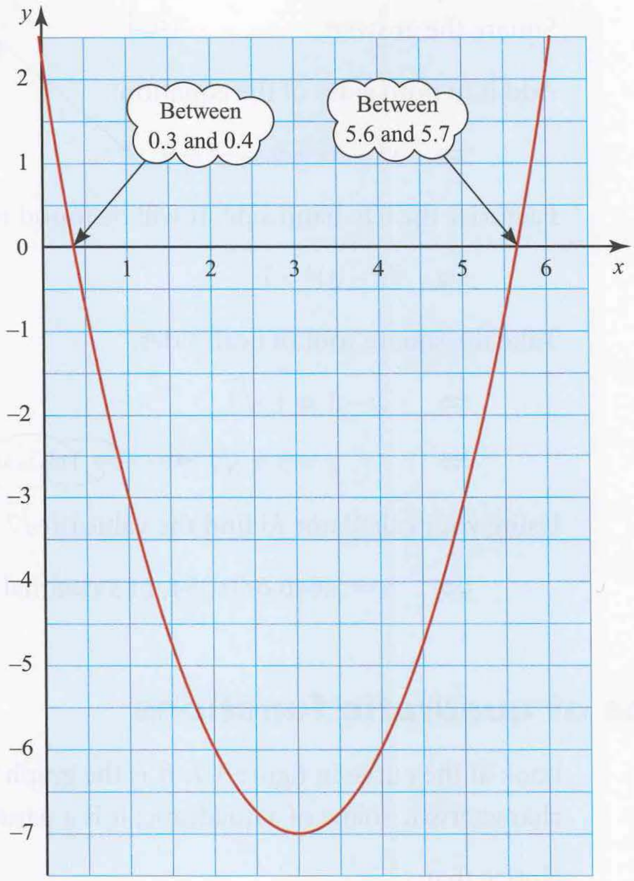


Figure 1.6

From figure 1.6, x is between 0.3 and 0.4 so approximately 0.35, or between 5.6 and 5.7 so approximately 5.65.

Clearly the accuracy of the answer is dependent on the scale of the graph but, however large a scale you use, your answer will never be completely accurate.

Completing the square

If a quadratic equation has a solution, this method will give it accurately. It involves adjusting the left-hand side of the equation to make it a perfect square. The steps involved are shown in the following example.

EXAMPLE 1.30 Solve the equation $x^2 - 6x + 2 = 0$ by completing the square.

SOLUTION

Subtract the constant term from both sides of the equation:

$$\Rightarrow \quad x^2 - 6x = -2$$

Take the coefficient of x: -6
Halve it: -3
Square the answer: $+9$

Explain why this makes the left-hand side a perfect square.

Add it to both sides of the equation:

$$\Rightarrow \quad x^2 - 6x + 9 = -2 + 9$$

Factorise the left-hand side. It will be found to be a perfect square:

$$\Rightarrow \quad (x-3)^2 = 7$$

Take the square root of both sides:

$$\Rightarrow \quad x - 3 = \pm\sqrt{7}$$

$$\Rightarrow \quad x = 3 \pm \sqrt{7}$$

This is an exact answer.

Using your calculator to find the value of $\sqrt{7}$

This is an approximate answer.

$$\Rightarrow \quad x = 5.646 \text{ or } 0.354, \text{ to 3 decimal places.}$$

The graphs of quadratic functions

Look at the curve in figure 1.7. It is the graph of $y = x^2 - 4x + 5$ and it has the characteristic shape of a quadratic; it is a parabola.

Notice that:

- it has a minimum point (or *vertex*) at (2, 1)
- it has a line of symmetry, $x = 2$.

It is possible to find the vertex and the line of symmetry without plotting the points by using the technique of completing the square.

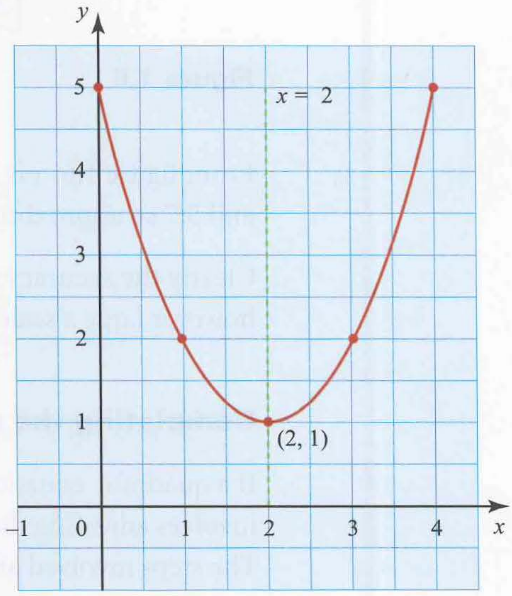


Figure 1.7

Rewrite the expression with the constant term moved to the right

$$x^2 - 4x \qquad + 5.$$

Take the coefficient of x: $\quad -4$
Divide it by 2: $\quad -2$
Square the answer: $\quad +4$

Add this to the left-hand part and compensate by subtracting it from the constant term on the right

$$x^2 - 4x + 4 \qquad + 5 - 4.$$

This is the completed square form.

This can now be written as $(x-2)^2 + 1$.

The line of symmetry is $x - 2 = 0$ or $x = 2$.

The minimum value is 1, so the vertex is (2, 1).

EXAMPLE 1.31

Write $x^2 + 5x + 4$ in completed square form.

Hence state the equation of the line of symmetry and the co-ordinates of the vertex of the curve $y = x^2 + 5x + 4$.

SOLUTION

$x^2 + 5x \qquad + 4$
$x^2 + 5x + 6.25 \quad + 4 - 6.25$ (5 ÷ 2 = 2.5; $2.5^2 = 6.25$)
$(x + 2.5)^2 - 2.25$ (This is the completed square form.)

The line of symmetry is $x + 2.5 = 0$, or $x = -2.5$.

The vertex is (–2.5, –2.25).

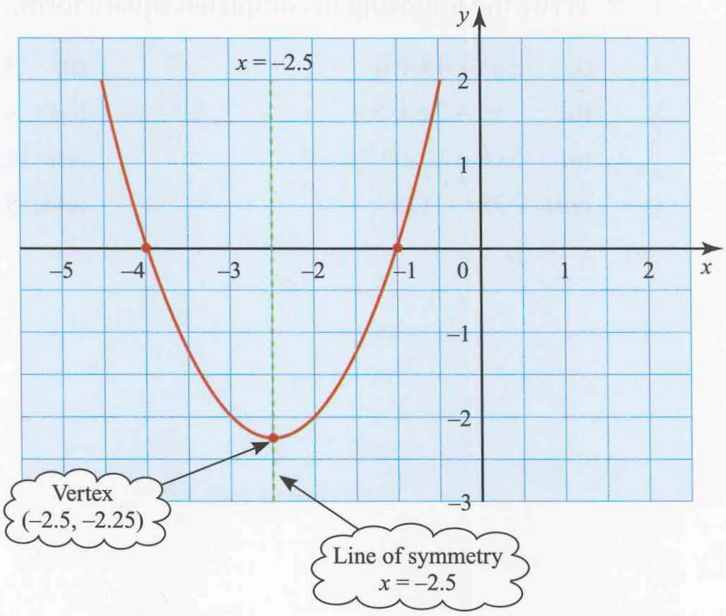


Figure 1.8

⚠ For this method, the coefficient of x^2 must be 1. To use it on, say, $2x^2 + 6x + 5$, you must write it as $2(x^2 + 3x + 2.5)$ and then work with $x^2 + 3x + 2.5$. In completed square form, it is $2(x + 1.5)^2 + 0.5$. Similarly treat $-x^2 + 6x + 5$ as $-1(x^2 - 6x - 5)$ and work with $x^2 - 6x - 5$. In completed square form it is $-1(x - 3)^2 + 14$.

Completing the square is an important technique. Knowing the symmetry and least (or greatest) value of a quadratic function will often give you valuable information about the situation it is modelling.

EXERCISE 1E

1 For each of the following equations:

(a) write it in completed square form

(b) hence write down the equation of the line of symmetry and the co-ordinates of the vertex

(c) sketch the curve.

(i) $y = x^2 + 4x + 9$	**(ii)** $y = x^2 - 4x + 9$
(iii) $y = x^2 + 4x + 3$	**(iv)** $y = x^2 - 4x + 3$
(v) $y = x^2 + 6x - 1$	**(vi)** $y = x^2 - 10x$
(vii) $y = x^2 + x + 2$	**(viii)** $y = x^2 - 3x - 7$
(ix) $y = x^2 - \frac{1}{2}x + 1$	**(x)** $y = x^2 + 0.1x + 0.03$

2 Write the following as quadratic expressions in descending powers of x.

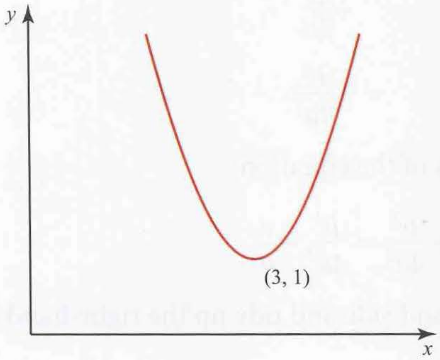
(i) $(x + 2)^2 - 3$	**(ii)** $(x + 4)^2 - 4$
(iii) $(x - 1)^2 + 2$	**(iv)** $(x - 10)^2 + 12$
(v) $\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$	**(vi)** $(x + 0.1)^2 + 0.99$

3 Write the following in completed square form.

(i) $2x^2 + 4x + 6$	**(ii)** $3x^2 - 18x - 27$
(iii) $-x^2 - 2x + 5$	**(iv)** $-2x^2 - 2x - 2$
(v) $5x^2 - 10x + 7$	**(vi)** $4x^2 - 4x - 4$
(vii) $-3x^2 - 12x$	**(viii)** $8x^2 + 24x - 2$

4 The curves below all have equations of the form $y = x^2 + bx + c$.
In each case find the values of b and c.

(i)



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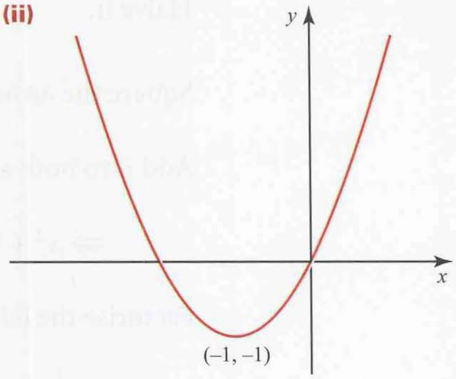
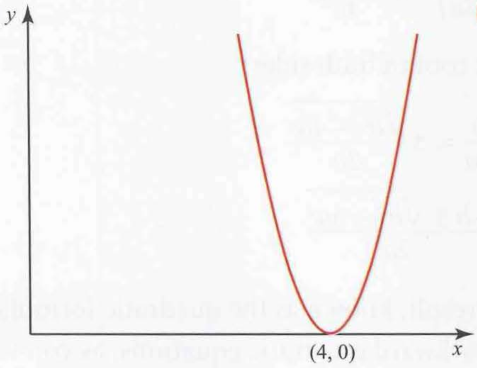
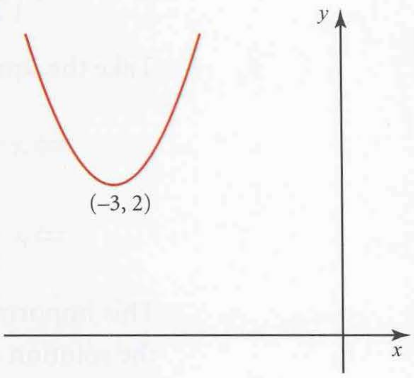


(iii)



(iv)



5 Solve the following equations by completing the square.

(i) $x^2 - 6x + 3 = 0$

(ii) $x^2 - 8x - 1 = 0$

(iii) $x^2 - 3x + 1 = 0$

(iv) $2x^2 - 6x + 1 = 0$

(v) $5x^2 + 4x - 2 = 0$

The quadratic formula

Completing the square is a powerful method because it can be used on any quadratic equation. However it is seldom used to solve an equation in practice because it can be generalised to give a formula which is used instead. The derivation of this follows exactly the same steps.

To solve a general quadratic equation $ax^2 + bx + c = 0$ by completing the square:

First divide both sides by a: $\Rightarrow \; x^2 + \frac{bx}{a} + \frac{c}{a} = 0.$

Subtract the constant term from both sides of the equation:

$$\Rightarrow x^2 + \frac{bx}{a} = -\frac{c}{a}$$

Take the coefficient of x: $+\frac{b}{a}$

Halve it: $+\frac{b}{2a}$

Square the answer: $+\frac{b^2}{4a^2}$

Add it to both sides of the equation:

$$\Rightarrow x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Factorise the left-hand side and tidy up the right-hand side:

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides:

$$\Rightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This important result, known as the quadratic formula, has significance beyond the solution of awkward quadratic equations, as you will see later. The next two examples, however, demonstrate its use as a tool for solving equations.

EXAMPLE 1.32

Use the quadratic formula to solve $3x^2 - 6x + 2 = 0$.

SOLUTION

Comparing this to the form $ax^2 + bx + c = 0$

gives $a = 3$, $b = -6$ and $c = 2$.

Substituting these values in the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

gives $x = \frac{6 \pm \sqrt{36 - 24}}{6}$

$= 0.423$ or 1.577 (to 3 d.p.).

EXAMPLE 1.33

Solve $x^2 - 2x + 2 = 0$.

SOLUTION

The first thing to notice is that this cannot be factorised. The only two whole numbers which multiply to give 2 are 2 and 1 (or −2 and −1) and they cannot be added to get −2.

Comparing $x^2 - 2x + 2$ to the form $ax^2 + bx + c = 0$
gives $a = 1$, $b = -2$ and $c = 2$.

Substituting these values in $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

gives $\dfrac{2 \pm \sqrt{4-8}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

Trying to find the square root of a negative number creates problems.
A positive number multiplied by itself is positive: $+2 \times +2 = +4$.
A negative number multiplied by itself is also positive: $-2 \times -2 = +4$.
Since $\sqrt{-4}$ can be neither positive nor negative, no such number exists, and so you can find no real solution.

Note

It is not quite true to say that a negative number has no square root. Certainly it has none among the real numbers but mathematicians have invented an imaginary number, denoted by i, with the property that $i^2 = -1$. Numbers like $1 + i$ and $-1 - i$ (which are in fact the solutions of the equation above) are called complex numbers. Complex numbers are extremely useful in both pure and applied mathematics; they are covered in P3.

To return to the problem of solving the equation $x^2 - 2x + 2 = 0$, look what happens if you draw the graph of $y = x^2 - 2x + 2$. The table of values is given below and the graph is shown in figure 1.9. As you can see, the graph does not cut the x axis and so there is indeed no real solution to this equation.

x	−1	0	1	2	3
x^2	+1	0	+1	+4	+9
$-2x$	+2	0	−2	−4	−6
+2	+2	+2	+2	+2	+2
y	+5	+2	+1	+2	+5

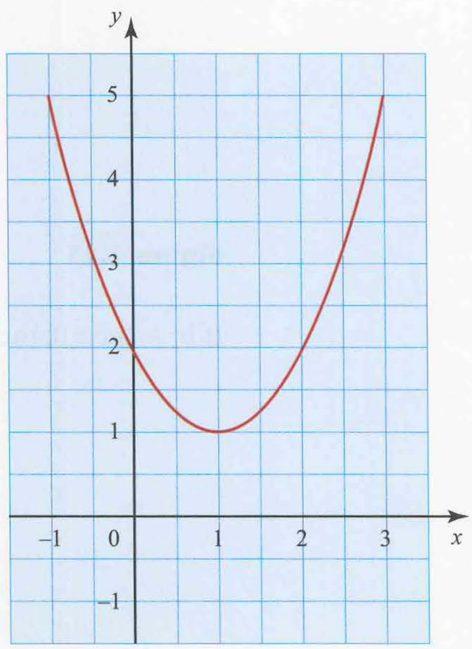


Figure 1.9

The part of the quadratic formula which determines whether or not there are real roots is the part under the square root sign. This is called the *discriminant.*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant, $b^2 - 4ac$

If $b^2 - 4ac > 0$, the equation has two real roots (see figure 1.10).

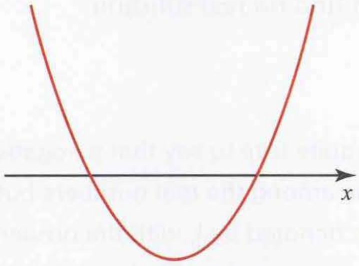


Figure 1.10

If $b^2 - 4ac < 0$, the equation has no real roots (see figure 1.11).

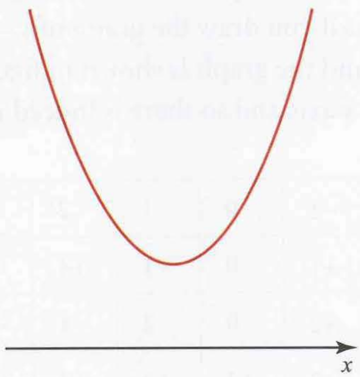


Figure 1.11

If $b^2 - 4ac = 0$, the equation has one repeated root (see figure 1.12).

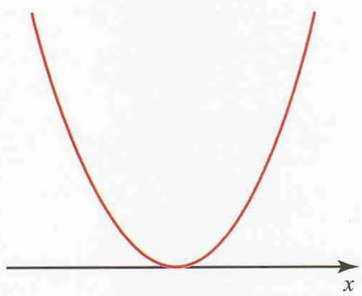


Figure 1.12

EXERCISE 1F

1 Use the quadratic formula to solve the following equations, where possible.

(i) $x^2 + 8x + 5 = 0$ (ii) $x^2 + 2x + 4 = 0$
(iii) $x^2 - 5x - 19 = 0$ (iv) $5x^2 - 3x + 4 = 0$
(v) $3x^2 + 2x - 4 = 0$ (vi) $x^2 - 12 = 0$

2 Find the value of the discriminant and use it to find the number of real roots for each of the following equations.

(i) $x^2 - 3x + 4 = 0$ (ii) $x^2 - 3x - 4 = 0$
(iii) $4x^2 - 3x = 0$ (iv) $3x^2 + 8 = 0$
(v) $3x^2 + 4x + 1 = 0$ (vi) $x^2 + 10x + 25 = 0$

3 Show that the equation $ax^2 + bx - a = 0$ has real roots for all values of a and b.

4 Find the value(s) of k for which these equations have one repeated root.

(i) $x^2 - 2x + k = 0$ (ii) $3x^2 - 6x + k = 0$
(iii) $kx^2 + 3x - 4 = 0$ (iv) $2x^2 + kx + 8 = 0$
(v) $3x^2 + 2kx - 3k = 0$

5 The height h metres of a ball at time t seconds after it is thrown up in the air is given by the expression

$$h = 1 + 15t - 5t^2.$$

(i) Find the times at which the height is 11 m.
(ii) Use your calculator to find the time at which the ball hits the ground.
(iii) What is the greatest height the ball reaches?

Simultaneous equations

There are many situations which can only be described mathematically in terms of more than one variable. When you need to find the values of the variables in such situations, you need to solve two or more equations simultaneously (i.e. at the same time). Such equations are called *simultaneous equations*. If you need to find values of two variables, you will need to solve two simultaneous equations; if three variables, then three equations, and so on. The work here is confined to solving two equations to find the values of two variables, but most of the methods can be extended to more variables if required.