MATH 221 - Finding a unique row space solution. October 23, 2015

Experience is another word for mistakes.

The general solution to the system $A\vec{x}_g = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is obtained from the row reduction $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & -1 & -3 \end{array}\right]$. This row reduction shows that a basis for the row space is $R_A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$. Further, from the row reduction $x_3 = 3$ and $x_1 = -2x_2 - 3x_3 + 2 = -2x_2 - 7$. Hence, the general solution $\vec{x}_g$, in vector form, reads

$$\vec{x}_g = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 3 \end{bmatrix} \equiv x_2\vec{\eta} + \vec{x}_p \tag{1}$$

where x_2 is arbitrary. The form of the general solution shows $\mathcal{N}(A) = \text{span}\{\vec{\eta}\}$. The Orthogonal Subspace Decomposition Theorem (**OSDT** (pronounced ohst) is on page 146) guarantees that there is a solution $\vec{x}_r = \alpha\vec{\eta} + \vec{r}$ where $\vec{\eta} \in \mathcal{N}(A)$ and $\vec{r} \in R_A$. In addition, these vectors are orthogonal, $\vec{\eta} \cdot \vec{r} = 0$, and they vectors are unique. Equating $\vec{x}_g$ and $\vec{x}_r$ and solving for the unknown row space solution yields

$$\vec{r} = (x_2 - \alpha)\vec{\eta} + \vec{x}_p \in R_A. \tag{2}$$

Using the orthogonality of R_A with $\mathcal{N}(A)$ leads to the equation

$$0 = \vec{\eta} \cdot \vec{r} = \vec{\eta} \cdot \{(x_2 - \alpha)\vec{\eta} + \vec{x}_p\} = (x_2 - \alpha)||\vec{\eta}||^2 + \vec{\eta} \cdot \vec{x}_p = (x_2 - \alpha)5 + 14$$

which implies $x_2 - \alpha = -14/5$. Substituting this into (2) shows

$$\begin{aligned} \vec{r} &= (x_2 - \alpha)\vec{\eta} + \vec{x}_p = \left(\frac{-14}{5}\right) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 28/5 - 7 \\ -14/5 \\ 3 \end{bmatrix} = \begin{bmatrix} -7/5 \\ -14/5 \\ -21/5 \end{bmatrix} + (36/5) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \left\{ (-7/5) \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (36/5) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \in R_A. \end{aligned}$$

If you wish to make "doubly sure" you have a particular solution, check that $A\vec{r} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. To find $\vec{x}_r$ or $\vec{x}_g$ (they are the same through the connection

$x_2 = \alpha - 14/5$), write

$$\vec{x}_g = x_2\vec{\eta} + \vec{x}_p \;=\; (\alpha - 14/5)\vec{\eta} + \vec{x}_p = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 14/5 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 3 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \left\{ (-7/5) \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (36/5) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \alpha\vec{\eta} + \vec{r} = \vec{x}_r.$$

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Repeat the above for $A\vec{x}_g = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The row reduction reads $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This gives

$$\vec{x}_g = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \equiv x_2\vec{\eta} + x_3\vec{\xi} + \vec{x}_p$$

where x_2 and x_3 are arbitrary. The form of the general solution implies $\mathcal{N}(A) = \text{span}\left\{\vec{\eta}, \vec{\xi}\right\}$ and the row reduction shows that $R_A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$. Once again, **OSDT** guarantees a solution $\vec{x}_r = \alpha_1\vec{\eta} + \alpha_2\vec{\xi} + \vec{r}$ where $\alpha_1\vec{\eta} + \alpha_2\vec{\xi} \in \mathcal{N}(A)$ and $\vec{r} \in R_A$. These vectors are orthogonal so that they satisfy $\left[\alpha_1\vec{\eta} + \alpha_2\vec{\xi}\right] \cdot \vec{r} = 0$. Equating these two solutions and solving for the unknown row space solution gives

$$\vec{r} = (x_2 - \alpha_1)\vec{\eta} + (x_3 - \alpha_2)\vec{\xi} + \vec{x}_p \tag{3}$$

Using the orthogonality ($R_A \perp \mathcal{N}(A)$), it follows that $0 = \vec{r}\cdot\vec{\eta} = \vec{r}\cdot\vec{\xi}$. This leads to two equations in the unknows $x_2 - \alpha_1$ and $x_3 - \alpha_2$. Solve to find $x_2 - \alpha_1 = 1/7$ and $x_3 - \alpha_2 = 3/14$ which, when substituted into (3) gives

$$\vec{r} = \left\{ 1/7 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + (3/14) \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/7 - 9/14 + 1 \\ 1/7 \\ 3/14 \end{bmatrix} = \begin{bmatrix} 1/14 \\ 2/14 \\ 3/14 \end{bmatrix}.$$

Hence, the ”row space” solution reads

$$\vec{x}_r = \alpha_1\vec{\eta} + \alpha_2\vec{\xi} + \vec{r} = \left\{ \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} + (1/14) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

By definition $\alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \in \mathcal{N}(A)$ and $(1/14) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in R_A$