Instantaneous Rate of Change — Lecture 8. The Derivative.

Recall that the **average rate of change** of a function $y = f(x)$ on an interval from x_1 to x_2 is just the ratio of the change in y to the change in x:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

For example, if f measures distance traveled with respect to time x, then this average rate of change is the average velocity over that interval. But that leaves us with the question of what is the **instantaneous velocity** at some moment x_0, the velocity that the speedometer in a car is claimed to give us?

The answer is in some sense quite easy to give: The **instantaneous rate of change** of the function $y = f(x)$ at the point x_0 in its domain is:

$$\lim_{x \to x_0} \frac{\Delta y}{\Delta x} = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}.$$

provided this limit exists.

Example 1. Let $f(x) = 1/x$ and let's find the instantaneous rate of change of f at $x_0 = 2$. The first step is to compute the average rate of change over some interval $x_0 = 2$ to x; and in order for this to make sense we need $x \neq 2$. So that average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(2) - f(x)}{2 - x} = \frac{\left(\frac{1}{2} - \frac{1}{x}\right)}{2 - x} = \frac{x - 2}{2x(2 - x)} = -\frac{1}{2x}.$$

Thus, the instantaneous rate of change at $x_0 = 2$ is

$$\lim_{x \to 2} \frac{\Delta y}{\Delta x} = \lim_{x \to 2} \frac{-1}{2x} = -\frac{1}{4}.$$

The instantaneous rate of change at some point $x_0 = a$ involves first the average rate of change from a to *some other value x*. So if we set $h = a - x$, then $h \neq 0$ and the average rate of change from $x = a + h$ to $x = a$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}.$$

Either of these last two ratios is known as a **difference quotient**, a term we shall us repeatedly. With this notation **the instantaneous rate of change of f at $x = a$** is the limit, if it exists,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This has a brief official name:

The derivative of f at $x = a$, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

the instantaneous rate of change of f at a, if it exists.

Example 2. Let's calculate the derivative of $f(x) = x^2$ at $x = 3$. From the above definition we have

$$
\begin{aligned}
f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} \\
&= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \lim_{h \to 0} (6 + h) = 6.
\end{aligned}
$$

It is definitely important to understand the geometric significance of the derivative or instantaneous rate of change. The key is to remember that the average rate of change of a function $y = f(x)$ from some value a to some other value $a + h$ is just the change in $f(x)$ divided by the (non-zero!!!) change in x, and this is just the *slope* of the line L_h between the two points

$$(a, f(a)) \quad \text{and} \quad (a + h, f(a + h)).$$

Now let's assume the the graph of $y = f(x)$ near $x = 1$ is smooth and not too wiggly. Then the smaller we choose h the closer the point $a + h$ is to a, and the closer to the line L through these points is to a line that just touches the graph at the point $(a, f(a)$ on the graph.
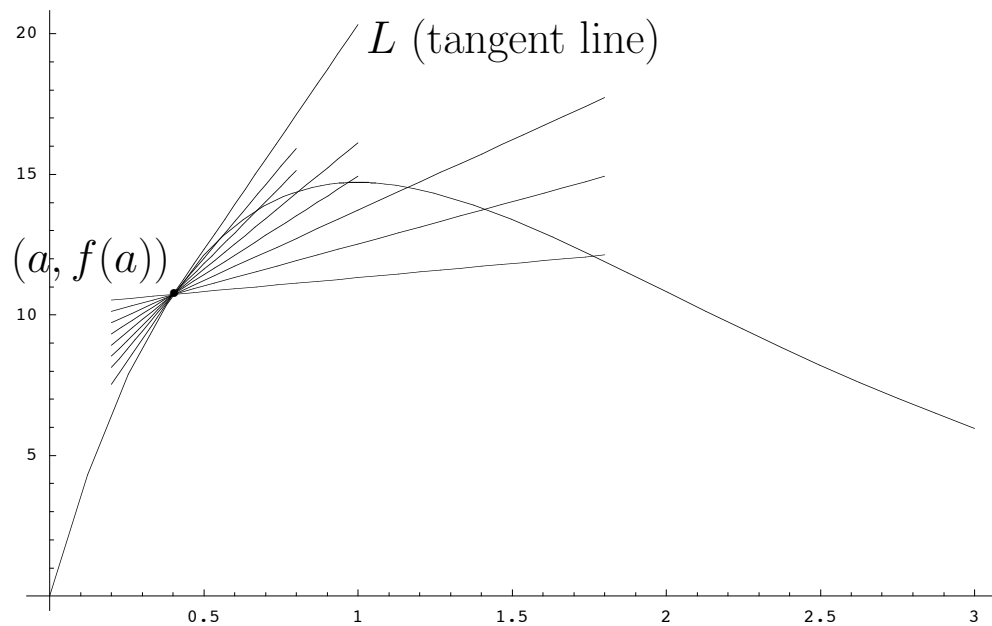
This limiting line L is called the **tangent line** to the graph at

the point $(a, f(a))$, and the punch line is

The slope of the line tangent to the graph $y = f(x)$ at the point $(a, f(a))$ is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

of $y = f(x)$ at $(a, f(a))$, if it exists.

Example 3. We saw in Example 2 that the derivative of $f(x) = x^2$ at $x = 3$ is $f'(3) = 6$, so the line tangent to the parabola $y = x^2$ at the point $(3, f(3)) = (3, 9)$ is

$$y = 6(x - 3) + 9 = 6x - 9.$$

Example 4. Next let's look at a case where there is no derivative and no tangent line. Consider the function $f(x) = |x|$ and let's see what, if anything, its derivative is at $x = 0$. Remember

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

So the appropriate difference quotient at $x = 0$ is

$$\frac{\Delta y}{\Delta x} = \frac{|0 + h| - |0|}{h} = \frac{|h|}{h} = \begin{cases} 1, & \text{if } h > 0; \\ -1, & \text{if } h < 0. \end{cases}$$

But

$$\lim_{h \to 0} \frac{|h|}{h}$$

simply does not exist! So at $x = 0$ the function $f(x) = |x|$ has no derivative and the graph of $y = |x|$ has no tangent at $x = 0$ — something that is quite clear from a glance at the graph!

Most of the functions we will encounter will have derivatives at most points in their domains. Generally speaking, derivatives will exist at points on the graph where the function is continuous and there is no sharp corner (as in the absolute value function). There are some pretty nasty functions out there with quite bizarre behavior, but fortunately, we won't have to deal with them here!

So, let $y = f(x)$ be one of our reasonably nice functions with derivatives at most points in its domain. Then at each such point there *is* a derivative, and hence there is a new function that assigns to each such nice point x in the domain of f a value, $f'(x)$. Not surprisingly, we call this new function **the derivative of $f(x)$.** Thus,

> The **derivative of a function** $y = f(x)$ is the function defined by
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So the derivative $f'(x)$ of a function $y = f(x)$ spews out the slope of the tangent to the graph $y = f(x)$ at each x in the domain of f where there *is* a tangent line. One thing we will have to deal with is that there is quite a variety of notational versions of the derivative of a function $y = f(x)$. Here are the ones we are most likely to meet:

$$f'(x), \quad \frac{d}{dx}f, \quad \frac{d}{dx}f(x), \quad \frac{df}{dx}, \quad \frac{dy}{dx}.$$

Example 4. Let's find the derivative of the function $f(x) = x^2$.
Well, from the definition above

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) \\
&= 2x.
\end{aligned}
$$

Thus at each point x in $\mathbb{R}$, the domain of f, the tangent to the parabola $y = x^2$ has slope $f'(x) = 2x$.

Example 5. Given $y = 1/x$, let's find the equation of the line tangent to the graph of this equation at the point $(4, 1/4)$. First step is to calculate the "slope" function, the derivative $\frac{dy}{dx}$. Well,

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{1/(x+h) - 1/x}{h} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x^2 + xh} = \frac{-1}{x^2}.
\end{aligned}
$$

So when $x = 4$, the derivative is $\left. \frac{dy}{dx} \right|_{x=4} = -\frac{1}{16}$. Therefore, the tangent line at $(4, 1/4)$ is

$$
y = -1/16(x - 4) + 1/4 = -1/16x + 1/2.
$$

Practice Problems.

1. Find the derivative $f'(x)$ of the function $f(x) = 5x + 2$.

2. Find the derivative dy/dx of the constant function $y = 4$.

3. Find the tangent line to the graph $y = \sqrt{x}$ at the point $(4, 2)$.

4. Find all points on the graph of $f(x) = 3x^2 + 1$ where the tangent line has slope 1.

5. Find the derivative of the function $y = f(x) = |x - 2|$ at the point $x = 2$.

6. A car starting from a dead stop is $s(t) = t^2$ feet from the starting point t seconds after it begins to move. What is the velocity of the car 20 seconds after it begins its journey? How long does it take for the car to reach a speed of 60 mph? Of 80 mph?