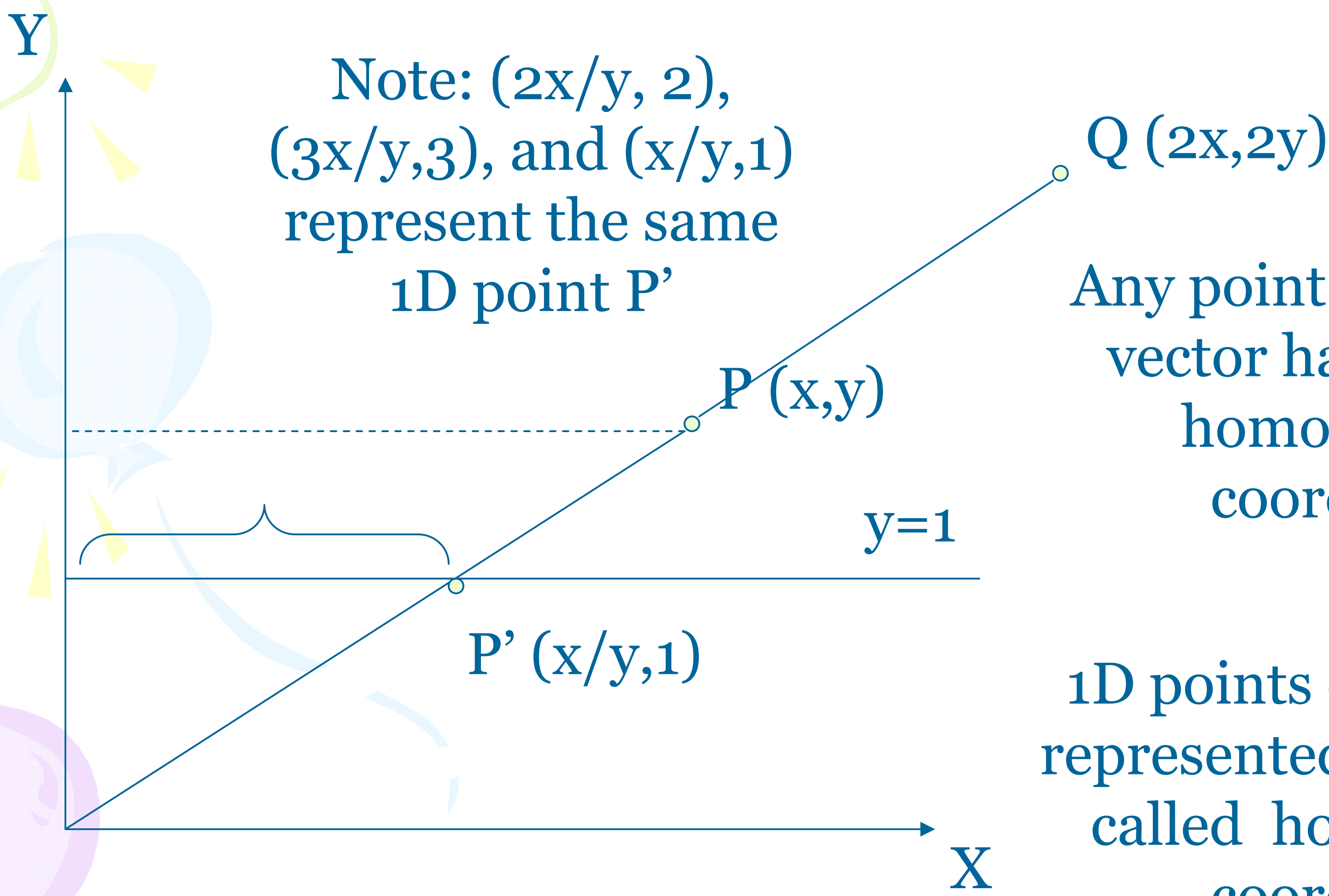
Geometric Transformation

CS 211A

What is transformation?

- Moving points
- (x,y) moves to (x+t, y+t)
- Can be in any dimension
  - 2D – Image warps
  - 3D – 3D Graphics and Vision
- Can also be considered as a movement to the coordinate axes

Homogeneous Coordinates

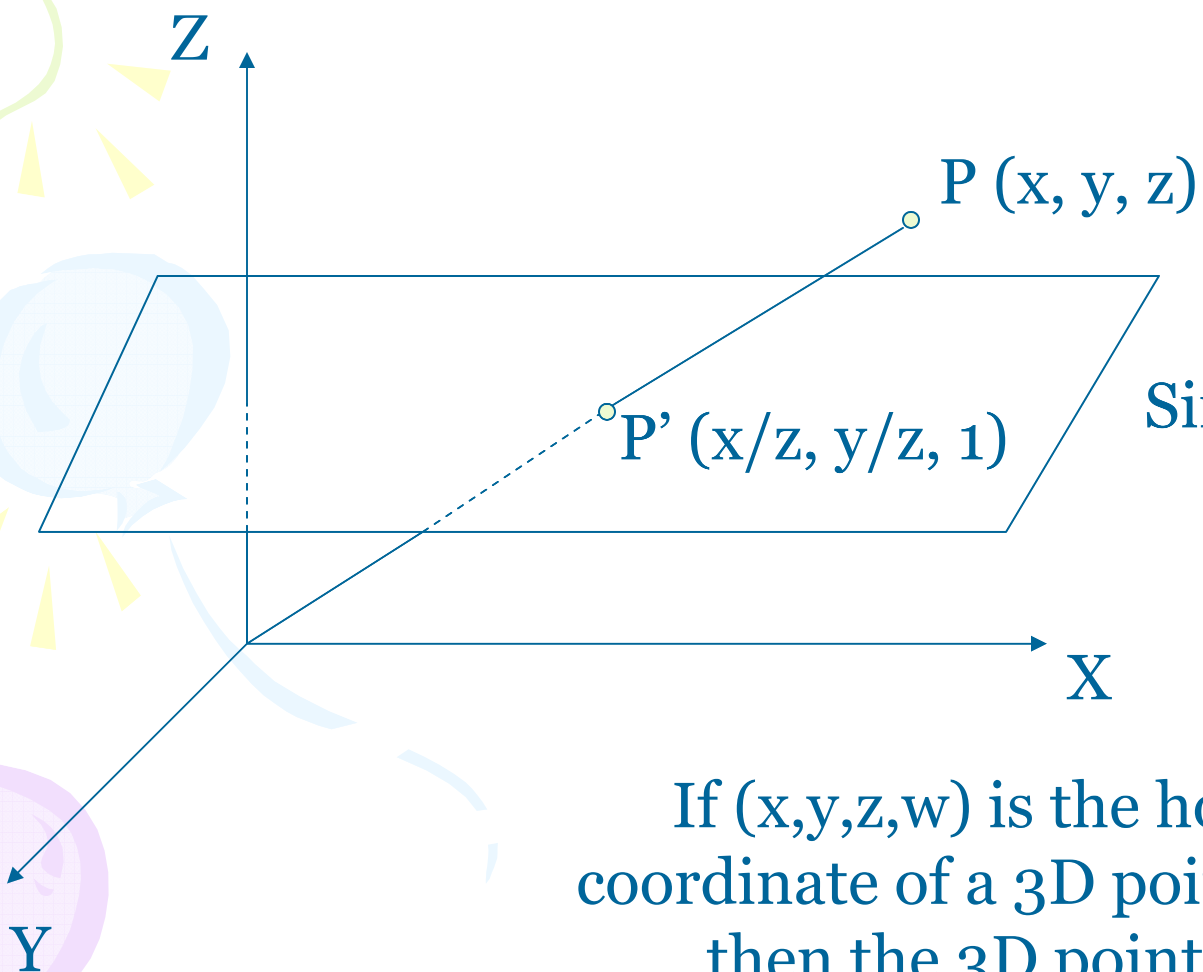


Note: (2x/y, 2), (3x/y,3), and (x/y,1) represent the same 1D point P'

Any point on the same vector has the same homogeneous coordinates

1D points on the line is represented by 2D array, called homogeneous coordinates

Generalize to Higher Dimensions



2D points represented by homogeneous coordinates

Similarly, 3D points are represented by homogeneous coordinates

If (x,y,z,w) is the homogeneous coordinate of a 3D point, where $w \neq 1$, then the 3D point is given by (x/w,y/w,z/w,1)

Practically

- [x y z w], $w \neq 1$
- Then, [x/w, y/w, z/w, 1]
- Try to put it the w=1 hyperplane
- Why?
  - Can represent pts at infinity
  - 2D – [∞, ∞]
  - 3D homogeneous – [2, 3, 0]
    - Point at infinity in the direction of [2, 3]
  - Distinguish between points and vectors
    - [2, 3, 1] vs [2, 3, 0]

Linear Transformation

- **L**(ap+bq) = a**L**(p) + b**L**(q)
- Lines/planes transform to lines/planes
- If transformation of vertices are known, transformation of linear combination of vertices can be achieved
- p and q are points or vectors in (n+1)x1 homogeneous coordinates
  - For 2D, 3x1 homogeneous coordinates
  - For 3D, 4x1 homogeneous coordinates
- **L** is a (n+1)x(n+1) square matrix
  - For 2D, 3x3 matrix
  - For 3D, 4x4 matrix

Linear Transformations

- Euclidian
  - Length and angles are preserved
- Affine
  - Ratios of lengths and angles are preserved
- Projective
  - Can move points at infinity in range and finite points to infinity

Euclidian Transformations

- Lengths and angles are preserved
  - Translation
  - Rotation

2D Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Line between two points is transformed to a line between the transformed points

3D Translation

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$P' = \mathbf{T}P$$

$x' = x + t_x$
$y' = y + t_y$
$z' = z + t_z$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denoted by $T(t_x, t_y, t_z)$

Inverse Translation

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$P = \mathbf{T}^{-1}P'$$

$$x = x' - t_x$$
$$y = y' - t_y$$
$$z = z' - t_z$$

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = T(-t_x, -t_y, -t_z)$$

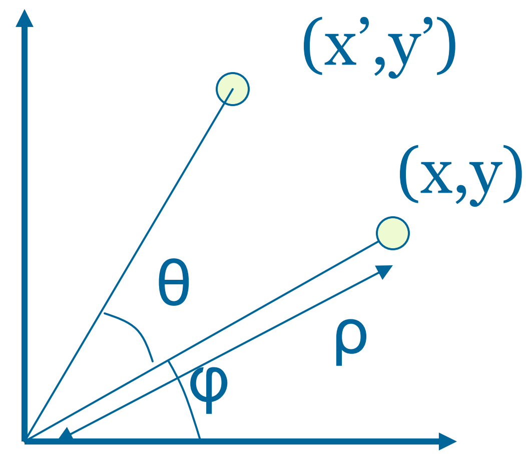
2D Rotation

$x = \rho \cos \varphi$

$y = \rho \sin \varphi$

$x' = \rho \cos(\theta + \varphi)$

$y' = \rho \sin(\theta + \varphi)$



$x' = \rho \cos\theta \cos\varphi - \rho \sin\theta \sin\varphi = x \cos\theta - y \sin\theta$

$y' = \rho \sin\theta \cos\varphi + \rho \cos\theta \sin\varphi = x \sin\theta + y \cos\theta$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{R} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

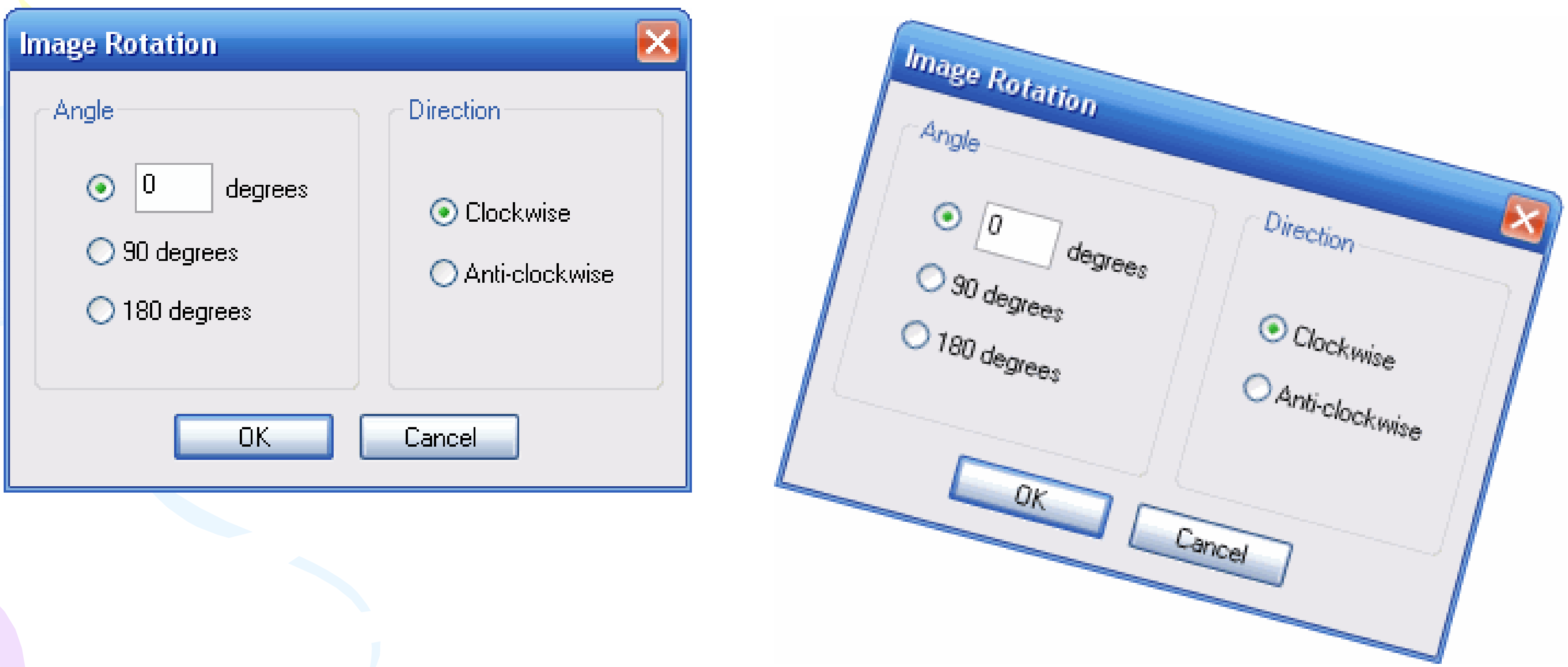


Image Rotation
Angle
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Image Rotation
Angle
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Anti-clockwise
OK
Cancel

Rotation in 3D about z axis

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

$z' = z$

$P' = \mathbf{R}_z P$

$$\mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denoted by $\mathbf{R}(\theta)$

$\mathbf{R}^{-1} = \mathbf{R}(-\theta) = \mathbf{R}^T(\theta)$

Where $\mathbf{R} = \mathbf{R}_x$ or $\mathbf{R}_y$ or $\mathbf{R}_z$

Affine Transformations

- Ratio of lengths and angles are preserved
  - Scale
    - Lengths are not preserved
  - Shear
    - Angles are not preserved

Scaling

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$P' = \mathbf{S}P$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x' = s_x\, x$

$y' = s_y\, y$

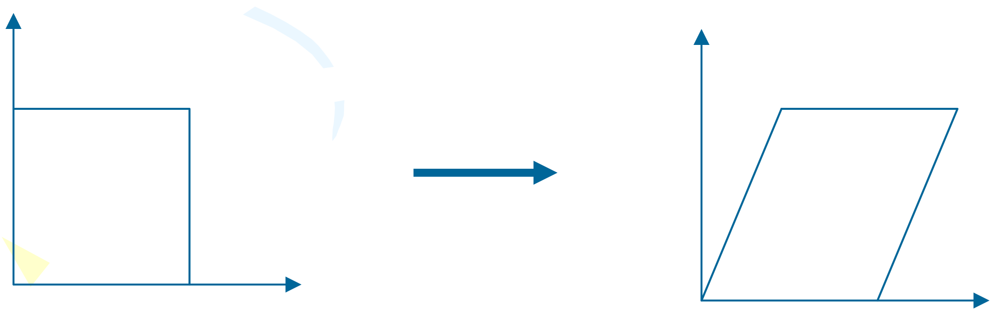
$z' = s_z\, z$

Denoted by $\mathbf{S}(s_x, s_y, s_z)$

$$\mathbf{S}^{-1} = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$$

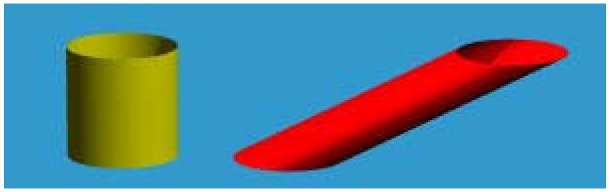
Shear

- Translation of one coordinate of a point is proportional to the 'value' of the other coordinate of the same point.
  - Point : (x,y)
  - After 'y-shear': (x+ay,y)
  - After 'x-shear': (x,y+bx)
- Changes the shape of the object.



Using matrix for Shear

- Example: Z-shear (Z coordinate does not change)

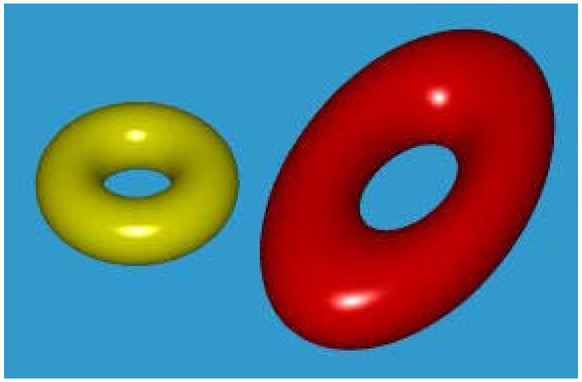


$$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+az \\ y+bz \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

General Affine Transformation

- The last row is fixed
- Has 12 degrees of freedom
- Does not change degrees of polynomial
- Parallel and intersecting lines/planes to the same

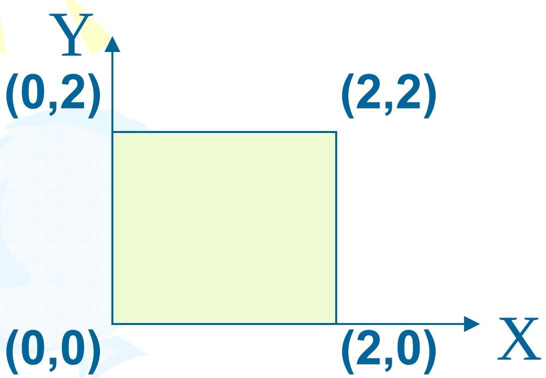
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

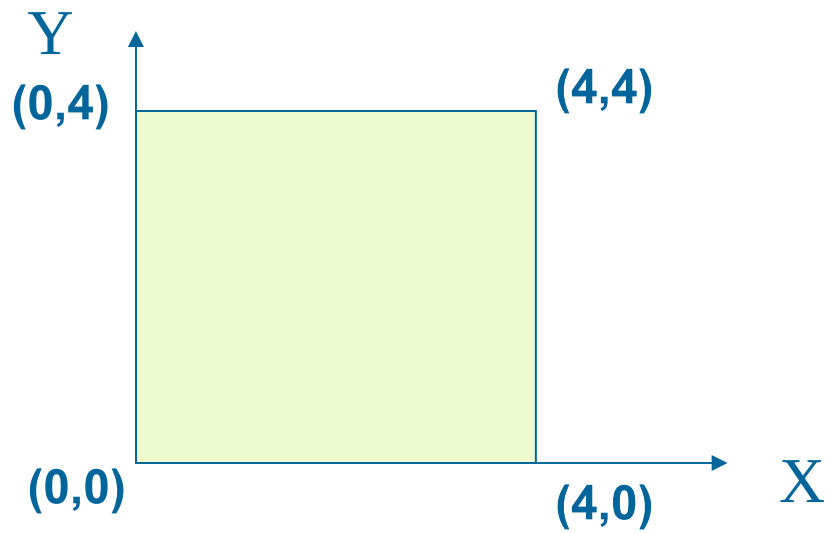


Fixed Points and Lines

- Some points and lines can be fixed under a transformation
- Scaling – Point
  - Origin
- Rotation – Line
  - Axis of rotation

Scaling About a point



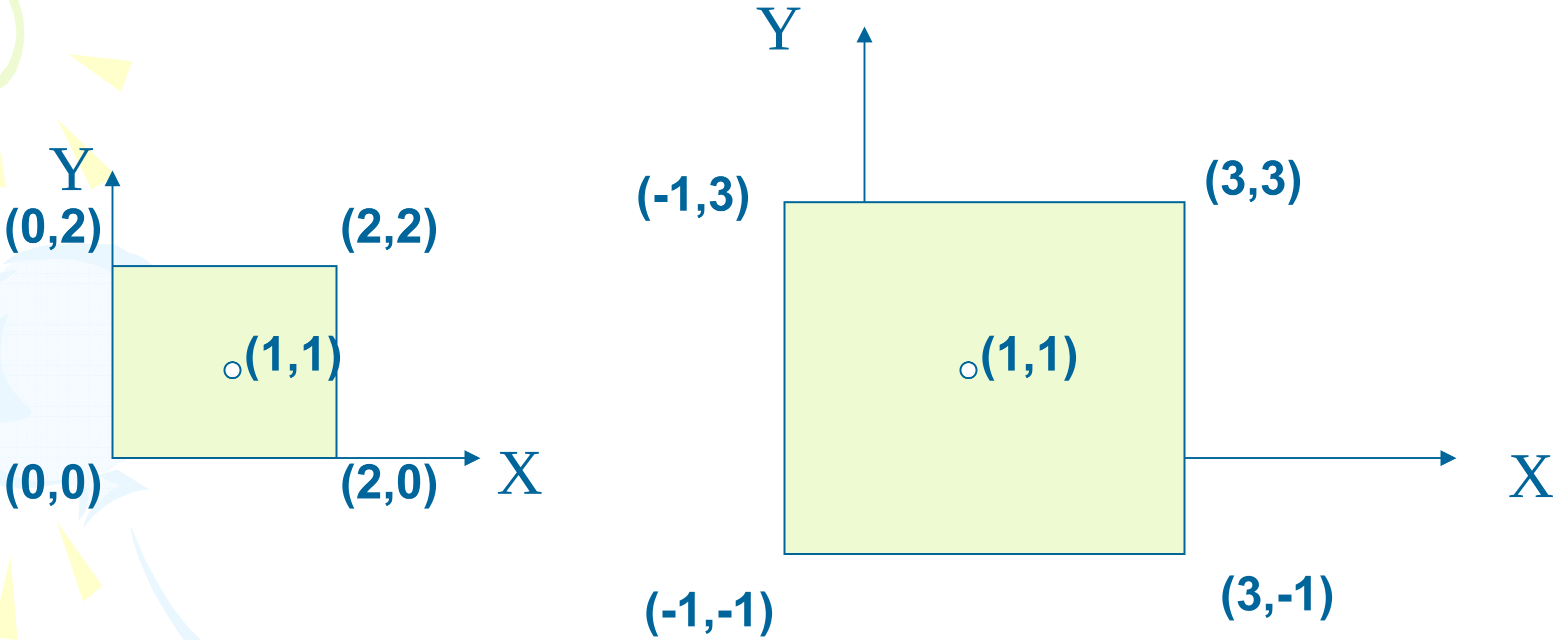


Origin is fixed with transformation -> Scaling about origin

Concatenation of Transformations

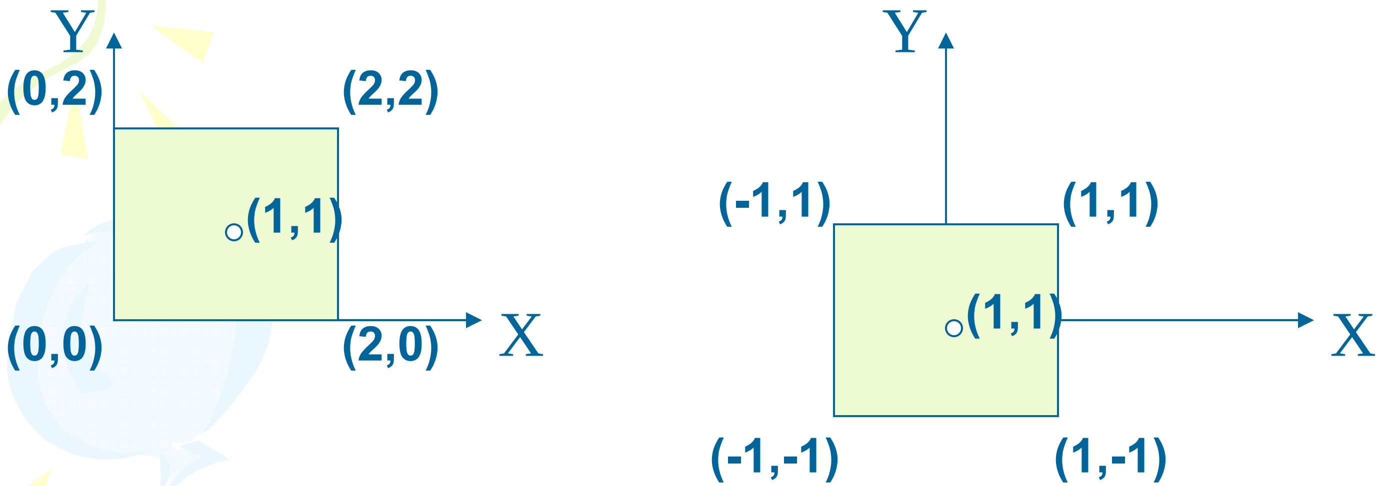
- How do we use multiple transformation?
- Apply **F**
  - Get to a known situation
- Apply the required **L**
- Apply $\mathbf{F}^{-1}$

Scaling About a point



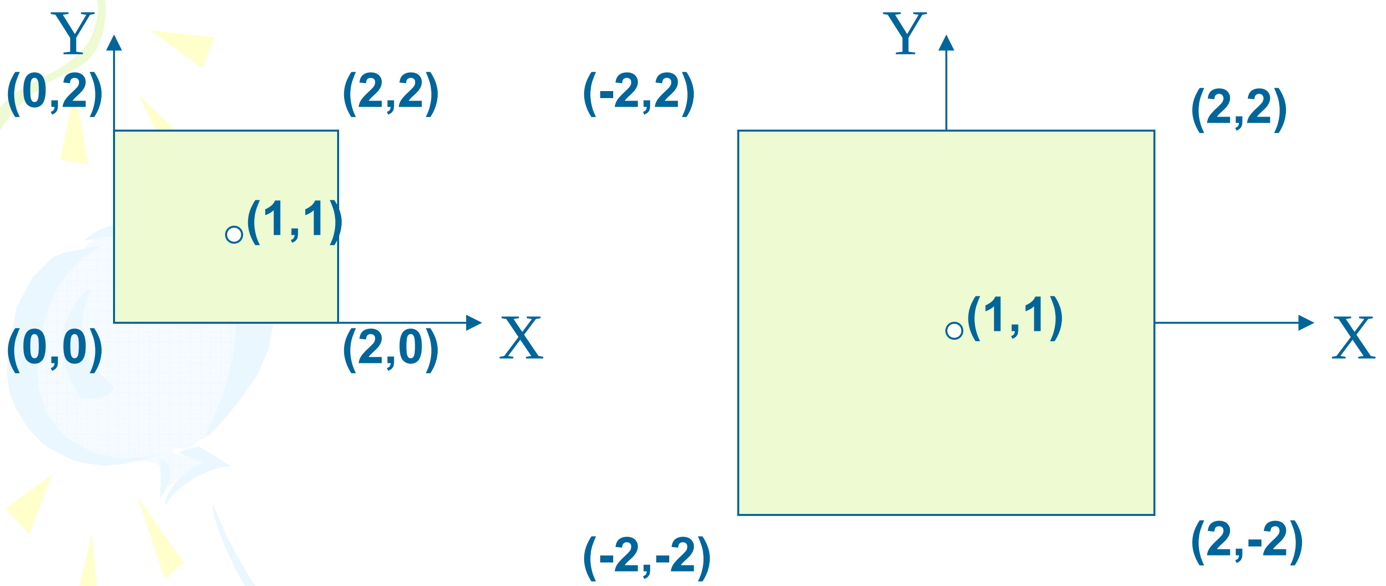
Scaling about center -> Center is fixed with transformation

Done by concatenation



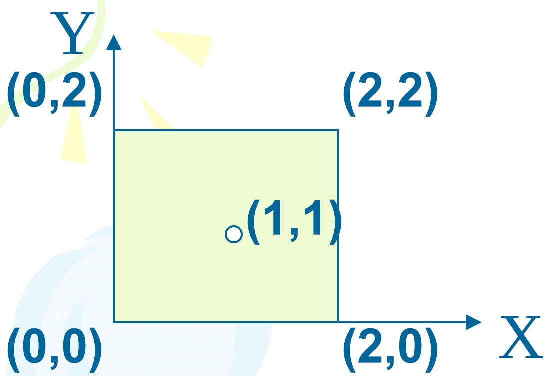
Translate so that center coincides with origin - T(-1,-1).

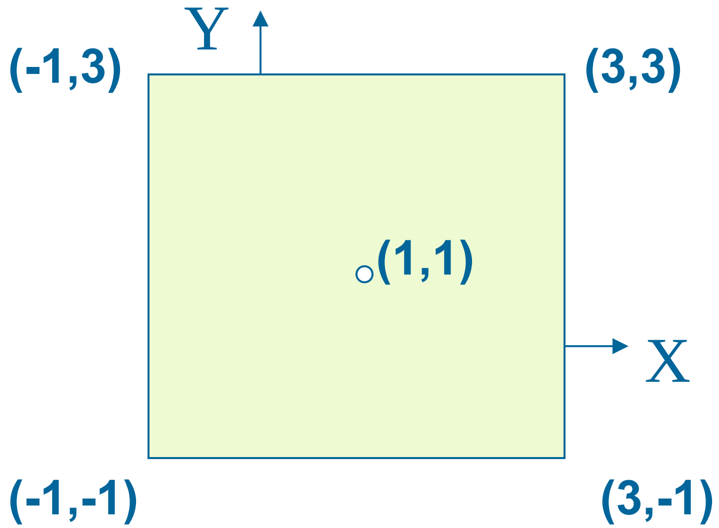
Done by concatenation



Scale the points about the center – S(2,2)

Done by concatenation





Translate it back by reverse parameters – T(1,1)

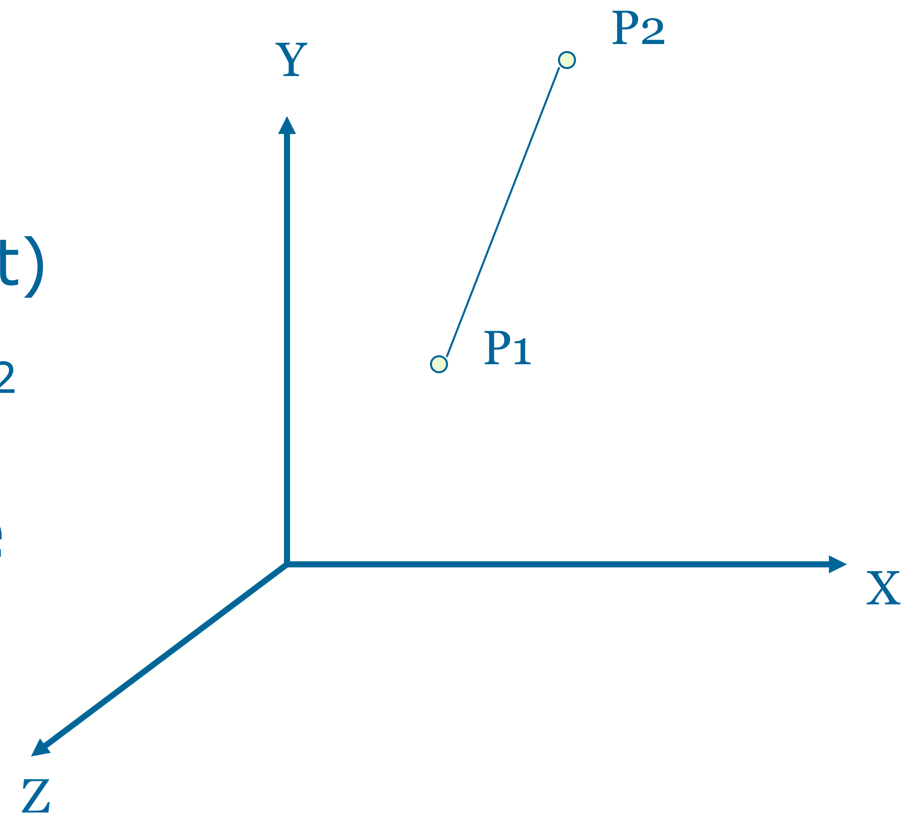
Total Transformation: T(1,1) S(2,2) T(-1,-1) P

Rotation about a fixed point

- z-axis rotation of θ about its center P_f
- Translate by $-P_f$: $T(-P_f)$
- Rotate about z-axis : $R_z(\theta)$
- Translate back by P_f : $T(P_f)$
- Total Transformation $M = T(P_f)R_z(\theta)T(-P_f)$

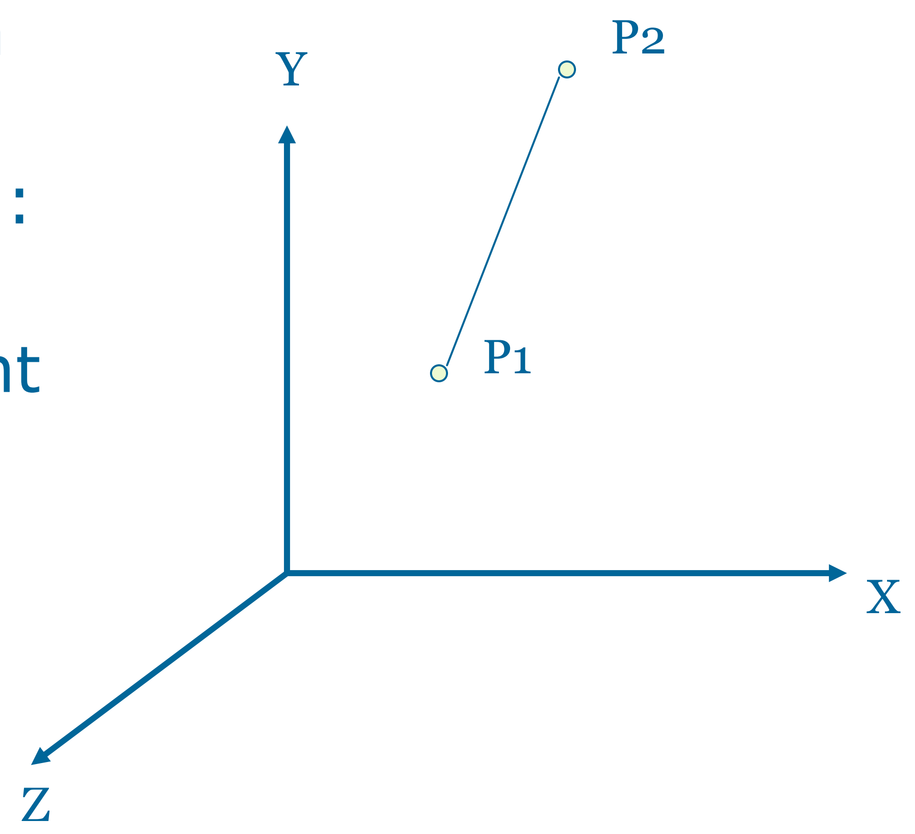
Rotation About an Arbitrary Axis

- Axis given by two points
  - P_1 (starting point) and P_2 (ending point)
  - P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2)
- Anticlockwise angle of rotation is θ
- Rotate all points to around P_1P_2 by θ



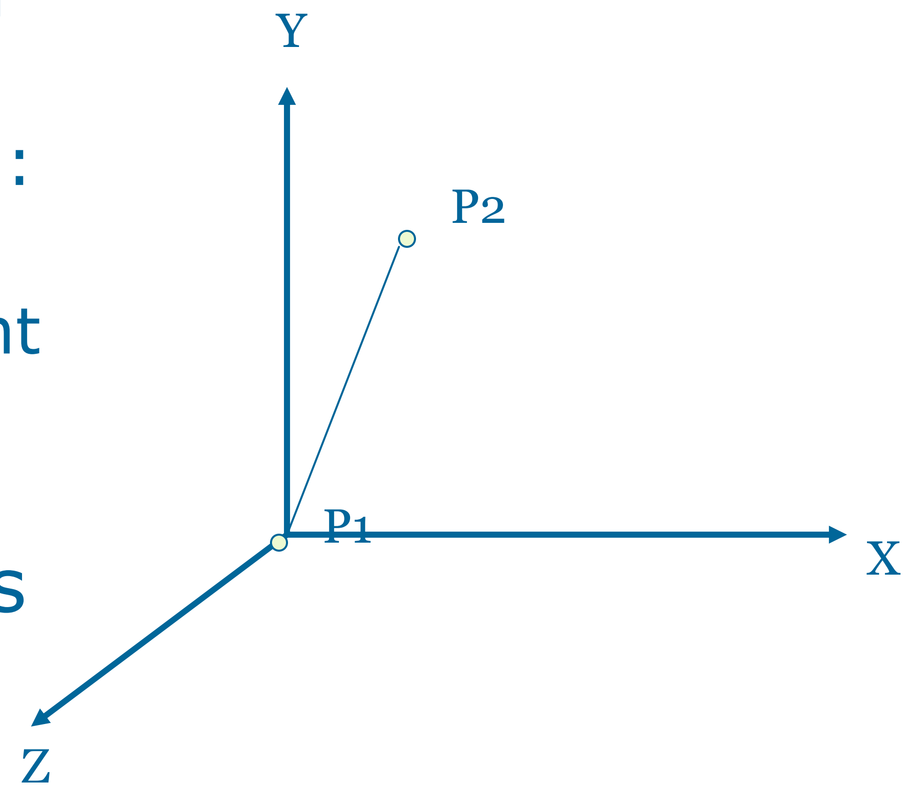
Rotation about an Arbitrary Axis

- Make P_1P_2 coincide with Z-axis
  - Translate P_1 to origin: $T(-x_1,-y_1,-z_1)$
    - Coincides one point of the axis with origin



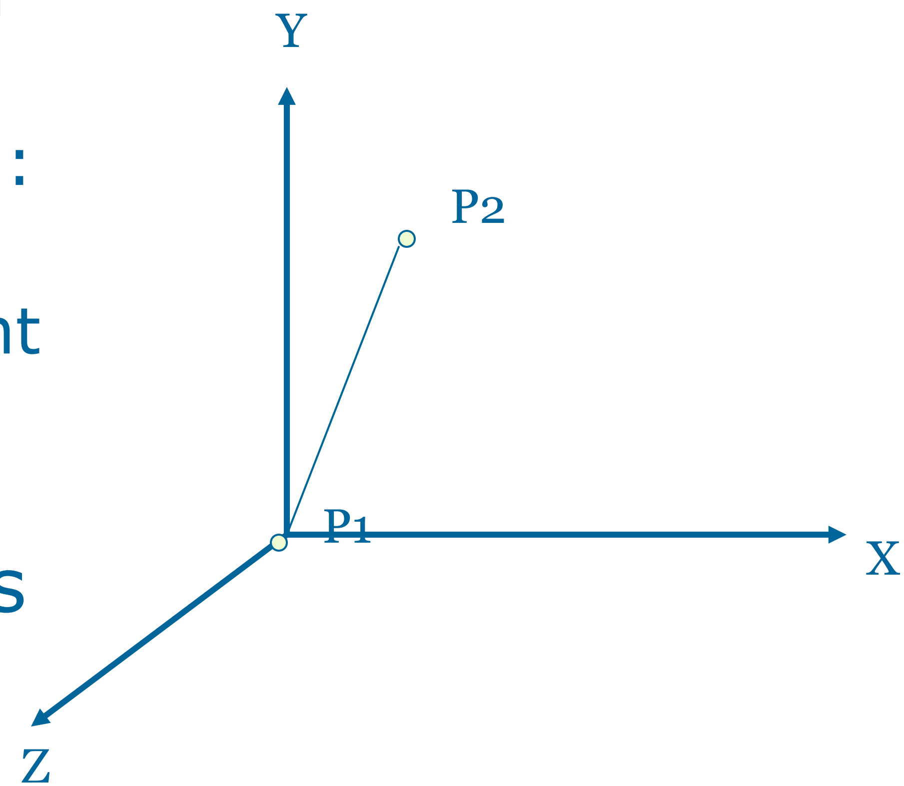
Rotation about an Arbitrary Axis

- Make P_1P_2 coincide with Z-axis
  - Translate P_1 to origin: $T(-x_1,-y_1,-z_1)$
    - Coincides one point of the axis with origin
  - Rotate shifted axis to coincide with Z axis



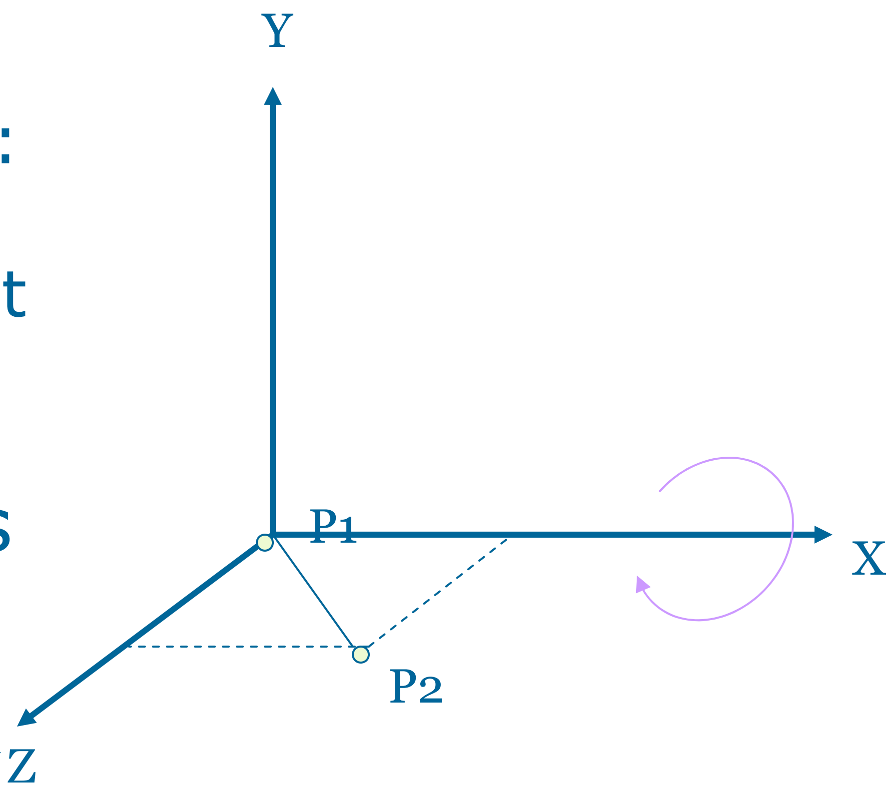
Rotation about an Arbitrary Axis

- Make P_1P_2 coincide with Z-axis
  - Translate P_1 to origin: $T(-x_1,-y_1,-z_1)$
    - Coincides one point of the axis with origin
  - Rotate shifted axis to coincide with Z axis
    - R_1: Rotate about X to lie on XZ plane



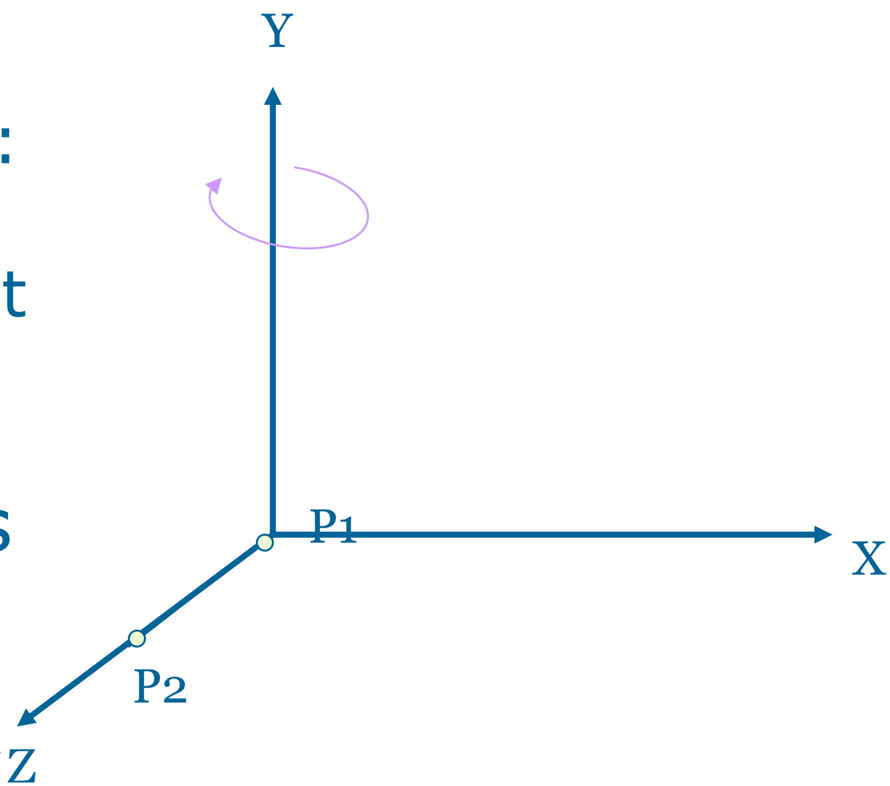
Rotation about an Arbitrary Axis

- Make P_1P_2 coincide with Z-axis
  - Translate P_1 to origin: $T(-x_1,-y_1,-z_1)$
    - Coincides one point of the axis with origin
  - Rotate shifted axis to coincide with Z axis
    - R_1: Rotate about X to lie on XZ plane
    - R_2: Rotate about Y to lie on Z axis



Rotation about an Arbitrary Axis

- Make P_1P_2 coincide with Z-axis
  - Translate P_1 to origin: $T(-x_1,-y_1,-z_1)$
    - Coincides one point of the axis with origin
  - Rotate shifted axis to coincide with Z axis
    - R_1: Rotate about X to lie on XZ plane
    - R_2: Rotate about Y to lie on Z axis

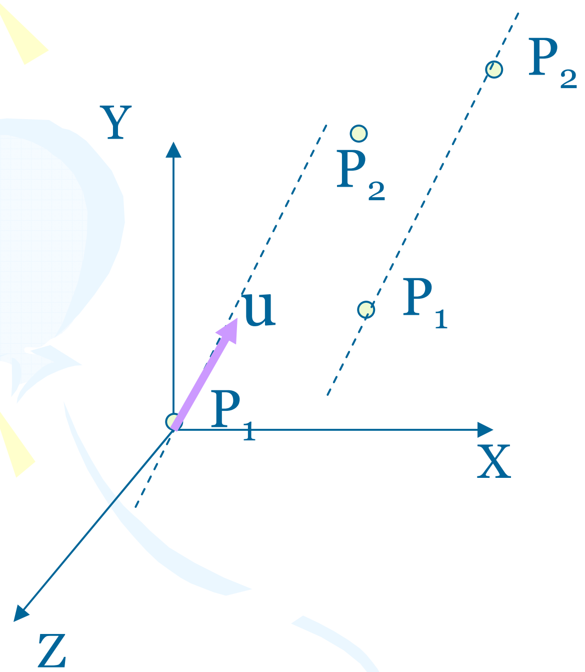


Rotation about an Arbitrary Axis

- Make the axis P_1P_2 coincide with the Z-axis
  - Translation to move P_1 to the origin: $T(-x_1,-y_1,-z_1)$
    - Coincides one point of the axis with origin
  - Rotation to coincide the shifted axis with Z axis
    - R_1: Rotation around X such that the axis lies on the XZ plane.
    - R_2: Rotation around Y such that the axis coincides with the Z axis
- R_3: Rotate the scene around the Z axis by an angle θ
- Inverse transformations of R_2, R_1 and T_1 to bring back the axis to the original position
- $M = T^{-1} R_1^{-1} R_2^{-1} R_3 R_2 R_1 T$

Translation

- After translation

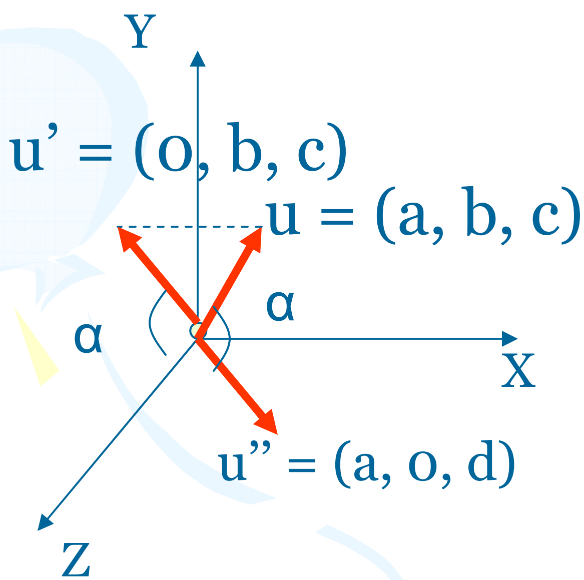


$$\text{Axis } V = P_2 - P_1$$
$$= (x_1\text{-}x_2, y_1\text{-}y_2, z_1\text{-}z_2)$$

$$u = \frac{V}{|V|} = (a, b, c)$$

Rotation about X axis

- Rotate u about X so that it coincides with XZ plane



Project u on YZ plane : u' (o, b, c)

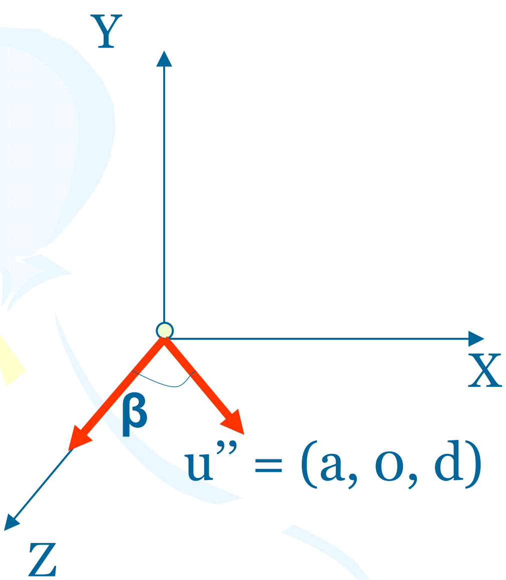
α is the angle made by u' with Z axis

$\cos\alpha = c/\sqrt{b^2+c^2} = c/d$
$\sin\alpha = b/d$

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Y axis

- Rotate u'' about Y so that it coincides with Z axis



$\text{Cos}\,\beta = d/\sqrt{a^2+d^2} = d/\sqrt{a^2+b^2+c^2} = d$

$\text{Sin}\,\beta = a$

$$\mathbf{R}_2 = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Z axis

- Rotate by θ about Z axis

$$\mathbf{R}_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

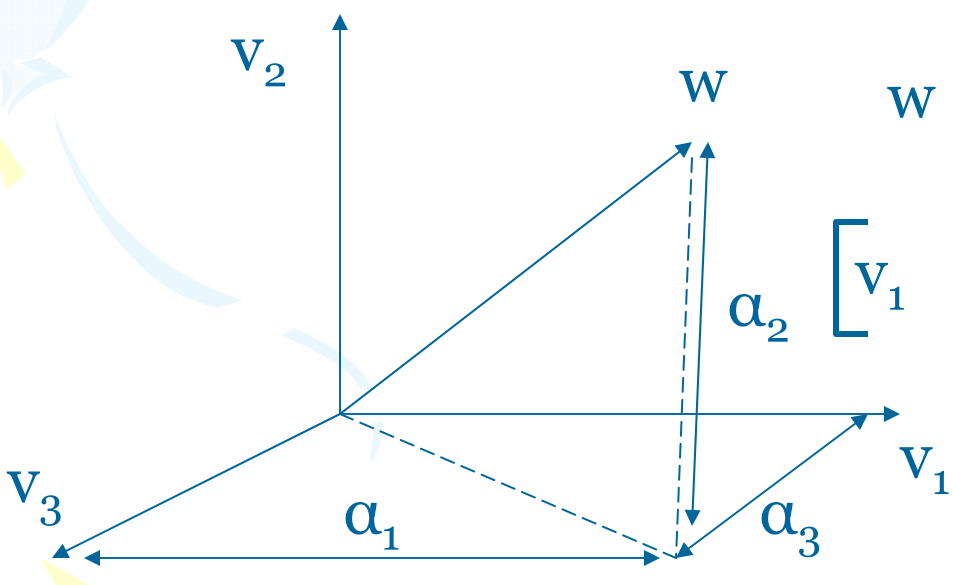
- $M = T^{-1} R_1^{-1} R_2^{-1} R_3(\theta) R_2(\beta) R_1(\alpha) T$

  $= T^{-1} R_x^{-1} R_y^{-1} R_z(\theta) R_y(\beta) R_x(\alpha) T$

  $= T^{-1} R_x(-\alpha) R_y(-\beta) R_z(\theta) R_y(\beta) R_x(\alpha) T$

Coordinate Systems

- Represent a point as a linear combination of three vectors and the origin
- Linearly independent vectors – *basis*
  - Orthogonal vectors are linearly independent



$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + R$$

$$\begin{bmatrix} v_1 & v_2 & v_3 & R \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix} = w$$

Coordinate Systems

- Represent a point as a linear combination of three vectors and the origin
- Linearly independent vectors – *basis*
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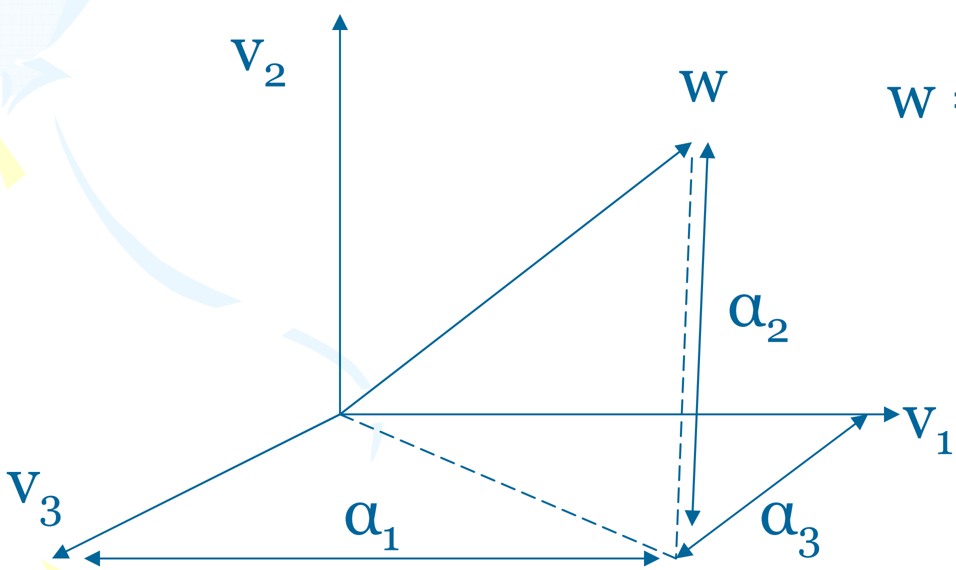
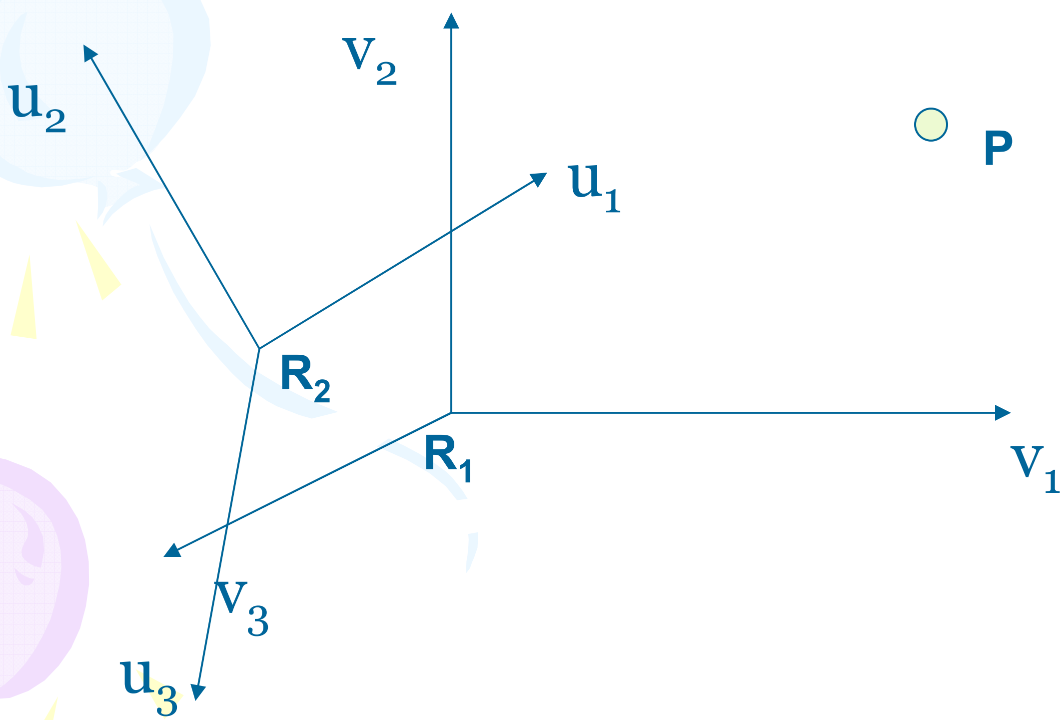


$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + R$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix} = w$$

Change of Coordinates

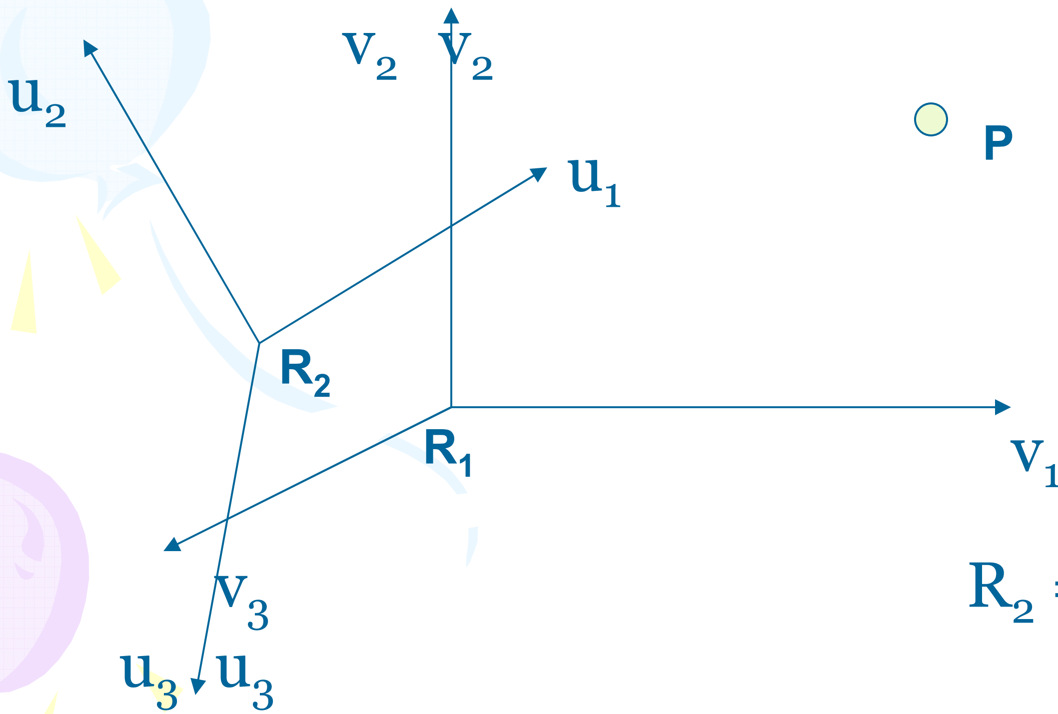
- First coordinate – v_1, v_2, v_3, R_1
- Second coordinate – u_1, u_2, u_3, R_2



$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix} = C_1$$

Change of Coordinates

- First coordinate – v_1, v_2, v_3, R_1
- Second coordinate – u_1, u_2, u_3, R_2



$$u_1 = \gamma_{11}v_1 + \gamma_{21}v_2 + \gamma_{31}v_3$$

$$u_2 = \gamma_{12}v_1 + \gamma_{22}v_2 + \gamma_{32}v_3$$

$$u_3 = \gamma_{13}v_1 + \gamma_{23}v_2 + \gamma_{33}v_3$$

$$R_2 = \gamma_{14}v_1 + \gamma_{24}v_2 + \gamma_{34}v_3 + R_1$$

Change of Coordinates

$$u_1 = \gamma_{11}v_1 + \gamma_{21}v_2 + \gamma_{31}v_3$$

$$u_2 = \gamma_{12}v_1 + \gamma_{22}v_2 + \gamma_{32}v_3$$

$$u_3 = \gamma_{13}v_1 + \gamma_{23}v_2 + \gamma_{33}v_3$$

$$R_2 = \gamma_{14}v_1 + \gamma_{24}v_2 + \gamma_{34}v_3 + R_1$$

$$\begin{bmatrix} u_1 & u_2 & u_3 & R_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & R_1 \end{bmatrix} \mathbf{M} \qquad \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Change of Coordinates

If $R_1 = R_2$,

then $\gamma_{14} = \gamma_{24} = \gamma_{34} = 0$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Change of Coordinates

$$P = \begin{bmatrix} u_1 & u_2 & u_3 & R_2 \end{bmatrix} C_2$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & R_1 \end{bmatrix} M\, C_2$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & R_1 \end{bmatrix} C_1$$

Hence, $C_1 = MC_2$

What is this matrix?

- You need translate
  - Coincide origins
- You need to rotate
  - To make the axis match
- This is a rotation and translation concatenated

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is this concatenation?

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

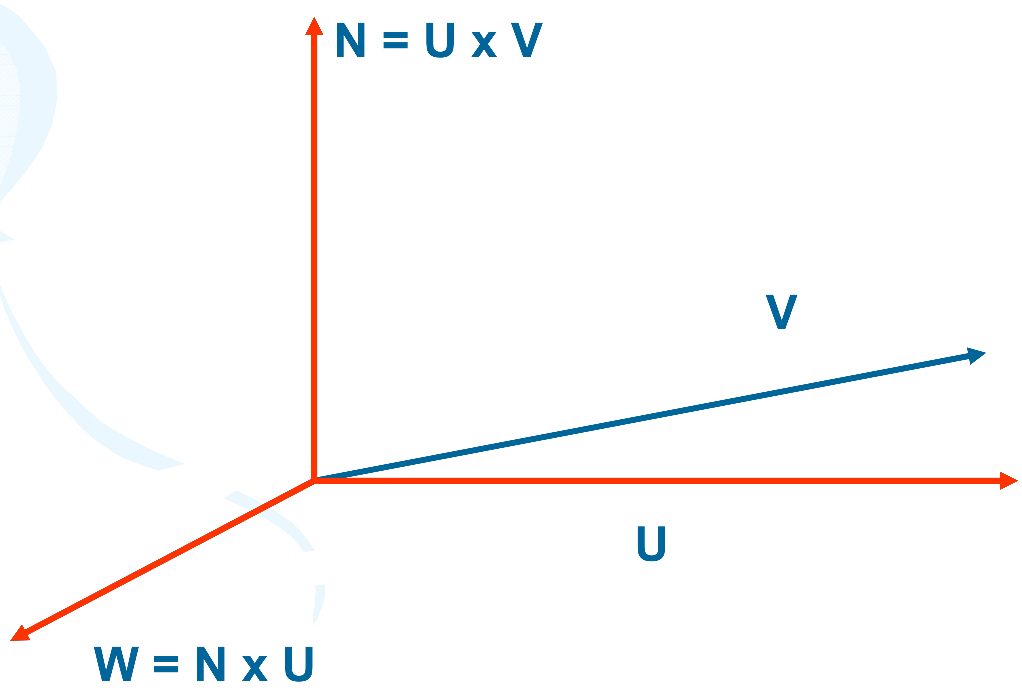
X

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t'_x \\ r_{21} & r_{22} & r_{23} & t'_y \\ r_{31} & r_{32} & r_{33} & t'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = [R \mid RT]$$

How to simplify rotation about arbitrary axis?

- Translation goes in the last column
- The rotation matrix defined by finding a new coordinate with the arbitrary axis as a X, Y or Z axis

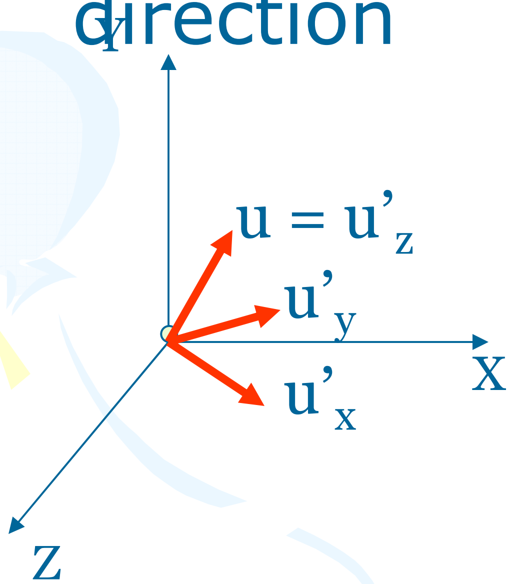
How to find coordinate axes?



Faster Way

- Faster way to find R_2R_1
  - u_x, u_y, u_z are unit vectors in the X, Y, Z direction

Set up a coordinate system where $u = u'_z$



$$u'_z = \frac{u}{|u|}$$

$$u'_y = \frac{u \times u_x}{|u \times u_x|}$$

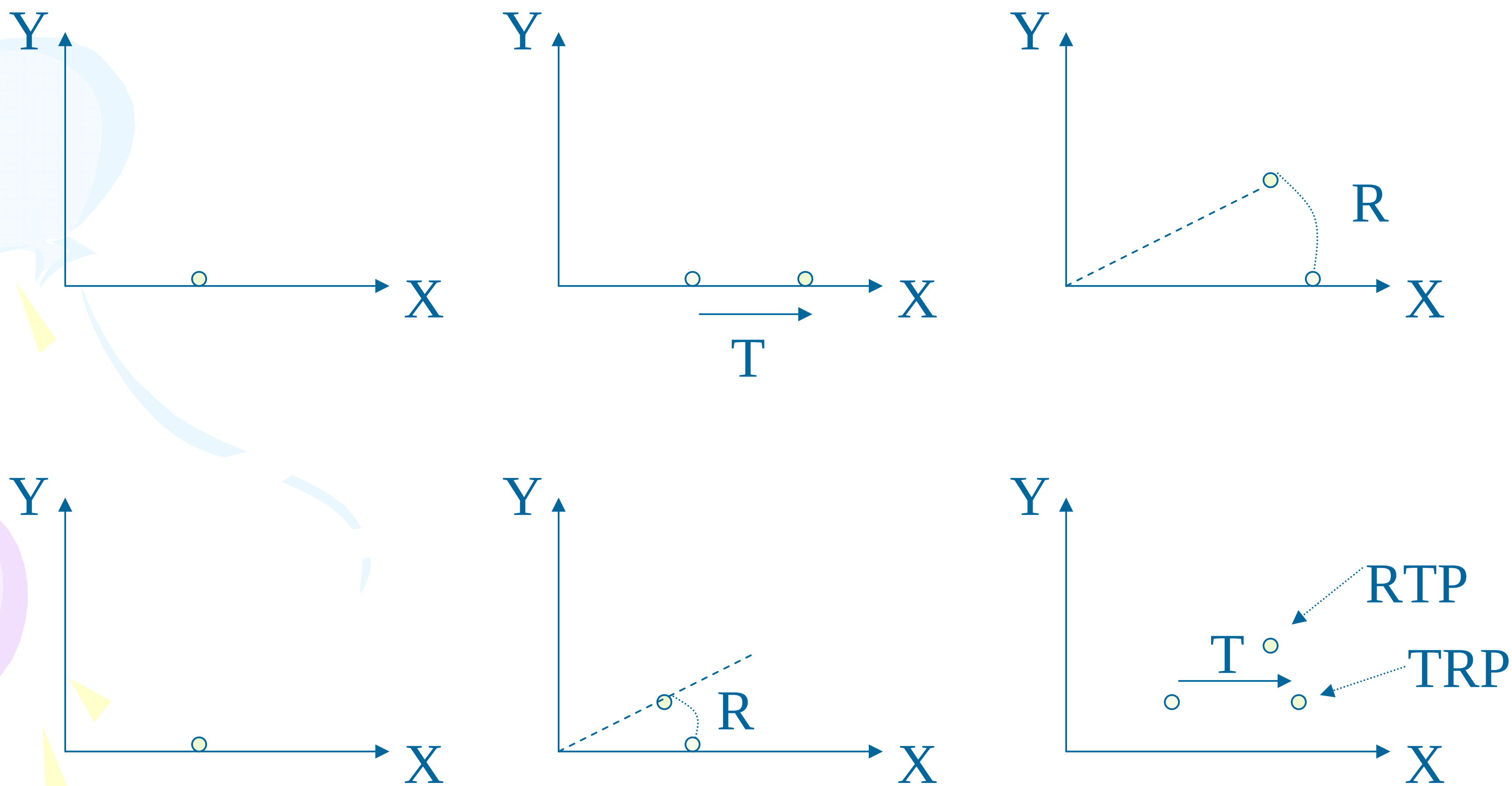
$$u'_x = u'_y \times u'_z$$

$$R = R_2R_1 = \begin{bmatrix} u'_{x1} & u'_{x2} & u'_{x3} & 0 \\ u'_{y1} & u'_{y2} & u'_{y3} & 0 \\ u'_{z1} & u'_{z2} & u'_{z3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1^{-1}R_2^{-1} = R^{-1}$$

Properties of Concatenation

- Not commutative



Properties of Concatenation

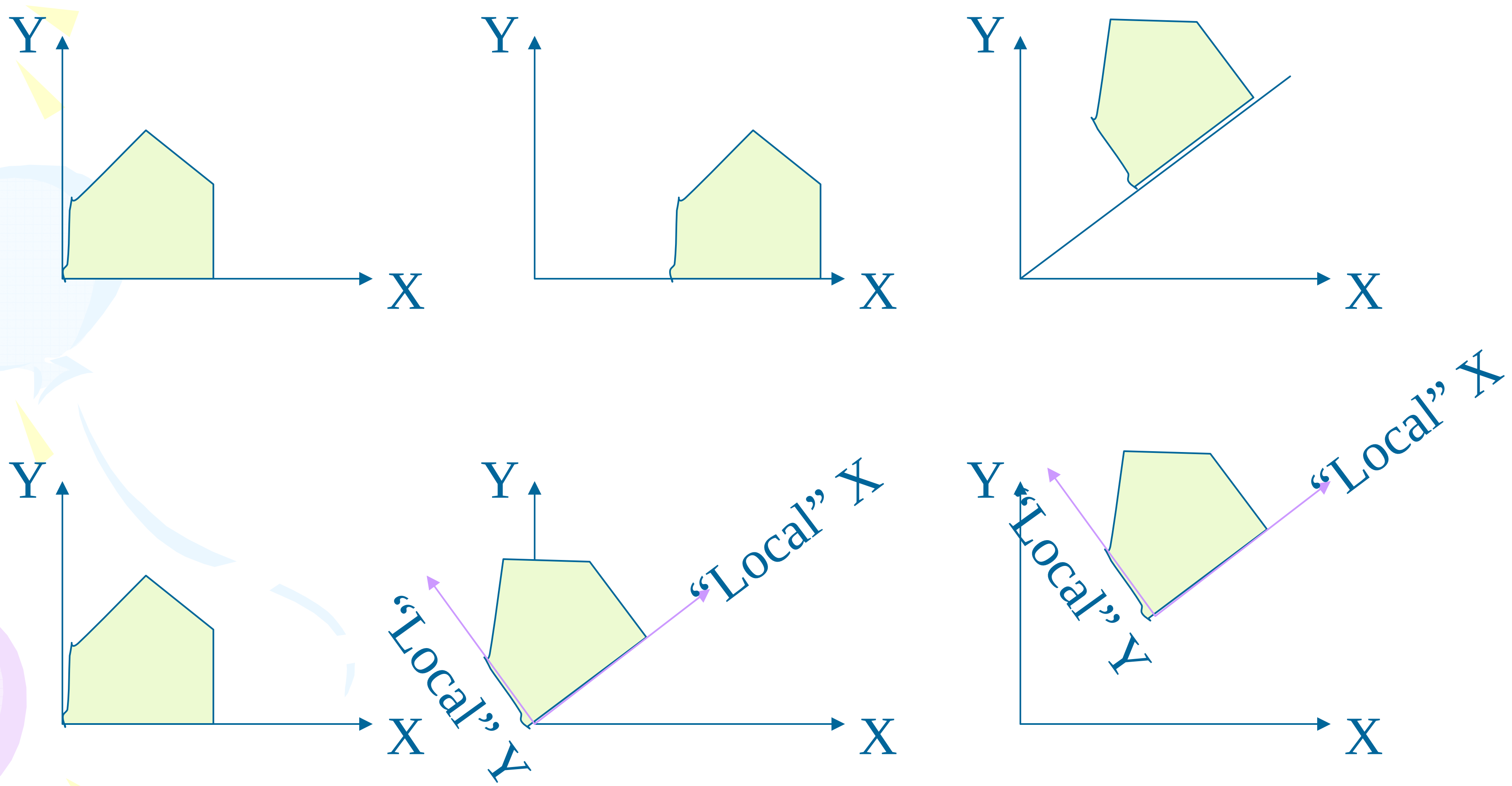
- Associative
- Does not matter how to multiply
- ((AB)C)P = (A(BC))P
- What is the interpretation of these two?
- Till now we were doing (A(BC))P
  - Right to left
  - Transforming points
  - Coordinate axes same across A, B and C
  - GLOBAL COORDINATES

Properties of Concatenation

- What is the geometric interpretation of (AB)C
  - Left to right
  - Transforms the axes (not the points)
  - LOCAL COORDINATES
- Results are the same as long as the matrix is (ABC)

Local/Global Coordinate Systems

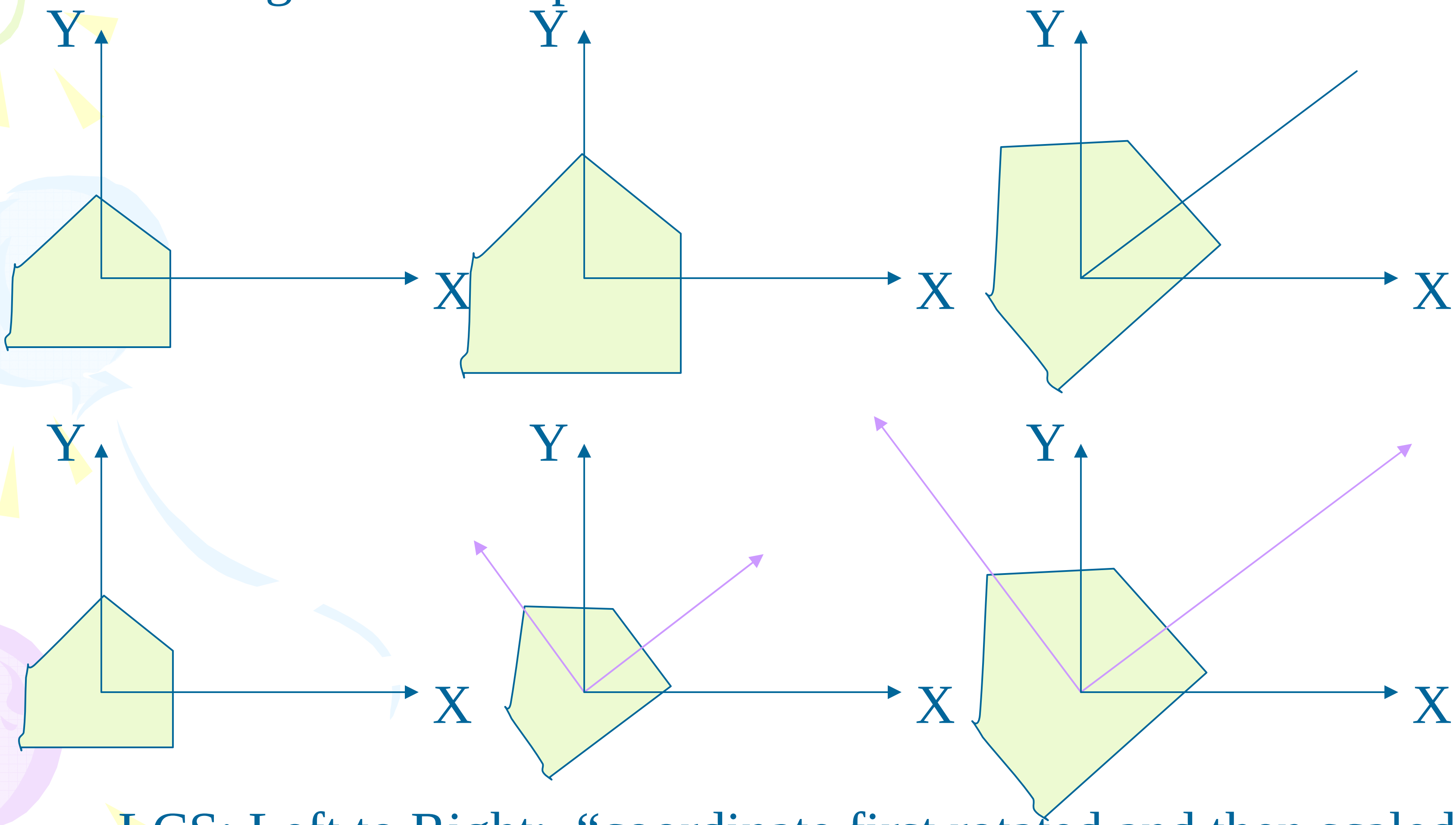
GCS: Right to Left: “point is first translated and then rotated”



LCS: Left to Right: “coordinate first rotated and then translated”

Local / Global Coordinate Systems

GCS: Right to Left: “point is first scaled and then rotated”



LCS: Left to Right: “coordinate first rotated and then scaled”

Projective Transformation

- Most general form of linear transformation
- Note that we are interested in points such that w'=1
- Takes finite points to infinity and vice versa

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Example

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ w \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

- Take a circle in (x,y) – $x^2+y^2=1$
  - Show that it goes to parabola
- Take two parallel lines
  - Show that they go to intersecting lines
- Intersection lines can become parallel
- Degree of the polynomials are preserved
  - Since linear

Images are two-dimensional patterns of brightness values.

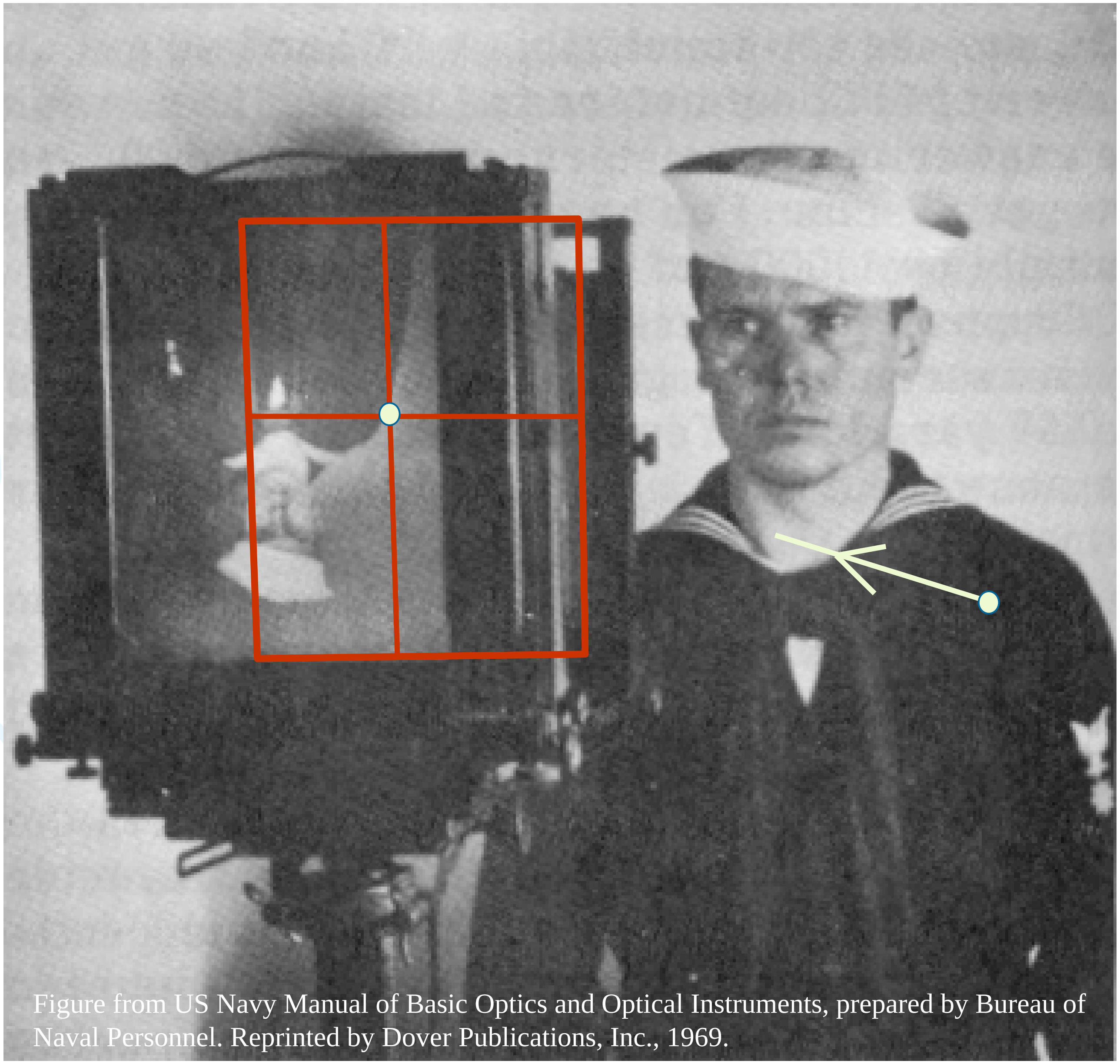
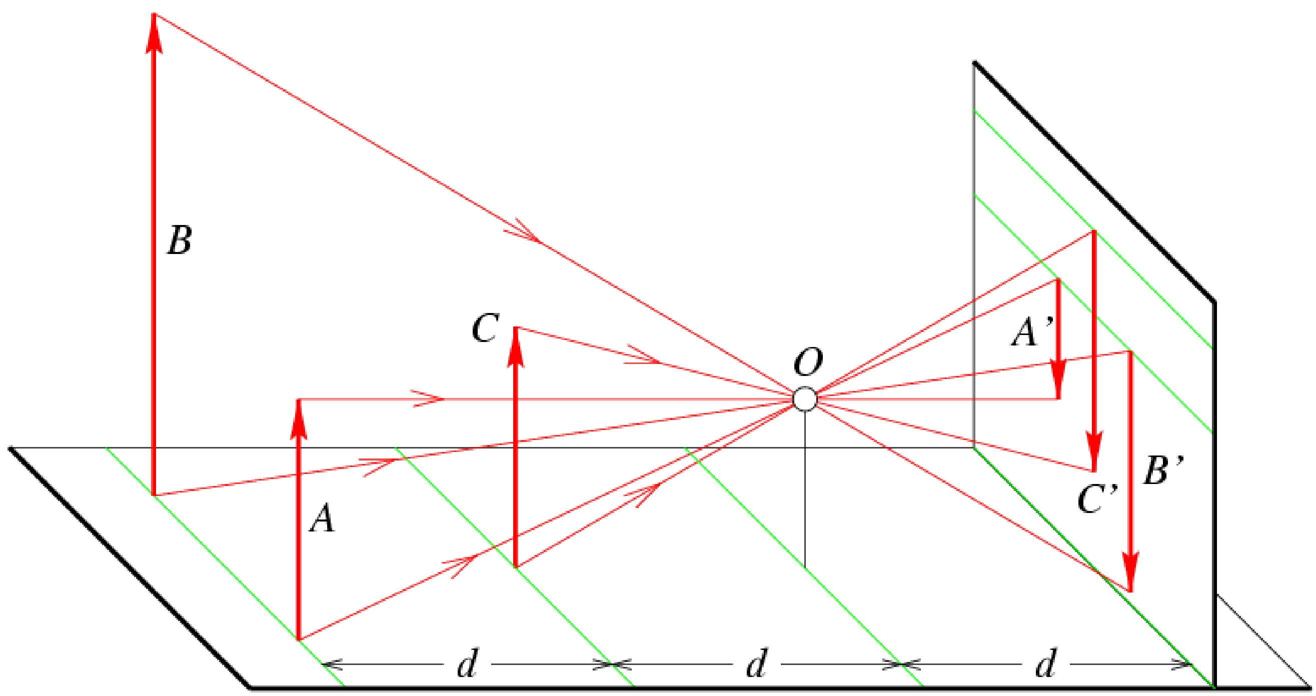


Figure from US Navy Manual of Basic Optics and Optical Instruments, prepared by Bureau of Naval Personnel. Reprinted by Dover Publications, Inc., 1969.

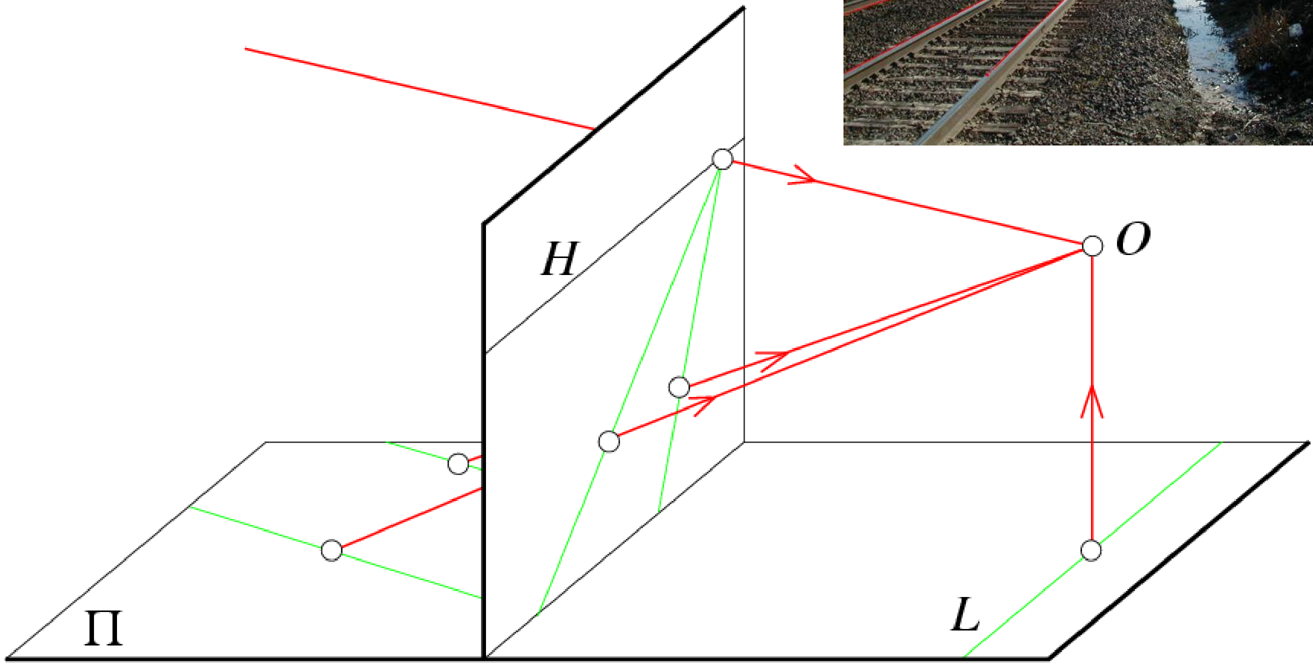
They are formed by the projection of 3D objects.

Distant objects appear smaller

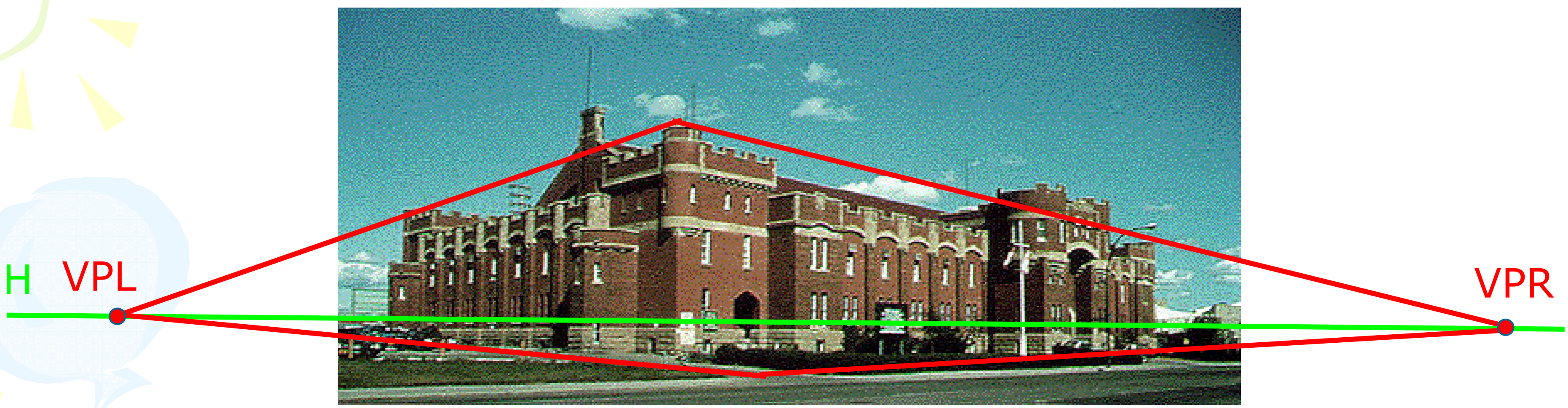


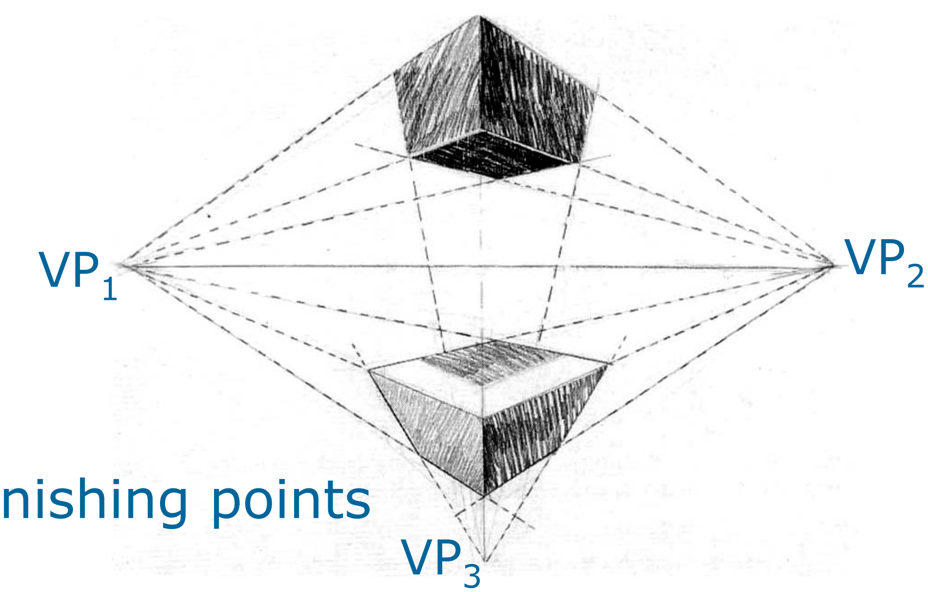
Parallel lines meet

- vanishing point



Vanishing points





To different directions
correspond different vanishing points