NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Derivatives and Their Applications

Student Handout

2017-2018 EDITION

Derivatives and Their Applications

Students should be able to compute derivatives of the following:

- Polynomials
- Rational functions (quotients)
- Radical functions
- Trigonometric functions
- Exponential and logarithmic functions

Students should be able to apply various techniques and rules including:

- The power rule for integer, rational (fractional) exponents, expressions with radicals.
- Derivatives of sum, differences, products, and quotients
- The chain rule for composite functions
- Implicit differentiation
- Fundamental Theorems of Calculus (FTC)
- The derivative of expression in functional form $\left(h(x) = f(g(x)),\ q(x) = \dfrac{f(x)}{g(x)},\ \text{etc.} \right)$ where the functions are not given but values are taken from table or graph.

Students should be able to apply the concept of derivative to

- Solve related rate problems
- Solve optimization problems
- Determine the slope of tangent line to a curve at a point
- Determine the equations of tangent lines
- Approximate a value on a function using a tangent line and determine if the estimate is an over- or under- approximation based on concavity of the function

Students need to be able to recognize different ways that a tangent line approximation can appear on the AP exam:

- Tangent line approximation
- Linear approximation
- Linearization
- Euler's method (BC)

Multiple Choice

1. (calculator not allowed)

   If $f(x) = (x^2 - 2x - 1)^{\frac{2}{3}}$, then $f'(0)$ is

   (A) $\frac{4}{3}$

   (B) 0

   (C) $-\frac{2}{3}$

   (D) $-\frac{4}{3}$

   (E) -2

2. (calculator not allowed)

   If $f(x) = \left(2x+1\right)^4$, then the 4th derivative of $f(x)$ at $x = 0$ is

   (A) 0
   (B) 24
   (C) 48
   (D) 240
   (E) 384

For questions 3-5: The table below shows some of the values of two differentiable functions f and g and their derivatives.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	-3	6	-5	2
4	0	3	-3	9
5	3	-2	4	5

3. (calculator allowed)

   If $h(x) = f(x)g(x) + \dfrac{f(x)}{g(x)}$, then find the value of $h'(5)$.

   (A) $\frac{5}{4}$

   (B) $\frac{35}{4}$

   (C) $\frac{84}{5}$

   (D) $\frac{37}{2}$

4. (calculator not allowed)

If $h(x) = f(g(2x))$, then find the value of $h'(2)$.

(A) -90
(B) -45
(C) -5
(D) -3

5. (calculator not allowed)

If $h(x) = [g(x)]^2$, then find the value of $h'(3)$.

(A) 4
(B) 12
(C) 24
(D) 36

6. (calculator not allowed)

If $y = 2\cos\left(\frac{x}{2}\right)$, then $\frac{d^2y}{dx^2}$ is

(A) $-8\cos\left(\frac{x}{2}\right)$

(B) $-2\cos\left(\frac{x}{2}\right)$

(C) $-\sin\left(\frac{x}{2}\right)$

(D) $-\cos\left(\frac{x}{2}\right)$

(E) $-\frac{1}{2}\cos\left(\frac{x}{2}\right)$

7. (calculator not allowed)

The cost, in dollars, to shred the confidential documents of a company is modeled by C, a differentiable function of the weight of documents in pounds. Of the following, which is the best interpretation of $C'(500) = 80$?

(A) The cost to shred 500 pounds of documents is \$80.
(B) The average cost to shred documents is $\frac{80}{500}$ dollar per pound.
(C) Increasing the weight of the documents by 500 pounds will increase the cost to shred the documents by approximately \$80.
(D) The cost to shred documents is increasing at a rate of \$80 per pound when the weight of the documents is 500 pounds.

8. (calculator not allowed)

What is the slope of the tangent line to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point $(3, 2)$?

(A) 0

(B) $\frac{4}{9}$

(C) $\frac{7}{9}$

(D) $\frac{6}{7}$

(E) $\frac{5}{3}$

9. (calculator allowed)

The temperature, in degrees Fahrenheit $\left({}^{o}F\right)$, of water in a pond is modeled by the function H given by $H(t) = 55 - 9\cos\left(\frac{2\pi}{365}(t+10)\right)$, where t is the number of days since January 1 $\left(t = 0\right)$. What is the instantaneous rate of change of the temperature of the water at time $t = 90$ days?

(A) $0.114\,{}^{o}F\,/\,day$

(B) $0.153\,{}^{o}F\,/\,day$

(C) $50.252\,{}^{o}F\,/\,day$

(D) $56.350\,{}^{o}F\,/\,day$

10. (calculator not allowed)

If $\sin(xy) = x$, then $\frac{dy}{dx}$ is

(A) $\frac{1}{\cos(xy)}$

(B) $\frac{1}{x\cos(xy)}$

(C) $\frac{1-\cos(xy)}{\cos(xy)}$

(D) $\frac{1-y\cos(xy)}{x\cos(xy)}$

(E) $\frac{y(1-\cos(xy))}{x}$

11. (calculator not allowed)

If f and g are twice differentiable and if $h(x) = f(g(x))$ then $h''(x) =$

(A) $f''\left(g(x)\right)\left[g'(x)\right]^2 + f'\left(g(x)\right)g''(x)$

(B) $f''(g(x))g'(x) + f'(g(x))g''(x)$

(C) $f''(g(x))\left[g'(x)\right]^2$

(D) $f''(g(x))g''(x)$

(E) $f''(g(x))$

12. (calculator not allowed)

If $f(x) = \tan^{-1} x$, then $f'\left(\sqrt{3}\right) =$

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{3}$

13. (calculator not allowed)

Let $f(x) = \sqrt{e^x + 3}$. What is the approximation for $f(0.5)$ found by using the line tangent to the graph of f at $x = 0$?

(A) 1.25
(B) 1.5
(C) 2.125
(D) 2.25

14. (calculator not allowed)

If $f(x) = \ln\left(x + 4 + e^{-3x}\right)$, then $f'(0)$ is

(A) $-\frac{2}{5}$

(B) $\frac{1}{5}$

(C) $\frac{1}{4}$

(D) $\frac{2}{5}$

(E) nonexistent

15. (calculator not allowed)

If $f(x)=(x-1)(x^2+2)^3$, then $f'(x)=$

(A) $6x(x^2+2)^2$

(B) $6x(x-1)(x^2+2)^2$

(C) $(x^2+2)^2(x^2+3x-1)$

(D) $(x^2+2)^2(7x^2-6x+2)$

(E) $-3(x-1)(x^2+2)^2$

16. (calculator not allowed)

Let f be a differentiable function such that $f(3)=15$, $f(6)=3$, $f'(3)=-8$, and $f'(6)=-2$. The function g is differentiable and $g(x)=f^{-1}(x)$ for all x. What is the value of $g'(3)$?

(A) $-\frac{1}{2}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

17. (calculator not allowed)

The function $h(x)=x^3+kx^2$ where k is a constant. If the tangent line to the graph of h at $x=-2$ is parallel to the line that contains the points $(1, 3)$ and $(4,15)$, what is the value of k?

(A) 4

(B) $\frac{49}{16}$

(C) $\frac{47}{16}$

(D) 2

18. (calculator not allowed)

The local linear approximation to the function g at $x = \frac{1}{2}$ is $y = 4x + 1$. What is the value of $g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right)$?

(A) 4
(B) 5
(C) 6
(D) 7

19. (calculator not allowed)

An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic meters per hour. At what rate, in square meters per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 meters?

(Note: For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $\frac{4}{3}\pi r^3$.)

(A) $\frac{4\pi}{5}$
(B) 40π
(C) $80\pi^2$
(D) 100π

20. (calculator not allowed)

The graph of $f(x) = e^{\sin x} - 1$ crosses the x-axis at one point on the interval $[1, 4]$. What is the instantaneous rate of change of $f(x)$ at this point?

(A) -2
(B) -1
(C) 0
(D) 1

21. (calculator not allowed)

In the xy-plane, the line $x + y = k$, where k is a constant, is tangent to the graph of $y = x^2 + 3x + 1$. What is the value of k?

(A) -3
(B) -2
(C) -1
(D) 0
(E) 1

22. (calculator not allowed)

If the base b of a triangle is increasing at a rate of 3 inches per minute while its height h is decreasing at a rate of 3 inches per minute, which of the following must be true about the area A of the triangle?

(A) A is always increasing.
(B) A is always decreasing.
(C) A is decreasing only when $b < h$.
(D) A is decreasing only when $b > h$.
(E) A remains constant.

23. (calculator not allowed)

An equation of the line tangent to the graph of $f(x) = x(1-2x)^3$ at the point $(1, -1)$ is

(A) $y = -7x + 6$
(B) $y = -6x + 5$
(C) $y = -2x + 1$
(D) $y = 2x - 3$
(E) $y = 7x - 8$

24. (calculator not allowed)

If $f(x) = \sin\left(\ln(2x)\right)$, then $f'(x) =$

(A) $\dfrac{\sin(\ln(2x))}{2x}$

(B) $\dfrac{\cos(\ln(2x))}{x}$

(C) $\dfrac{\cos(\ln(2x))}{2x}$

(D) $\cos\left(\dfrac{1}{2x}\right)$

25. (calculator not allowed)

What is the maximum area of a rectangle that can be inscribed into the region bounded by the graph of $f(x) = 9 - x^2$ and the x-axis?

(A) $\sqrt{3}$
(B) 6
(C) $6\sqrt{3}$
(D) $12\sqrt{3}$

26. (calculator not allowed)

Let $f(x)$ be a twice differentiable function such that $f(3) = 1$ and $\frac{dy}{dx} = y^2 + y$. What is the value of $\frac{d^2y}{dx^2}$ at $x = 3$?

(A) 3
(B) 6
(C) 7
(D) 84

Free Response

27. (calculator not allowed)

Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x, where $f(3) = 25$.
(a) Find $f''(3)$

28. (calculator not allowed)

Consider the curve given by $xy^2 - x^3y = 6$.
(a) Show that $\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

29. (calculator allowed)

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where *A(t)* is measured in pounds and *t* is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

30. (calculator allowed)

t (minutes)	0	4	9	15	20
W(t) (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time *t* is modeled by a strictly increasing, twice-differentiable function *W,* where *W(t)* is measured in degrees Fahrenheit and *t* is measured in minutes. At time $t = 0$, the temperature of the water is $55^O F$. The water is heated for 30 minutes, beginning at time $t = 0$. Values of *W(t)* at selected times *t* for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

31. (calculator allowed)

t (days)	$W(t)$ (^{O}C)
0	20
3	31
6	28
9	24
12	22
15	21

The temperature, in degrees Celsius (^{O}C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

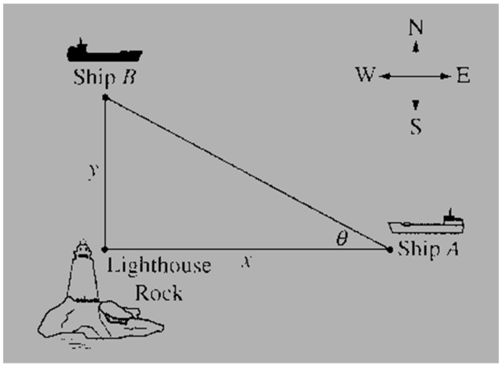
32. (calculator not allowed)

Solutions to the differentiable equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1+3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differentiable equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part(a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

33. (calculator not allowed)



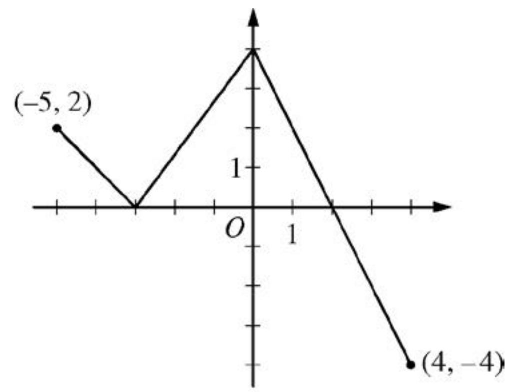
Ship A is travelling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is travelling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

34. (calculator not allowed)

The function f is defined on the closed interval $[-5,4]$. The graph of f consists of three line segments and is shown in the figure above.



Graph of f

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

35. (calculator allowed)

A 12,000- liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where r is given by the piecewise-defined function.

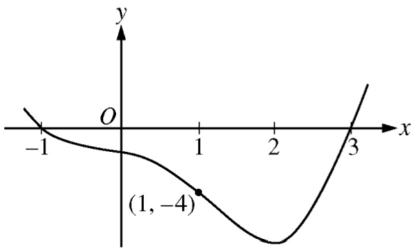
$$r(t) = \begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(c) Find $r'(3)$. Using correct units, explain the meaning of that value in the context of this problem.

36. (calculator not allowed)

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f' the derivative of f, is shown below. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.

(a) Write an equation for the line tangent to the graph of g at $x = 1$.



Graph of f'

37. (calculator not allowed)

The function h is defined by

$$h(x) = \begin{cases} x^4 + 3x \text{ for } x \le 0 \\ e^{3x} + 5 \text{ for } x > 0 \end{cases}$$

Find $h'(0)$. If it does not exist, explain why.

Derivatives and Their Applications Reference Page

General Rules

Definition of Derivative:

$\frac{d}{dx}\left(f(x)\right) = f'(x) =$ ____________________

Sum and Difference Rule:

$\frac{d}{dx}\left(f(x) \pm g(x)\right) =$ ____________________

Constant Multiple Rule:

$\frac{d}{dx}\left(k \cdot f(x)\right) =$ ____________________

Product Rule:

$\frac{d}{dx}\left(f(x) \cdot g(x)\right) =$ ____________________

Quotient Rule:

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =$ ____________________

Chain Rule:

$\frac{d}{dx}\left(f(g(x))\right) =$ ____________________

Particular Rules

Power Rule:

$\frac{d}{dx}\left(x^n\right) =$ ____________________

Exponential functions:

$\frac{d}{dx}\left(e^u\right) =$ ____________________

Bases other than e: $\frac{d}{dx}(a^u) =$ __________

Trigonometric Functions:

$\frac{d}{dx}\left(\sin(u)\right) =$ ____________________

$\frac{d}{dx}\left(\cos(u)\right) =$ ____________________

$\frac{d}{dx}\left(\tan(u)\right) =$ ____________________

$\frac{d}{dx}\left(\csc(u)\right) =$ ____________________

$\frac{d}{dx}\left(\sec(u)\right) =$ ____________________

$\frac{d}{dx}\left(\cot(u)\right) =$ ____________________

Logarithmic Functions:

$\frac{d}{dx}\left(\ln(u)\right) =$ ____________________

$\frac{d}{dx}(\log_a u) =$ ____________________

Inverse Trigonometric Functions:

$\frac{d}{dx}\left(\sin^{-1}(u)\right) =$ ____________________

$\frac{d}{dx}\left(\tan^{-1}(u)\right) =$ ____________________

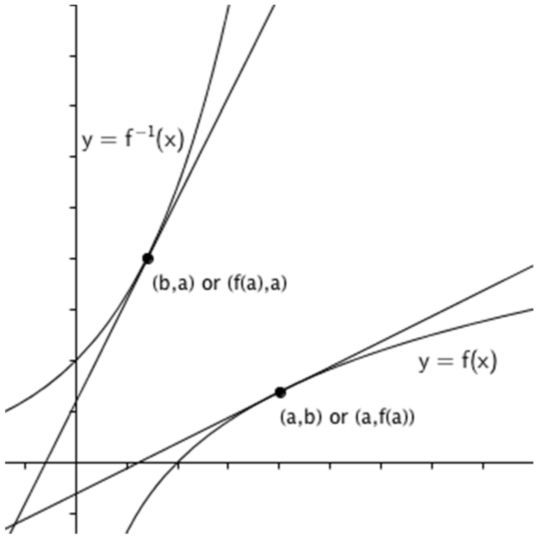
The derivative of the inverse of a function

$$\frac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=f(a)} = \frac{1}{f'(a)}$$

or

$$f^{-1\prime}(b) = \frac{1}{f'(\mathrm{a})}$$

That is to say - "The derivative of the *inverse* of f at the point $(b,\, a)$ is the reciprocal of the derivative of f at $(a,\, b)$ "



Point-Slope Form of the Equation of the Tangent Line

$$y - y_0 = m(x - x_0)$$

Replacing $(x_0,\, y_0)$ with $(c,\, f(c))$:

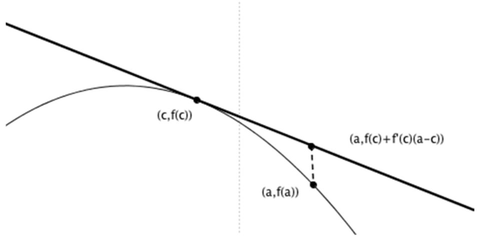
$$y = f(c) + f'(c)(x - c)$$

which leads to the form for linear approximation:

$$f(x) \approx f(c) + f'(c)(x - c)$$

It is also important to know if the linear approximation is an over- or under-approximation:

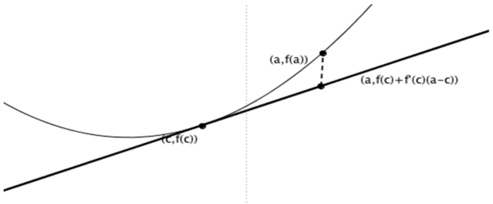
over-approximation



$f(c) + f'(c)(a - c) > f(a)$ if the graph of $y = f(x)$ is concave downward near $x = c$.

This will occur if $f''(x) < 0$ near $x = c$.

under-approximation



$f(c) + f'(c)(a - c) < f(a)$ if the graph of $y = f(x)$ is concave upward near $x = c$. This will occur if $f''(x) > 0$ near $x = c$.

Unlike a Riemann Sum, determining whether a tangent line is an over/under approximation is not related to whether a function is increasing or decreasing. When determining (or justifying) whether a tangent line is an over or under approximation, the concavity of the function must be discussed. It is important to look at the concavity on the interval from the point of tangency to the x-value of the approximation, not just the concavity at the point of tangency.

Example Justification: The approximation of $f(1.1)$ using the tangent line of f(x) at the point $x = 1$ is an over-approximation of the function because $f''(x) < 0$ on the interval $1 < x < 1.1$

Suggested steps for solving related rate problems

1. Draw a picture of the situation described in the problem (if applicable).
2. List all given information and the quantities to be determined.
3. Label each quantity that varies with time with an appropriate variable.
4. Label each quantity that does not vary with its constant value.
5. Relate the variables in an equation and make appropriate substitutions.
6. Differentiate both sides of the equation with respect to time.
7. Substitute known quantities into the result and solve for the unknown quantity.
8. Include units with your answers.

