6.4 Transformations of Exponential and Logarithmic Functions

Essential Question

How can you transform the graphs of exponential and logarithmic functions?

EXPLORATION 1 Identifying Transformations

Work with a partner. Each graph shown is a transformation of the parent function

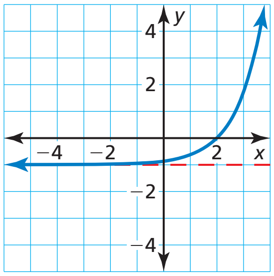
$f(x) = e^x$ or $f(x) = \ln x.$

Match each function with its graph. Explain your reasoning. Then describe the transformation of f represented by g.

a. $g(x) = e^{x+2} - 3$ **b.** $g(x) = -e^{x+2} + 1$ **c.** $g(x) = e^{x-2} - 1$

d. $g(x) = \ln(x + 2)$ **e.** $g(x) = 2 + \ln x$ **f.** $g(x) = 2 + \ln(-x)$

A.



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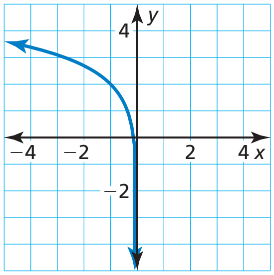

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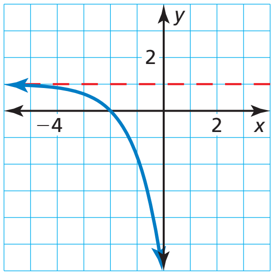

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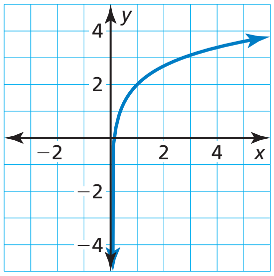

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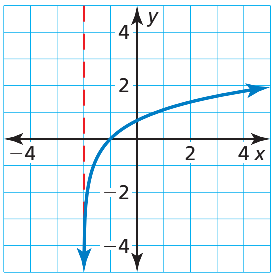

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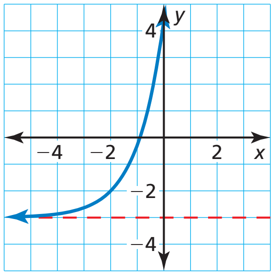


EXPLORATION 2 Characteristics of Graphs

Work with a partner. Determine the domain, range, and asymptote of each function in Exploration 1. Justify your answers.

REASONING QUANTITATIVELY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

3. How can you transform the graphs of exponential and logarithmic functions?

4. Find the inverse of each function in Exploration 1. Then check your answer by using a graphing calculator to graph each function and its inverse in the same viewing window.

6.4 Lesson

Core Vocabulary

Previous
exponential function
logarithmic function
transformations

What You Will Learn

- Transform graphs of exponential functions.
- Transform graphs of logarithmic functions.
- Write transformations of graphs of exponential and logarithmic functions.

Transforming Graphs of Exponential Functions

You can transform graphs of exponential and logarithmic functions in the same way you transformed graphs of functions in previous chapters. Examples of transformations of the graph of $f(x) = 4^x$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = 4^{x-3}$ $g(x) = 4^{x+2}$	3 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = 4^x + 5$ $g(x) = 4^x - 1$	5 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = 4^{-x}$ $g(x) = -4^x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = 4^{2x}$ $g(x) = 4^{x/2}$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 3(4^x)$ $g(x) = \frac{1}{4}(4^x)$	stretch by a factor of 3 shrink by a factor of $\frac{1}{4}$

EXAMPLE 1 Translating an Exponential Function

Describe the transformation of $f(x) = \left(\frac{1}{2}\right)^x$ represented by $g(x) = \left(\frac{1}{2}\right)^x - 4$. Then graph each function.

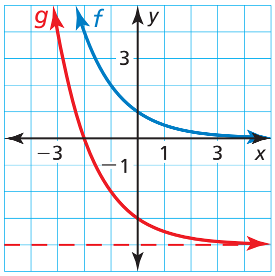
SOLUTION

Notice that the function is of the form $g(x) = \left(\frac{1}{2}\right)^x + k$.

Rewrite the function to identify k.

$$g(x) = \left(\frac{1}{2}\right)^x + (\underbrace{-4}_{k})$$

- Because $k = -4$, the graph of g is a translation 4 units down of the graph of f.



STUDY TIP

Notice in the graph that the vertical translation also shifted the asymptote 4 units down, so the range of g is $y > -4$.

EXAMPLE 2 Translating a Natural Base Exponential Function

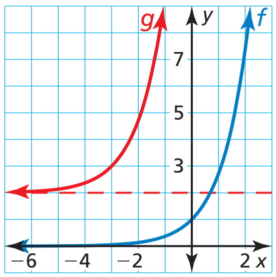
Describe the transformation of $f(x) = e^x$ represented by $g(x) = e^{x+3} + 2$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = e^{x-h} + k$. Rewrite the function to identify h and k.

$$g(x) = e^{x - \overset{\uparrow}{(-3)}} + \overset{\uparrow}{2}$$

h $\quad$ k



▶ Because $h = -3$ and $k = 2$, the graph of g is a translation 3 units left and 2 units up of the graph of f.

STUDY TIP

Notice in the graph that the vertical translation also shifted the asymptote 2 units up, so the range of g is $y > 2$.

EXAMPLE 3 Transforming Exponential Functions

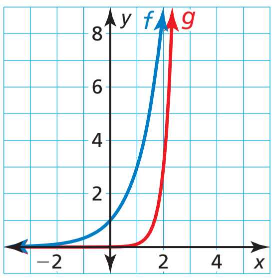
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = 3^x$, $g(x) = 3^{3x-5}$ **b.** $f(x) = e^{-x}$, $g(x) = -\frac{1}{8}e^{-x}$

SOLUTION

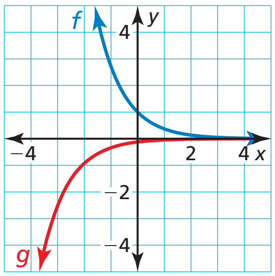
a. Notice that the function is of the form $g(x) = 3^{ax-h}$, where $a = 3$ and $h = 5$.

▶ So, the graph of g is a translation 5 units right, followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of f.



b. Notice that the function is of the form $g(x) = ae^{-x}$, where $a = -\frac{1}{8}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{8}$ of the graph of f.



LOOKING FOR STRUCTURE

In Example 3(a), the horizontal shrink follows the translation. In the function $h(x) = 3^{3(x-5)}$, the translation 5 units right follows the horizontal shrink by a factor of $\frac{1}{3}$.

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Describe the transformation of f represented by g. Then graph each function.

1. $f(x) = 2^x$, $g(x) = 2^{x-3} + 1$
2. $f(x) = e^{-x}$, $g(x) = e^{-x} - 5$
3. $f(x) = 0.4^x$, $g(x) = 0.4^{-2x}$
4. $f(x) = e^x$, $g(x) = -e^{x+6}$

Transforming Graphs of Logarithmic Functions

Examples of transformations of the graph of $f(x) = \log x$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = \log(x - 4)$ $g(x) = \log(x + 7)$	4 units right 7 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = \log x + 3$ $g(x) = \log x - 1$	3 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = \log(-x)$ $g(x) = -\log x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = \log(4x)$ $g(x) = \log\left(\frac{1}{3}x\right)$	shrink by a factor of $\frac{1}{4}$ stretch by a factor of 3
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 5 \log x$ $g(x) = \frac{2}{3} \log x$	stretch by a factor of 5 shrink by a factor of $\frac{2}{3}$

EXAMPLE 4 Transforming Logarithmic Functions

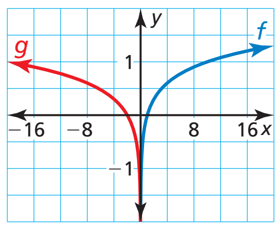
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = \log x$, $g(x) = \log\left(-\frac{1}{2}x\right)$ **b.** $f(x) = \log_{1/2} x$, $g(x) = 2 \log_{1/2}(x + 4)$

SOLUTION

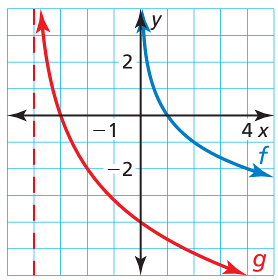
a. Notice that the function is of the form $g(x) = \log(ax)$, where $a = -\frac{1}{2}$.

▶ So, the graph of g is a reflection in the y-axis and a horizontal stretch by a factor of 2 of the graph of f.



b. Notice that the function is of the form $g(x) = a \log_{1/2}(x - h)$, where $a = 2$ and $h = -4$.

▶ So, the graph of g is a horizontal translation 4 units left and a vertical stretch by a factor of 2 of the graph of f.



STUDY TIP

In Example 4(b), notice in the graph that the horizontal translation also shifted the asymptote 4 units left, so the domain of g is $x > -4$.

Monitoring Progress

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Describe the transformation of f represented by g. Then graph each function.

5. $f(x) = \log_2 x,\ g(x) = -3\log_2 x$

6. $f(x) = \log_{1/4} x,\ g(x) = \log_{1/4}(4x) - 5$

Writing Transformations of Graphs of Functions

EXAMPLE 5 Writing a Transformed Exponential Function

Let the graph of g be a reflection in the x-axis followed by a translation 4 units right of the graph of $f(x) = 2^x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the reflection of f.

$h(x) = -f(x)$ — Multiply the output by -1.

$= -2^x$ — Substitute 2^x for $f(x)$.

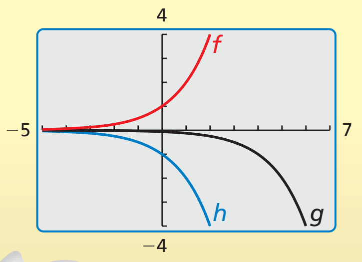
Step 2 Then write a function g that represents the translation of h.

$g(x) = h(x - 4)$ — Subtract 4 from the input.

$= -2^{x-4}$ — Replace x with $x - 4$ in $h(x)$.

▶ The transformed function is $g(x) = -2^{x-4}$.

Check



EXAMPLE 6 Writing a Transformed Logarithmic Function

Let the graph of g be a translation 2 units up followed by a vertical stretch by a factor of 2 of the graph of $f(x) = \log_{1/3} x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the translation of f.

$h(x) = f(x) + 2$ — Add 2 to the output.

$= \log_{1/3} x + 2$ — Substitute $\log_{1/3} x$ for $f(x)$.

Step 2 Then write a function g that represents the vertical stretch of h.

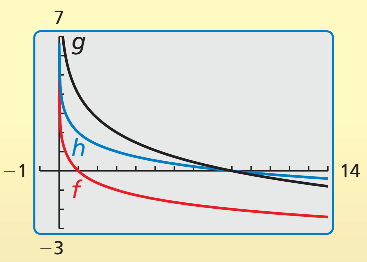
$g(x) = 2 \cdot h(x)$ — Multiply the output by 2.

$= 2 \cdot (\log_{1/3} x + 2)$ — Substitute $\log_{1/3} x + 2$ for $h(x)$.

$= 2\log_{1/3} x + 4$ — Distributive Property

▶ The transformed function is $g(x) = 2\log_{1/3} x + 4$.

Check



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7. Let the graph of g be a horizontal stretch by a factor of 3, followed by a translation 2 units up of the graph of $f(x) = e^{-x}$. Write a rule for g.

8. Let the graph of g be a reflection in the y-axis, followed by a translation 4 units to the left of the graph of $f(x) = \log x$. Write a rule for g.

6.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Given the function $f(x) = ab^{x-h} + k$, describe the effects of a, h, and k on the graph of the function.

2. **COMPLETE THE SENTENCE** The graph of $g(x) = \log_4(-x)$ is a reflection in the __________ of the graph of $f(x) = \log_4 x$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the function with its graph. Explain your reasoning.

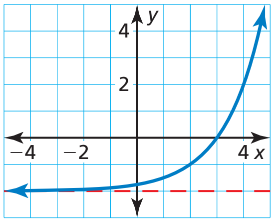
3. $f(x) = 2^{x+2} - 2$

4. $g(x) = 2^{x+2} + 2$

5. $h(x) = 2^{x-2} - 2$

6. $k(x) = 2^{x-2} + 2$

A.



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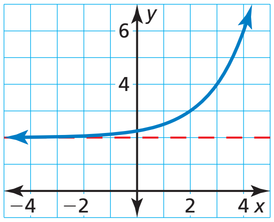

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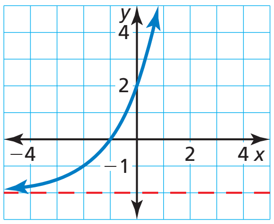

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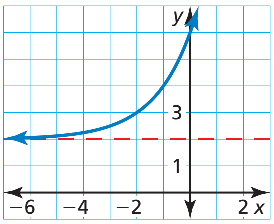


In Exercises 7–16, describe the transformation of f represented by g. Then graph each function. *(See Examples 1 and 2.)*

7. $f(x) = 3^x$, $g(x) = 3^x + 5$

8. $f(x) = 4^x$, $g(x) = 4^x - 8$

9. $f(x) = e^x$, $g(x) = e^x - 1$

10. $f(x) = e^x$, $g(x) = e^x + 4$

11. $f(x) = 2^x$, $g(x) = 2^{x-7}$

12. $f(x) = 5^x$, $g(x) = 5^{x+1}$

13. $f(x) = e^{-x}$, $g(x) = e^{-x} + 6$

14. $f(x) = e^{-x}$, $g(x) = e^{-x} - 9$

15. $f(x) = \left(\frac{1}{4}\right)^x$, $g(x) = \left(\frac{1}{4}\right)^{x-3} + 12$

16. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x+2} - \frac{2}{3}$

In Exercises 17–24, describe the transformation of f represented by g. Then graph each function. *(See Example 3.)*

17. $f(x) = e^x$, $g(x) = e^{2x}$

18. $f(x) = e^x$, $g(x) = \frac{4}{3}e^x$

19. $f(x) = 2^x$, $g(x) = -2^{x-3}$

20. $f(x) = 4^x$, $g(x) = 4^{0.5x-5}$

21. $f(x) = e^{-x}$, $g(x) = 3e^{-6x}$

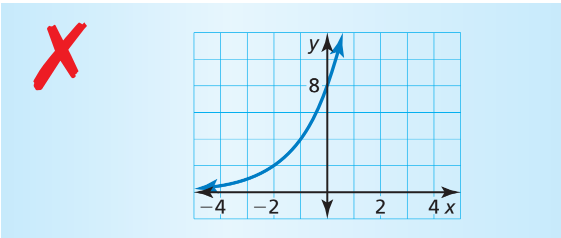
22. $f(x) = e^{-x}$, $g(x) = e^{-5x} + 2$

23. $f(x) = \left(\frac{1}{2}\right)^x$, $g(x) = 6\left(\frac{1}{2}\right)^{x+5} - 2$

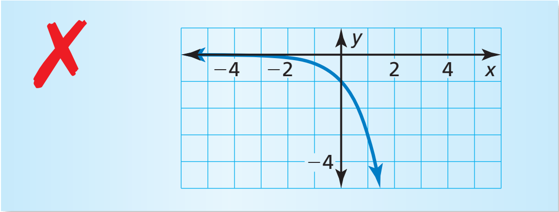
24. $f(x) = \left(\frac{3}{4}\right)^x$, $g(x) = -\left(\frac{3}{4}\right)^{x-7} + 1$

ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in graphing the function.**

25. $f(x) = 2^x + 3$



26. $f(x) = 3^{-x}$



In Exercises 27–30, describe the transformation of *f* represented by *g*. Then graph each function. *(See Example 4.)*

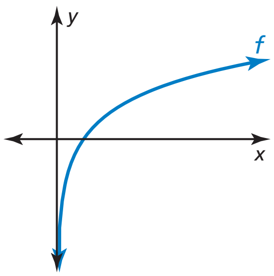
27. $f(x) = \log_4 x$, $g(x) = 3 \log_4 x - 5$

28. $f(x) = \log_{1/3} x$, $g(x) = \log_{1/3}(-x) + 6$

29. $f(x) = \log_{1/5} x$, $g(x) = -\log_{1/5}(x - 7)$

30. $f(x) = \log_2 x$, $g(x) = \log_2(x + 2) - 3$

ANALYZING RELATIONSHIPS **In Exercises 31–34, match the function with the correct transformation of the graph of *f*. Explain your reasoning.**



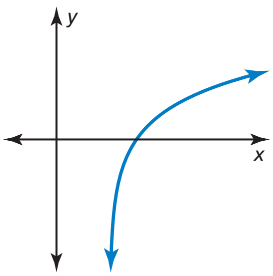
31. $y = f(x - 2)$

32. $y = f(x + 2)$

33. $y = 2f(x)$

34. $y = f(2x)$

A.



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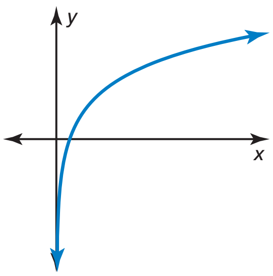

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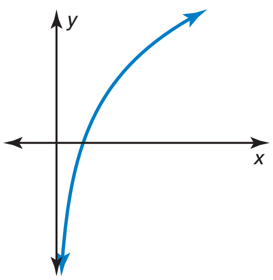

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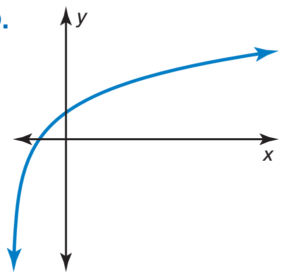


In Exercises 35–38, write a rule for *g* that represents the indicated transformations of the graph of *f*. *(See Example 5.)*

35. $f(x) = 5^x$; translation 2 units down, followed by a reflection in the y-axis

36. $f(x) = \left(\frac{2}{3}\right)^x$; reflection in the x-axis, followed by a vertical stretch by a factor of 6 and a translation 4 units left

37. $f(x) = e^x$; horizontal shrink by a factor of $\frac{1}{2}$, followed by a translation 5 units up

38. $f(x) = e^{-x}$; translation 4 units right and 1 unit down, followed by a vertical shrink by a factor of $\frac{1}{3}$

In Exercises 39–42, write a rule for *g* that represents the indicated transformation of the graph of *f*. *(See Example 6.)*

39. $f(x) = \log_6 x$; vertical stretch by a factor of 6, followed by a translation 5 units down

40. $f(x) = \log_5 x$; reflection in the x-axis, followed by a translation 9 units left

41. $f(x) = \log_{1/2} x$; translation 3 units left and 2 units up, followed by a reflection in the y-axis

42. $f(x) = \ln x$; translation 3 units right and 1 unit up, followed by a horizontal stretch by a factor of 8

JUSTIFYING STEPS **In Exercises 43 and 44, justify each step in writing a rule for *g* that represents the indicated transformations of the graph of *f*.**

43. $f(x) = \log_7 x$; reflection in the x-axis, followed by a translation 6 units down

$$h(x) = -f(x)$$
$$= -\log_7 x$$
$$g(x) = h(x) - 6$$
$$= -\log_7 x - 6$$

44. $f(x) = 8^x$; vertical stretch by a factor of 4, followed by a translation 1 unit up and 3 units left

$$h(x) = 4 \cdot f(x)$$
$$= 4 \cdot 8^x$$
$$g(x) = h(x + 3) + 1$$
$$= 4 \cdot 8^{x+3} + 1$$

USING STRUCTURE In Exercises 45–48, describe the transformation of the graph of f represented by the graph of g. Then give an equation of the asymptote.

45. $f(x) = e^x$, $g(x) = e^x + 4$

46. $f(x) = 3^x$, $g(x) = 3^{x-9}$

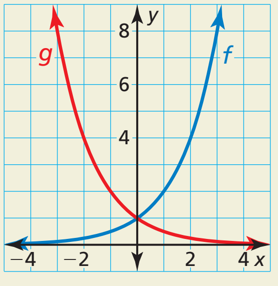
47. $f(x) = \ln x$, $g(x) = \ln(x + 6)$

48. $f(x) = \log_{1/5} x$, $g(x) = \log_{1/5} x + 13$

49. MODELING WITH MATHEMATICS The slope S of a beach is related to the average diameter d (in millimeters) of the sand particles on the beach by the equation $S = 0.159 + 0.118 \log d$. Describe the transformation of $f(d) = \log d$ represented by S. Then use the function to determine the slope of a beach for each sand type below.

Sand particle	Diameter (mm), d
fine sand	0.125
medium sand	0.25
coarse sand	0.5
very coarse sand	1

50. HOW DO YOU SEE IT? The graphs of $f(x) = b^x$ and $g(x) = \left(\frac{1}{b}\right)^x$ are shown for $b = 2$.



a. Use the graph to describe a transformation of the graph of f that results in the graph of g.

b. Does your answer in part (a) change when $0 < b < 1$? Explain.

51. MAKING AN ARGUMENT Your friend claims a single transformation of $f(x) = \log x$ can result in a function g whose graph never intersects the graph of f. Is your friend correct? Explain your reasoning.

52. THOUGHT PROVOKING Is it possible to transform the graph of $f(x) = e^x$ to obtain the graph of $g(x) = \ln x$? Explain your reasoning.

53. ABSTRACT REASONING Determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

a. A vertical translation of the graph of $f(x) = \log x$ changes the equation of the asymptote.

b. A vertical translation of the graph of $f(x) = e^x$ changes the equation of the asymptote.

c. A horizontal shrink of the graph of $f(x) = \log x$ does not change the domain.

d. The graph of $g(x) = ab^{x-h} + k$ does not intersect the x-axis.

54. PROBLEM SOLVING The amount P (in grams) of 100 grams of plutonium-239 that remains after t years can be modeled by $P = 100(0.99997)^t$.

a. Describe the domain and range of the function.

b. How much plutonium-239 is present after 12,000 years?

c. Describe the transformation of the function if the initial amount of plutonium were 550 grams.

d. Does the transformation in part (c) affect the domain and range of the function? Explain your reasoning.

55. CRITICAL THINKING Consider the graph of the function $h(x) = e^{-x-2}$. Describe the transformation of the graph of $f(x) = e^{-x}$ represented by the graph of h. Then describe the transformation of the graph of $g(x) = e^x$ represented by the graph of h. Justify your answers.

56. OPEN-ENDED Write a function of the form $y = ab^{x-h} + k$ whose graph has a y-intercept of 5 and an asymptote of $y = 2$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the indicated operation. *(Section 5.5)*

57. Let $f(x) = x^4$ and $g(x) = x^2$. Find $(fg)(x)$. Then evaluate the product when $x = 3$.

58. Let $f(x) = 4x^6$ and $g(x) = 2x^3$. Find $\left(\frac{f}{g}\right)(x)$. Then evaluate the quotient when $x = 5$.

59. Let $f(x) = 6x^3$ and $g(x) = 8x^3$. Find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

60. Let $f(x) = 2x^2$ and $g(x) = 3x^2$. Find $(f - g)(x)$. Then evaluate the difference when $x = 6$.