

9. Work and Potential Energy

A) Overview

This unit is concerned with two topics. We will first discuss the relationship between the real work done by kinetic friction on a deformable body and the calculation that we can perform using the work-kinetic energy theorem to determine the change in the mechanical energy due to kinetic friction. We will then determine how to describe a conservative force directly in terms of its potential energy function. We will use this understanding to develop a description of the equilibrium conditions for objects acted on by conservative forces.

B) Box Sliding Down a Ramp

Figure 9.2 shows a box sliding down a ramp. In Unit 6, we identified the forces acting on the box as shown in the figure and applied Newton's second law to determine that the acceleration of the box was proportional to the acceleration due to gravity, with the constant of proportionality being determined by the angle of the ramp and the coefficient of kinetic friction.

$$a = g(\sin \theta - \mu_K \cos \theta)$$

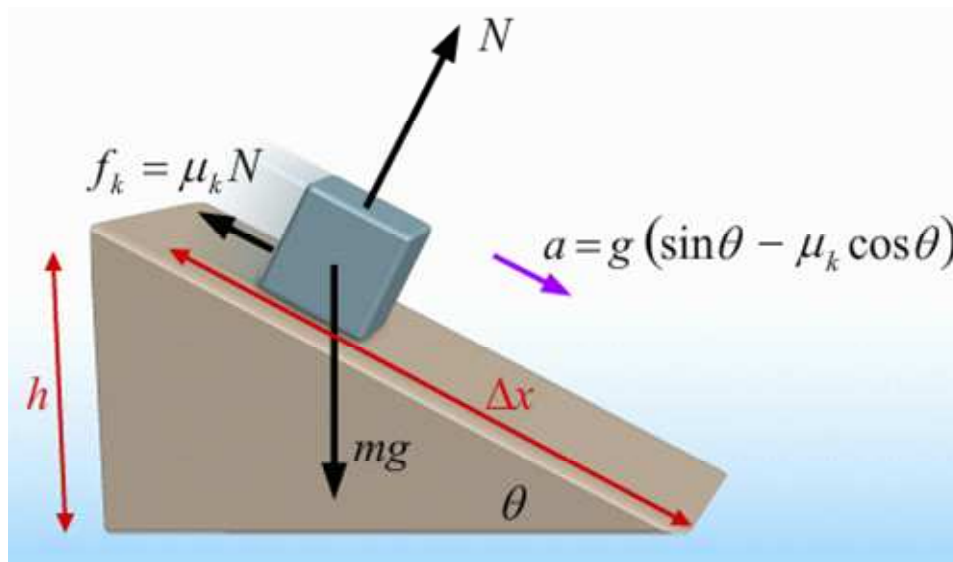


Figure 9.1

A box slides down a ramp. Three forces act on the box. Newton's second law can be used to determine the acceleration of the box.

Since this acceleration is constant, we can apply our kinematics equations we derived for such a motion to determine the speed of the box at it reaches the end of the ramp, if it were released from rest.

$$v^2 = 2a\Delta x$$
$$v^2 = 2gh(1 - \mu_K \cot \theta)$$

We can also apply the work-kinetic energy theorem here, treating the box as a particle. In this case, the change in the mechanical energy of the box is equal to the work done by

the non-conservative forces. The normal force does no work since it is perpendicular to the motion. Therefore, the change in mechanical energy of the box is just equal to the the integral of the frictional force over the displacement.

$$\frac{1}{2} m_{\text{box}} v^2 - m_{\text{box}} g h = \int \vec{f}_K \cdot d\vec{x}$$

Writing the friction force in terms of the coefficient of kinetic friction and the normal force and applying Newton's second law to relate the normal force to the weight, we can obtain an expression for the final velocity.

$$\frac{1}{2} m_{\text{box}} v^2 - m_{\text{box}} g h = \mu_K N \Delta x = \mu_K m_{\text{box}} g \cos \theta \Delta x = \mu_K m_{\text{box}} g \cos \theta \left(\frac{h}{\sin \theta} \right)$$

$$v^2 = 2gh(1 - \mu_K \cot \theta)$$

This expression that we have obtained from the work-kinetic energy theorem is identical to that obtained by using Newton's second law as it must be! In the next section we will discuss the connection between the work done by kinetic friction when we consider the box to be a deformable body and the calculation we can make to correctly determine the change in the mechanical energy of such an object.

C) Work Done by Kinetic Friction

Figure 9.2 shows a box with an initial velocity v_0 skidding across the floor,

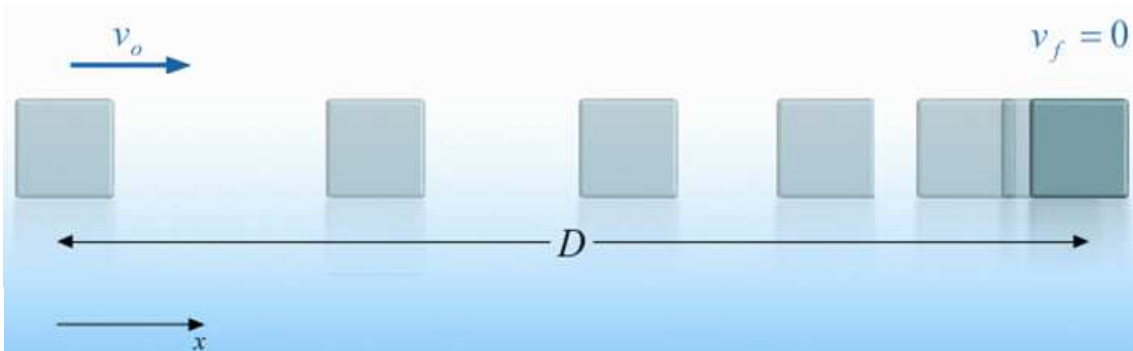


Figure 9.2

A box, with initial velocity v_0 skids across a floor and comes to rest a distance D from its initial position.

coming to rest a distance D from its initial position. We can use the free-body diagram shown in Figure 9.2 to determine the acceleration of the box. We can then use a constant acceleration kinematic equation to determine the distance D .

$$a = -\frac{\mu_K N}{m_{\text{box}}} = -\mu_K g$$

$$v_f^2 - v_0^2 = 2aD$$

$$D = \frac{v_0^2}{2\mu_K g}$$

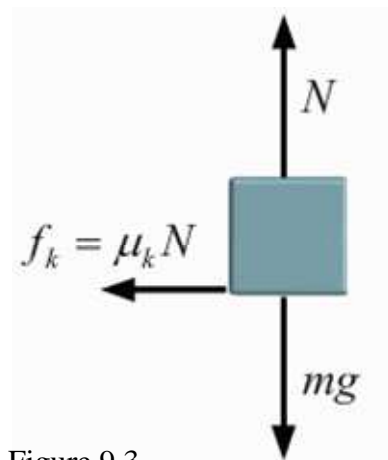


Figure 9.3

The free-body diagram for the sliding box in Fig 9.1

We can also calculate this distance using the work-kinetic energy theorem by setting the change in kinetic energy equal to the macroscopic work done by kinetic friction.

$$-\frac{1}{2}m_{\text{box}}v_o^2 = -\mu_K m_{\text{box}}gD$$

We've used the words "macroscopic work" here to indicate that we are interested only in the macroscopic motion of the box. To obtain this motion, we have treated the box as a *single object*. At the microscopic level, the box does have deformable surfaces and a calculation of the microscopic work done by friction must account for the interactions at these surfaces. The good news is that we do not need to know anything about these microscopic details to correctly calculate the macroscopic motion of the box. In the next unit we will discuss systems of particles and introduce the concept of the *center of mass* that will justify this claim. Indeed, throughout this course we will be concerned only with *macroscopic mechanical energy*; this energy can be understood solely in terms of Newton's second law.

The underlying physics in this example that *cannot* be understood solely in terms of Newton's second law is the thermodynamics needed to understand why, as the box comes to rest, it actually gets hotter. Namely, friction converts the macroscopic kinetic energy of the box into microscopic (thermal) energy of the molecules in the box and the floor. Understanding the details of this process is well beyond the scope of this course, yet we can get a qualitative picture of the mechanism involved by considering a simple model in which the atoms that make up the materials are thought of as little balls connected by springs. The frictional force then is modeled as the interactions of protrusions as the surfaces move past each other. The protrusions deform as they make contact, compressing and stretching some of the springs and causing a force opposing the relative motion – this is the force we call friction. Eventually the protrusions snap back past each other, causing vibrations of the balls that propagate to neighboring balls via the springs. These increased vibrations imply an increase in temperature. In other words, the deformation and release of the points of contact as the surfaces move past each other cause both the frictional force and the heating of the box

D) Forces and Potential Energy

It may be useful at this point to summarize what we have learned so far. We can calculate the work done on an object by a force as it moves through some displacement by integrating the dot product of the force and the displacement. The total macroscopic work done on an object by all forces is equal to the change in the kinetic energy of the object.

For conservative forces such as gravity and springs, we defined a quantity called the change in the potential energy (ΔU) as minus the work done by that conservative force. The potential energy U for an object at a particular location was defined as the change in potential energy of the object between an arbitrary point that defines the zero of potential energy and the specified location.

We obtain the potential energy associated with a force by calculating the work done by that force.

$$\Delta U_{\text{conservative } F} = -W_{\text{conservative } F} = -\int \vec{F} \cdot d\vec{x}$$

Can we invert this process? That is, can we start with the potential energy of an object at a particular location and determine the force that is acting on the object at that location?

To answer this question, let's consider the one-dimensional case. If we look at the work done by the force as the object moves from x to $x + dx$, we see that the potential energy changes by an amount $dU = -Fdx$. Consequently, we see that the force acting on the object at a given point is just equal to minus the derivative of the potential energy function at that same point.

$$F(x) = -\frac{dU(x)}{dx}$$

The force is a measure of how fast the potential energy is changing!

Of course, this result is not too surprising; if we find the potential energy from the force by evaluating an integral, then it's reasonable that we can find the force by performing the inverse operation, namely by taking the derivative of the potential. Indeed, this result can be generalized to more than one dimension with the use of the gradient operator, but we will only deal with examples that are essentially one-dimensional here. In the next section, we will verify directly that differentiating the expressions we have derived for the potential energy associated with gravity and the spring force yield the familiar force expressions.

E) Examples: Force from Potential Energy

We will now consider a simple example to illustrate our result that the force acting on an object at a particular location is related to the spatial derivative of the potential energy of the object at that location. Figure 9.4 shows a mass attached to a spring. Defining x to be the displacement of the mass from its equilibrium position, we see that the potential energy function for the mass is a simple parabola.

The spatial derivative of the potential energy of the mass at point x is represented on the graph as the slope of the tangent to the curve at point x . *The steeper the slope, the bigger the magnitude of the force.* Taking the derivative of the potential function,

$$F(x) = -\frac{dU(x)}{dx} = -kx$$

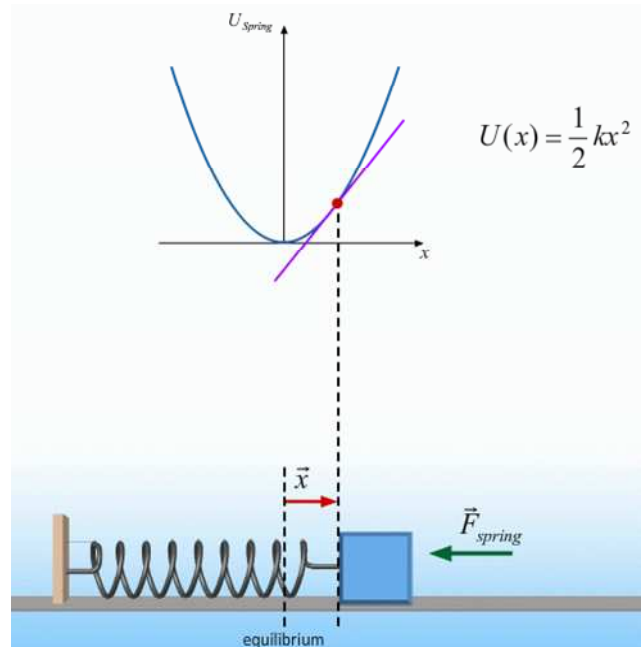


Figure 9.4

Defining the potential energy of a spring to be equal to zero at its equilibrium position results in the potential energy function being a parabola.

we find that the force is proportional to the displacement from equilibrium, in agreement with our expression for the force law for springs.

The same result also applies for the gravitational force as shown in Figure 9.5. Namely, we know that the potential energy of an object out in space some distance R from the center of Earth is inversely proportional to R . The slope of the tangent for this $1/R$ potential as a function of R tells us that the magnitude of the gravitational force is largest at the surface of the Earth and decreases as $1/R^2$ as we move further away. Since this slope is always positive, the force will always be negative, indicating that the direction of the force always points toward the center of the Earth. This result is just what we expect for Newton's universal law of gravitation.

Note that shifting the potential energy function up or down in either example does *not* change the force since adding a constant to a function does not change its slope. This result affirms our understanding that we are always free to add a constant to the potential energy function without changing the underlying physics.

F) Equilibrium

So far we have used the word "equilibrium" to mean the position or orientation where the net force on an object is zero. For conservative forces, we can get an equivalent condition for equilibrium in terms of the potential energy. Namely, since we can express a conservative force in terms of the spatial derivative of its potential energy function, we see that the locations for equilibrium will be at points at which the slope of this potential energy function is zero!

For example, if we refer back to Figure 9.4, we see the potential energy function for a mass on a spring. The slope of the tangent to this curve is zero at only one point, namely, the point at which the potential energy itself is at its minimum. .

Indeed, the slope of the tangent to *any* function will be zero at *any* minima or maxima of the function. Figure 9.6 shows a roller coaster track. Since the potential energy due to gravity for an object near the surface of the earth is just equal to the product of the weight of the object and its height above a position defined to be zero, we see that the height of the track (measured from the zero of potential energy) is directly proportional the gravitational potential energy of a car at that point. If a car is placed at

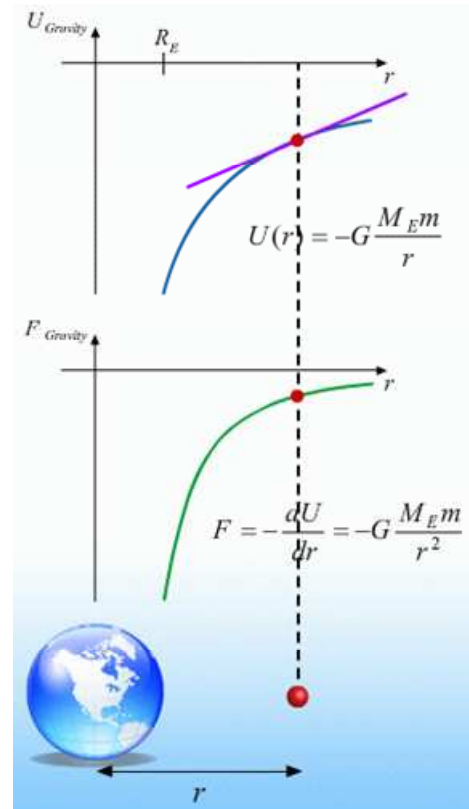


Figure 9.5
Differentiating the gravitational potential function recovers the familiar inverse square form of the Newton's universal gravitational force.

rest at the minima (B) or either of the two maxima (A) and (C), it will remain at rest since the net horizontal force on the car at each of these positions is zero. If the car at position

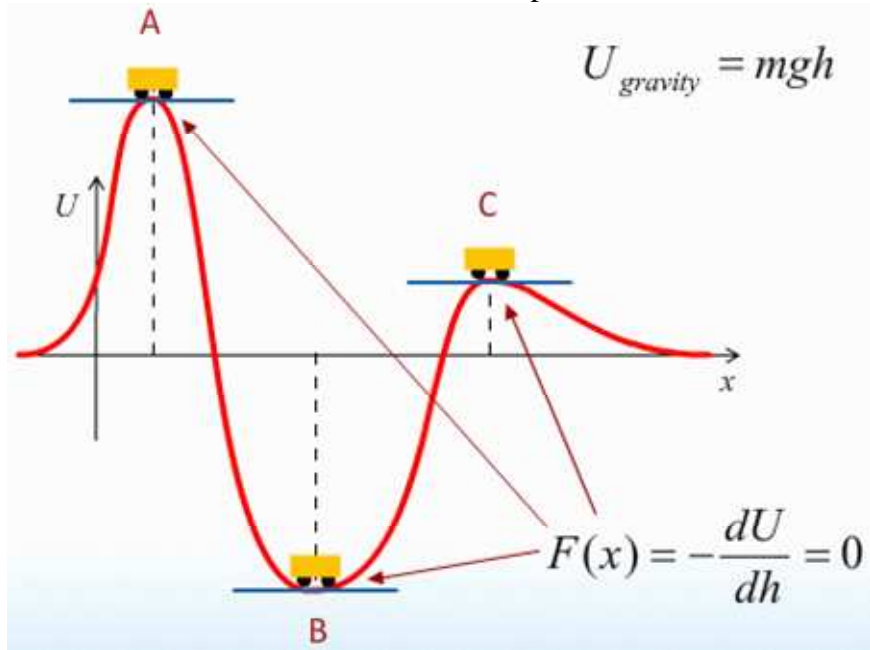


Figure 9.6

The points of equilibrium on a roller coaster track are determined as the points in which the derivative of the potential energy function is zero.

(B) is given a small push in either direction it will return to the equilibrium position. If the car is given a small push in either direction at positions (A) or (C), however, it will roll down the track, moving farther away from the equilibrium position. We say that the equilibrium is *stable* at the minima and is *unstable* at the maxima.

To generalize this observation, we can say that an object which is initially at rest tends to move toward the configuration where its potential energy is minimized. Note that there is no new physics in this statement – this statement in terms of potential energy is equivalent to the statement that an object accelerates in the direction of the net force acting on it.

Main Points

- **Macroscopic Work Done by Friction**

The work done by a kinetic friction force acting on a deformable object cannot be calculated without an understanding of the nature of the interactions at the surfaces.

However, the work-kinetic energy theorem can still be applied if we consider all forces to act at the center of mass of the object. The integral of the friction force through the full displacement of the object is called the macroscopic work done by friction. This macroscopic work does determine the change in the mechanical energy of the object.

$$W_{friction}^{macroscopic} \equiv \int \vec{f}_K \cdot d\vec{x}$$

$$W_{friction}^{macroscopic} = \Delta E$$

if friction is only non-conservative force acting

- **Force from Potential Energy**

For conservative forces, the force acting on any object at any point is equal to minus the spatial derivative of the potential energy function of the object at that point

(1-D)

$$F(x) = -\frac{dU(x)}{dx}$$

(3-D)

$$\vec{F} = -\vec{\nabla}U$$

Equilibrium positions for a particle acted on by conservative forces are those locations where the slope of the potential energy function is zero

