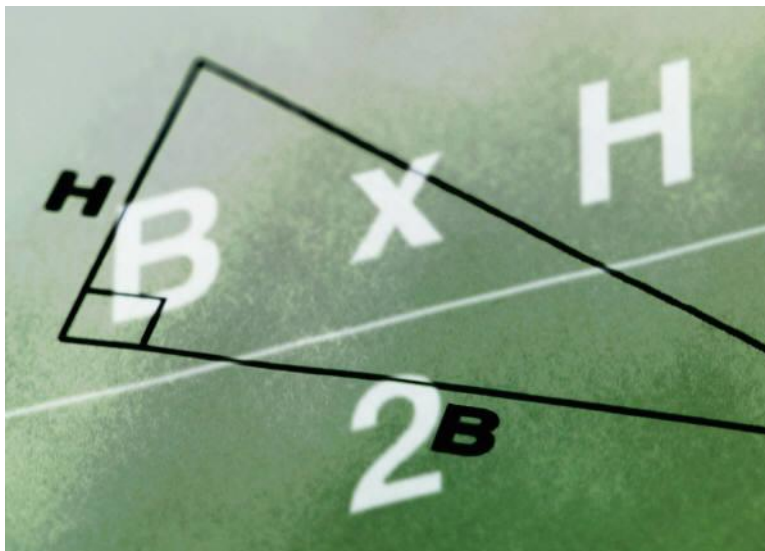


SUMMER

KNOWHOW
STUDY AND LEARNING CENTRE



Indices & Logarithms



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INDICES

Index notation

Consider the following:

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

$$5^3 = 5 \times 5 \times 5 = 125$$

$$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$$

In general $a^n = \underbrace{a \times a \times a \times a \times a \times a \dots \times a}_{n \text{ factors}}$

The letter 'n' in a^n is called one of three things:

- n is the index in a^n with a as the base
- n is the exponent or power to which the base a is raised
- n is the logarithm, with a as the base

When a number such as 125 is written in the form 5^3 we say it is in exponential or index notation.

First Index Law

Expressions in index form can be multiplied or divided only if they have the same base.

To multiply index expressions add the indices.

$$2^3 \times 2^2 = 2 \cdot 2 \cdot 2 \times 2 \cdot 2 = 2^5 \quad [\text{5 factors of 2 multiplied together}]$$

Therefore $2^3 \times 2^2 = 2^5$

In general:

$$a^m \times a^n = a^{m+n}$$

Second Index Law

To divide index expressions subtract the indices

$$\begin{aligned} 3^5 \div 3^3 &= \frac{3^5}{3^3} \\ &= \frac{3 \times 3 \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3}} \quad [\text{Cancelling three lots of 3}] \\ &= 3^2 \end{aligned}$$

Therefore $3^5 \div 3^3 = 3^{5-3} = 3^2$

In general:

$$a^m \div a^n = a^{m-n}, \quad a \neq 0$$

Third Index Law

When an expression in index form is raised to a power, multiply the indices.

$$(5^2)^3 \text{ means } 5^2 \times 5^2 \times 5^2$$

From the First Index Law $5^2 \times 5^2 \times 5^2 = 5^{2+2+2} = 5^6$

Therefore $(5^2)^3 = 5^{2 \times 3} = 5^6$

In general:

$$(a^m)^n = a^{m \times n} \quad \text{and} \quad (a^m \times b^p)^n = a^{mn} \times b^{pn}$$

Examples

- 1) $x^5 \times x^6 = x^{11}$ [using first index law]
- 2) $y^8 \div y^3 = y^5$ [using second index law]
- 3) $(2x^2)^3 = 2^3x^6 = 8x^6$ [using third index law]

Zero Index

So far we have only considered expressions in which the indices are positive whole numbers. The index laws also apply when the index is zero or negative.

Any expression with a zero index is equal to 1.

Consider $2^3 \div 2^3$:

Using the second index law this is $2^3 \div 2^3 = \frac{2^3}{2^3} = 2^{3-3} = 2^0$

$$\text{But } 2^3 \div 2^3 = \frac{2 \times 2 \times 2}{2 \times 2 \times 2} = \frac{\cancel{2} \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2} \times \cancel{2}} = 1$$

So $2^3 \div 2^3 = 2^0$ AND $2^3 \div 2^3 = 1$

Therefore $2^0 = 1$

In general: $a^0 = 1, a \neq 0$

Examples

- 1) $7^0 = 1$
- 2) $(xy)^0 = 1$
- 3) $(\frac{1}{2})^0 = 1$
- 4) $(28x^2)^0 = 1$

Negative Indices

Consider $\frac{1}{2^4} = 1 \div 2^4$
 $= 2^0 \div 2^4$ [because $2^0 = 1$]
 $= 2^{-4}$ [using the second index law]

Therefore $2^{-4} = \frac{1}{2^4}$

In general: $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n, a \neq 0$

Examples

Express the following with positive indices:

- 1) $2^{-3} = \frac{1}{2^3}$
- 2) $\frac{1}{x} = x^{-1}$
- 3) $2y^{-1} = \frac{2}{y}$
- 4) $\frac{1}{3x^{-2}} = \frac{x^2}{3}$
- 5) $\frac{1}{(-2a)^{-3}} = (-2a)^3$
- 6) $\frac{2x^2}{y^{-3}} = 2x^2y^3$
- 7) $(\frac{3}{4})^{-2} = \frac{3^{-2}}{4^{-2}} = \frac{4^2}{3^2}$

Combining index laws

Index laws are used to simplify complex expressions.

Examples

1. Simplify $(4a^2b)^3 \div b^2$

$$(4a^2b)^3 \div b^2 = \frac{4^3 a^6 b^3}{b^2} = 4^3 a^6 b \quad [\text{using a combination of the second and third laws}]$$

2. Simplify $\left(\frac{3a^3b}{c^2}\right)^2 \div \left(\frac{ab}{3c^{-2}}\right)^{-3}$

$$\begin{aligned} \left(\frac{3a^3b}{c^2}\right)^2 \div \left(\frac{ab}{3c^{-2}}\right)^{-3} &= \frac{3^2 a^6 b^2}{c^4} \div \frac{a^{-3} b^{-3}}{3^{-3} c^6} && [\text{remove brackets}] \\ &= \frac{3^2 a^6 b^2}{c^4} \times \frac{3^{-3} c^6}{a^{-3} b^{-3}} && [\text{convert division to multiplication}] \\ &= \frac{3^2 a^6 b^2}{c^4} \times \frac{a^3 b^3 c^6}{3^3} && [\text{express with positive indices}] \\ &= 3^{2-3} \times a^{6+3} \times b^{2+3} \times c^{6-4} && [\text{apply index laws}] \\ &= 3^{-1} \times a^9 \times b^5 \times c^2 \\ &= \frac{a^9 b^5 c^2}{3} && [\text{express with positive indices}] \end{aligned}$$

3. Write $x^{-1} + x^2$ as a single fraction

$$\begin{aligned} x^{-1} + x^2 &= \frac{1}{x} + x^2 \\ &= \frac{1+x^3}{x} \quad [\text{add using a common denominator, } x] \end{aligned}$$

Exercise

1. Simplify the following

a) $c^5 \times c^3 \times c^7$ b) $3 \times 2^2 \times 2^3$ c) $a^3 \times a^2 b^3 \times ab^4$ d) $3^6 \div 3^4$
e) $a^8 \div a^3$ f) $x^4 y^6 \div x^2 y^3$ g) $(x^3)^4$ h) $(x^m y^n)^5$

2. Write with positive indices and evaluate if possible:

a) x^{-5} b) 250^0 c) $3ab^{-5}$ d) $(pq)^{-2}$
e) $(5xy)^{-3}$ f) $\frac{2y}{z^{-5}}$ g) 2^{-5} h) $(-2)^{-3}$
i) $-(3^{-2})$ j) $2 \times (-5)^{-2}$

3. Simplify these expressions giving your answer in positive index form:

a) $2a^3 b^2 \times a^{-1} b^3$ b) $(5x^{-2}y)^{-3}$ c) $\frac{a^7 \times a^8 \times a^3}{a^2 \times a^5}$ d) $x(x - x^{-1})$
e) $\frac{a^2 b^3 c^{-4}}{a^4 b c^5}$ f) $\left(\frac{3a^4 b^{-3}}{c^3}\right) \div \left(\frac{b^5}{ac^3}\right)$

Answers

1. a) c^{15} b) 96 c) $a^6 b^7$ d) 9 e) a^5 f) $x^2 y^3$ g) x^{12} h) $x^5 m y^5 n$
2. a) $\frac{1}{x^5}$ b) 1 c) $\frac{3a}{b^5}$ d) $\frac{1}{p^2 q^2}$ e) $\frac{1}{125 x^3 y^3}$ f) $2yz^5$ g) $\frac{1}{32}$ h) $-\frac{1}{8}$ i) $-\frac{1}{9}$ j) $\frac{2}{25}$
3. (a) $2a^2 b^5$ b) $\frac{x^6}{125 y^3}$ c) a^{11} d) $x^2 - 1$ e) $\frac{b^2}{a^2 c^9}$ f) $\frac{3a^5}{b^8}$

FRACTIONAL INDICES

Expressions of the form $a^{\frac{1}{n}}$

The index laws apply to fractional indices as well as positive and negative integer indices.

Using the first index law we know that

$$3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^1 = 3$$

That is $3^{\frac{1}{2}}$ multiplied by itself equals 3.

The square root of 3, $\sqrt{3}$ is also a number that, when multiplied by itself, equals 3:

$$\sqrt{3} \times \sqrt{3} = 3$$

Since $3^{\frac{1}{2}}$ behaves like $\sqrt{3}$ we say that $3^{\frac{1}{2}} = \sqrt{3}$.

Similarly $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$ and $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 2$ using the first index law

Since $2^{\frac{1}{3}}$ behaves like $\sqrt[3]{2}$ we say that $2^{\frac{1}{3}} = \sqrt[3]{2}$

In general:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \text{ (nth root of } a \text{ where } n \text{ is a positive integer)}$$

Examples

1) $4^{\frac{1}{2}} = \sqrt{4} = 2$

2) $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

3) $3^{\frac{1}{4}} = \sqrt[4]{3}$

4) $b^{\frac{1}{5}} = \sqrt[5]{b}$

5) $x^{\frac{1}{2}} = \sqrt{x}$

6) $32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{32}} = \frac{1}{2}$

In most cases the root of a number will not be able to be written as a fraction and will be an irrational number. For example $\sqrt{2} = 1.414\dots\dots$

Expressions of the form $a^{\frac{m}{n}}$

If $a^{\frac{1}{3}} = \sqrt[3]{a}$, what meaning can be given to $a^{\frac{2}{3}}$?

$a^{\frac{2}{3}}$ can be written $(a^2)^{\frac{1}{3}}$ using the third index law

So $a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}} = \sqrt[3]{a^2}$

In general:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ where } m \text{ and } n \text{ are both integers}$$

Examples

1) $5^{\frac{3}{4}} = \sqrt[4]{5^3}$

2) $7^{\frac{5}{2}} = \sqrt{7^5}$

3) $a^{\frac{7}{5}} = \sqrt[5]{a^7}$

4) $y^{-\frac{3}{4}} = \frac{1}{y^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{y^3}}$

5) $\sqrt[4]{x^3} = x^{\frac{3}{4}}$

The index laws can be used to evaluate and simplify expressions with fractional indices.

Examples

Simplify the following, and evaluate if possible.

$$1) (x^3y^9)^{\frac{2}{3}} = x^{3 \times \frac{2}{3}} y^{9 \times \frac{2}{3}} = x^2y^6 \quad [\text{using the third index law}]$$

$$2) 3^{\frac{1}{3}} \div 3^{\frac{4}{3}} = 3^{\frac{1}{3} - \frac{4}{3}} = 3^{-1} = \frac{1}{3} \quad [\text{using second index law}]$$

$$3) 32^{\frac{3}{5}} = (2^5)^{\frac{3}{5}} = 2^3 = 8 \quad [\text{write 32 as } 2^5]$$

$$\begin{aligned} 4) (a^2b^5)^{\frac{1}{3}} \times a^{\frac{1}{3}}b^{-\frac{2}{3}} &= a^{\frac{2}{3}}b^{\frac{5}{3}} \times a^{\frac{1}{3}}b^{-\frac{2}{3}} \\ &= a^{\frac{2}{3} + \frac{1}{3}} b^{\frac{5}{3} - \frac{2}{3}} \\ &= ab \end{aligned}$$

Exercise

1. Evaluate the following expressions. If the answer is not exact give the decimal approximation to two decimal places.

(a) $64^{\frac{1}{2}}$

(b) $125^{\frac{1}{3}}$

(c) $81^{-\frac{1}{2}}$

(d) $250^{\frac{1}{5}}$

2. Simplify the following expressions. Give your answer in index notation with positive indices.

(a) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

(b) $\sqrt{a} \times \sqrt[4]{a} \times \sqrt[3]{a^2}$

(c) $(125a^6b)^{-\frac{1}{3}} \times b^{\frac{2}{3}}$

(d) $\frac{(2xy^3)^{\frac{1}{2}}}{2} \times \left(\frac{x^{\frac{3}{2}}}{y^2}\right)^4$

Answers

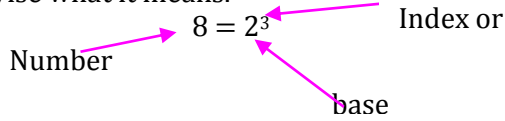
1. (a) 8 (b) 5 (c) $\frac{1}{9}$ (d) 3.02

2. (a) $\frac{2^2}{3^2}$ (b) $a^{\frac{17}{12}}$ (c) $\frac{b^{\frac{1}{3}}}{5a^2}$ (d) $\frac{x^{\frac{13}{2}}}{2^2y^{\frac{13}{2}}}$

Logarithms

The logarithm of a number is the power to which the base must be raised to give the number. This means that a logarithm is an index.

Now, let's revise what it means.



This familiar statement may be written as $\log_2 8 = 3$.
 Diagram illustrating the components of the equation $\log_2 8 = 3$. The base 2 is labeled 'base', the number 8 is labeled 'Number', and the result 3 is labeled 'Index or'.

Generally:

$$a^x = n \iff \log_a n = x \quad (\text{follow the arrows and read } a^x = n, a > 0)$$

Examples

Re-write these equations in logarithmic notation:

(a) $5^3 = 125$
 $\log_5 125 = 3$ Using the above general expression:

(b) $3^{-2} = \frac{1}{9}$
 $\log_3 \frac{1}{9} = -2$ Using the above general expression

(c) $25^{\frac{1}{2}} = 5$
 $\log_{25} 5 = \frac{1}{2}$ Using the above general expression

Examples

To evaluate a logarithm:
 write the number in index form, with same base as the logarithm.
 use the definition of a logarithm to evaluate.

Evaluate $\log_5 125$
 $125 = 5^3$ 125 in index form with base 5
 $\therefore \log_5 125 = 3$ equivalent logarithm statement

Evaluate $\log_{10} 0.0001$
 $0.0001 = 10^{-4}$ 0.0001 in index form with base 10
 $\therefore \log_{10} 0.0001 = -4$ equivalent logarithm statement

Two important logs to remember

$$a^0 = 1$$

$$\therefore a^0 = 1 \iff \log_a 1 = 0$$

In words: For any base, the log of 1 is zero.

$$a = a^1$$

$$\therefore a^1 = a \iff \log_a a = 1$$

In words: The log of any number with itself as base is 1.

Logarithm Laws

Corresponding to the three index laws, there are three laws of logarithms to help in manipulating logarithms.

First Logarithm Law

$$\log_a (mn) = \log_a m + \log_a n$$

Second Logarithm Law

$$\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$$

Third Logarithm Law

$$\log_a (m^p) = p \log_a m$$

Examples

$$\begin{aligned} \log_a 15 &= \log_a (5 \times 3) \\ &= \log_a 5 + \log_a 3 \end{aligned} \quad \text{Using the first logarithm law}$$

$$\log_a \left(\frac{12}{5} \right) = \log_a 12 - \log_a 5 \quad \text{Using the second logarithm law}$$

$$\begin{aligned} \log_a (9^2) &= 2 \log_a 9 \\ &= 2 \log_a (3^2) \\ &= 4 \log_a 3 \end{aligned} \quad \text{Using the third logarithm law}$$

Note: To use the logarithm laws, all logarithms must have the same base.

These laws together with the definition of a logarithm can be used to simplify and evaluate logarithmic expressions and to solve equations involving logarithms.

Examples

1. Solve

$$\begin{aligned} \log_2 x &= 5 \\ \log_2 x = 5 &\Rightarrow 2^5 = x \quad \text{using the definition of a logarithm} \\ \text{Therefore } x &= 32 \end{aligned}$$

2. Simplify

$$\begin{aligned} \log_{10} 6 + \log_{10} 2 \\ \log_{10} 6 + \log_{10} 2 &= \log_{10} (6 \times 2) \quad \text{using first log. law} \\ &= \log_{10} 12 \end{aligned}$$

3. Simplify $\log_3 6a + \log_3 b - \log_3 2a$ Use the laws for adding and subtracting logarithms.

$$\begin{aligned} &= \log_3 (6a \times b) - \log_3 2a \\ &= \log_3 \left(\frac{6a \times b}{2a} \right) \\ &= \log_3 3b \end{aligned}$$

4. Simplify

$$\frac{1}{2} \log_{10} 36 - \log_{10} 15 + 2 \log_{10} 5$$

$$\begin{aligned} &= \log_{10} 36^{\frac{1}{2}} - \log_{10} 15 + \log_{10} 5^2 \\ &= \log_{10} 6 - \log_{10} 15 + \log_{10} 25 \\ &= \log_{10} \frac{6 \times 25}{15} \\ &= \log_{10} 10 \\ &= 1 \end{aligned}$$

NB: $\frac{1}{2} \log_{10} 36$ and $2 \log_{10} 5$ must be written as $\log_{10} 36^{\frac{1}{2}}$ and $\log_{10} 5^2$ before using addition and subtraction laws.

[use laws for adding and subtracting logs]

5. Solve for x

$$\log_{10}(2x+1) = \log_{10} 3$$

Therefore

$$2x+1 = 3$$

$$x = 1 \quad \text{both sides of the equation are log10 so can equate } (2x+1) \text{ and } 3$$

Exercises

Exercise 1

Write in logarithm form.

- (a) $3^2 = 9$ (b) $10^4 = 10\,000$
 (c) $10^{-2} = 0.01$ (d) $e^a = b$

Exercise 2

Evaluate without using a calculator.

- (a) $\log_7 49$ (b) $\log_{10} \sqrt{10}$ (c) $\log_5 1$ (d) $\log_{10} 100\,000$ (e) $\log_5 5$

Exercise 3

1. Simplify.

- (a) $\log_4 8 + \log_4 3 - \log_4 2$ (b) $\frac{1}{2} \log_{10} 25 - \log_{10} 4 + 2 \log_{10} 3$
 (c) $\log_e 2e^3 + 2 \log_e 3 - \log_e 18$ (d) $\log_a 4 + 2 \log_a 3 - 2 \log_a 6$
 (e) $\frac{1}{2} \log_{10} a^2 + 3 \log_{10} b - \log_{10} ab^2$

2. Solve for x

- (a) $\log_{10} x = \log_{10} 4 - \log_{10} 2$ (b) $\log_2 2x - 2 \log_2 3 = \log_2 6$

Answers

Exercise 1

- (a) $\log_3 9 = 2$ (b) $\log_{10} 10,000 = 4$ (c) $\log_{10} 0.01 = -2$ (d) $\log_e b = a$

Exercise 2

- (a) 2 (b) $1/2$ (c) 0 (d) 5 (e) 1

Exercise 3

- 1.(a) $\log_4 12$ (b) $\log_{10} \frac{45}{4}$ (c) 3 (d) 0 (e) $\log_{10} b$
 2. (a) $x = 2$ (b) $x = 27$

EXPONENTIAL EQUATIONS

Exponential equations have a term in which the variable is a power (index or exponent).

For example: $a^x = b$ and $2^{1-x} = 4^{3x}$

If two equal numbers can be expressed with the same base then the indices must be equal:

$$\text{ie. } m^x = m^y \Rightarrow x = y \text{ [equating indices]}$$

We can use this property to solve exponential equations.

Examples

1. $3^x = 27$

$$3^x = 3^3 \quad [27 \text{ is written in index form so that both sides have a base of } 3].$$

$$x = 3 \quad [\text{Equate indices}]$$

2. $2^{1-x} = \frac{1}{8}$

$$2^{1-x} = 2^{-3} \quad \left[\frac{1}{8} \text{ is written in index form with a base of } 2 \right]$$

$$1 - x = -3 \quad [\text{equate indices and solve for } x]$$

$$x = 4$$

3. $5^{\frac{1}{x}} = 25$

$$5^{\frac{1}{x}} = 5^2 \quad [25 \text{ is written in index form, with a base of } 5]$$

$$\frac{1}{x} = 2 \quad [\text{Equate indices and solve for } x]$$

$$x = \frac{1}{2}$$

Not all equations have integer solutions.

For example: $3^x = 10$ has a solution between 2 and 3 since $3^2 = 9$ and $3^3 = 27$

Logarithms with base 10 or base 'e' can be used to solve equations like this with the calculator.

On the calculator use the log button to evaluate logarithms to base 10 and the ln button to evaluate logarithms to base 'e'.

Examples

1. $3^x = 10$

$$\log_{10}(3^x) = \log_{10}(10) \quad [\text{Take logs (base 10) of both sides}]$$

$$x \log_{10}(3) = 1 \quad [\text{using log laws}]$$

$$x = \frac{1}{\log_{10}(3)}$$

$$x = \frac{1}{0.4771} \quad [\text{using calculator}]$$

$$x = 2.095$$

$$\begin{aligned}
2. \quad & 2 \times 5^{x+1} = 15 \\
& 5^{x+1} = 7.5 \\
& \log_{10}(5^{x+1}) = \log_{10}(7.5) \quad [\text{Take logs of both sides}] \\
& (x+1)\log_{10}(5) = \log_{10}(7.5) \quad [\text{using log laws}] \\
& (x+1) = \frac{\log_{10}(7.5)}{\log_{10}(5)} \\
& x+1 = 1.252 \quad [\text{using calculator}] \\
& x = 0.25
\end{aligned}$$

$$\begin{aligned}
3. \quad & 2^{2x+1} = 5^{2-x} \\
& \log 2^{2x+1} = \log 5^{2-x} \quad [\text{Take logs of both sides}] \\
& (2x+1)\log 2 = (2-x)\log 5 \\
& (2x+1)0.301 = (2-x)0.699 \\
& 0.602x + 0.301 = 1.398 - 0.699x \quad [\text{removing brackets}] \\
& 1.301x = 1.097 \\
& x = 0.843
\end{aligned}$$

See Exercise 1

Growth and decay

Example

The number of bacteria present in a sample is given by $N = 800e^{0.2t}$, where t is in seconds by
Find (a) the initial number of bacteria
(b) the time when the number of bacteria reaches 10 000

(a) The initial number of bacteria occurs when $t = 0$

$$\begin{aligned}
N &= 800e^{0.2t} \\
&= 800e^{0.2 \times 0} \quad [\text{Substitute } t = 0 \text{ in the equation for } N] \\
&= 800e^0 \\
&= 800
\end{aligned}$$

The initial number of bacteria is 800.

(b) The number of bacteria, N , is equal to 10 000.

$$\begin{aligned}
N &= 800e^{0.2t} \\
10000 &= 800e^{0.2t} \quad [\text{Substitute } N = 10\,000 \text{ in the equation for } N] \\
\frac{10000}{800} &= e^{0.2t} \\
12.5 &= e^{0.2t} \\
\ln(12.5) &= \ln(e^{0.2t}) \quad [\text{Take the logarithm to base } e \text{ of both sides}] \\
\ln(12.5) &= 0.2t \\
t &= 12.6
\end{aligned}$$

It takes 12.6 sec. For the number of bacteria to reach 10 000

See Exercise 2

Exercises

Exercise 1

Solve for x:

1. (a) $5^x = 125$ (b) $3^{x-1} = 27$ (c) $2^{2x-1} = 128$ (d) $9^{\frac{1}{x}} = 3^{-4}$ (e) $9^{2x+1} = 27^x$
2. (a) $5^x = 12$ (b) $2^{x-3} = 9$ (c) $7^{-2x} = 5$ (d) $\frac{1}{2^{x+1}} = 4^{x+2}$

Exercise 2

1. The decay rate for a radio-active element is given by $R = 400e^{0.03t}$ where R is the decay rate in counts/s at time t(s).

Find: (i) the initial decay rate

(ii) the time for the decay rate to reduce to half the initial decay rate.

2. The charge Q units on a plate of a condenser t seconds after it starts to discharge is given by the formula $Q = Q_0 10^{-kt}$

If the initial charge is 5076 units and $Q = 1840$ when $t = 0.5$ s

find: (i) the value of k

(ii) the time needed for the charge to fall to 1000 units.

(iii) the charge after 2 sec.

Answers

Exercise 1

- 1 (a) $x = 3$ (b) $x = 4$ (c) $x = 4$ (d) $x = -\frac{1}{2}$ (e) $x = -2$
2 (a) $x = 1.54$ (b) $x = 6.17$ (c) $x = -0.41$ (d) $x = -1.67$

Exercise 2.

1 (i) $R_0 = 400$ (ii) $t = 23.1$ s

2 (i) $k = 0.881$ (ii) $t = 0.8$ s (iii) $Q = 87.8$ units

SIMPLIFYING SURDS

Definition

A **surd** is an irrational number resulting from a radical expression that **cannot** be evaluated directly.

For example $\sqrt{2}$, $\sqrt{3}$, $\frac{1}{\sqrt{8}}$, $\sqrt{20}$, $\sqrt[3]{2}$, $\sqrt[3]{9}$, $\sqrt[4]{8}$, etc are all surds,

but $\sqrt{1}$, $\sqrt{9}$, $\frac{1}{\sqrt{100}}$, $\sqrt[3]{8}$, $\sqrt[4]{16}$, $\sqrt[3]{64}$ are not.

Surds **cannot** be expressed exactly in the form $\frac{a}{b}$, and can only be approximated by a decimal.

eg: $\sqrt{2} \sim 1.414$

See Exercise 1

Simplifying Surds

If a surd has a square factor ie 4, 9, 16, 25, 36..., then it can be simplified.

Examples

$$\begin{aligned} 1. \sqrt{200} &= \sqrt{100 \times 2} \\ &= \sqrt{100} \times \sqrt{2} \quad [\text{NB: It is best to find the } \textit{largest} \text{ square number factor!}] \\ &= 10\sqrt{2} \end{aligned}$$

$$\begin{aligned} 2. 3\sqrt{48} &= 3\sqrt{16 \times 3} \\ &= 3\sqrt{16} \times \sqrt{3} \\ &= 3 \times 4 \times \sqrt{3} \\ &= 12\sqrt{3} \end{aligned}$$

$$\begin{aligned} 3. \frac{\sqrt[3]{24}}{4} &= \frac{\sqrt[3]{8 \times 3}}{4} \\ &= \frac{\sqrt[3]{8} \times \sqrt[3]{3}}{4} \\ &= \frac{2 \times \sqrt[3]{3}}{4} \\ &= \frac{\sqrt[3]{3}}{2} \end{aligned}$$

See Exercise 2

Addition and Subtraction

Only like surds can be added or subtracted

Examples

$$1. \quad 8\sqrt{5} - 3\sqrt{5} = 5\sqrt{5}$$

$$2. \quad 2\sqrt{3} + 5\sqrt{7} - 10\sqrt{3} = 5\sqrt{7} - 8\sqrt{3}$$

Sometimes it is necessary to simplify each surd first

$$\begin{aligned} 3. \quad \sqrt{18} - \sqrt{8} - \sqrt{20} &= \sqrt{9 \times 2} - \sqrt{4 \times 2} - \sqrt{4 \times 5} \\ &= 3\sqrt{2} - 2\sqrt{2} - 2\sqrt{5} \\ &= \sqrt{2} - 2\sqrt{5} \end{aligned}$$

NB: $\sqrt{5} + \sqrt{7} \neq \sqrt{12}$ because $\sqrt{5}$ and $\sqrt{7}$ are not 'like' surds.

See Exercise 3

Multiplication of Surds

To multiply surds the whole numbers are multiplied together, as are the numbers enclosed by the radical sign

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$$

Examples:

1. $\sqrt{2} \times \sqrt{3} = \sqrt{6}$

2. $3\sqrt{3} \times 4\sqrt{5} = 12\sqrt{15}$

Sometimes it is possible to simplify after multiplication

3. $2\sqrt{10} \times 7\sqrt{6} = 14\sqrt{60}$
 $= 14\sqrt{4 \times 15}$
 $= 14 \times 2\sqrt{15}$
 $= 28\sqrt{15}$

See Exercise 4

Division of Surds

Expressions such as $\sqrt{a} \div \sqrt{b}$ may be expressed as fractions and simplified using

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Examples

1. $\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}$

2. $\frac{4}{\sqrt{2}} = \frac{\sqrt{16}}{\sqrt{2}} = \sqrt{\frac{16}{2}} = \sqrt{8} = 2\sqrt{2}$

See Exercise 5

Exercises

Exercise 1

Decide whether the following radical expressions are rational or irrational and evaluate exactly if rational:

(a) $\sqrt{25}$ (b) $\sqrt{2.25}$ (c) $\sqrt{\frac{1}{10}}$ (d) $\sqrt[3]{64}$ (e) $\sqrt{12 \times 18}$

Exercise 2

Simplify

(a) $\sqrt{20}$ (b) $\sqrt{48}$ (c) $\sqrt{500}$ (d) $2\sqrt{12}$ (e) $\frac{2}{\sqrt{20}}$

Exercise 3

Simplify

(a) $2\sqrt{3} + 5\sqrt{3}$ (b) $3\sqrt{7} + 5\sqrt{7} - 3\sqrt{7}$
(c) $\sqrt{2} + \sqrt{7} + 3\sqrt{2} - 4\sqrt{7}$ (d) $\sqrt{54} - \sqrt{24}$

Exercise 4

Simplify

(a) $\sqrt{3} \times \sqrt{10}$ (b) $2\sqrt{7} \times 5\sqrt{3}$
(c) $\sqrt{10} \times 3\sqrt{10}$ (d) $2\sqrt{8} \times 2\sqrt{50} \times \sqrt{2}$

Exercise 5

Simplify

(a) $\frac{\sqrt{12}}{\sqrt{6}}$ (b) $\frac{\sqrt{32}}{\sqrt{2}}$
(c) $\frac{3\sqrt{28}}{\sqrt{7}}$ (d) $\frac{4\sqrt{2} \times 3\sqrt{3}}{6\sqrt{6}}$

Answers

Exercise 1

(a) 5 rational (b) 1.5 rational (c) irrational (d) 4 rational (e) irrational

Exercise 2

(a) $2\sqrt{5}$ (b) $4\sqrt{3}$ (c) $10\sqrt{5}$ (d) $4\sqrt{3}$ (e) $\frac{1}{\sqrt{5}}$

Exercise 3

(a) $7\sqrt{3}$ (b) $5\sqrt{7}$ (c) $4\sqrt{2} - 3\sqrt{7}$ (d) $\sqrt{6}$

Exercise 4

(a) $\sqrt{30}$ (b) $10\sqrt{21}$ (c) 30 (d) $80\sqrt{2}$

Exercise 5

(a) $\sqrt{2}$ (b) 4 (c) 6 (d) 2

OPERATIONS ON SURDS

Expansion of Brackets

The usual algebraic rules for expansion of brackets apply to brackets containing surds

$$\begin{aligned} a(b + c) &= ab + ac \\ \text{and} \\ (a + b)(c + d) &= ac + bc + ad + bd \end{aligned}$$

Examples

- $\sqrt{2}(\sqrt{2} + 5) = 2 + 5\sqrt{2}$
- $2\sqrt{3}(\sqrt{3} - 3\sqrt{2}) = 2\sqrt{9} - 6\sqrt{6}$
 $= 6 - 6\sqrt{6}$
- $(7 - \sqrt{5})^2 = (7 - \sqrt{5})(7 - \sqrt{5})$
 $= 49 - 7\sqrt{5} - 7\sqrt{5} + 5$
 $= 54 - 14\sqrt{5}$
- $(\sqrt{6} - 4\sqrt{3})(2\sqrt{2} - 3\sqrt{5}) = 2\sqrt{12} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15}$
 $= 2\sqrt{4 \times 3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15}$
 $= 2 \times 2 \times \sqrt{3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15}$
 $= 4\sqrt{3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15}$

See Exercise 1

Rationalizing surds

Sometimes fractions containing surds are required to be expressed with a *rational denominator*.

Examples

- $\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$
 $= \frac{2\sqrt{5}}{5}$
- $\frac{\sqrt{5}}{3\sqrt{2}} = \frac{\sqrt{5}}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
 $= \frac{\sqrt{10}}{6}$

Conjugate surds

The pair of expressions $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called *conjugate surds*. Each is the conjugate of the other.

The product of two conjugate surds does **NOT** contain any surd term!

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$$

$$\begin{aligned} \text{Eg: } (\sqrt{10} + \sqrt{3})(\sqrt{10} - \sqrt{3}) &= (\sqrt{10})^2 - (\sqrt{3})^2 \\ &= 10 - 3 \\ &= 7 \end{aligned}$$

We make use of this property of conjugates to rationalize denominators of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$

Example

$$\begin{aligned}\frac{\sqrt{3}}{5+\sqrt{2}} &= \frac{\sqrt{3}}{5+\sqrt{2}} \times \frac{5-\sqrt{2}}{5-\sqrt{2}} \\ &= \frac{\sqrt{3}(5-\sqrt{2})}{25-2} \\ &= \frac{5\sqrt{3}-\sqrt{6}}{23}\end{aligned}$$

See Exercise 2

Operations on fractions that contain surds

When adding and subtracting fractions containing surds it is generally advisable to first rationalize each fraction:

Example

$$\begin{aligned}\frac{2}{3\sqrt{2}+1} + \frac{1}{\sqrt{3}-\sqrt{2}} &= \frac{2}{3\sqrt{2}+1} \times \frac{3\sqrt{2}-1}{3\sqrt{2}-1} + \frac{1}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\ &= \frac{6\sqrt{2}-2}{18-1} + \frac{\sqrt{3}+\sqrt{2}}{3-2} \\ &= \frac{6\sqrt{2}-2}{17} + \frac{\sqrt{3}+\sqrt{2}}{1} \\ &= \frac{6\sqrt{2}-2}{17} + \frac{17(\sqrt{3}+\sqrt{2})}{17} \\ &= \frac{6\sqrt{2}-2+17(\sqrt{3}+\sqrt{2})}{17} \\ &= \frac{23\sqrt{2}-2+17\sqrt{3}}{17}\end{aligned}$$

See Exercise 3

Exercises

Exercise 1

Expand the brackets and simplify if possible

(a) $\sqrt{2}(\sqrt{2}-8)$ (b) $(2+\sqrt{3})(\sqrt{3}-4)$ (c) $(1+\sqrt{10})^2$ (d) $(\sqrt{11}+3)(\sqrt{11}-3)$

Exercise 2

Express the following fractions with a rational denominator in simplest form:

1. (a) $\frac{\sqrt{5}}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{10}}$ (c) $\frac{2\sqrt{18}}{\sqrt{8}}$
2. (a) $\frac{2+\sqrt{3}}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}-\sqrt{2}}$ (c) $\frac{\sqrt{3}+2}{\sqrt{3}-2}$

Exercise 3

Evaluate and express with a rational denominator $\frac{2}{\sqrt{3}-1} + \frac{3}{2-\sqrt{3}}$

Answers

Exercise 1

(a) $2-8\sqrt{2}$ (b) $-5-2\sqrt{3}$ (c) $11+2\sqrt{10}$ (d) 2

Exercise 2

1. (a) $\frac{\sqrt{10}}{2}$ (b) $\frac{\sqrt{10}}{10}$ (c) 3

2. (a) $\frac{2\sqrt{2}+\sqrt{6}}{2}$ (b) $\sqrt{3}+\sqrt{2}$ (c) $-7-4\sqrt{3}$

Exercise 3

$4\sqrt{3}+7$

SCIENTIFIC NOTATION

Scientific Notation (or Standard Form) is a way of writing numbers in a compact form.

A number written in Scientific Notation is expressed as a number from 1 to less than 10, multiplied by a power of 10 eg 2.3×10^{-3}

To write a number in Scientific Notation:

- move the decimal point so that there is one digit (which cannot be zero), before the decimal point.
- multiply by a power of 10, equal to the number of places the decimal point has been moved.

If the decimal point is moved to the left, the power of 10 is positive

If the decimal point is moved to the right the power of 10 is negative

Examples

1. Write 5630 in Scientific Notation.

$$5630 = 5.63 \times 1000 = 5.63 \times 10^3 \quad [\text{remember: } 1000 = 10^3]$$

Move the decimal point three places to the left. The number becomes 5.63 and the power of 10 is +3.

2. Write 0.00725 in Scientific Notation.

$$0.00725 = 7.25 \times 0.001 = 7.25 \times 10^{-3} \quad [\text{remember } 0.001 = 10^{-3}]$$

Move the decimal point three places to the right. The number becomes 7.25 and the power of 10 is -3.

3. Calculate $3.5 \times 10^5 \times 5 \times 10^{18}$

$$\begin{aligned} 3.5 \times 10^5 \times 5 \times 10^{18} &= 3.5 \times 5 \times 10^{5+18} \quad [\text{Calculations can be simplified by using index laws}] \\ &= 17.5 \times 10^{23} \\ &= 1.75 \times 10^{24} \end{aligned}$$

Always check your answer:

- If the magnitude of the number is less than one, then the power of 10 will be negative.
- If the magnitude of the number is greater than or equal to 10 then the power of 10 will be positive.

The mass of the moon is: 73 600 000 000 000 000 000 000 kg. = 7.36×10^{22} kg.

The charge on one electron is: 0.000 000 000 000 000 000 160 2 C = 1.602×10^{-19} C

Note how the size of a number is more easily seen when written in scientific notation.

Errors are less likely when writing very big and very small numbers if they are in scientific notation.

See Exercises

Exercises

Exercise 1

Write the following numbers in Scientific Notation.

- (a) 58000 (b) 0.0026 (c) 70.6 (d) 0.3 (e) 2 400 000
(f) 0.000 000 684 (g) 0.0704 (h) 0.260 (i) 17600 (j) 0.04080
(k) 0.000500 (l) 357 000 000 000

Exercise 2

Write the following numbers in Scientific Notation

- (a) 6 370 000 m (the mean radius of the Earth)
(b) 86 400 s (the time for the Earth to orbit the sun)
(c) 0.0018 s (the half-life of a radioactive isotope)
(d) 380 000 000 m (the average distance of the moon from the Earth)
(e) 0.000 000 000 001 1 s (the time for a ray of light to pass through a window pane)
(f) 637 000 000 000 000 000 000 000 kg (the mass of the planet Mars)

Exercise 3

Write the following numbers in decimal form or in whole numbers.

- (a) 7×10^2 (b) 4×10^0 (c) 3.46×10^3 (d) 5.96×10^{-5}
(e) 9×10^7 (f) 3.98×10^1 (g) 2.78×10^5 (h) 6.78×10^{-1}

Answers

Exercise 1

- (a) 5.8×10^4 (b) 2.6×10^{-3} (c) 7.06×10 (d) 3×10^{-1} (e) 2.4×10^6 (f) 6.84×10^{-7}
(g) 7.04×10^{-2} (h) 2.60×10^{-1} (i) 1.76×10^4 (j) 4.08×10^{-2} (k) 5.00×10^{-4} (l) 3.57×10^{11}

Exercise 2

- (a) 6.37×10^6 m (b) 8.64×10^4 s (c) 1.8×10^{-3} m (d) 3.8×10^8 m (e) 1.1×10^{-12} s (f) 6.37×10^{23} kg

Exercise 3

- (a) 700 (b) 4 (c) 3460 (d) 0.0000596
(e) 90000000 (f) 39.8 (g) 278000 (h) 0.678

SIGNIFICANT FIGURES

In scientific measurements, significant figures are used to indicate the accuracy of a measurement. The last digit in a measurement is often an estimate, a good guess.

All the digits in a measured value, including the last estimated digit, are called significant figures or significant digits.

The number of Significant Figures in a measured value

Rule 1: Any non-zero digit is significant. The position of a decimal point makes no difference.

Example 1	15.7	3 sig. figs
	157	3 sig. figs
	1.57	3 sig. figs
	2.7942	5 sig. figs

Rule 2: Zeros between numbers **are** significant.

Example 2	1.05	3 sig. figs
	10.51	4 sig. figs
	200.708	6 sig. figs

Rule 3: Zeros at the right hand end of whole numbers are **not** significant, unless otherwise stated.

Example 3	70	1 sig. figs
	2860	3 sig. figs
	15090	4 sig. figs

Rule 4: Zeros at the left hand end of decimal numbers are **not** significant.

Example 4	0.28	2 sig. figs
	0.0039	2 sig. figs
	0.0604	3 sig. figs

Rule 5: Zeros at the right hand end of decimal numbers **are** significant.

Example 5	12.0	3 sig. figs
	0.760	3 sig. figs
	0.48300	5 sig. figs
	2.07090	6 sig. figs

Significant Figures and Scientific Notation

The problem of deciding how many significant figures a value has is simplified by writing its value in Scientific Notation.

Examine the number of digits in the first number below, not in the power of ten.

(a)	$0.0003 = 3 \times 10^{-4}$	1 sig fig	Rule 4
(b)	$720000 = 7.2 \times 10^5$	2 sig figs	Rule 3
(c)	$660 = 6.6 \times 10^2$	2 sig figs	Rule 3
(d)	$660.0 = 6.600 \times 10^2$	4 sig figs	Rule 5
(e)	$0.66000 = 6.6000 \times 10^{-1}$	5 sig figs	Rule 5
(f)	$808.01 = 8.0801 \times 10^2$	5 sig figs	Rule 2

See Exercise 1

Rounding Off

Sometimes, numbers must be “rounded off” so that values do not appear to have more significant figures than would be appropriate or realistic.

Rule 1: If the last digit is greater than 5, the second last digit increases by 1 when rounding off.

Example 1 5.78 rounded off to 2 sig figs becomes 5.8
5.8 rounded off to 1 sig fig becomes 6 (not 6.0)

Rule 2: If the last digit is less than 5, the second last digit does not change.

Example 2 1.304 rounded off to 3 sig figs becomes 1.30
1.30 rounded off to 2 sig figs becomes 1.3
1.3 rounded off to 1 sig fig becomes 1.

Rule 3: If the last digit is 5, the number is rounded off to the nearest even number.

Example 3 4.85 rounded off to 2 sig figs becomes 4.8
4.55 rounded off, becomes 4.6

See Exercise 2

Exercise 1

Write each of the following measurements in Scientific Notation and state the number of significant digits in the value.

- (a) 345 (b) 0.000306 (c) 870 (d) 20000
(e) 140.600 (f) 710.0 (g) 0.04080 (h) 0.0050

Exercise 2

Round off each of the following numbers to one less significant figure and then to two less significant figures.

Example: 6.549 → 6.55 → 6.6

- (a) 7.668 (b) 0.0854 (c) 21.092
(d) 255.6 (e) 6.75 (f) 2.85

Answers

Exercise 1

- | | | | |
|-----|----------|------------------------|------------|
| (a) | 345 | 3.45×10^2 | 3 sig figs |
| (b) | 0.000306 | 3.06×10^{-4} | 3 sig figs |
| (c) | 870 | 8.7×10^2 | 2 sig figs |
| (d) | 20000 | 2×10^4 | 1 sig fig |
| (e) | 140.600 | 1.40600×10^2 | 6 sig figs |
| (f) | 710.0 | 7.100×10^2 | 4 sig figs |
| (g) | 0.04080 | 4.080×10^{-2} | 4 sig figs |
| (h) | 0.0050 | 5.0×10^{-3} | 2 sig figs |

Exercise 2

- | | | | | | |
|-----|--------|---|-------|---|------|
| (a) | 7.668 | → | 7.67 | → | 7.7 |
| (b) | 0.0854 | → | 0.085 | → | 0.08 |
| (c) | 21.092 | → | 21.09 | → | 21.1 |
| (d) | 255.6 | → | 256 | → | 260 |
| (e) | 6.75 | → | 6.8 | → | 7 |
| (f) | 2.85 | → | 2.8 | → | 3 |