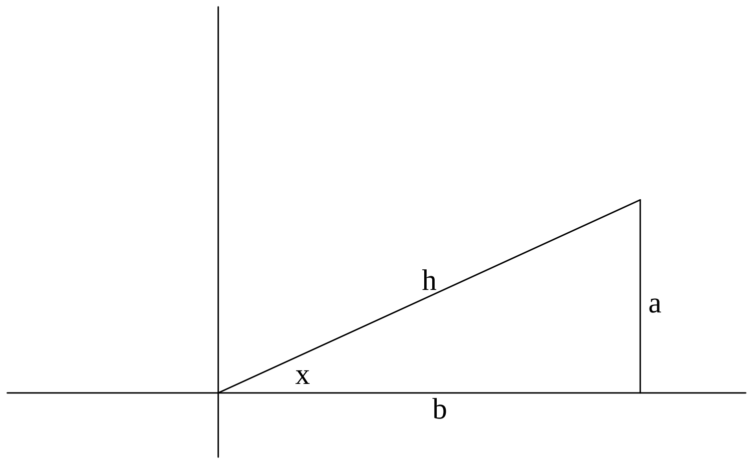
Trigonometric Identities

An identity is an equation that is satisfied by all the values of the variable(s) in the equation. We have already introduced the following:

(a) $\tan x = \dfrac{\sin x}{\cos x}$ (b) $\sec x = \dfrac{1}{\cos x}$ (c) $\csc x = \dfrac{1}{\sin x}$ (d) $\cot x = \dfrac{1}{\tan x}$

The trigonometric identity $\sin^2 x + \cos^2 x = 1$

The trig identity $\sin^2 x + \cos^2 x = 1$ follows from the Pythagorean theorem. In the figure below, an angle x is drawn. The side opposite the angle has length a, the adjacent side has length b, and the hypotenuse has length h.



Therefore, $\sin x = \dfrac{a}{h}$ and $\cos x = \dfrac{b}{h}$. It follows that

$$(\sin x)^2 + (\cos x)^2 = \left(\frac{a}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = \frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2+b^2}{h^2}$$

The Pythagorean theorem asserts that $a^2 + b^2 = h^2$. It follows that

$$(\sin x)^2 + (\cos x)^2 = \frac{a^2+b^2}{h^2} = \frac{h^2}{h^2} = 1$$

For convenience, $(\sin x)^2$ and $(\cos x)^2$ are written more briefly as $\sin^2 x$ and $\cos^2 x$ respectively. Therefore we have verified that if x is a given angle then

$$\sin^2 x + \cos^2 x = 1$$

You are expected to have this identity on your finger tips.

Two useful identities are derived from $\sin^2 x + \cos^2 x = 1$ by dividing both sides of the identity by $\sin^2 x$ or $\cos^2 x$.

- If we divide both sides of $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$, the result is

  $$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{OR} \quad \left(\frac{\sin x}{\cos x}\right)^2 + 1 = \left(\frac{1}{\cos x}\right)^2$$

  Since $\dfrac{\sin x}{\cos x} = \tan x$ and $\dfrac{1}{\cos x} = \sec x$, it follows that $(\tan x)^2 + 1 = (\sec x)^2$. For convenience, we write $(\tan x)^2$ and $(\sec x)^2$ more briefly as $\tan^2 x$ and $\sec^2 x$. Therefore we have verified that

  $$\tan^2 x + 1 = \sec^2 x$$

- If we divide both sides of $\sin^2 x + \cos^2 x = 1$ by $\sin^2 x$, the result is

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \quad \text{OR} \quad 1 + \left(\frac{\cos x}{\sin x}\right)^2 = \left(\frac{1}{\sin x}\right)^2$$

Since $\dfrac{\cos x}{\sin x} = \cot x$ and $\dfrac{1}{\sin x} = \csc x$, it follows that $1 + (\cot x)^2 = (\csc x)^2$. For convenience, we write $(\cot x)^2$ and $(\csc x)^2$ more briefly as $\cot^2 x$ and $\csc^2 x$. Therefore we have verified that

$$\cot^2 x + 1 = \csc^2 x$$

You will be required to verify identities. One way to do so is to start with one side of the identity and through a series of algebraic operations, derive the other side of the identity. Here is an example:

Example 1 *To verify the identity* $\dfrac{\sin^2 x}{\cos x} + \cos x = \sec x$.

We start with the left hand side because there is more we can do with it than the right hand side. Denote the left hand side by LHS and the right hand side by RHS. Then

$$\begin{aligned}
LHS &= \frac{\sin^2 x}{\cos x} + \cos x \\
&= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \quad \textit{(since we need a common denominator in order to add the two fractions)} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} \quad \textit{(since } \sin^2 x + \cos^2 x = 1\textit{)} \\
&= \sec x = RHS
\end{aligned}$$

We have verified that both sides of the identity are equal. This verifies the identity.

Exercise 2

1. *The trigonometric expression* $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x}$ *can be reduced to:*

   (A) $2 \sin x$ *(B)* $2 \cos x$ *(C)* $2 \cos^2 x$ *(D)* $2 \sec^2 x$

2. *Verify each identity:*

   a) $\dfrac{\cos x \sec x}{\cot x} = \tan x$ *b)* $\tan x \cos x = \sin x$

   c) $\tan x + \cot x = \sec x \csc x$ *d)* $\sec^2 x \left(1 - \sin^2 x\right) = 1$

   e) $\dfrac{\sec^2 x}{\tan x} = \sec x \csc x$ *f)* $\cos x + \sin x \tan x = \sec x$

   g) $\sin x + \cos x \cot x = \csc x$ *h)* $\sec x - \cos x = \tan x \sin x$

   i) $\dfrac{\csc^2 x}{\cot x} = \sec x \csc x$ *j)* $\sin^2 x \left(1 + \cot^2 x\right) = 1$

   k) $\dfrac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \cos^2 x$ *l)* $\dfrac{1 - \cos^2 x}{\cos x} = \sin x \tan x$

m) $\sec x - \cos x = \tan x \sin x$

n) $\dfrac{\sin x}{\tan x} + \dfrac{\cos x}{\cot x} = \sin x + \cos x$

o) $\dfrac{\tan x}{\sin x} + \dfrac{\cot x}{\sin x} = \sec x \csc x$

p) $\dfrac{\sin x + \cos x}{\sin x} - \dfrac{\cos x - \sin x}{\cos x} = \sec x \csc x$

q) $\dfrac{\sin x}{1 - \cot x} - \dfrac{\cos x}{\tan x - 1} = \sin x + \cos x$

r) $\dfrac{\tan x}{1 + \tan^2 x} = \sin x \cos x$

s) $1 - \dfrac{\cos^2 x}{1 + \sin x} = \sin x$

t) $1 - \dfrac{\sin^2 x}{1 + \cos x} = \cos x$

u) $\tan x + \dfrac{\cos x}{1 + \sin x} = \sec x$

v) $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$

w) $\dfrac{\cos x}{1 - \sin x} + \dfrac{1 - \sin x}{\cos x} = 2 \sec x$

x) $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x} = 0$

y) $\dfrac{\sin y}{1 - \cot y} - \dfrac{\cos y}{\tan y - 1} = \sin y + \tan y$

z) $(3\cos x - 4\sin x)^2 + (4\cos x + 3\sin x)^2 = 25$

3. *Write* $\cos^4 x - \sin^4 x$ *as a difference of two squares, factor it and use the result to show that* $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$.

Identities for differences or sums of angles

In this section we walk you through a derivation of the identity for $\cos(y - x)$. This will be used to derive others involving sums or differences of angles. Contrary to what one would expect, it is NOT TRUE that $\cos(y - x) = \cos y - \cos x$ for all angles x and y. Almost any pair of angles x and y gives $\cos(y - x) \neq \cos y - \cos x$. For example, the choice $y = 120°$ and $x = 30°$ gives $\cos(y - x) = \cos(90°) = 0$ which is not equal to $\cos 120° - \cos 30°$, (you can easily check that).

To derive the identity, start with a circle of radius 1 and center $(0, 0)$, labelled P in Figure 1 below.

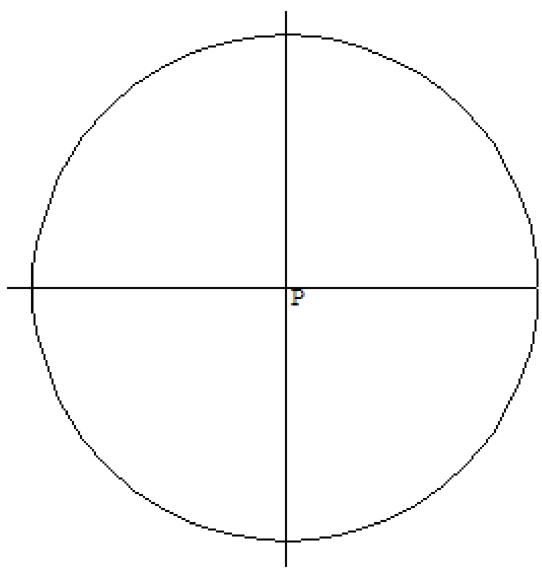


Figure 1

Consider a line PQ that makes an angle y, in the second quadrant, with the positive horizontal axis.

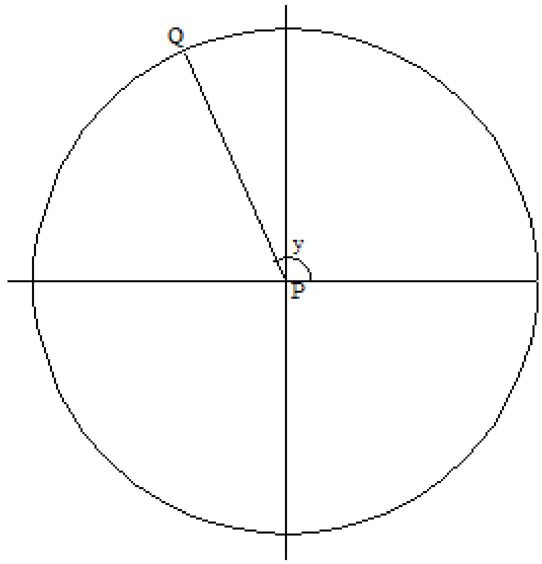


Figure 2

Can you see why the coordinates of Q must be $(\cos y, \sin y)$?

The figure below includes a line PR making an angle x in the first quadrant, with the positive horizontal axis.

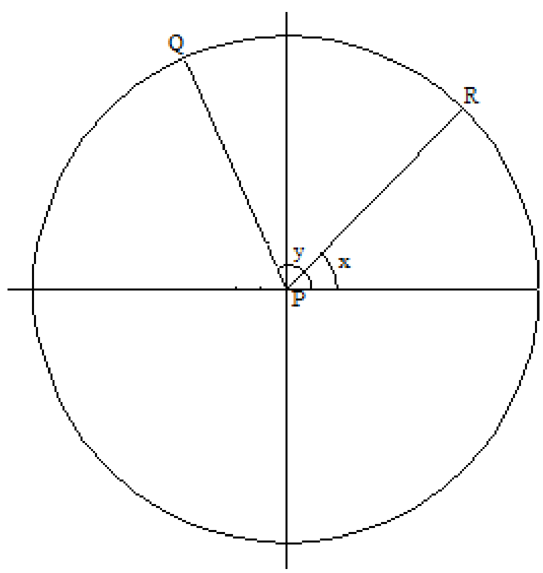


Figure 3

The coordinates of R are $(\cos x, \sin x)$. A standard notation for the length of the line segment QR is $||QR||$. Use the distance formula to show that $||QR||^2$ simplifies to

$$||QR||^2 = 2 - 2(\cos y \cos x + \sin y \sin x) \tag{1}$$

If you rotate the circle in Figure 3 counter-clockwise until the ray PR merges into the positive horizontal axis the result is Figure 4 below.

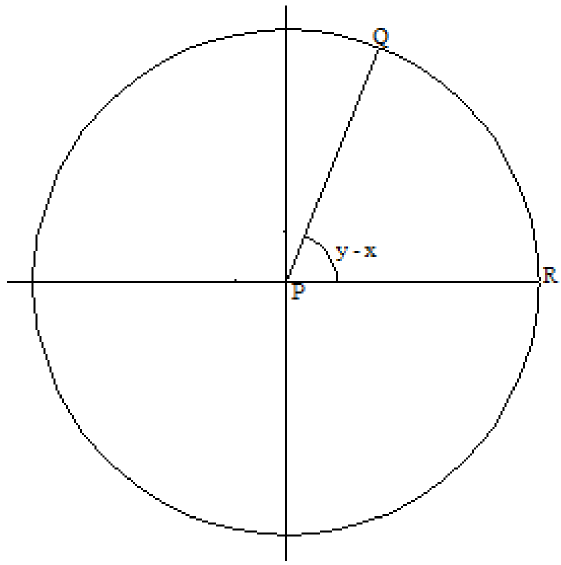


Figure 4

Angle QPR is $y-x$, therefore Q has coordinates $(\cos(y-c), \sin(y-x))$. Clearly R has coordinates $(1,0)$. It follows that the length of QR is also given by

$$||QR||^2 = [\cos(y-x)-1]^2 + [\sin(y-1)]^2$$

Show that this simplifies to

$$||QR||^2 = 2 - 2\cos(y-x) \tag{2}$$

Comparing (1) and (2) reveals that

$$2 - 2\cos(y-x) = 2 - 2(\cos y \cos x + \sin y \sin x)$$

What do you conclude about $\cos(y-x)$?

You should obtain

$$\cos(y-x) = \cos y \cos x + \sin y \sin x$$

This is our first identity for the cosine of the difference of two angles.

Example 3 *We know the exact values of* $\cos 30°$, $\sin 30°$, $\cos 45°$ *and* $\sin 45°$. *We may use them to calculate* $\cos 15°$. *The result:*

$$\begin{aligned}\cos 15° &= \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{2}\sqrt{3}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example 4 *We are given that x is an angle in the first quadrant with* $\cos x = \frac{1}{4}$, *y is an angle in the third quadrant with* $\cos y = -\frac{1}{3}$ *and we have to determine* $\cos(y-x)$. *Since* $\cos(y-x) = \cos y \cos x + \sin y \sin x$, *we have to find* $\sin x$ *and* $\sin y$. *Diagrams will help. In Figure 5 we have a right triangle with a hypotenuse of length 4 and a horizontal side of length 1 because*

$$\cos x = \frac{\textit{horizontal coordinate}}{\textit{length of hypotenuse}} = \frac{1}{4}$$

The vertical side of the triangle is labelled b because it is not known. We may determine b using the Pythagorean theorem:

$$1^2 + b^2 = 4^2$$

Therefore $b^2 = 15$*, hence* $b = \pm\sqrt{15}$*. We take the positive sign because the vertical coordinate is positive in the first quadrant. Therefore* $b = \sqrt{15}$ *and so* $\sin x = \frac{\sqrt{15}}{4}$*. In Figure 6 we have drawn a rectangle with a hypotenuse with length 3 and a horizontal coordinate* -1 *because*

$$\cos y = \frac{\textit{horizontal coordinate}}{\textit{length of hypotenuse}} = \frac{-1}{3} = -\frac{1}{3}$$

The vertical coordinate is unknown, therefore we labelled it a. By the Pythagorean theorem

$$3^2 = (-1)^2 + a^2 = 1 + a^2$$

Therefore $a^2 = 8$*, hence* $a = \pm\sqrt{8}$*. In this case we must take the negative sign because the vertical coordinate is negative in the third quadrant. Therefore* $\sin y = \frac{-\sqrt{8}}{3}$

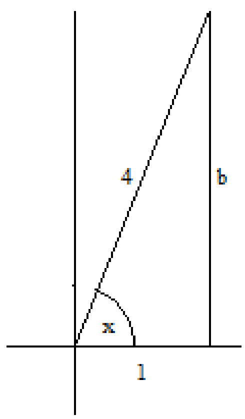


Figure 5

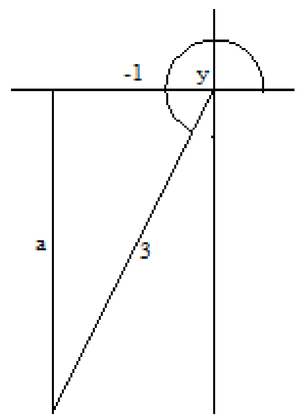


Figure 6

It follows that

$$\cos(y - x) = \cos y \cos x + \sin y \sin x = \left(\frac{1}{4}\right)\left(-\frac{1}{3}\right) + \left(\frac{\sqrt{15}}{4}\right)\left(-\frac{\sqrt{8}}{3}\right) = \frac{-1 - \sqrt{120}}{12}$$

Other sum/difference identities

- $\cos(y + x) = \cos y \cos x - \sin y \sin x$. To derive this, we use the fact that $\cos(-A) = \cos A$ and $\sin(-B) = -\sin B$ for all angles A and B. Therefore

$$\cos(y + x) = \cos(y - (-x)) = \cos y \cos(-x) + \sin y \sin(-x) = \cos y \cos x - \sin y \sin x$$

Example 5 $\cos 75^\circ = \cos(30^\circ + 45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

- $\sin(y - x) = \sin y \cos x + \cos y \sin x$. To derive it, we use the fact that $\cos(90^\circ - A) = \sin A$ and $\sin(90^\circ - B) = \cos B$ for any angles A and B. Therefore

$$\begin{aligned} \sin(y - x) &= \cos(90^\circ - (y - x)) = \cos((90^\circ - y) + x) = \cos(90^\circ - y)\cos x - \sin(90^\circ - y)\sin x \\ &= \sin y \cos x - \cos y \sin x \end{aligned}$$

Example 6 $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

- $\sin(y+x) = \sin y \sin x + \cos y \sin x$. We derive this using the fact that $\cos(-A) = \cos A$ and $\sin(-B) = -\sin B$ for all angles A and B. Therefore

$$\begin{aligned}\sin(y+x) &= \sin(y-(-x)) = \sin y\cos(-x) - \cos y\sin(-x) = \sin y\cos x - \cos y(-\sin x)\\ &= \sin y\cos x + \cos y\sin x\end{aligned}$$

Example 7 $\sin 75° = \sin(30° + 45°) = \sin 30°\cos 45° + \cos 30°\sin 45° = \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

- $\tan(y+x) = \dfrac{\tan y + \tan x}{1 - \tan y\tan x}$ and $\tan(y-x) = \dfrac{\tan y - \tan x}{1 + \tan y \tan x}$. These follows from the fact that $\tan(y \pm x) = \dfrac{\sin(x\pm y)}{\cos(x \pm y)}$.

Example 8 $\tan 75° = \tan(30° + 45°) = \dfrac{\tan 30° + \tan 45°}{1 - \tan 30°\tan 45°} = \dfrac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}} = \dfrac{\sqrt{3}+3}{3-\sqrt{3}}$

Exercise 9

1. *Given that x is an angle in the second quadrant with $\sin x = \frac{3}{5}$, and y is an angle in the fourth quadrant with $\cos y = \frac{4}{5}$, the exact value of $\sin(x+y)$ is*

   (A) $\dfrac{3}{25}$ *(B)* $\dfrac{8}{25}$ *(C)* $\dfrac{17}{25}$ *(D)* $\dfrac{24}{25}$

2. *Given that x is an angle in the first quadrant with $\sin x = \frac{3}{5}$, and y is an angle in the third quadrant with $\cos y = -\frac{5}{13}$, the exact value of $\cos(x-y)$ is*

   (A) $\dfrac{14}{65}$ *(B)* $\dfrac{56}{65}$ *(C)* $-\dfrac{64}{65}$ *(D)* $-\dfrac{56}{65}$

3. *You are given that x and y are angles in the first and fourth quadrants respectively, with $\sin x = \frac{12}{13}$ and $\cos y = \frac{4}{5}$. Draw the two angles then determine the exact values of the following expressions:*

   a) $\cos x$ *b)* $\sin y$ *c)* $\tan x$ *d)* $\tan y$ *e)* $\sin(x+y)$ *f)* $\sin(x-y)$

   g) $\cos(x+y)$ *h)* $\cos(x-y)$ *i)* $\tan(x-y)$ *j)* $\tan(x-y)$ *k)* $\csc(x-y)$ *l)* $\cot(x+y)$

4. *If x is an angle in the first quadrant with $\sin x = \frac{4}{5}$, and y is an angle in the fourth quadrant with $\cos y = \frac{2}{3}$, determine the exact value of each expression:*

   a. $\sin(x-y)$ *b.* $\cos(x+y)$ *c.* $\sin(x+y)$ *d.* $\tan(x-y)$ *e.* $\sec(x-y)$ *f.* $\cot(x+y)$

The Double Angle Formulas

These are formulas that enable us to calculate $\sin 2x$, $\cos 2x$, $\tan 2x$ and their reciprocals, once we know the values of $\sin x$, $\cos x$ and $\tan x$, hence the term "double angle formulas". They are derived from the identities for sums of angles.

- If we replace y by x in the identity $\sin(x+y) = \sin x\cos y + \cos x \sin y$, we get

$$\sin(x+x) = \sin 2x = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$$

Thus

$$\sin 2x = 2\sin x\cos x \tag{3}$$

This is our first "double angle formula"

- If we replace y by x in the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$, we get

$$\cos(x+x) = \cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

Thus

$$\cos 2x = \cos^2 x - \sin^2 x \tag{4}$$

There are two other versions of this formula obtained by using the identity $\sin^2 x + \cos^2 x = 1$. If we solve for $\sin^2 x$ to get $\sin^2 x = 1 - \cos^2 x$ then substitute into (4) we get

$$\cos 2x = \cos^2 x - \sin^2 x = \cos 2x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

I.e.

$$\cos 2x = 2\cos^2 x - 1$$

If, on the other hand, we solve for $\cos^2 x$ to get $\cos^2 x = 1 - \sin^2 x$ then substitute into (4) we get

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$$

I.e.

$$\cos 2x = 1 - 2\sin^2 x$$

Which one of these three formulas for $\cos 2x$ to use depends on the problem at hand.

- If we replace y by x In the identity $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ we get

$$\tan(x+x) = \tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2\tan x}{1 - \tan^2 x}$$

Thus

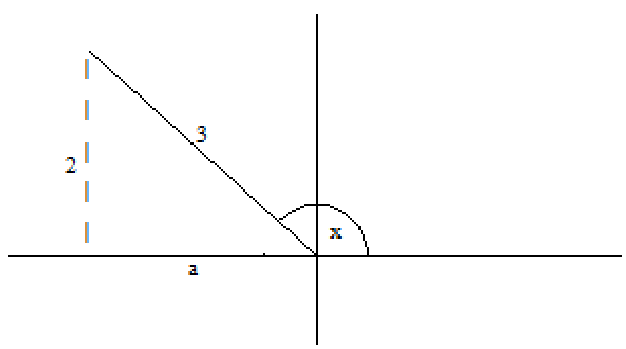
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Example 10 *To determine* $\sin 2x$, $\cos 2x$ *and* $\tan 2x$ *given that* x *is an angle in the second quadrant with* $\sin x = \frac{2}{3}$.

The angle is shown below. The horizontal coordinate a *is unknown but we easily calculate it using the Pythagorean theorem. Thus*

$$2^2 + a^2 = 3^2$$

which translates into $a^2 = 5$. *This implies that* $a = \pm\sqrt{5}$. *Since the horizontal coordinate must be negative, we must choose* $a = -\sqrt{5}$.



From the figure, $\cos x = \frac{a}{3} = -\frac{\sqrt{5}}{3}$ *and* $\tan x = \frac{2}{a} = -\frac{2}{\sqrt{5}}$. *Therefore*

$\sin 2x = 2\sin x \cos x = 2\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) = -\frac{4\sqrt{5}}{9}$.

$\cos 2x = 2\cos^2 x - 1 = 2\left(-\frac{\sqrt{5}}{3}\right)^2 - 1 = \frac{10}{9} - 1 = \frac{1}{9}$.

$$\tan 2x = \tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\left(-\frac{2}{\sqrt{5}}\right)}{1 - \left(-\frac{2}{\sqrt{5}}\right)^2} = \frac{-\frac{4}{\sqrt{5}}}{1 - \frac{4}{5}} = -\frac{4}{\sqrt{5}} \times \frac{5}{1} = -\frac{20}{\sqrt{5}} = -4\sqrt{5}$$

Rules for Changing Product to Sum/Differences

In some problems, it is easier to deal with sums of two trigonometric functions than their product. In such cases, we use the "*Product to Sum*" rules to convert products like $\sin x \cos y$, $\sin x \sin y$ and $\cos x \cos y$ into sums or differences involving sine or cosine terms. We use the sum/difference rules we have already derived which are:

$$\sin A \cos B + \cos A \sin B = \sin(A+B) \tag{5}$$

$$\sin A \cos B - \cos A \sin B = \sin(A-B) \tag{6}$$

$$\cos A \cos B + \sin A \sin B = \cos(A-B) \tag{7}$$

$$\cos A \cos B - \sin A \sin B = \cos(A+B) \tag{8}$$

Adding (5) to (6) gives

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

Adding (7) to (8) gives

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Subtracting (8) from (7) gives

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Example 11 *To write* $\sin 2x \sin 5x$*,* $\cos 4x \cos 5x$ *and* $\cos x \sin 2x$ *as sums/differences of trig functions.*

We use $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ *to write* $\sin 2x \sin 5x$ *as a sum of trig functions. Take* $A = 2x$ *and* $B = 5x$*. Then*

$$\sin 2x \sin 5x = \frac{1}{2}\left[\cos(-3x) - \cos 8x\right] = \frac{1}{2}\left[\cos 3x - \cos 8x\right]$$

For $\cos 4x \cos 5x$*, we use* $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$ *with* $A = 4x$ *and* $B = 5x$*. The result is*

$$\cos 4x \cos 5x = \frac{1}{2}\left[\cos x + \cos 9x\right]$$

For $\cos x \sin 2x$*, we first write it as* $\sin 2x \cos x$ *and use* $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$ *with* $A = 2x$ *and* $B = x$*. The result is*

$$\cos x \sin 2x = \frac{1}{2}\left[\sin 3x + \sin x\right] = \frac{1}{2}\left[\sin 3x + \sin x\right]$$

Sum/Difference to Products

There are problems that require us to change a sum or difference of sine or cosine terms into a product of sine and/or cosine terms. In other words, we may want to do the opposite of what we did in the above section. We use the same identities as above but now we write them as:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \tag{9}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \tag{10}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{11}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \tag{12}$$

We then introduce variables $P = A+B$ and $Q = A-B$. We have to write A and B in terms of these new variables. It turns out that

$$A = \tfrac{1}{2}(P+Q) \text{ and } B = \tfrac{1}{2}(P-Q)$$

Adding (9) to 10 gives a sum of sines which is

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$

Using the new variables P and Q this result may be written as

$$\sin P+\sin Q=2\left[\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)\right]$$

Subtracting 10 from (9) to gives a difference of sines which is

$$\sin P-\sin Q=2\left[\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)\right]$$

Adding (11) to 12 gives a sum of cosines which is

$$\cos P+\cos Q=2\left[\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)\right]$$

Subtracting 12 from (11) to gives a difference of cosines which is

$$\cos P-\cos Q=-2\left[\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)\right]$$

Example 12 *To write* $\cos 3x+\cos 5x$, $\sin 3x+\sin x$, $\sin 5x-\sin 2x$ *and* $\cos 2x-\cos 3x$ *as products of trig functions:*

Using the above identities:

$$3x+\cos 5x=2\cos\tfrac{1}{2}\left(3x+5x\right)\cos\tfrac{1}{2}(3x-5x)=2\cos 4x\cos 2x$$

$$\sin 3x+\sin x=2\sin\tfrac{1}{2}\left(3x+x\right)\cos\tfrac{1}{2}(3x-x)=2\sin 2x\cos x$$

$$\sin 5x-\sin 2x=2\cos\tfrac{1}{2}\left(5x+2x\right)\sin\tfrac{1}{2}(5x-2x)=-2\cos\tfrac{7}{2}x\sin\tfrac{3}{2}x$$

$$\cos 2x-\cos 3x=-2\sin\tfrac{1}{2}\left(2x+3x\right)\sin\tfrac{1}{2}(2x-3x)=2\sin\tfrac{5}{2}x\sin\tfrac{1}{2}x$$

The Half Angle Formulas

These are formulas that enable us to calculate $\sin\frac{1}{2}x$, $\cos\frac{1}{2}x$, $\tan\frac{1}{2}x$ and their reciprocals, once we know the values of $\sin x$, $\cos x$ and $\tan x$, hence the term "half angle formulas". They are derived from the two identities.

$$\cos 2y=1-2\sin^2 y \qquad \text{and} \qquad \cos 2y=2\cos^2 y-1$$

Take $\cos 2y=1-2\sin^2 y$. Rearrange it as $2\sin^2 y=1-\cos 2y$. Now solve for $\sin y$. The result is

$$\sin y=\pm\sqrt{\frac{1-\cos 2y}{2}}$$

Finally, replace y with $\frac{1}{2}x$ to get

$$\sin\tfrac{1}{2}x=\pm\sqrt{\frac{1-\cos x}{2}}$$

The sign is dictated by the position of the angle $\frac{1}{2}x$. If it is in the first or second quadrant, (where the sine function is positive), take the positive sign. If it is in the third or fourth quadrant, take the negative sign.

To derive the half angle formula for the cosine function, take the identity $\cos 2y=2\cos^2 y-1$ and solve for $\cos y$. The result is

$$\cos y=\pm\sqrt{\frac{1+\cos 2y}{2}}$$

As you would expect, replace y by $\frac{1}{2}x$ to get

$$\cos \tfrac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$$

Again the sign is dictated by the position of the angle $\frac{1}{2}x$. If it is in the first or fourth quadrant, (where the cosine function is positive), take the positive sign. If it is in the second or third quadrant, take the negative sign.

We do not need to do any heavy lifting to determine the half angle formula for the tangent function. We simply use the identity

$$\tan \tfrac{1}{2}x = \frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} = \pm\frac{\sqrt{\frac{1-\cos x}{2}}}{\sqrt{\frac{1+\cos x}{2}}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}.$$

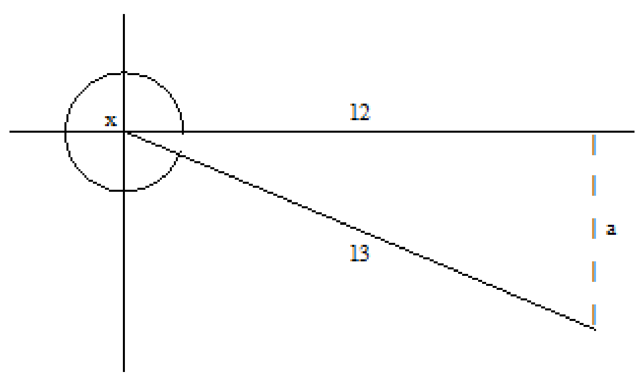
Likewise, you should expect the sign to be dictated by the position of $\frac{1}{2}x$. If it is in the first or third quadrant, take the positive sign. If it is in the second or fourth quadrant, take the negative sign.

Example 13 *To determine* $\sin 2x$, $\cos 2x$, $\tan 2x$, $\sin \frac{1}{2}x$, $\cos \frac{1}{2}x$ *and* $\tan \frac{1}{2}x$ *given that* x *is an angle in the fourth quadrant with* $\cos x = \frac{12}{13}$.

The angle is shown below. The vertical coordinate a *may be obtained from*

$$12^2 + a^2 = 13^2$$

which translates into $a^2 = 25$. *This implies that* $a = \pm 5$. *Since the coordinate must be negative, we must take* $a = -5$.



From the figure, $\sin x = \frac{a}{3} = -\frac{5}{13}$ *and* $\tan x = -\frac{5}{12}$. *Therefore*

$\sin 2x = 2\sin x \cos x = 2\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) = -\frac{120}{169}$.

$\cos 2x = \cos^2 x - \sin^2 x = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$.

$$\tan 2x = \tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\left(-\frac{2}{12}\right)}{1-\left(-\frac{2}{12}\right)^2} = \frac{-\frac{4}{12}}{1-\frac{4}{144}} = -\frac{1}{3} \times \frac{144}{140} = -\frac{48}{140} = -\frac{12}{35}$$

Before calculating $\sin \frac{1}{2}x$, $\cos \frac{1}{2}x$ *and* $\tan \frac{1}{2}x$, *we note that* $\frac{1}{2}x$ *is an angle between* $135°$ *and* $180°$, *(because* x *is between* $270°$ *and* $360°$*), therefore it is in the second quadrant. This implies that* $\sin \frac{1}{2}x$ *is positive but* $\cos \frac{1}{2}x$ *and* $\tan \frac{1}{2}x$ *are both negative. Therefore*

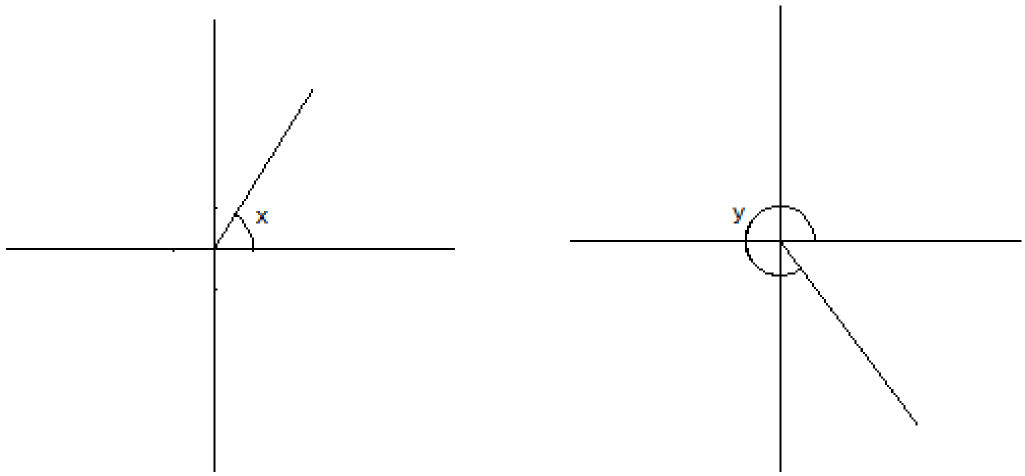
$$\sin \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\frac{12}{13}}{2}} = \sqrt{\frac{1}{26}}$$

$$\cos \tfrac{1}{2}x = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1+\frac{12}{13}}{2}} = -\sqrt{\tfrac{25}{26}}$$

$$\tan \tfrac{1}{2}x = -\sqrt{\frac{1-\cos x}{1+\cos x}} = -\sqrt{\frac{1-\frac{12}{13}}{1+\frac{12}{13}}} = -\sqrt{\tfrac{1}{25}} = -\tfrac{1}{5}$$

Exercise 14

1. *In the figure below, x and y are angles in the first and fourth quadrants respectively, with $\sin x = \frac{15}{17}$ and $\cos y = \frac{5}{13}$.*



   Determine, without using a calculator, the exact value of:

   (a) $\cos x$
   (b) $\sin y$
   (c) $\cos(y-x)$
   (d) $\sin(x+y)$
   (e) $\sin 2x$
   (f) $\cos \frac{1}{2}y$

2. *You should know the exact value of $\cos 45°$. Use it and a half angle formula to determine the exact value of $\cos 22.5°$.*

3. *x is an angle in the third quadrant with $\cos x = \frac{1}{3}$. Draw the angle then use a half angle formula to determine the exact values of $\sin\left(\frac{1}{2}x\right)$ and $\cos\left(\frac{1}{2}x\right)$.*

4. *If x is an angle in the first quadrant with $\sin x = \frac{1}{2}$, and y is an angle in the fourth quadrant with $\cos y = \frac{3}{5}$, determine the exact value of each expression:*

   (a) $\sin(x-y)$ *(b)* $\cos 2x$ *(c)* $\tan(x+y)$ *(d)* $tan\frac{1}{2}y$

5. *If x is an angle in the first quadrant with $\sin x = \frac{1}{2}$, and y is an angle in the fourth quadrant with $\cos y = \frac{3}{5}$, determine the exact value of each expression:*

   (a) $\cos(x-y)$ *(b)* $\sin 2x$ *(c)* $\tan 2y$

6. *Verify the identity $(\sin x + \cos x)^2 = 1 + \sin 2x$*

7. *You are given that x is an angle in the second quadrant and $\cos x = -\dfrac{5}{13}$*

(a) *Draw the angle and calculate the exact values of* $\sin x$ *and* $\tan x$.

(b) *Now calculate the exact values of the following:*

$$\text{(a) } \sin 2x \qquad \text{(b) } \cos 2x \qquad \text{(c) } \tan 2x \qquad \text{(d) } tan\tfrac{1}{2}x$$

(c) *You have enough information to calculate* $\sec 2x$. *Calculate it.*

8. *You are given that* y *is an angle in the third quadrant and* $\tan y = \dfrac{4}{3}$.

   (a) *Draw the angle and calculate the exact value of* $\cos y$ *and* $\sin y$.

   (b) *Calculate the exact value, (no calculator), of the following:*

   $$\text{(a) } \sin \tfrac{1}{2}y \qquad \text{(b) } \cos \tfrac{1}{2}y \qquad \text{(c) } \cos 2y$$

   (c) *You have enough information to calculate the exact value of* $\sec \frac{1}{2}y$. *Calculate it.*

9. *You are given that* y *is an angle in the third quadrant and* $\tan y = \dfrac{12}{5}$.

   (a) *Draw the angle and calculate the exact value of* $\cos y$ *and* $\sin y$.

   (b) *Now calculate the exact value, (no calculator), of the following:*

   (i) $\sin \frac{1}{2}y$
   (ii) $\cos \frac{1}{2}y$
   (iii) $\tan \frac{1}{2}y$

   You have enough information to calculate the exact value of $\csc \frac{1}{2}y$. *Calculate it.*

10. *You are given that* y *is an angle in the third quadrant and* $\tan y = \dfrac{4}{3}$.

    (a) *Draw the angle and calculate the exact value of* $\cos y$ *and* $\sin y$.

    (b) *Calculate the exact value, (no calculator), of the following:*

    (i) $\sin \frac{1}{2}y$
    (ii) $\cos \frac{1}{2}y$
    (iii) $\tan \frac{1}{2}y$

    You have enough information to calculate the exact value of $\cot \frac{1}{2}y$. *Calculate it.*

11. *If* x *is an angle in the first quadrant with* $\sin x = \frac{1}{3}$, *and* y *is an angle in the fourth quadrant with* $\cos y = \frac{2}{3}$, *determine the exact value of each expression:*

    $$\text{(a) } \sin(x-y) \qquad \text{(b) } \cos \tfrac{1}{2}y \qquad \text{(c) } \sin 2x \qquad \text{(d) } \tan(x+y)$$