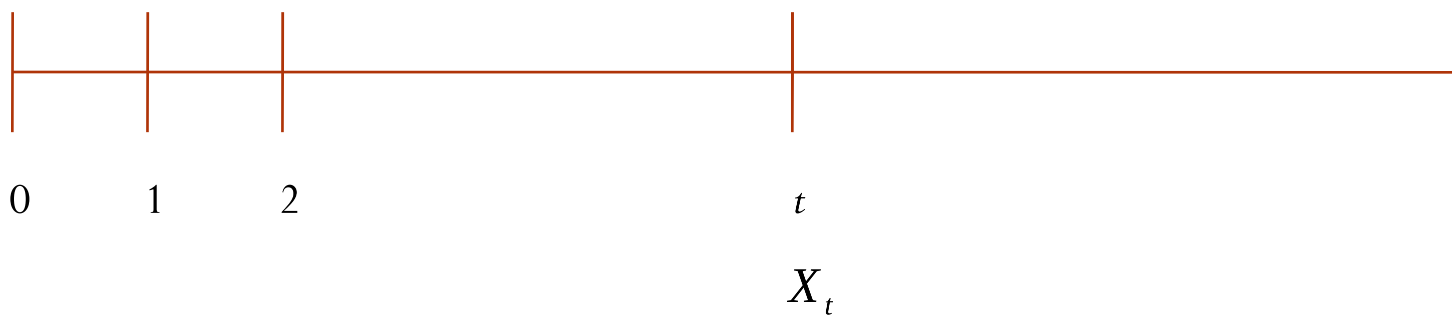
Markov chain

Carles Sitompul

What is a stochastic process?

- Observe some characteristic of a system **at discrete points** in time.



X_t be the value of the system characteristic at time t, is not known with certainty, hence viewed as a **random variable**.

- **A discrete-time stochastic process** is a description of relation between the random variable, X_0, $X_1 \ldots X_t$.

The gambler's ruin

At time 0, I have \$2. At times 1, 2, . . . , I play a game in which I bet \$1.

- With probability p, I win the game, and
- with probability $1 - p$, I lose the game.

My goal is to increase my capital to \$4, and as soon as I do, the game is over. The game is also over if my capital is reduced to \$0.

Define:

X_t **= capital position after the time *t* game *(if any)* is played (in dollars)**,

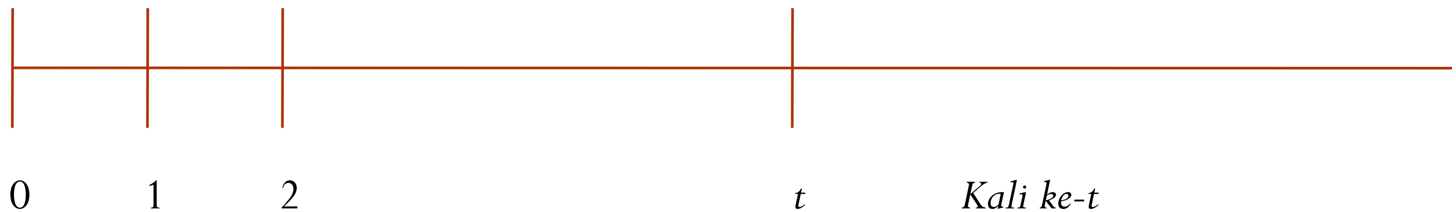
then $X_0, X_1, \ldots, X_t$ may be viewed as **a discrete-time stochastic process**.

- Note that $X_0 = 2$ is a known constant, but X_1 and later X_t's are random. For example, with probability p, $X_1 = \$3$, and with probability 1 - p, $X_1 = 1$.
- Note that if $X_t = 4$, then X_{t+1} and all later X_t's will also equal 4. Similarly, if $X_t = 0$, then X_{t+1} and all later X_t's will also equal 0.

Choosing ball from an Urn

An urn contains two unpainted balls at present. We choose a ball at random and flip a coin. If the chosen ball is unpainted and the coin comes up heads, we paint the chosen unpainted ball red; if the chosen ball is unpainted and the coin comes up tails, we paint the chosen unpainted ball black. If the ball has already been painted, then (whether heads or tails has been tossed) we change the color of the ball (from red to black or from black to red).

- Define time t to be the time after the coin has been flipped for the t-*th* time and the chosen ball has been painted.



- The state at any time may be described by the vector [u r b],
  - u is the number of unpainted balls in the urn,
  - r is the number of red balls in the urn, and
  - b is the number of black balls in the urn

$$X_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$X_t \rightarrow X_{t+1}$$

$$X_t = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \longrightarrow X_{t+1} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

CSL computer stock

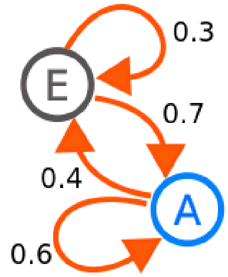
- Let $\mathbf{X_0}$ be the price of a share of CSL Computer stock at the beginning of the current trading day.
- Let $\mathbf{X_t}$ be the price of a share of CSL stock at the beginning of the *t*-th trading day in the future.

- Clearly, knowing the values of $\mathbf{X_0}, \mathbf{X_1}, \ldots, \mathbf{X_t}$ tells us something about the probability distribution of $\mathbf{X_{t+1}}$; the question is, what does the past (stock prices up to time *t*) tell us about $\mathbf{X_{t+1}}$?

Continuous time stochastic process

- **A continuous-time stochastic process** is simply a stochastic process in which the state of the system can be **viewed at any time**, not just at discrete instants in time.
- For example, the number of people in a supermarket *t* minutes after the store opens for business may be viewed as a continuous-time stochastic process.

What is a Markov Chain?



A discrete-time stochastic process is a **Markov chain** if, for $t = 0, 1, 2, \ldots$ and all states,

$$P(\mathbf{X}_{t+1} = i_{t+1} | \mathbf{X}_t = i_t, \mathbf{X}_{t-1} = i_{t-1}, \ldots, \mathbf{X}_1 = i_1, \mathbf{X}_0 = i_0) = P(\mathbf{X}_{t+1} = i_{t+1} | \mathbf{X}_t = i_t) \quad \blacksquare \tag{1}$$

- The probability distribution of the state at time $t+1$ depends on the state at time t (i_t) and
- does not depend on the states the chain passed through on the way to it at time t.

Stationarity Assumption

- Further assumption that for all states i and j and all t, $P(\boldsymbol{X}_{t+1} = j \mid \boldsymbol{X}_t = i)$ is independent of t.

$$\boxed{P(\boldsymbol{X}_{t+1} = j \mid X_t = i) = p_{ij}}$$

p_{ij} is the probability that given the system is in state i at time t, it will be in a state j at time $t + 1$.

= **transition probabilities** for the Markov Chain

→ **Stationary Markov Chain**

Initial probability distribution

- Define q_i to be the probability that the chain is in state i at time 0;

$$\boxed{P(\boldsymbol{X_0} = i) = q_i.}$$

- We call the vector q $[q_1\ q_2 \dots q_s]$ the **initial probability distribution** for the Markov chain.

Transition probability matrix

- The transition probabilities are displayed as an *s* x *s* **transition probability matrix**

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}$$

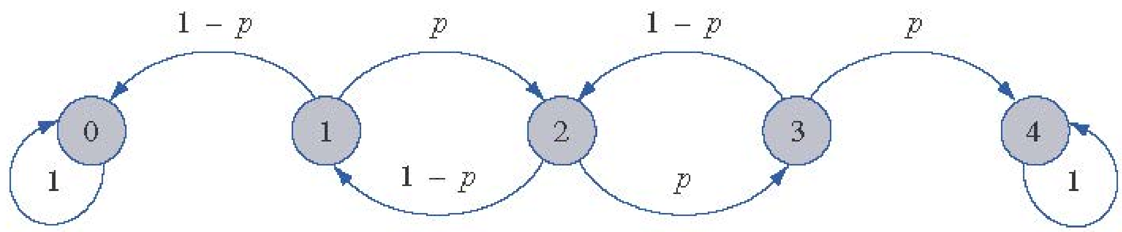
- Hence, all entries in the transition probability matrix are nonnegative, and the entries in each row must sum to 1.

$$\sum_{j=1}^{j=s} P(\mathbf{X}_{t+1} = j | P(\mathbf{X}_t = i)) = 1$$

$$\sum_{j=1}^{j=s} p_{ij} = 1$$

Gambler's ruin

$$
\begin{array}{c}
\text{State} \\
P = \begin{array}{c c}
 & \begin{array}{ccccc} \$0 & \$1 & \$2 & \$3 & \$4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1-p & 0 & p & 0 & 0 \\
0 & 1-p & 0 & p & 0 \\
0 & 0 & 1-p & 0 & p \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\end{array}
$$



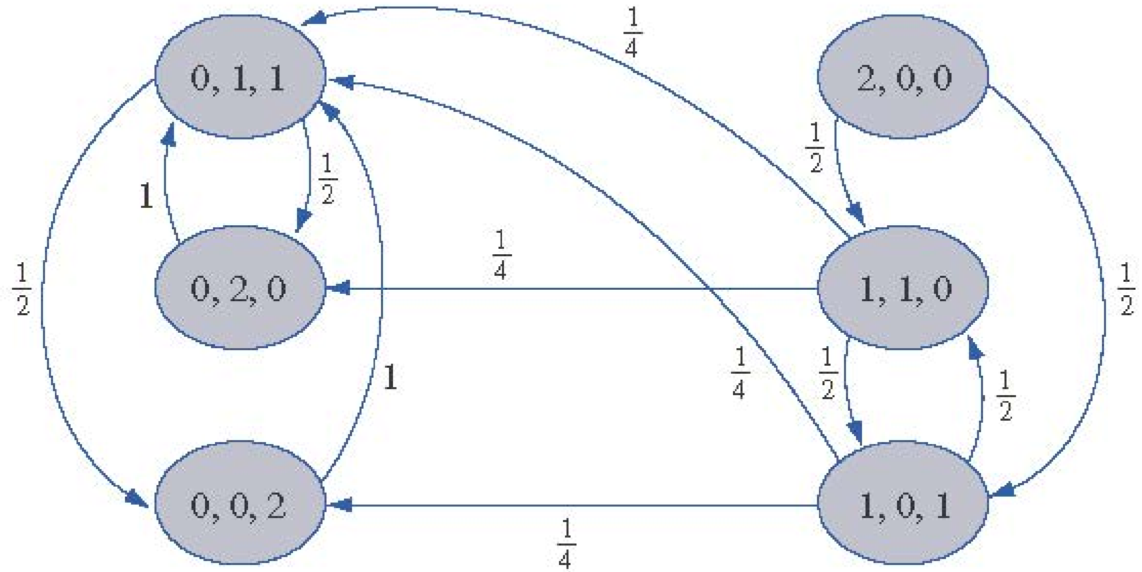
Choosing Balls

$$P = \begin{array}{c} \\ [0\ 1\ 1] \\ [0\ 2\ 0] \\ [0\ 0\ 2] \\ [2\ 0\ 0] \\ [1\ 1\ 0] \\ [1\ 0\ 1] \end{array} \begin{array}{c} \text{State} \\ \begin{array}{cccccc} [0\ 1\ 1] & [0\ 2\ 0] & [0\ 0\ 2] & [2\ 0\ 0] & [1\ 1\ 0] & [1\ 0\ 1] \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

TABLE 1

Computations of Transition Probabilities If Current State Is [1 1 0]

Event	Probability	New State
Flip heads and choose unpainted ball	$\frac{1}{4}$	[0 2 0]
Choose red ball	$\frac{1}{2}$	[1 0 1]
Flip tails and choose unpainted ball	$\frac{1}{4}$	[0 1 1]



0, 1, 1
2, 0, 0
0, 2, 0
1, 1, 0
0, 0, 2
1, 0, 1
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{2}$
1
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{2}$
1
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{4}$

Problems

1. In Smalltown, 90% of all sunny days are followed by sunny days, and 80% of all cloudy days are followed by cloudy days. Use this information to model Smalltown's weather as a Markov chain.
2. Consider an inventory system in which the sequence of events during each period is as follows.
    1. We observe the inventory level (call it *i)* at the beginning of the period.
    2. If $i \le 1$, $4 - i$ units are ordered. If $i \ge 2$, 0 units are ordered. Delivery of all ordered units is immediate.
    3. With probability 1/3, 0 units are demanded during the period; with probability 1/3, 1 unit is demanded during the period; and with probability 1/3, 2 units are demanded during the period.
    4. We observe the inventory level at the beginning of the next period.

    Define a period's state to be the period's beginning inventory level. Determine the transition matrix that could be used to model this inventory system as a Markov chain.

3. A company has two machines. During any day, each machine that is working at the beginning of the day has a 1/3 chance of breaking down. If a machine breaks down during the day, it is sent to a repair facility and will be working two days after it breaks down. (Thus, if a machine breaks down during day 3, it will be working at the beginning of day 5.) Letting the state of the system be the number of machines working at the beginning of the day, formulate a transition probability matrix for this situation.

n-step transition probabilities

- If a Markov chain is in state *i* at time *m*, what is the probability that *n* periods later the Markov chain will be in state *j*?
- Since we are dealing with a stationary Markov chain, this probability will be independent of *m*,

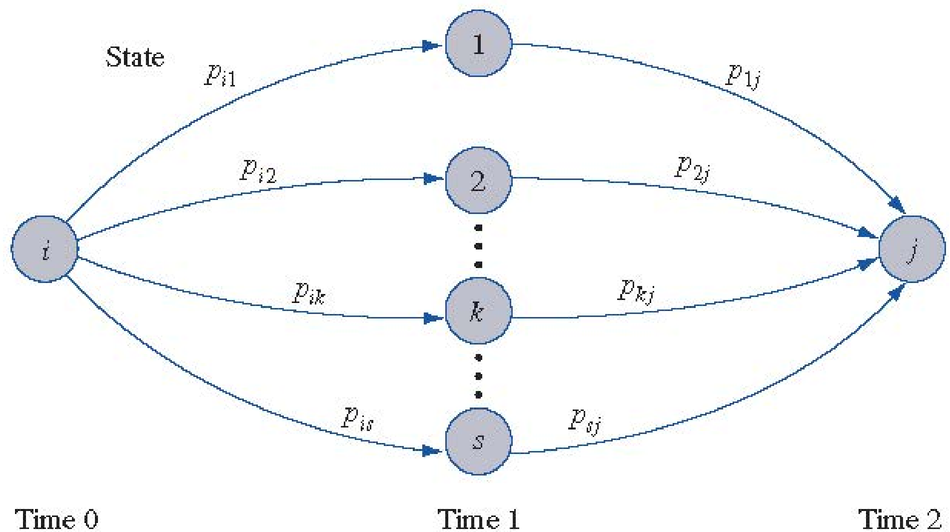
$$\boxed{P(\mathbf{X}_{m+n} = j \,|\, \mathbf{X}_m = i) = P(\mathbf{X}_n = j \,|\, \mathbf{X}_0 = i) = P_{ij}(n)}$$

$P_{ij}(n)$ called the ***n-step probability of a transition*** from state i to state j.

$$p_{ij}(1) = p_{ij}$$

$$P_{ij}(2) = \sum_{k=1}^{k=s} (\text{probability of transition from } i \text{ to } k) \times (\text{probability of transition from } k \text{ to } j)$$

$$P_{ij}(2) = \sum_{k=1}^{k=s} p_{ik}p_{kj}$$



- By extending this reasoning, it can be shown that for $n > 1$,

$$\boxed{P_{ij}(n) = ij\text{-th element of } P^n}$$

Probability of being in state j at time n

$$= \sum_{i=1}^{i=s} (\text{probability that state is originally } i)$$

$$\times (\text{probability of going from } i \text{ to } j \text{ in } n \text{ transitions})$$

$$= \sum_{i=1}^{i=s} q_i P_{ij}(n)$$

$$= \mathbf{q}(\text{column } j \text{ of } P^n)$$

$$\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_s].$$

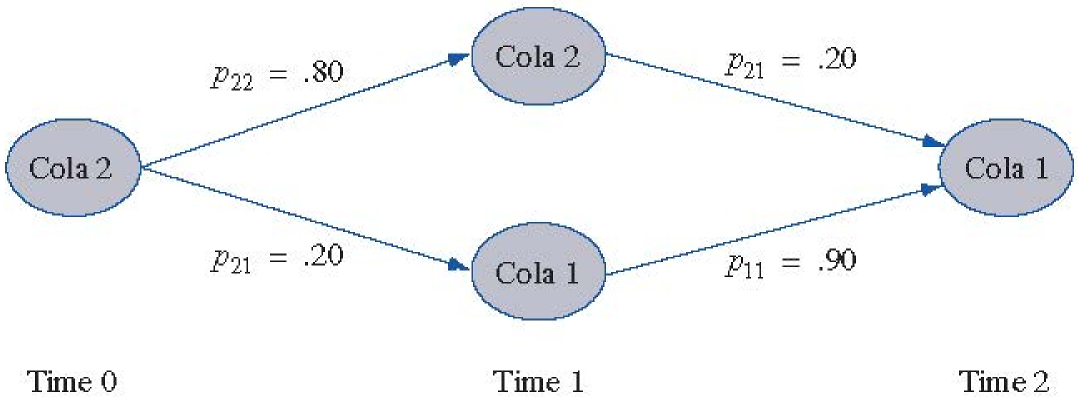
The cola example

- Suppose the entire cola industry produces only two colas. Given that a person last purchased cola 1, there is a 90% chance that her next purchase will be cola 1. Given that a person last purchased cola 2, there is an 80% chance that her next purchase will be cola 2.
  1. If a person is currently a cola 2 purchaser, what is the probability that she will purchase cola 1 two purchases from now?
  2. If a person is currently a cola 1 purchaser, what is the probability that she will purchase cola 1 three purchases from now?

- State 1 = person has last purchased cola 1
- State 2 = person has last purchased cola 2

$$P = \begin{array}{c c} & \begin{array}{cc} \text{Cola 1} & \text{Cola 2} \end{array} \\ \begin{array}{c} \text{Cola 1} \\ \text{Cola 2} \end{array} & \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} = \begin{bmatrix} .83 & .17 \\ .34 & .66 \end{bmatrix}$$



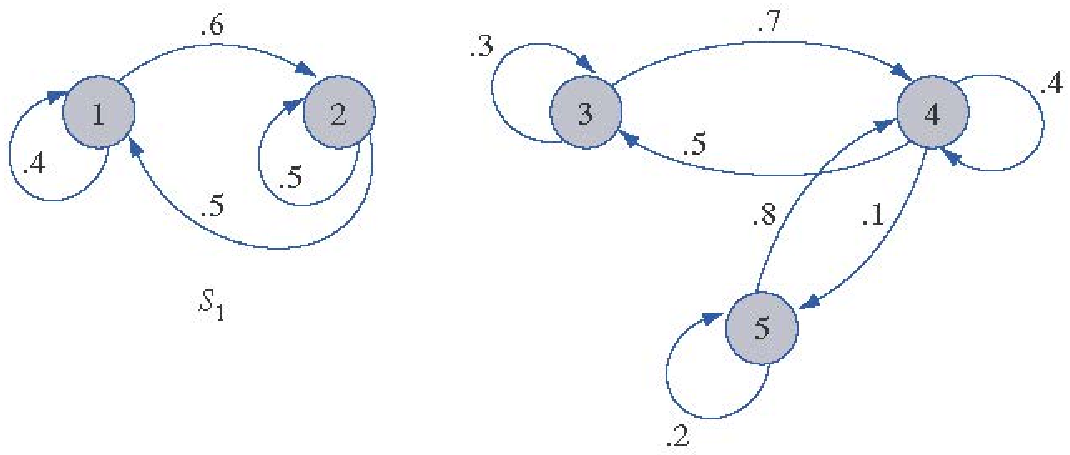
$$P^3 = P(P^2) = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} \begin{bmatrix} .83 & .17 \\ .34 & .66 \end{bmatrix} = \begin{bmatrix} .781 & .219 \\ .438 & .562 \end{bmatrix}$$

TABLE 2

n-Step Transition Probabilities for Cola Drinkers

n	$P_{11}(n)$	$P_{12}(n)$	$P_{21}(n)$	$P_{22}(n)$
1	.90	.10	.20	.80
2	.83	.17	.34	.66
3	.78	.22	.44	.56
4	.75	.25	.51	.49
5	.72	.28	.56	.44
10	.68	.32	.65	.35
20	.67	.33	.67	.33
30	.67	.33	.67	.33
40	.67	.33	.67	.33

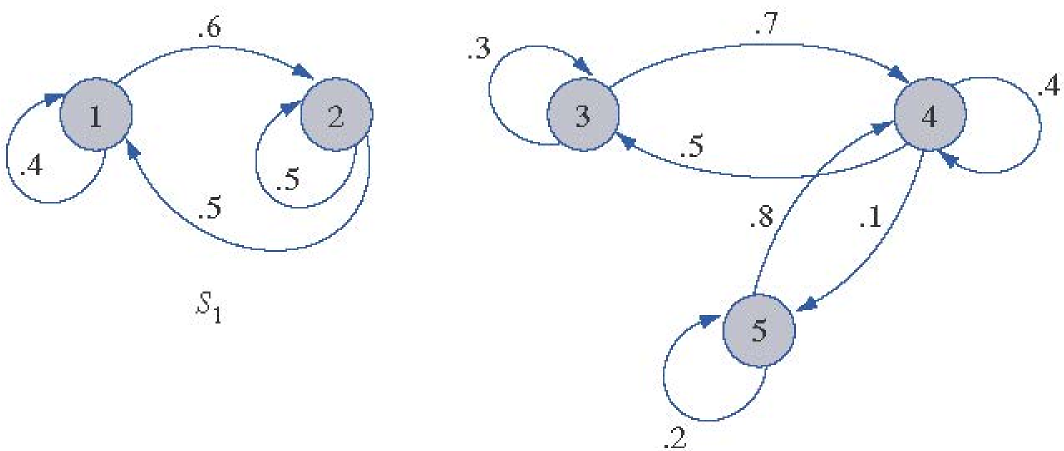
Classification of states



$$P = \begin{bmatrix} .4 & .6 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & 0 & .3 & .7 & 0 \\ 0 & 0 & .5 & .4 & .1 \\ 0 & 0 & 0 & .8 & .2 \end{bmatrix}$$

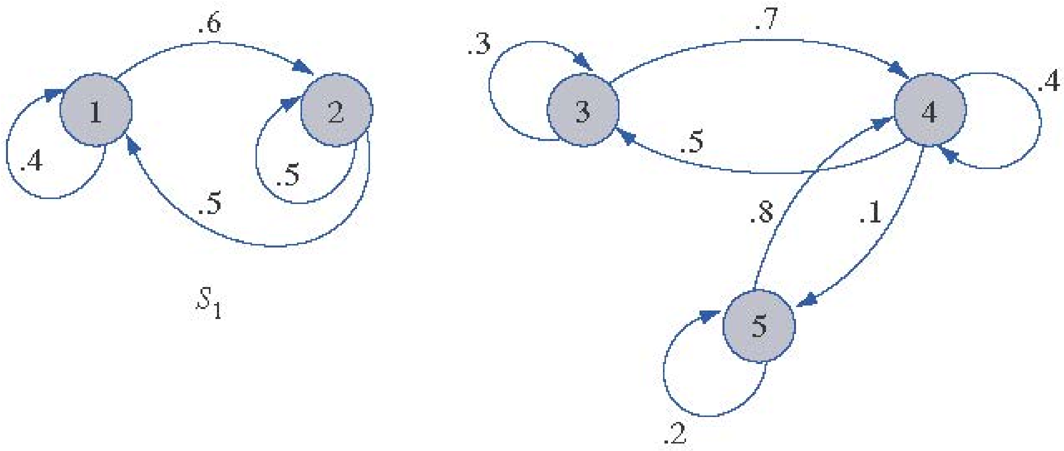
Given two states i and j, a **path** from i to j is a sequence of transitions that begins in i and ends in j, such that each transition in the sequence has a positive probability of occurring. ■

A state j is **reachable** from state i if there is a path leading from i to j. ■



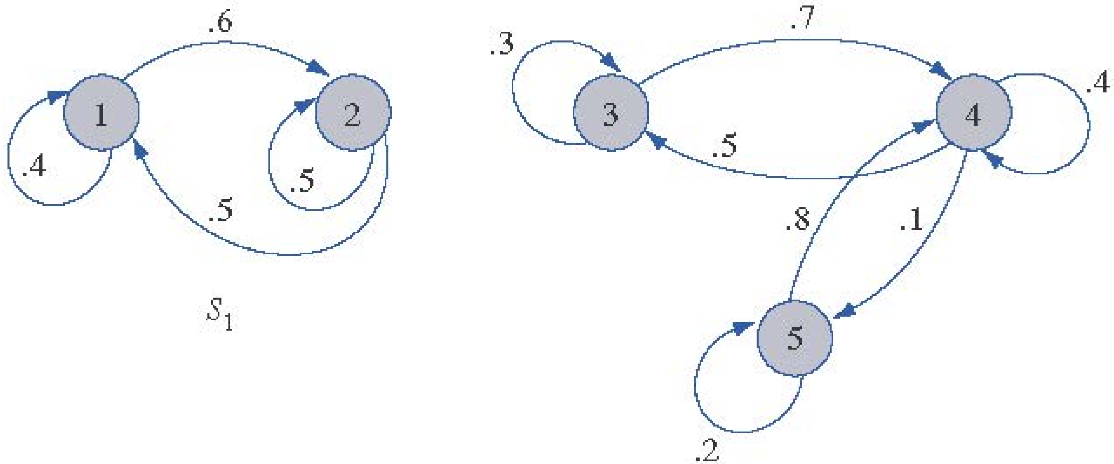
- state 5 is reachable from state 3 (via the path 3–4–5), but
- state 5 is not reachable from state 1 (there is no path from 1 to 5).

Two states i and j are said to **communicate** if j is reachable from i, and i is reachable from j. ■



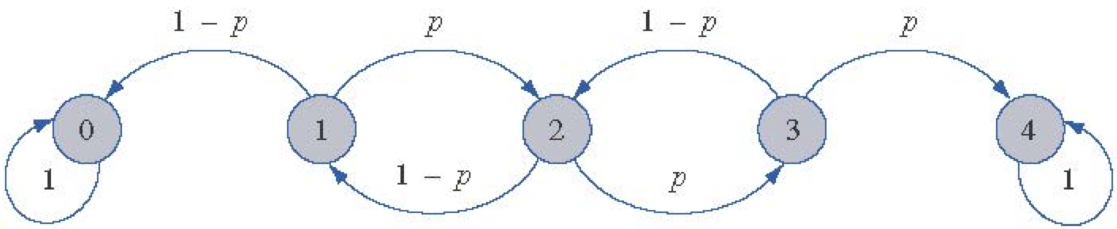
- States 1 and 2 communicate (we can go from 1 to 2 and from 2 to 1)

A set of states S in a Markov chain is a **closed set** if no state outside of S is reachable from any state in S. ■



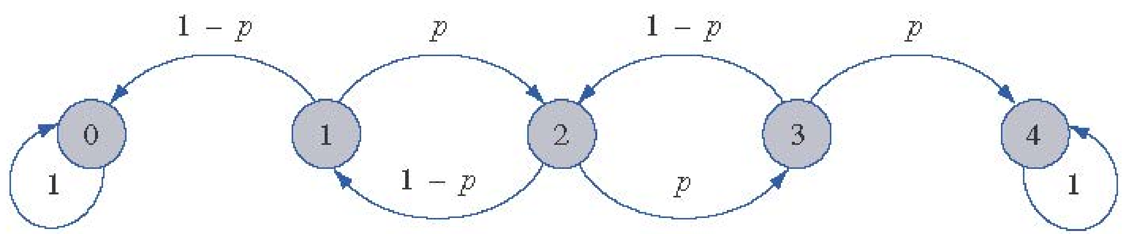
- $S_1 = \{1, 2\}$ and $S_2 = \{3, 4, 5\}$ are both closed sets.
- No arc begins in S_1 and ends in S2 or begins in S2 and ends in S1

A state i is an **absorbing state** if $p_{ii} = 1$. ■



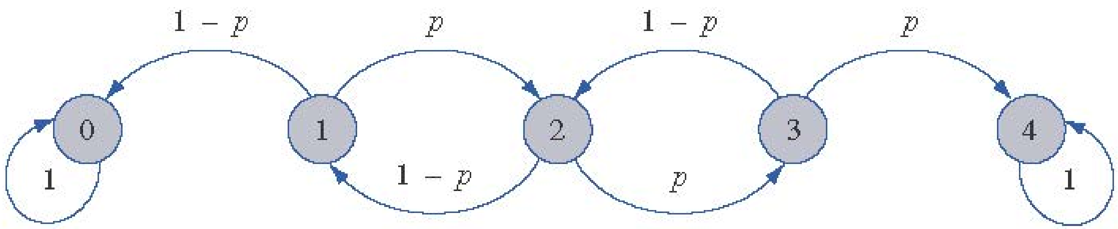
- In the gambler's ruin, states 0 and 4 are absorbing states.
- An absorbing state is a closed set containing only one state

A state i is a **transient state** if there exists a state j that is reachable from i, but the state i is not reachable from state j. ■



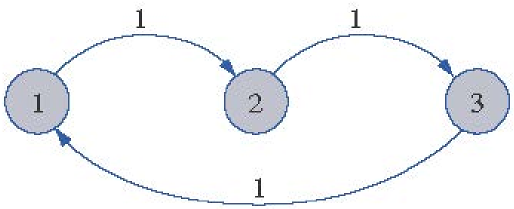
- A state i is transient if there is a way to leave state i that never returns to state i.
- In the gambler's ruin example, states 1, 2, and 3 are transient states
- For example from state 2, it is possible to go along the path 2–3–4, but there is no way to return to state 2 from state 4

If a state is not transient, it is called a **recurrent state.** ■



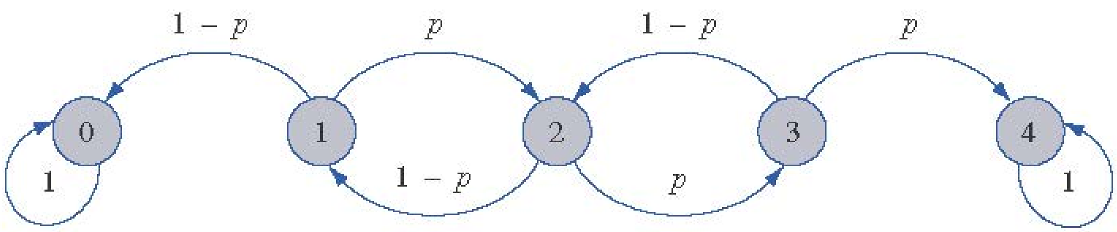
- States 0 and 4 are recurrent states (and also absorbing states), A state i is **periodic** with period $k > 1$ if k is the smallest number such that all paths leading from state i back to state i have a length that is a multiple of k. If a recurrent state is not periodic, it is referred to as **aperiodic.** ■

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



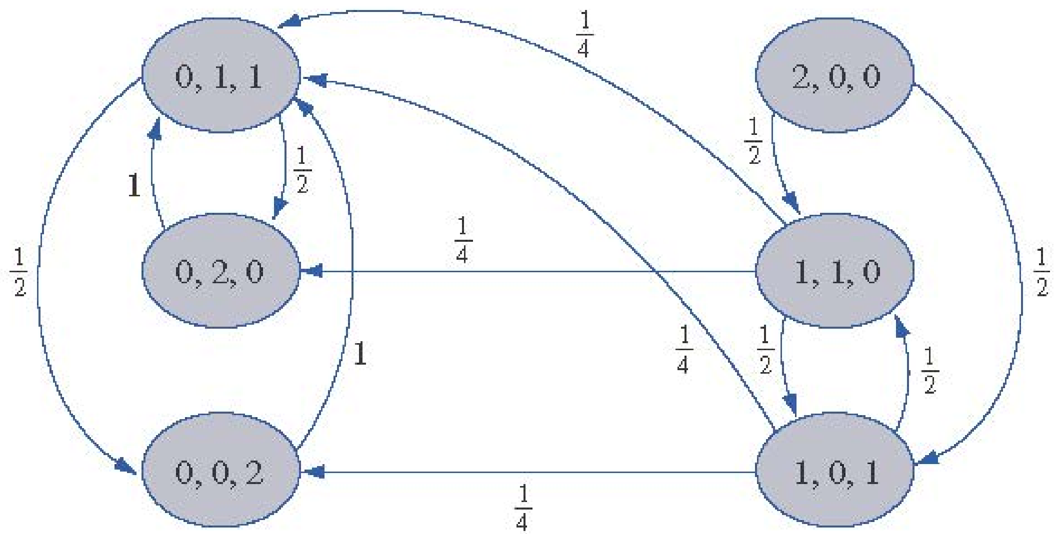
- For example, if we begin in state 1, the only way to return to state 1 is to follow the path 1–2–3–1 for some number of times (say, *m*). Hence, any return to state 1 will take *3m* transitions, so state 1 has period 3.

If all states in a chain are recurrent, aperiodic, and communicate with each other, the chain is said to be **ergodic.** ■



- The gambler's ruin example is not an ergodic chain, because (for example) states 3 and 4 do not communicate.

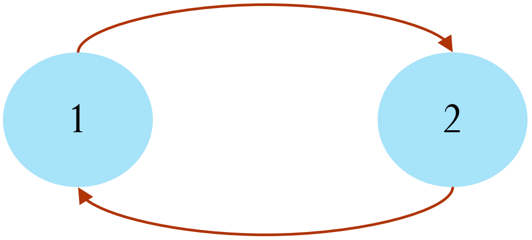
If all states in a chain are recurrent, aperiodic, and communicate with each other, the chain is said to be **ergodic.** ■



- Example 2 is also not an ergodic chain, because (for example) [2 0 0] and [0 1 1] do not communicate.

If all states in a chain are recurrent, aperiodic, and communicate with each other, the chain is said to be **ergodic.** ■

$$P = \begin{array}{c c} & \begin{array}{cc} \text{Cola 1} & \text{Cola 2} \end{array} \\ \begin{array}{c} \text{Cola 1} \\ \text{Cola 2} \end{array} & \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} \end{array}$$



- The cola example, is an ergodic Markov chain

- Ergodic

$$P_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

- Non ergodic

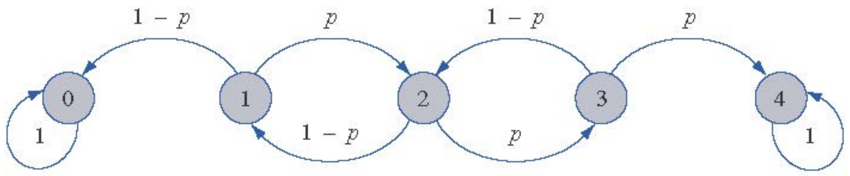
$$P_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

- Ergodic

$$P_3 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Problems

1. What is the period of states 1 and 3?



2. Each American family is classified as living in an urban,rural, or suburban location. During a given year, 15% of all urban families move to a suburban location, and 5% move to a rural location; also, 6% of all suburban families move to an urban location, and 4% move to a rural location; finally, 4% of all rural families move to an urban location, and 6% move to a suburban location. Is the Markov chain an ergodic chain?

3. Consider the following transition matrix:
   a. Which states are transient
   b. Which states are recurrent
   c. Identify all closed sets of states

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

4. For each of the following chains, determine whether the Markov chain is ergodic.

$$P_1 = \begin{bmatrix} 0 & .8 & .2 \\ .3 & .7 & 0 \\ .4 & .5 & .1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} .2 & .8 & 0 & 0 \\ 0 & 0 & .9 & .1 \\ .4 & .5 & .1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answers

1. 2.
2. Yes.
3.
    a. State 4.
    b. States 1, 2, 3, 5, and 6.
    c. $\{1, 3, 5\}$ and $\{2, 6\}$.
4. P_1 is ergodic; P_2 is not ergodic.

Steady state probabilities and mean first passage times

THEOREM 1

Let P be the transition matrix for an s-state ergodic chain.† Then there exists a vector $\pi = [\pi_1 \quad \pi_2 \quad \cdots \quad \pi_s]$ such that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1\ \pi_2\ \ldots.\ \pi_s \\ \pi_1\ \pi_2\ \ldots.\ \pi_s \\ \vdots \\ \pi_1\ \pi_2\ \ldots.\ \pi_s \end{bmatrix}$$

$$\pi = [\pi_1\ \pi_2\ \ldots \pi_s] = \text{steady state distribution/}$$
$$\text{equilibrium distribution}$$

- Then

$$P_{ij}(n+1) \cong P_{ij}(n) \cong \pi_j$$

$$P_{ij}(n+1) = \sum_{k=1}^{k=s} P_{ik}(n) p_{kj}$$

$$\pi_j = \sum_{k=1}^{k=s} \pi_k p_{kj}$$

$$\boxed{\pi = \pi P}$$

$$\boxed{\pi_1 + \pi_2 + \cdots + \pi_s = 1}$$

- Example:

$$P = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] = [\pi_1 \quad \pi_2] \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix}$$

$$\pi_1 = .90\pi_1 + .20\pi_2$$

$$\cancel{\pi_2 = .10\pi_1 + .80\pi_2}$$

$$\pi_1 + \pi_2 = 1$$

$$\pi_1 = 2/3$$

$$\pi_2 = 1/3$$

Transient analysis

- The behavior of a Markov chain before the steady state = transient (short run) behavior.
- To study the transient behavior, use the formula:

$$P_{ij}(n) = ij\text{-th element of } P^n$$

Probability of being in state j at time n

$$= \sum_{i=1}^{i=s} (\text{probability that state is originally } i)$$

$$\times (\text{probability of going from } i \text{ to } j \text{ in } n \text{ transitions})$$

$$= \sum_{i=1}^{i=s} q_i P_{ij}(n)$$

$$= \mathbf{q}(\text{column } j \text{ of } P^n)$$

$$\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_s].$$

Intuitive interpretation of steady state probabilities

$$\pi_j = \sum_{k=1}^{k=s} \pi_k p_{kj}$$

Each side substracted by $\pi_j p_{jj}$

$$\pi_j(1 - p_{jj}) = \sum_{k \neq j} \pi_k p_{kj}$$

Probability that a particular transition leaves state j

= probability that a particular transition enters state j

Probability that a particular transition leaves state j

$=$ (probability that the current period begins in j)

$\times$ (probability that the current transition leaves j)

$= \pi_j(1 - p_{jj})$

=

Probability that a particular transition enters state j

$= \sum_k$ (probability that the current period begins in $k \neq j$)

$\times$ (probability that the current transition enters j)

Equilibrium

Use of steady state for decision making

- In Example 4, suppose that each customer makes one purchase of cola during any week (52 weeks 1 year). Suppose there are 100 million cola customers. One selling unit of cola costs the company $1 to produce and is sold for $2. For $500 million per year, an advertising firm guarantees to decrease from 10% to 5% the fraction of cola 1 customers who switch to cola 2 after a purchase. Should the company that makes cola 1 hire the advertising firm?

- Present: $\pi_1 = 2/3$
- Current annual profit: 2/3 x \$1 x 52 week (100,000,000) = \$3,466,666,667
- The advertising firm offers to change the matrix:

$$P_1 = \begin{bmatrix} .95 & .05 \\ .20 & .80 \end{bmatrix}$$

- Calculate the new steady state:

$$\pi_1 = .95\pi_1 + .20\pi_2$$
$$\pi_2 = .05\pi_1 + .80\pi_2$$

$$\pi_1 + \pi_2 = 1$$

$$\pi_1 = 0.8$$

- The new cola 1 annual profit:

  = 0.8 (52,000,000,000) - 500,000,000

  = 3,660,000,000

- Conclusion: Cola 1 should hire the advertising company.

Mean first passage time

- For an ergodic chain, let m_{ij} expected number of transitions before we first reach state j, given that we are currently in state i; m_{ij} is called the **mean first passage time from** state i to state j.

$$m_{ij} = p_{ij}(1) + \sum_{k \neq j} p_{ik}(1 + m_{kj})$$

$$p_{ij} + \sum_{k \neq j} p_{ik} = 1$$

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

$$m_{ii} = \frac{1}{\pi_i}$$

- Example

$$\pi_1 = 2/3, \pi_2 = 1/3$$

$$m_{11} = 1.5, m_{22} = 3$$

$$m_{12} = 1 + p_{11}m_{12} = 1 + 0.9m_{12},$$

$$m_{21} = 1 + p_{22}m_{21} = 1 + 0.8m_{21}$$

$$m_{12} = 10, m_{21} = 5$$

Absorbing chains

- A Markov chain in which some of the states are absorbing and the rest are transient states is called an **absorbing chain.**
- If we begin in a transient state, then eventually we are sure to leave the transient state and end up in one of the absorbing states.

- The accounts receivable situation of a firm is often modeled as an absorbing Markov chain. Suppose a firm assumes that an account is uncollectable if the account is more than three months overdue. Then at the beginning of each month, each account may be classified into one of the following states

  State 1 New account

  State 2 Payment on account is one month overdue.

  State 3 Payment on account is two months overdue.

  State 4 Payment on account is three months overdue

  State 5 Account has been paid

  State 6 Account is written off as bad debt

$$\begin{array}{l|l}
 & \begin{array}{cccccc} \text{New} & \text{1 month} & \text{2 months} & \text{3 months} & \text{Paid} & \text{Bad debt} \end{array} \\
\begin{array}{l} \text{New} \\ \text{1 month} \\ \text{2 months} \\ \text{3 months} \\ \text{Paid} \\ \text{Bad debt} \end{array} & \left[\begin{array}{cccc|cc} 0 & 0.6 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]
\end{array}$$

- For example, if an account is two months overdue at the beginning of a month, there is a 40% chance that at the beginning of next month, the account will not be paid up (and therefore be three months overdue) and a 60% chance that the account will be paid up.
- *What is the probability that a new account will eventually be collected?*

Workforce planning

- The law firm of Mason and Burger employs three types of lawyers: junior lawyers, senior lawyers, and partners. During a given year, there is a .15 probability that a junior lawyer will be promoted to senior lawyer and a .05 probability that he or she will leave the firm. Also, there is a .20 probability that a senior lawyer will be promoted to partner and a .10 probability that he or she will leave the firm. There is a .05 probability that a partner will leave the firm. The firm never demotes a lawyer.

1. *What is the probability that a newly hired junior lawyer will leave the firm before becoming a partner?*
2. *On the average, how long does a newly hired junior lawyer stay with the firm?*

Transient states | **Absorbing states**

$$
\begin{array}{ll}
 & \begin{array}{ccccc} \text{Junior} & \text{Senior} & \text{Partner} & \text{Leave as NP} & \text{Leave as P} \end{array} \\
\begin{array}{l} \text{Junior} \\ \text{Senior} \\ \text{Partner} \\ \text{Leave as nonpartner} \\ \text{Leave as partner} \end{array} &
\begin{bmatrix}
.80 & .15 & 0 & .05 & 0 \\
0 & .70 & .20 & .10 & 0 \\
0 & 0 & .95 & 0 & .05 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

1. *If the chain begins in a given transient state, and before we reach an absorbing state, what is the expected number of times that each state will be entered? How many periods do we expect to spend in a given transient state before absorption takes place?*
2. *If a chain begins in a given transient state, what is the probability that we end up in each absorbing state?*

$$P = \begin{array}{rl} & \begin{array}{cc} \begin{array}{c} s-m \\ \text{columns} \end{array} & \begin{array}{c} m \\ \text{columns} \end{array} \end{array} \\ \begin{array}{r} s-m \text{ rows} \\ m \text{ rows} \end{array} & \left[\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right] \end{array}$$

- Q = transition matrix between transient states
- R = transistion matrix from transient states to observing states

$$
\begin{array}{r|cccc|cc}
 & \text{New} & \text{1 month} & \text{2 months} & \text{3 months} & \text{Paid} & \text{Bad debt} \\
\text{New} & 0 & .6 & 0 & 0 & .4 & 0 \\
\text{1 month} & 0 & 0 & .5 & 0 & .5 & 0 \\
\text{2 months} & 0 & 0 & 0 & .4 & .6 & 0 \\
\text{3 months} & 0 & 0 & 0 & 0 & .7 & .3 \\
\hline
\text{Paid} & 0 & 0 & 0 & 0 & 1 & 0 \\
\text{Bad debt} & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
Q = \begin{bmatrix} 0 & .6 & 0 & 0 \\ 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & .4 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} \qquad R = \begin{bmatrix} .4 & 0 \\ .5 & 0 \\ .6 & 0 \\ .7 & .3 \end{bmatrix}_{4\times 2}
$$

$$
\begin{array}{l}
\\
\text{Junior} \\
\text{Senior} \\
\text{Partner} \\
\text{Leave as nonpartner} \\
\text{Leave as partner}
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
\text{Junior} & \text{Senior} & \text{Partner} & \text{Leave as NP} & \text{Leave as P}
\end{array} \\
\left[\begin{array}{ccc|cc}
.80 & .15 & 0 & .05 & 0 \\
0 & .70 & .20 & .10 & 0 \\
0 & 0 & .95 & 0 & .05 \\
\hline
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$
Q = \begin{bmatrix} .80 & .15 & 0 \\ 0 & .70 & .20 \\ 0 & 0 & .95 \end{bmatrix}_{3\times 3} \qquad R = \begin{bmatrix} .05 & 0 \\ .10 & 0 \\ 0 & .05 \end{bmatrix}_{3\times 2}
$$

- *If the chain begins in a given transient state, and before we reach an absorbing state, what is the expected number of times that each state will be entered? How many periods do we expect to spend in a given transient state before absorption takes place?*

- Answer*:* If we are at present in transient state t_i, the expected number of periods that will be spent in transient state t_j before absorption is the *ij*-th element of the matrix $(I - Q)^{-1}$.

- If a chain begins in a given transient state, what is the probability that we end up in each absorbing state?

- Answer: If we are at present in transient state t_i, the probability that we will eventually be absorbed in absorbing state a_j is the ij-th element of the matrix $(I - Q)^{-1} R$.

$(I - Q)^{-1}$ = **Markov chains fundamental matrix**

1. What is the probability that a new account will eventually be collected?
2. What is the probability that a one-month-overdue account will eventually become a bad debt?
3. If the firm's sales average \$100,000 per month, how much money per year will go uncollected?

$$Q = \begin{bmatrix} 0 & .6 & 0 & 0 \\ 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & .4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} .4 & 0 \\ .5 & 0 \\ .6 & 0 \\ .7 & .3 \end{bmatrix}$$

- Then :

$$(I-Q)^{-1} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{bmatrix} 1 & .60 & .30 & .12 \\ 0 & 1 & .50 & .20 \\ 0 & 0 & 1 & .40 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- And

$$(I-Q)^{-1}R = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{bmatrix} .964 & .036 \\ .940 & .060 \\ .880 & .120 \\ .700 & .300 \end{bmatrix} \end{array}$$

1. t_1 = *New*, a_1 = *Paid*. Thus, the probability that a new account is eventually collected is element 11 of $(I - Q)^{-1}R$ = ***0.964***.
2. t_2 =*1 Month*, a_2 = *Bad Debt*. Thus, the probability that a one-month overdue account turns into a bad debt is element 22 of $(I - Q)^{-1}R$ = **0.06**.
3. From Answer.1: 0.036 debts are uncollected. Since yearly accounts payable are $1,200,000, on the average, (.036)(1,200,000) = **$43,200** per year will be uncollected.