FIXED POINT ITERATION

The idea of the fixed point iteration methods is to first reformulate a equation to an equivalent fixed point problem:

$$f(x) = 0 \quad \Longleftrightarrow \quad x = g(x)$$

and then to use the iteration: with an initial guess x_0 chosen, compute a sequence

$$x_{n+1} = g(x_n), \quad n \geq 0$$

in the hope that $x_n \to \alpha$.

There are infinite many ways to introduce an equivalent fixed point problem for a given equation; e.g., for any function $G(t)$ with the property

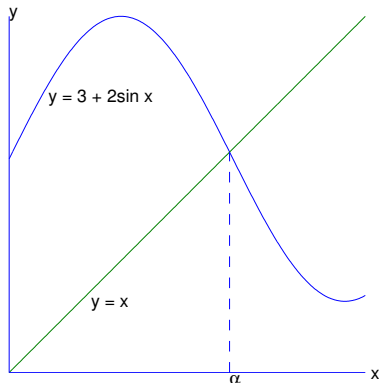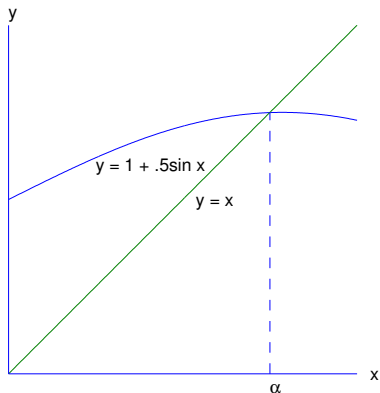$$G(t) = 0 \quad \Longleftrightarrow \quad t = 0,$$

we can take $g(x) = x + G(f(x))$. The resulting iteration method may or may not converge, though.

Example

We begin with an example. Consider solving the two equations

$$\text{E1:} \quad x = 1 + .5 \sin x$$
$$\text{E2:} \quad x = 3 + 2 \sin x$$

$$\text{E1:} \quad x = 1 + .5 \sin x$$
$$\text{E2:} \quad x = 3 + 2 \sin x$$

The solutions are

$$\text{E1:} \quad \alpha = 1.49870113351785$$
$$\text{E2:} \quad \alpha = 3.09438341304928$$

We are going to use a numerical scheme called 'fixed point iteration'. It amounts to making an initial guess of x_0 and substituting this into the right side of the equation. The resulting value is denoted by x_1; and then the process is repeated, this time substituting x_1 into the right side. This is repeated until convergence occurs or until the iteration is terminated.

$$\text{E1:} \quad x_{n+1} = 1 + .5 \sin x_n$$
$$\text{E2:} \quad x_{n+1} = 3 + 2 \sin x_n$$

for $n = 0, 1, 2, ...$

	E1	*E2*
n	x_n	x_n
0	0.00000000000000	3.00000000000000
1	1.00000000000000	3.28224001611973
2	1.42073549240395	2.71963177181556
3	1.49438099256432	3.81910025488514
4	1.49854088439917	1.74629389651652
5	1.49869535552190	4.96927957214762
6	1.49870092540704	1.06563065299216
7	1.49870112602244	4.75018861639465
8	1.49870113324789	1.00142864236516
9	1.49870113350813	4.68448404916097
10	1.49870113351750	1.00077863465869

We show the results of the first 10 iterations in the table. Clearly convergence is occurring with E1, but not with E2. Why?

Fixed point iteration methods

In general, we are interested in solving the equation

$$x = g(x)$$

by means of fixed point iteration:

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, ...$$

It is called 'fixed point iteration' because the root α of the equation $x - g(x) = 0$ is a fixed point of the function $g(x)$, meaning that α is a number for which $g(\alpha) = \alpha$.
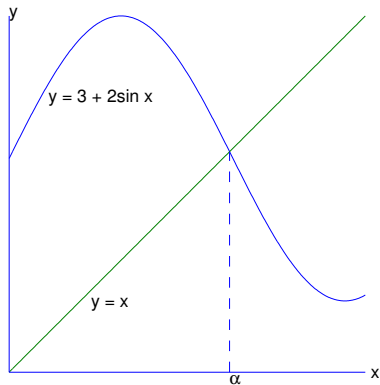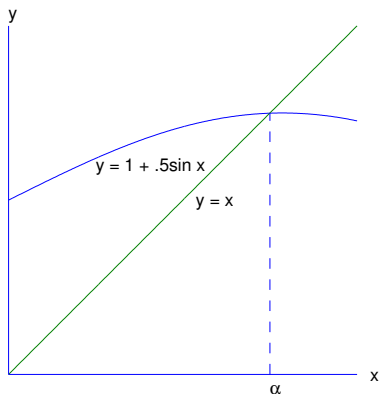
The Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is also an example of fixed point iteration, for the equation

$$x = x - \frac{f(x)}{f'(x)}$$

EXISTENCE THEOREM

We begin by asking whether the equation $x = g(x)$ has a solution. For this to occur, the graphs of $y = x$ and $y = g(x)$ must intersect, as seen on the earlier graphs.

Solution Existence

Lemma: Let $g(x)$ be a continuous function on the interval $[a, b]$, and suppose it satisfies the property

$$a \leq x \leq b \quad \Rightarrow \quad a \leq g(x) \leq b \qquad (\#)$$

Then the equation $x = g(x)$ has at least one solution α in the interval $[a, b]$.

The proof of this is fairly intuitive. Look at the function

$$f(x) = x - g(x), \qquad a \leq x \leq b$$

Evaluating at the endpoints,

$$f(a) \leq 0, \qquad f(b) \geq 0$$

The function $f(x)$ is continuous on $[a, b]$, and therefore it contains a zero in the interval.

Examples

Example 1. Consider the equation

$$x = 1 + 0.5 \sin x.$$

Here

$$g(x) = 1 + 0.5 \sin x.$$

Note that $0.5 \le g(x) \le 1.5$ for any $x \in \mathbb{R}$. Also, $g(x)$ is a continuous function. Applying the existence lemma, we conclude that the equation $x = 1 + 0.5 \sin x$ has a solution in $[a, b]$ with $a \le 0.5$ and $b \ge 1.5$.

Example 2. Similarly, the equation

$$x = 3 + 2 \sin x$$

has a solution in $[a, b]$ with $a \le 1$ and $b \ge 5$.

Theorem

Assume $g(x)$ and $g'(x)$ exist and are continuous on the interval $[a, b]$; and further, assume

$$a \leq x \leq b \quad \Rightarrow \quad a \leq g(x) \leq b$$

$$\lambda \equiv \max_{a \leq x \leq b} \left| g'(x) \right| < 1$$

Then:

S1. The equation $x = g(x)$ has a unique solution α in $[a, b]$.

S2. For any initial guess x_0 in $[a, b]$, the iteration

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots$$

will converge to α.

S3.

$$|\alpha - x_n| \leq \frac{\lambda^n}{1 - \lambda} |x_1 - x_0|, \quad n \geq 0$$

S4.

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

Thus for x_n close to α, $\alpha - x_{n+1} \approx g'(\alpha)(\alpha - x_n)$.

The following general result is useful in the proof. For any two points w and z in $[a, b]$,

$$g(w) - g(z) = g'(c)(w - z)$$

for some unknown point c between w and z. Therefore,

$$|g(w) - g(z)| \leq \lambda |w - z| \qquad (\textcircled{0})$$

for any $a \leq w, z \leq b$.

For S1, suppose there are two solutions α and β:

$$\alpha = g(\alpha), \quad \beta = g(\beta).$$

By $(\textcircled{0})$,

$$|\alpha - \beta| = |g(\alpha) - g(\beta)| \leq \lambda |\alpha - \beta|$$

implying $|\alpha - \beta| = 0$ since $\lambda < 1$.

For S2, note that from $(\#)$, if x_0 is in $[a, b]$, then $x_1 = g(x_0)$ is also in $[a, b]$. Repeat the argument to show that every x_n belongs to $[a, b]$.

Subtract $x_{n+1} = g(x_n)$ from $\alpha = g(\alpha)$ to get

$$\alpha - x_{n+1} = g(\alpha) - g(x_n) = g'(c_n)\,(\alpha - x_n) \qquad (\$)$$
$$|\alpha - x_{n+1}| \le \lambda\,|\alpha - x_n| \qquad (*)$$

with c_n between α and x_n. From $(*)$, we have that the error is guaranteed to decrease by a factor of λ with each iteration. This leads to

$$|\alpha - x_n| \le \lambda^n\,|\alpha - x_0|\,, \qquad n \ge 0 \qquad (\%)$$

Convergence follows from the condition that $\lambda < 1$.

For S3, use (*) for $n = 0$,

$$|\alpha - x_0| \leq |\alpha - x_1| + |x_1 - x_0| \leq \lambda |\alpha - x_0| + |x_1 - x_0|$$
$$|\alpha - x_0| \leq \frac{1}{1 - \lambda} |x_1 - x_0|$$

Combine this with (%) to get the error bound.

For S4, use ($) to write

$$\frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(c_n)$$

Since $x_n \to \alpha$ and c_n is between α and x_n, we have $g'(c_n) \to g'(\alpha)$.

The statement

$$\alpha - x_{n+1} \approx g'(\alpha)\,(\alpha - x_n)$$

tells us that when near to the root α, the errors will decrease by a constant factor of $g'(\alpha)$. If $g'(\alpha)$ is negative, then the errors will oscillate between positive and negative, and the iterates will be approaching from both sides. When $g'(\alpha)$ is positive, the iterates will approach α from only one side.

The statements

$$\alpha - x_{n+1} = g'(c_n)\,(\alpha - x_n)$$

$$\alpha - x_{n+1} \approx g'(\alpha)\,(\alpha - x_n)$$

also tell us a bit more of what happens when

$$\big|g'(\alpha)\big| > 1$$

Then the errors will increase as we approach the root rather than decrease in size.

Look at the earlier example. First consider

$$\text{E1:} \quad x = 1 + .5 \sin x$$

Here

$$g(x) = 1 + .5 \sin x$$

We can take $[a, b]$ with any $a \leq 0.5$ and $b \geq 1.5$. Note that

$$g'(x) = .5 \cos x, \qquad |g'(x)| \leq \frac{1}{2}$$

Therefore, we can apply the theorem and conclude that the fixed point iteration

$$x_{n+1} = 1 + .5 \sin x_n$$

will converge for E1.

Then we consider the second equation

$$\text{E2:} \quad x = 3 + 2\sin x$$

Here

$$g(x) = 3 + 2\sin x$$

Note that

$$g(x) = 3 + 2\sin x, \qquad g'(x) = 2\cos x$$
$$g'(\alpha) = 2\cos(3.09438341304928) \doteq -1.998$$

Therefore the fixed point iteration

$$x_{n+1} = 3 + 2\sin x_n$$

will diverge for E2.

Often, the theorem is not easy to apply directly due to the need to identify an interval $[a, b]$ on which the conditions on g and g' are valid. So we turn to a localized version of the theorem.

Assume $x = g(x)$ has a solution α, both $g(x)$ and $g'(x)$ are continuous for all x in some interval about α, and

$$\left| g'(\alpha) \right| < 1 \qquad (**)$$

Then for any sufficiently small number $\varepsilon > 0$, the interval $[a, b] = [\alpha - \varepsilon, \alpha + \varepsilon]$ will satisfy the hypotheses of the theorem.

This means that if (**) is true, and if we choose x_0 sufficiently close to α, then the fixed point iteration $x_{n+1} = g(x_n)$ will converge and the earlier results S1–S4 will all hold. The result does not tell us how close x_0 needs to be to α in order to have convergence.

NEWTON'S METHOD

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is a fixed point iteration with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Check its convergence by checking the condition (**).

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$g'(\alpha) = 0$$

Therefore the Newton method will converge if x_0 is chosen sufficiently close to α.

HIGHER ORDER METHODS

What happens when $g'(\alpha) = 0$? We use Taylor's theorem to answer this question.

Begin by writing

$$g(x) = g(\alpha) + g'(\alpha)(x - \alpha) + \frac{1}{2}g''(c)(x - \alpha)^2$$

with c between x and α. Substitute $x = x_n$ and recall that $g(x_n) = x_{n+1}$ and $g(\alpha) = \alpha$. Also assume $g'(\alpha) = 0$. Then

$$x_{n+1} = \alpha + \frac{1}{2}g''(c_n)(x_n - \alpha)^2$$

$$\alpha - x_{n+1} = -\frac{1}{2}g''(c_n)(\alpha - x_n)^2$$

with c_n between α and x_n. Thus if $g'(\alpha) = 0$, the fixed point iteration is quadratically convergent or better. In fact, if $g''(\alpha) \neq 0$, then the iteration is exactly quadratically convergent.

ANOTHER RAPID ITERATION

Newton's method is rapid, but requires use of the derivative $f'(x)$. Can we get by without this? The answer is yes! Consider the method

$$D_n = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{D_n}$$

This is an approximation to Newton's method, with $f'(x_n) \approx D_n$. To analyze its convergence, regard it as a fixed point iteration with

$$D(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$$

$$g(x) = x - \frac{f(x)}{D(x)}$$

Then we can show that $g'(\alpha) = 0$ and $g''(\alpha) \neq 0$. So this new iteration is quadratically convergent.

Recall the result

$$\lim_{n \to \infty} \frac{\alpha - x_n}{\alpha - x_{n-1}} = g'(\alpha)$$

for the iteration

$$x_n = g(x_{n-1}), \quad n = 1, 2, \ldots$$

Thus

$$\alpha - x_n \approx \lambda \left(\alpha - x_{n-1} \right) \qquad (***)$$

with $\lambda = g'(\alpha)$ and $|\lambda| < 1$.

If we were to know λ, then we could solve (***) for α:

$$\alpha \approx \frac{x_n - \lambda x_{n-1}}{1 - \lambda}$$

Usually, we write this as a modification of the currently computed iterate x_n:

$$\alpha \approx \frac{x_n - \lambda x_{n-1}}{1 - \lambda}$$
$$= \frac{x_n - \lambda x_n}{1 - \lambda} + \frac{\lambda x_n - \lambda x_{n-1}}{1 - \lambda}$$
$$= x_n + \frac{\lambda}{1 - \lambda} (x_n - x_{n-1})$$

The formula

$$x_n + \frac{\lambda}{1 - \lambda} (x_n - x_{n-1})$$

is said to be an *extrapolation* of the numbers x_{n-1} and x_n. But what is λ?

From

$$\lim_{n \to \infty} \frac{\alpha - x_n}{\alpha - x_{n-1}} = g'(\alpha)$$

we have

$$\lambda \approx \frac{\alpha - x_n}{\alpha - x_{n-1}}$$

Unfortunately this also involves the unknown root α which we seek; and we must find some other way of estimating λ. To calculate λ consider the ratio

$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$$

To see this is approximately λ as x_n approaches α, write

$$\frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} = \frac{g(x_{n-1}) - g(x_{n-2})}{x_{n-1} - x_{n-2}} = g'(c_n)$$

with c_n between x_{n-1} and x_{n-2}. As the iterates approach α, the number c_n must also approach α. Thus λ_n approaches λ as $x_n \to \alpha$.

Combine these results to obtain the estimation

$$\widehat{x}_n = x_n + \frac{\lambda_n}{1 - \lambda_n} \left(x_n - x_{n-1} \right), \qquad \lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$$

We call $\widehat{x}_n$ the *Aitken extrapolate* of $\{x_{n-2}, x_{n-1}, x_n\}$; and $\alpha \approx \widehat{x}_n$.

We can also rewrite this as

$$\alpha - x_n \approx \widehat{x}_n - x_n = \frac{\lambda_n}{1 - \lambda_n} \left(x_n - x_{n-1} \right)$$

This is called *Aitken's error estimation formula*.

The accuracy of these procedures is tied directly to the accuracy of the formulas

$$\alpha - x_n \approx \lambda \left(\alpha - x_{n-1} \right), \qquad \alpha - x_{n-1} \approx \lambda \left(\alpha - x_{n-2} \right)$$

If this is accurate, then so are the above extrapolation and error estimation formulas.

Consider the iteration

$$x_{n+1} = 6.28 + \sin(x_n), \qquad n = 0, 1, 2, ...$$

for solving

$$x = 6.28 + \sin x$$

Iterates are shown on the accompanying sheet, including calculations of λ_n, the error estimate

$$\alpha - x_n \approx \widehat{x}_n - x_n = \frac{\lambda_n}{1 - \lambda_n} (x_n - x_{n-1}) \qquad \text{(Estimate)}$$

The latter is called "Estimate" in the table. In this instance,

$$g'(\alpha) \doteq .9644$$

and therefore the convergence is very slow. This is apparent in the table.

n	x_n	$x_n - x_{n-1}$	λ_n	$\alpha - x_n$	Estimate
0	6.0000000			$1.55E-2$	
1	6.0005845	$5.845E-4$		$1.49E-2$	
2	6.0011458	$5.613E-4$	.9603	$1.44E-2$	$1.36E-2$
3	6.0016848	$5.390E-4$	.9604	$1.38E-2$	$1.31E-2$
4	6.0022026	$5.178E-4$	.9606	$1.33E-2$	$1.26E-2$
5	6.0027001	$4.974E-4$	.9607	$1.28E-2$	$1.22E-2$
6	6.0031780	$4.780E-4$	.9609	$1.23E-2$	$1.17E-2$
7	6.0036374	$4.593E-4$	.9610	$1.18E-2$	$1.13E-2$

AITKEN'S ALGORITHM

Step 1: Select x_0

Step 2: Calculate

$$x_1 = g(x_0), \quad x_2 = g(x_1)$$

Step3: Calculate

$$x_3 = x_2 + \frac{\lambda_2}{1 - \lambda_2}(x_2 - x_1), \qquad \lambda_2 = \frac{x_2 - x_1}{x_1 - x_0}$$

Step 4: Calculate

$$x_4 = g(x_3), \quad x_5 = g(x_4)$$

and calculate x_6 as the extrapolate of $\{x_3, x_4, x_5\}$. Continue this procedure, ad infinatum.

Of course in practice we will have some kind of error test to stop this procedure when believe we have sufficient accuracy.

EXAMPLE

Consider again the iteration

$$x_{n+1} = 6.28 + \sin(x_n), \qquad n = 0, 1, 2, ...$$

for solving

$$x = 6.28 + \sin x$$

Now we use the Aitken method, and the results are shown in the accompanying table. With this we have

$$\alpha - x_3 \doteq 7.98 \times 10^{-4}, \quad \alpha - x_6 \doteq 2.27 \times 10^{-6}$$

In comparison, the original iteration had

$$\alpha - x_6 \doteq 1.23 \times 10^{-2}$$

GENERAL COMMENTS

Aitken extrapolation can greatly accelerate the convergence of a linearly convergent iteration

$$x_{n+1} = g(x_n)$$

This shows the power of understanding the behaviour of the error in a numerical process. From that understanding, we can often improve the accuracy, thru extrapolation or some other procedure.

This is a justification for using mathematical analysis to understand numerical methods. We will see this repeated at later points in the course, and it holds with many different types of problems and numerical methods for their solution.