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GRADE 10 MATHEMATICS TEACHER'S GUIDE

WRITTEN BY SIYAVULA AND VOLUNTEERS

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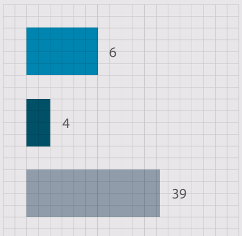

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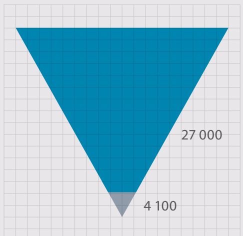
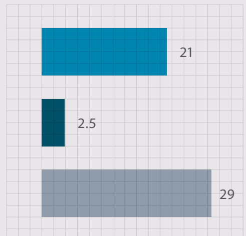


- Litres of ink used in the production of all the grade 10, 11 and 12 textbooks
- Litres of glue used in the production of all the grade 10, 11 and 12 textbooks



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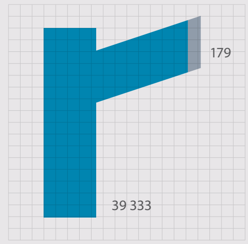

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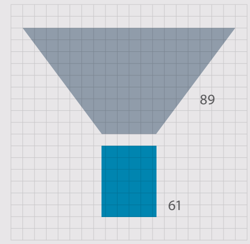
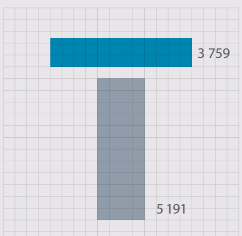
- Number of pages



- Hours spent being taught this book
- Hours spent doing homework from this book



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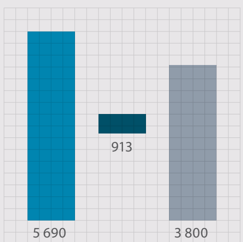

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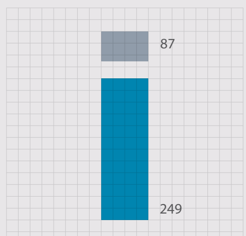
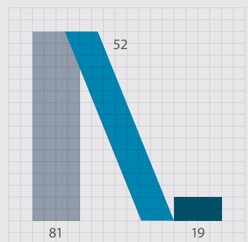
- How many times student clicks pen while reading this book



- Number of females who helped write this book
- Number of males who helped write this book



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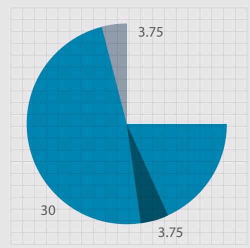
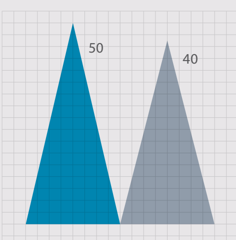
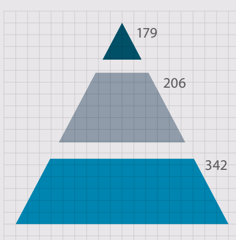
- Honours students who contributed to this book
- Undergraduate students who contributed to this book



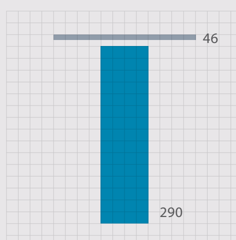
- Hours spent getting book to school per week
- Hours spent getting book home per week
- Hours spent with book in class per week



- Average size of class being taught from this book
- Average age of a maths teacher teaching from this book



- Number of pages in Grade 12 Maths textbook
- Number of pages in Grade 11 Maths textbook
- Number of pages in Grade 10 Maths textbook



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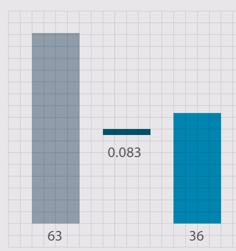

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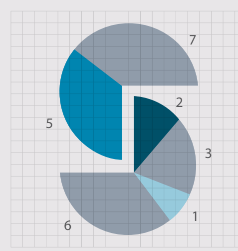

- Weekly UCT hackathons that contributed to this book
- Small office hackathons that contributed to this book
- Afrikaans hackathons that contributed to this book
- Virtual hackathons that contributed to this book

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GRADE 10 MATHEMATICS

TEACHER'S GUIDE

VERSION 1.1 CAPS

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www.siyavula.com
info@siyavula.com
021 469 4771

Siyavula Authors

Luke Kannemeyer; Alison Jenkin; Marina van Zyl; Dr Carl Scheffler

Siyavula and DBE team

Heather Williams; Nkosilathi Vundla; Bridget Nash; Ewald Zietsman; William Buthane Chauke; Leonard Gumani Mudau; Sthe Khanyile; Josephine Mamaroke Phatlane

Siyavula and Free High School Science Text contributors

Dr Mark Horner; Dr Samuel Halliday; Dr Sarah Blyth; Dr Rory Adams; Dr Spencer Wheaton

Iesrafeel Abbas; Sarah Abel; Taskeen Adam; Ross Adams; Tracey Adams; Dr Rory Adams; Andrea Africa; Wiehan Agenbag; Ismail Akhalwaya; Matthew Amundsen; Ben Anhalt; Prashant Arora; Bianca Bôhmer; Amos Baloyi; Bongani Baloyi; Raymond Barbour; Caro-Joy Barendse; Katherine Barry; Dr Ilsa Basson; Richard Baxter; Tara Beckerling; Tim van Beek; Lisette de Beer; Prof Margot Berger; Jessie Bester; Mariaan Bester; Jennifer de Beyer; Dr Sarah Blyth; Sebastian Bodenstein; Martin Bongers; Dr Thinus Booysen; Ena Bosman; Janita Botha; Pieter Botha; Gareth Boxall; Stephan Brandt; Hannes Breytenbach; Alexander Briell; Wilbur Britz; Graeme Broster; Craig Brown; Michail Brynard; Richard Burge; Christina Buys; Jan Buys; George Calder-Potts; Biddy Cameron; Eleanor Cameron; Mark Carolissen; Shane Carollisson; Richard Case; Sithembile Cele; Alice Chang; William Buthane Chauke; Faith Chaza; Richard Cheng; Fanny Cherblanc; Saymore Chifamba; Lizzy Chivaka; Dr Christine Chung; Dr Mareli Claasens; Brett Cocks; Zelmari Coetzee; Phillipa Colly; Roché Compaan; Willem Conradie; Stefaan Conradie; Deanne Coppejans; Rocco Coppejans; Gary Coppin; Tim Craib; Dr Andrew Craig; Tim Crombie; Dan Crytser; Jock Currie; Dr Anne Dabrowski; Laura Daniels; Gareth Davies; Mia de; Tariq Desai; Sandra Dickson; Sean Dobbs; Buhle Donga; William Donkin; Esmi Dreyer; Matthew Duddy; Christel Durie; Fernando Durrell; Dr Dan Dwyer; Frans van Eeden; Kobus Ehlers; Alexander Ellis; Tom Ellis; Dr Anthony Essien; Charl Esterhuysen; Andrew Fisher; Dr Philip Fourie; Giovanni Franzoni; Sanette Gildenhuys; Olivia Gillett; Ingrid von Glehn; Tamara von Glehn; Nicola Glenday; Lindsay Glesener; Kevin Godby; Dr Vanessa Godfrey; Terence Goldberg; Dr Johan Gonzalez; Saaligha Gool; Hemant Gopal; Dr Stephanie Gould; Umeshree Govender; Dr Ilse le Grange; Heather Gray; Lynn Greeff; Jaco Greyling; Martli Greyvenstein; Carine Grobbelaar; Suzanne Grové; Eric Gulbis; Dr Tom Gutierrez; Brooke Haag; Kate Hadley; Alex Hall; Dr Samuel Halliday; Asheena Hanuman; Dr Melanie Dymond Harper; Ebrahim Harris; Dr Nicholas Harrison; Neil Hart; Nicholas Hatcher; Jason Hayden; Laura Hayward; Thomas Haywood; Dr William P. Heal; Pierre van Heerden; Dr Fritha Hennessy; Dr Colleen Henning; Anna Herrington; Shaun Hewitson; Millie Hilgart; Grant Hillebrand; Malcolm Hillebrand; Gregory Hingle; Nick Hobbs; Chris Holdsworth; Dr Benne

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EVERYTHING MATHS

Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

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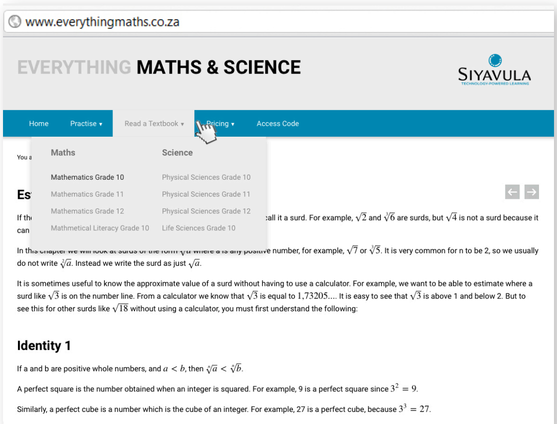

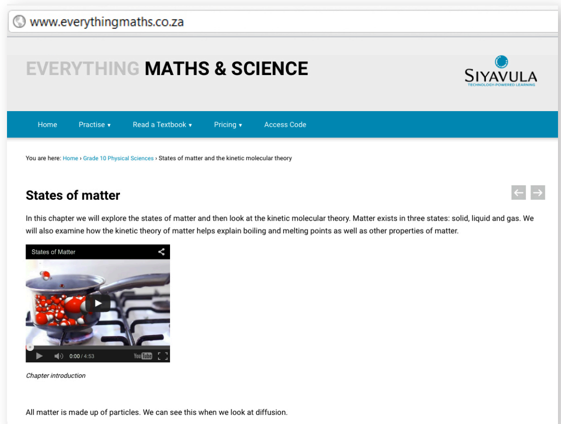

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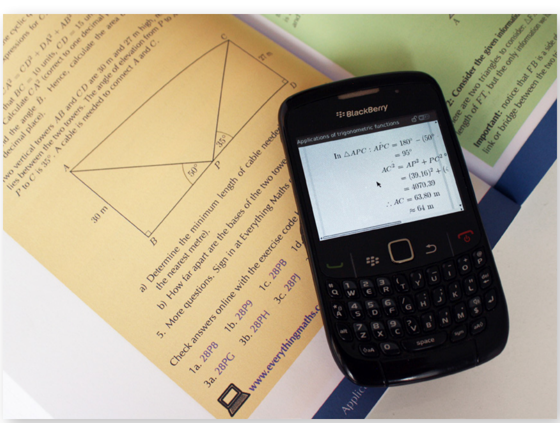

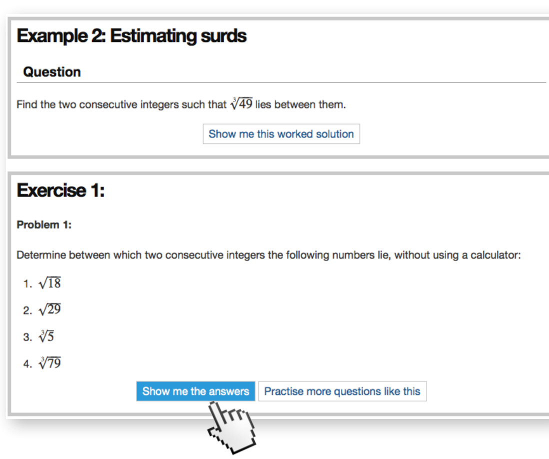

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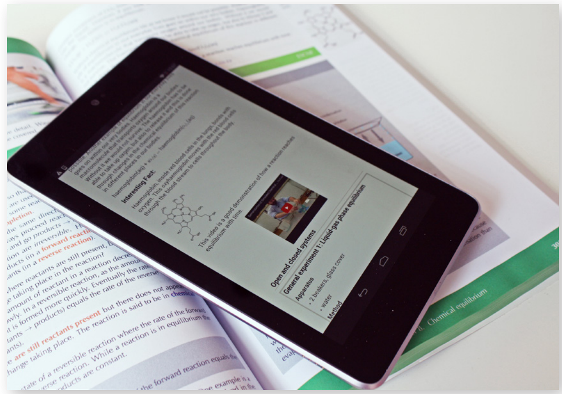

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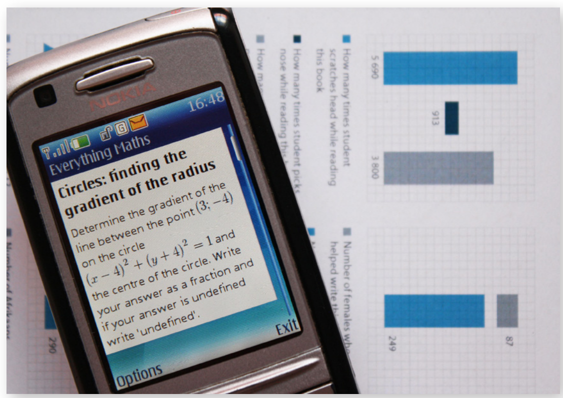

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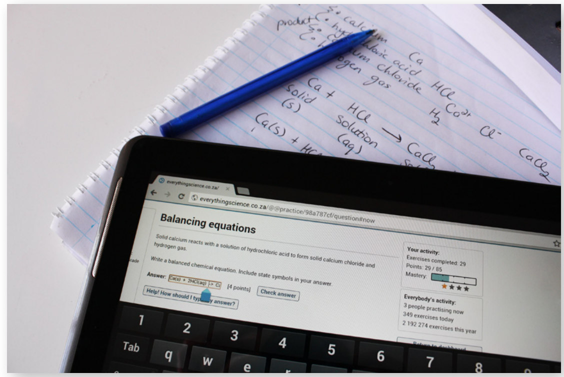

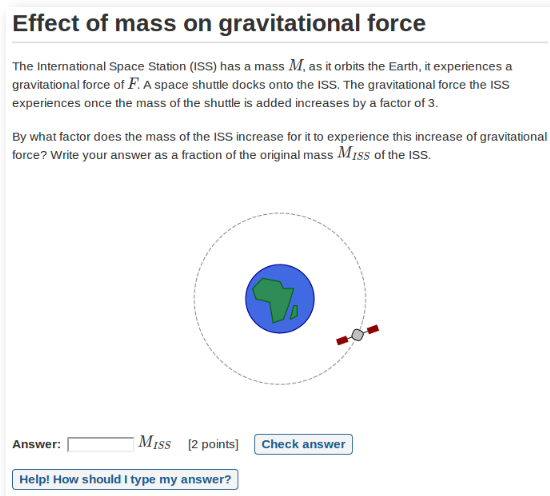

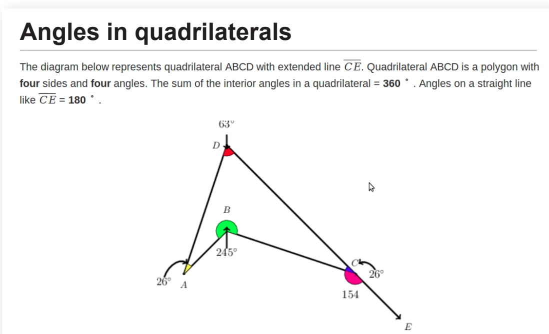

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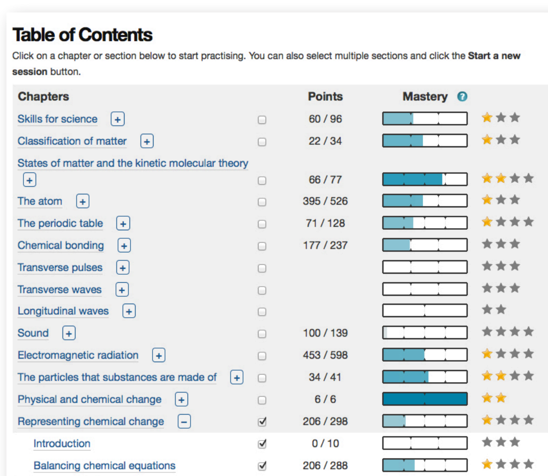


Contents

CHAPTER 0

Mathematics - Teachers guide

0.1 Blog posts

General blogs

- Educator's Monthly - Education News and Resources (http://www.teachersmonthly.com)
  - "We eat, breathe and live education! "
  - "Perhaps the most remarkable yet overlooked aspect of the South African teaching community is its enthusiastic, passionate spirit. Every day, thousands of talented, hard-working educators gain new insight from their work and come up with brilliant, inventive and exciting ideas. Educator's Monthly aims to bring educators closer and help them share knowledge and resources.
  - Our aim is twofold ...
    - To keep South African educators updated and informed.
    - To give educators the opportunity to express their views and cultivate their interests."
- Head Thoughts – Personal Reflections of a School Headmaster (http://headthoughts.co.za/)
  - blog by Arthur Preston
  - "Arthur is currently the headmaster of a growing independent school in Worcester, in the Western Cape province of South Africa. His approach to primary education is progressive and is leading the school through an era of new development and change."

Maths blog

- CEO: Circumspect Education Officer - Educating The Future
  - blog by Robyn Clark
  - "Mathematics teacher and inspirer."
  - http://clarkformaths.tumblr.com/
- dy/dan - Be less helpful
  - blog by Dan Meyer
  - "I'm Dan Meyer. I taught high school math between 2004 and 2010 and I am currently studying at Stanford University on a doctoral fellowship. My specific interests include curriculum design (answering the question, "how we design the ideal learning experience for students?") and teacher education (answering the questions, "how do teachers learn?" and "how do we retain more teachers?" and "how do we teach teachers to teach?")."
  - http://blog.mrmeyer.com
- Without Geometry, Life is Pointless - Musings on Math, Education, Teaching, and Research
  - blog by Avery
  - "I've been teaching some permutation (or is that combination?) of math and science to third through twelfth graders in private and public schools for 11 years. I'm also pursuing my EdD in education and will be both teaching and conducting research in my classroom this year."
  - http://mathteacherorstudent.blogspot.com/
- Overthinking my teaching - The Mathematics I Encounter in Classrooms
  - blog by Christopher Danielson
  - "I think a lot about my math teaching. Perhaps too much. This is my outlet. I hope you find it interesting and that you'll let me know how it's going."
  - http://christopherdanielson.wordpress.com
- A Recursive Process - Math Teacher Seeking Patterns
  - blog by Dan
  - "I am a High School math teacher in upstate NY. I currently teach Geometry, Computer Programming (Alice and Java), and two half year courses: Applied and Consumer Math. This year brings a new 21st century classroom (still not entirely sure what that entails) and a change over to standards based grades."

- http://dandersod.wordpress.com

- Think Thank Thunk – Dealing with the Fear of Being a Boring Teacher
  - blog by Shawn Cornally
  - "I am Mr. Cornally. I desperately want to be a good teacher. I teach Physics, Calculus, Programming, Geology, and Bioethics. Warning: I have problem with using colons. I proof read, albeit poorly."
  - http://101studiostreet.com/wordpress/

0.2 Overview

Before 1994 there existed a number of education departments and subsequent curriculum according to the segregation that was so evident during the apartheid years. As a result, the curriculum itself became one of the political icons of freedom or suppression. Since then the government and political leaders have sought to try and develop one curriculum that is aligned with our national agenda of democratic freedom and equality for all, in fore-grounding the knowledge, skills and values our country believes our learners need to acquire and apply, in order to participate meaningfully in society as citizens of a free country. The National Curriculum Statement (NCS) of Grades R – 12 (DBE, 2012) therefore serves the purposes of:

- equipping learners, irrespective of their socio-economic background, race, gender, physical ability or intellectual ability, with the knowledge, skills and values necessary for self-fulfilment, and meaningful participation in society as citizens of a free country;
- providing access to higher education;
- facilitating the transition of learners from education institutions to the workplace; and
- providing employers with a sufficient profile of a learner's competencies.

Although elevated to the status of political icon, the curriculum remains a tool that requires the skill of an educator in interpreting and operationalising this tool within the classroom. The curriculum itself cannot accomplish the purposes outlined above without the community of curriculum specialists, material developers, educators and assessors contributing to and supporting the process, of the intended curriculum becoming the implemented curriculum. A curriculum can succeed or fail, depending on its implementation, despite its intended principles or potential on paper. It is therefore important that stakeholders of the curriculum are familiar with and aligned to the following principles that the NCS (CAPS) is based on:

Principle	**Implementation**
Social Transformation	Redressing imbalances of the past. Providing equal opportunities for all.
Active and Critical Learning	Encouraging an active and critical approach to learning. Avoiding excessive rote and uncritical learning of given truths.
High Knowledge and Skills	Learners achieve minimum standards of knowledge and skills specified for each grade in each subject.
Progression	Content and context shows progression from simple to complex.
Social and Environmental Justice and Human Rights	These practices as defined in the Constitution are infused into the teaching and learning of each of the subjects.
Valuing Indigenous Knowledge Systems	Acknowledging the rich history and heritage of this country.
Credibility, Quality and Efficiency	Providing an education that is globally comparable in quality.

This guide is intended to add value and insight to the existing National Curriculum for Grade 10 Mathematics, in line with its purposes and principles. It is hoped that this will assist you as the educator in optimising the implementation of the intended curriculum.

Curriculum requirements and objectives

The main objectives of the curriculum relate to the learners that emerge from our educational system. While educators are the most important stakeholders in the implementation of the intended curriculum, the quality of learner coming through this curriculum will be evidence of the actual attained curriculum from what was intended and then implemented.

These purposes and principles aim to produce learners that are able to:

- identify and solve problems and make decisions using critical and creative thinking;
- work effectively as individuals and with others as members of a team;
- organise and manage themselves and their activities responsibly and effectively;
- collect, analyse, organise and critically evaluate information;
- communicate effectively using visual, symbolic and/or language skills in various modes;

- use science and technology effectively and critically showing responsibility towards the environment and the health of others; and
- demonstrate an understanding of the world as a set of related systems by recognising that problem solving contexts do not exist in isolation.

The above points can be summarised as an independent learner who can think critically and analytically, while also being able to work effectively with members of a team and identify and solve problems through effective decision making. This is also the outcome of what educational research terms the "reformed" approach rather than the "traditional" approach many educators are more accustomed to. Traditional practices have their role and cannot be totally abandoned in favour of only reform practices. However, in order to produce more independent and mathematical thinkers, the reform ideology needs to be more embraced by educators within their instructional behaviour. Here is a table that can guide you to identify your dominant instructional practice and try to assist you in adjusting it (if necessary) to be more balanced and in line with the reform approach being suggested by the NCS (CAPS).

	Traditional Versus Reform Practices
Values	**Traditional** – values content, correctness of learners' responses and mathematical validity of methods. **Reform** – values finding patterns, making connections, communicating mathematically and problem-solving.
Teaching Methods	**Traditional** – expository, transmission, lots of drill and practice, step by step mastery of algorithms. **Reform** – hands-on guided discovery methods, exploration, modelling. High level reasoning processes are central.
Grouping Learners	**Traditional** – dominantly same grouping approaches. **Reform** – dominantly mixed grouping and abilities.

The subject of mathematics, by the nature of the discipline, provides ample opportunities to meet the reformed objectives. In doing so, the definition of mathematics needs to be understood and embraced by educators involved in the teaching and the learning of the subject. In research it has been well documented that, as educators, our conceptions of what mathematics is, has an influence on our approach to the teaching and learning of the subject.

Three possible views of mathematics can be presented. The instrumentalist view of mathematics assumes the stance that mathematics is an accumulation of facts, rules and skills that need to be used as a means to an end, without there necessarily being any relation between these components. The Platonist view of mathematics sees the subject as a static but unified body of certain knowledge, in which mathematics is discovered rather than created. The problem solving view of mathematics is a dynamic, continually expanding and evolving field of human creation and invention that is in itself a cultural product. Thus mathematics is viewed as a process of enquiry, not a finished product. The results remain constantly open to revision. It is suggested that a hierarchical order exists within these three views, placing the instrumentalist view at the lowest level and the problem solving view at the highest.

According to the NCS (CAPS):

Mathematics is the study of quantity, structure, space and change. Mathematicians seek out patterns, formulate new conjectures, and establish axiomatic systems by rigorous deduction from appropriately chosen axioms and definitions. Mathematics is a distinctly human activity practised by all cultures, for thousands of years. Mathematical problem solving enables us to understand the world (physical, social and economic) around us, and, most of all, to teach us to think creatively.

This corresponds well to the problem solving view of mathematics and may challenge some of our instrumentalist or Platonistic views of mathematics as a static body of knowledge of accumulated facts, rules and skills to be learnt and applied. The NCS (CAPS) is trying to discourage such an approach and encourage mathematics educators to dynamically and creatively involve their learners as mathematicians engaged in a process of study, understanding, reasoning, problem solving and communicating mathematically.

Below is a check list that can guide you in actively designing your lessons in an attempt to embrace the definition of mathematics from the NCS (CAPS) and move towards a problem solving conception of the subject. Adopting such an approach to the teaching and learning of mathematics will in turn contribute to the intended curriculum being properly implemented and attained through the quality of learners coming out of the education system.

Practice	Example
Learners engage in solving contextual problems related to their lives that require them to interpret a problem and then find a suitable mathematical solution.	Learners are asked to work out which bus service is the cheapest given the fares they charge and the distance they want to travel.
Learners engage in solving problems of a purely mathematical nature, which require higher order thinking and application of knowledge (non-routine problems).	Learners are required to draw a graph; they have not yet been given a specific technique on how to draw (for example a parabola), but have learnt to use the table method to draw straight-line graphs.
Learners are given opportunities to negotiate meaning.	Learners discuss their understanding of concepts and strategies for solving problems with each other and the educator.
Learners are shown and required to represent situations in various but equivalent ways (mathematical modelling).	Learners represent data using a graph, a table and a formula to represent the same data.
Learners individually do mathematical investigations in class, guided by the educator where necessary.	Each learner is given a paper containing the mathematical problem (for instance to find the number of prime numbers less than 50) that needs to be investigated and the solution needs to be written up. Learners work independently.
Learners work together as a group/team to investigate or solve a mathematical problem.	A group is given the task of working together to solve a problem that requires them investigating patterns and working through data to make conjectures and find a formula for the pattern.
Learners do drill and practice exercises to consolidate the learning of concepts and to master various skills.	Completing an exercise requiring routine procedures.
Learners are given opportunities to see the interrelatedness of the mathematics and to see how the different outcomes are related and connected.	While learners work through geometry problems, they are encouraged to make use of algebra.
Learners are required to pose problems for their educator and peer learners.	Learners are asked to make up an algebraic word problem (for which they also know the solution) for the person sitting next to them to solve.

Overview of topics

Summary of topics and their relevance:

1. Functions – linear, quadratic, exponential, rational	Relevance
Work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and some quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions. Generate as many graphs as necessary, initially by means of point-to-point plotting, supported by available technology, to make and test conjectures and hence generalise the effect of the parameter which results in a vertical shift and that which results in a vertical stretch and/or reflection about the x-axis. Problem solving and graph work involving the prescribed functions.	Functions form a core part of learners' mathematical understanding and reasoning processes in algebra. This is also an excellent opportunity for contextual mathematical modelling questions.

2. Number Patterns, Sequences and Series	Relevance
Investigate number patterns leading to those where there is a constant difference between consecutive terms, and the general term is therefore linear.	Much of mathematics revolves around the identification of patterns.

3. Finance, Growth and Decay	Relevance
Use simple and compound growth formulae $A = P(1 + in)$ and $A = P(1 + i)^n$ to solve problems (including interest, hire purchase, inflation, population growth and other real life problems). The implications of fluctuating foreign exchange rates.	The mathematics of finance is very relevant to daily and long-term financial decisions learners will need to make in terms of investing, taking loans, saving and understanding exchange rates and their influence more globally.

4. Algebra	Relevance
Understand that real numbers can be rational or irrational. Simplify expressions using the laws of exponents for rational exponents. Establish between which two integers a given simple surd lies. Round real numbers to an appropriate degree of accuracy (to a given number of decimal digits). Manipulate algebraic expressions by: multiplying a binomial by a trinomial; factorising trinomials; factorising the difference and sums of two cubes; factorising by grouping in pairs; simplifying, adding and subtracting algebraic fractions with denominators of cubes (limited to sum and difference of cubes). Solve: linear equations; quadratic equations; literal equations (changing the subject of a formula); exponential equations; linear inequalities; systems of linear equations and word problems.	Algebra provides the basis for mathematics learners to move from numerical calculations to generalising operations, simplifying expressions, solving equations and using graphs and inequalities in solving contextual problems.

6. Probability	Relevance
Compare the relative frequency of an experimental outcome with the theoretical probability of the outcome. Venn diagrams as an aid to solving probability problems. Mutually exclusive and complementary events. The identity for any two events A and B: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.	This topic is helpful in developing good logical reasoning in learners and for educating them in terms of real-life issues such as gambling and the possible pitfalls thereof.

7. Euclidean Geometry and Measurement	Relevance
Revise basic results established in earlier grades. Investigate line segments joining the mid-points of two sides of a triangle. Properties of special quadrilaterals. Solve problems involving volume and surface area of solids studied in earlier grades as well as spheres, pyramids and cones and combinations of these objects.	The thinking processes and mathematical skills of proving conjectures and identifying false conjectures is more the relevance here than the actual content studied. The surface area and volume content studied in real-life contexts of designing kitchens, tiling and painting rooms, designing packages, etc. is relevant to the current and future lives of learners.

8. Trigonometry	Relevance
Definitions of the trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ in right-angled triangles. Extend the definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ to $0° \leq \theta \leq 360°$ Derive and use values of the trigonometric ratios (without using a calculator) for the special angles $\theta \in \{0° ; 30° ; 45° ; 60° ; 90°\}$ Define the reciprocals of trigonometric rations. Solve problems in two dimensions.	Trigonometry has several uses within society, including within navigation, music, geographical locations and building design and construction.

9. Analytical Geometry	Relevance
Represent geometric figures in a Cartesian co-ordinate system and derive and apply, for any two points $(x_1; y_1)$ and $(x_2; y_2)$, a formula for calculating: the distance between the two points; the gradient of a line segment joining the points; conditions for parallel and perpendicular lines and the co-ordinates of the mid-point of the line segment joining the two points.	This section provides a further application point for learners' algebraic and trigonometric interaction with the Cartesian plane. Artists and design and layout industries often draw on the content and thought processes of this mathematical topic.

10. Statistics	Relevance
Collect, organise and interpret univarate numerical data in order to determine: measures of central tendency; five number summary; box and whisker diagrams and measures of dispersion.	Citizens are daily confronted with interpreting data presented from the media. Often this data may be biased or misrepresented within a certain context. In any type of research, data collection and handling is a core feature. This topic also educates learners to become more socially and politically educated with regards to the media.

Mathematics educators also need to ensure that the following important specific aims and general principles are applied in mathematics activities across all grades:

- Calculators should only be used to perform standard numerical computations and verify calculations done by hand.
- Real-life problems should be incorporated into all sections to keep mathematical modelling as an important focal point of the curriculum.
- Investigations give learners the opportunity to develop their ability to be more methodical, to generalise and to make and justify and/or prove conjectures.

- Appropriate approximation and rounding skills should be taught and continuously included and encouraged in activities.
- The history of mathematics should be incorporated into projects and tasks where possible, to illustrate the human aspect and developing nature of mathematics.
- Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues where possible.
- Conceptual understanding of when and why should also feature in problem types.
- Mixed ability teaching requires educators to challenge able learners and provide remedial support where necessary.
- Misconceptions exposed by assessment need to be dealt with and rectified by questions designed by educators.
- Problem solving and cognitive development should be central to all mathematics teaching and learning so that learners can apply the knowledge effectively.

Allocation of teaching time:

Time allocation for Mathematics per week: 4 hours and 30 minutes e.g. six forty-five minute periods per week.

Term	Topic	No. of weeks
Term 1	Algebraic expressions	3
	Exponents	2
	Number patterns	1
	Equations and inequalities	2
	Trigonometry	3
Term 2	Functions	4
	Trigonometric functions	1
	Euclidean geometry	3
	Mid-year exams	3
Term 3	Analytical geometry	2
	Finance and growth	2
	Statistics	2
	Trigonometry	2
	Euclidean geometry	1
	Measurement	1
Term 4	Probability	2
	Revision	4
	Final exams	3

Please see page 18 of the Curriculum and Assessment Policy Statement for the sequencing and pacing of topics.

0.3 Assessment

"Educator assessment is part of everyday teaching and learning in the classroom. Educators discuss with learners, guide their work, ask and answer questions, observe, help, encourage and challenge. In addition, they mark and review written and other kinds of work. Through these activities they are continually finding out about their learners' capabilities and achievements. This knowledge then informs plans for future work. It is this continuous process that makes up educator assessment. It should not be seen as a separate activity necessarily requiring the use of extra tasks or tests."

As the quote above suggests, assessment should be incorporated as part of the classroom practice, rather than as a separate activity. Research during the past ten years indicates that learners get a sense of what they do and do not know, what they might do about this and how they feel about it, from frequent and regular classroom assessment and educator feedback. The educator's perceptions of and approach to assessment (both formal and informal assessment) can have an influence on the classroom culture that is created with regard to the learners' expectations of and performance in assessment tasks. Literature on classroom assessment distinguishes between two different purposes of assessment; assessment of learning and assessment for learning.

Assessment of learning tends to be a more formal assessment and assesses how much learners have learnt or understood at a particular point in the annual teaching plan. The NCS (CAPS) provides comprehensive guidelines on the types of and amount of formal assessment that needs to take place within the teaching year to make up the school-based assessment mark. The school-based assessment mark contributes 25% of the final percentage of a learner's promotion mark, while the end-of-year examination constitutes the other 75% of the annual promotion mark. Learners are expected to have 7 formal assessment tasks for their school-based assessment mark. The number of tasks and their weighting in the Grade 10 Mathematics curriculum is summarised below:

Tasks			Weight (percent)
School-Based Assessment	Term 1	Project/Investigation Test	20 10
	Term 2	Assignment/Test Mid-Year Examination	10 30
	Term 3	Test Test	10 10
	Term 4	Test	10
School-Based Assessment Mark			100
School-Based Assessment Mark (as a percent of Promotion Mark)			25%
End-of-Year Examination			75%
Promotion Mark			100%

The following provides a brief explanation of each of the assessment tasks included in the assessment programme above.

Tests

All mathematics educators are familiar with this form of formal assessment. Tests include a variety of items/questions covering the topics that have been taught prior to the test. The new NCS (CAPS) also stipulates that mathematics tests should include questions that cover the following four types of cognitive levels in the stipulated weightings:

Cognitive levels	Description	Weighting (percent)
Knowledge	Straight recall. Identification of correct formula on information sheet (no changing of the subject). Use of mathematical facts. Appropriate use of mathematical vocabulary.	20
Routine procedures	Estimation and appropriate rounding of numbers. Proofs and prescribed theorems and derivation of formulae. Identification and direct use of correct formula on the information sheet (no changing of the subject). Perform well known procedures. Simple applications and calculations which might involve a few steps. Derivation from given information may be involved. Identification and use (including changing the subject) of correct formula. Questions generally similar to those encountered in class.	35
Complex procedures	Problems involve complex calculations and/or higher reasoning. There is often not an obvious route to the solution. Problems need not be based on real world context. Could involve making significant connections between different representations. Require conceptual understanding.	30
Problem solving	Unseen, non-routine problems (which are not necessarily difficult). Higher order understanding and processes are often involved. Might require the ability to break the problem down into its constituent parts.	15

The breakdown of the tests over the four terms is summarised from the NCS (CAPS) assessment programme as follows:

Term 1: One test (of at least 50 marks and one hour).

Term 2: One test/assignment (of at least 50 marks and one hour).

Term 3: Two tests (of at least 50 marks and one hour).

Term 4: One test (of at least 50 marks and one hour).

Projects/Investigations

Investigations and projects consist of open-ended questions that initiate and expand thought processes. Acquiring and developing problem-solving skills are an essential part of doing investigations and projects. These tasks provide learners with the opportunity to investigate, gather information, tabulate results, make conjectures and justify or prove these conjectures. Examples of investigations and projects and possible marking rubrics are provided in the next section on assessment support. The NCS (CAPS) assessment programme indicates that only one project or investigation (of at least 50 marks) should be included per year. Although the project/investigation is scheduled in the assessment programme for the first term, it could also be done in the second term.

Assignments

The NCS (CAPS) includes the following tasks as good examples of assignments:

- Open book test
- Translation task
- Error spotting and correction
- Shorter investigation
- Journal entry
- Mind-map (also known as a metacog)
- Olympiad (first round)
- Mathematics tutorial on an entire topic
- Mathematics tutorial on more complex/problem solving questions

The NCS (CAPS) assessment programme requires one assignment in term 1 (of at least 50 marks) which could also be a combination of some of the suggested examples above. More information on these suggested examples of assignments and possible rubrics are provided in the following section on assessment support.

Examinations

Educators are also all familiar with this summative form of assessment that is usually completed twice a year: mid-year examinations and end-of-year examinations. These are similar to the tests but cover a wider range of topics completed prior to each examination. The NCS (CAPS) stipulates that each examination should also cover the four cognitive levels according to their recommended weightings as summarised in the section above on tests. The following table summarises the requirements and information from the NCS (CAPS) for the two examinations.

Examination	Marks	Breakdown	Content and Mark distribution
Mid-Year Exams	100 50 + 50	Mid-year exams can consist of either one paper of two hours (100 marks) **or** two papers, each of one hour (50 marks).	Topics completed
End-of-Year Exams	100	Paper 1: 3 hours	Algebra and equations (and inequalities) (30 ± 3) Patterns and sequences (15 ± 3) Finance and growth (10 ± 3) Functions and graphs (30 ± 3) Probability (15 ± 3)
End-of-Year Exams	100	Paper 2: 3 hours	Statistics (15 ± 3) Analytical geometry (15 ± 3) Trigonometry (40 ± 3) Euclidean geometry and measurement (30 ± 3)

In the annual teaching plan summary of the NCS (CAPS) in Mathematics for Grade 10, the pace setter section provides a detailed model of the suggested topics to be covered each week of each term and the accompanying formal assessment.

Assessment **for** learning tends to be more informal and focuses on using assessment in and of daily classroom activities that can include:

1. Marking homework
2. Baseline assessments
3. Diagnostic assessments
4. Group work
5. Class discussions
6. Oral presentations
7. Self-assessment
8. Peer-assessment

These activities are expanded on in the next section on assessment support and suggested marking rubrics are provided. Where formal assessment tends to restrict the learner to written assessment tasks, the informal assessment is necessary to evaluate and encourage the progress of the learners in their verbal mathematical reasoning and communication skills. It also provides a less formal assessment environment that allows learners to openly and honestly assess themselves and each other, taking responsibility for their own learning, without the heavy weighting of the performance (or mark) component. The assessment for learning tasks should be included in the classroom activities at least once a week (as part of a lesson) to ensure that the educator is able to continuously evaluate the learners' understanding of the topics covered as well as the effectiveness, and identify any possible deficiencies in his or her own teaching of the topics.

A selection of explanations, examples and suggested marking rubrics for the assessment of learning (formal) and the assessment for learning (informal) forms of assessment discussed in the preceding section are provided in this section.

Baseline assessment

Baseline assessment is a means of establishing:

- What prior knowledge a learner possesses
- What the extent of knowledge is that they have regarding a specific learning area?
- The level they demonstrate regarding various skills and applications
- The learner's level of understanding of various learning areas

It is helpful to educators in order to assist them in taking learners from their individual point of departure to a more advanced level and to thus make progress. This also helps avoid large "gaps" developing in the learners' knowledge as the learner moves through the education system. Outcomes-based education is a more learner-centered approach than we are used to in South Africa, and therefore the emphasis should now be on the level of each individual learner rather than that of the whole class.

The baseline assessments also act as a gauge to enable learners to take more responsibility for their own learning and to view their own progress. In the traditional assessment system, the weaker learners often drop from a 40% average in the first term to a 30% average in the fourth term due to an increase in workload, thus demonstrating no obvious progress. Baseline assessment, however, allows for an initial assigning of levels which can be improved upon as the learner progresses through a section of work and shows greater knowledge, understanding and skill in that area.

Diagnostic assessments

These are used to specifically find out if any learning difficulties or problems exist within a section of work in order to provide the learner with appropriate additional help and guidance. The assessment helps the educator and the learner identify problem areas, misunderstandings, misconceptions and incorrect use and interpretation of notation.

Some points to keep in mind:

- Try not to test too many concepts within one diagnostic assessment.
- Be selective in the type of questions you choose.
- Diagnostic assessments need to be designed with a certain structure in mind. As an educator, you should decide exactly what outcomes you will be assessing and structure the content of the assessment accordingly.
- The assessment is marked differently to other tests in that the mark is not the focus but rather the type of mistakes the learner has made.

An example of an understanding rubric for educators to record results is provided below:

0: indicates that the learner has not grasped the concept at all and that there appears to be a fundamental mathematical problem.

1: indicates that the learner has gained some idea of the content, but is not demonstrating an understanding of the notation and concept.

2: indicates evidence of some understanding by the learner but further consolidation is still required.

3: indicates clear evidence that the learner has understood the concept and is using the notation correctly.

Calculator worksheet - diagnostic skills assessment

1. Calculate:
   a) $242 + 63 =$
   b) $2 - 36 \times (114 + 25) =$
   c) $\sqrt{144 + 25} =$
   d) $\sqrt[4]{729} =$
   e) $-312 + 6 + 879 - 321 + 18\ 901 =$

2. Calculate:

a) $\frac{2}{7}+\frac{1}{3}=$

b) $2\frac{1}{5}-\frac{2}{9}=$

c) $-2\frac{5}{6}+\frac{3}{8}=$

d) $4-\frac{3}{4}\times\frac{5}{7}=$

e) $\left(\frac{9}{10}-\frac{8}{9}\right)\div\frac{3}{5}=$

f) $2\times\left(\frac{4}{5}\right)^2-\left(\frac{19}{25}\right)=$

g) $\sqrt{\frac{9}{4}-\frac{4}{16}}=$

Self-Assesment Rubric:

Name:

Question	**Answer**	$\surd$	**X**	**If X, write down sequence of keys pressed**
1a				
1b				
1c				
1d				
1e				
Subtotal				
2a				
2b				
2c				
2d				
2e				
Subtotal				
Total				

Educator Assessment Rubric:

Type of skill	**Competent**	**Needs practice**	**Problem**
Raising to a power			
Finding a root			
Calculations with Fractions			
Brackets and order of operations			
Estimation and mental control			

Guidelines for Calculator Skills Assessment:

Type of skill	**Sub-Division**	**Questions**
Raising to a Power	Squaring and cubing Higher order powers	1a, 2f 1b
Finding a Root	Square and cube roots Higher order roots	1c, 2g 1d
Calculations with Fractions	Basic operations Mixed numbers Negative numbers Squaring fractions Square rooting fractions	2a, 2d 2b, 2c 1e, 2c 2f 2g
Brackets and Order of Operations	Correct use of brackets or order of operations	1b, 1c, 2e, 2f, 2g
Brackets and Order of Operations	Estimation and Mental Control	All

Suggested guideline to allocation of overall levels

Level 1

- Learner is able to do basic operations on calculator.
- Learner is able to do simple calculations involving fractions.
- Learner does not display sufficient mental estimation and control techniques.

Level 2

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube whole numbers as well as find square and cube roots of numbers.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner displays some degree of mental estimation awareness.

Level 3

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube rational numbers as well as find square and cube roots of numbers.
- Learner is also able to calculate higher order powers and roots.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner works correctly with negative numbers.
- Learner is able to use brackets in certain calculations but has still not fully understood the order of operations that the calculator has been programmed to execute, hence the need for brackets.
- Learner is able to identify possible errors and problems in their calculations but needs assistance solving the problem.

Level 4

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube rational numbers as well as find square and cube roots.
- Learner is also able to calculate higher order powers and roots.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner works correctly with negative numbers.
- Learner is able to work with brackets correctly and understands the need and use of brackets and the "= key" in certain calculations due to the nature of a scientific calculator.
- Learner is able to identify possible errors and problems in their calculations and to find solutions to these in order to arrive at a "more viable" answer.

Other short diagnostic tests

These are short tests that assess small quantities of recall knowledge and application ability on a day-to-day basis. Such tests could include questions on one or a combination of the following:

- Definitions
- Theorems
- Riders (geometry)
- Formulae
- Applications
- Combination questions

Exercises

This entails any work from the textbook or other source that is given to the learner, by the educator, to complete either in class or at home. Educators should encourage learners not to copy each other's work and be vigilant when controlling this work. It is suggested that such work be marked/controlled by a check list (below) to speed up the process for the educator.

The marks obtained by the learner for a specific piece of work need not be based on correct and/or incorrect answers but preferably on the following:

1. the effort of the learner to produce answers.
2. the quality of the corrections of work that was previously incorrect.
3. the ability of the learner to explain the content of some selected examples (whether in writing or orally).

The following rubric can be used to assess exercises done in class or as homework:

Criteria	Performance indicators 2	1	0
Work Done	All the work	Partially completed	No work done
Work Neatly Done	Work neatly done	Some work not neatly done	Messy and muddled
Corrections Done	All corrections done consistently	At least half of the corrections done	No corrections done
Correct Mathematical Method	Consistently	Sometimes	Never
Understanding of Mathematical Techniques and Processes	Can explain concepts and processes precisely	Explanations are ambiguous or not focused	Explanations are confusing or irrelevant

Journal entries

A journal entry is an attempt by a learner to express in the written word what is happening in Mathematics. It is important to be able to articulate a mathematical problem, and its solution in the written word.

This can be done in a number of different ways:

- Today in Maths we learnt...
- Write a letter to a friend, who has been sick, explaining what was done in class today.
- Explain the thought process behind trying to solve a particular maths problem, e.g. sketch the graph of $y = x^2 - 2x^2 + 1$ and explain how to sketch such a graph.
- Give a solution to a problem, decide whether it is correct and if not, explain the possible difficulties experienced by the person who wrote the incorrect solution.

A journal is an invaluable tool that enables the educator to identify any mathematical misconceptions of the learners. The marking of this kind of exercise can be seen as subjective but a marking rubric can simplify the task.

The following rubric can be used to mark journal entries. The learners must be given the marking rubric before the task is done.

Task	Competent (2 marks)	Still developing (1 mark)	Not yet developed (0 marks)
Completion in time limit?			
Correctness of the explanation?			
Correct and relevant use of mathematical language?			
Has the concept been interpret correctly?			

Translations

Translations assess the learner's ability to translate from words into mathematical notation or to give an explanation of mathematical concepts in words. Often when learners can use mathematical language and notation correctly, they demonstrate a greater understanding of the concepts.

For example:

Write the letter of the correct expression next to the matching number:

x increased by 10	a)	xy
The product of x and y	b)	x^2
The sum of a certain number and	c)	x^2
double that number	d)	$29x$
Half of a certain number multiplied by itself	e)	$\frac{1}{2} \times 2$
Two less than x	f)	$x + x + 2$
A certain number multiplied by itself	g)	x^2

Group work

One of the principles in the NCS (CAPS) is to produce learners who are able to work effectively within a group. Learners generally find this difficult to do. Learners need to be encouraged to work within small groups. Very often it is while learning under peer assistance that a better understanding of concepts and processes is reached. Clever learners usually battle with this sort of task, and yet it is important that they learn how to assist and communicate effectively with other learners.

Mind maps or metacogs

A metacog or "mind map" is a useful tool. It helps to associate ideas and make connections that would otherwise be too unrelated to be linked. A metacog can be used at the beginning or end of a section of work in order to give learners an overall perspective of the work covered, or as a way of recalling a section already completed. It must be emphasised that it is not a summary. Whichever way you use it, it is a way in which a learner is given the opportunity of doing research in a particular field and can show that he/she has an understanding of the required section.

This is an open book form of assessment and learners may use any material they feel will assist them. It is suggested that this activity be practised, using other topics, before a test metacog is submitted for portfolio assessment purposes.

On completion of the metacog, learners must be able to answer insightful questions on the metacog. This is what sets it apart from being just a summary of a section of work. Learners must refer to their metacog when answering the questions, but may not refer to any reference material. Below are some guidelines to give to learners to adhere to when constructing a metacog as well as two examples to help you get learners started. A marking rubric is also provided. This should be made available to learners before they start constructing their metacogs. On the next page is a model question for a metacog, accompanied by some sample questions that can be asked within the context of doing a metacog about analytical geometry.

A basic metacog is drawn in the following way:

- Write the title/topic of the subject in the centre of the page and draw a circle around it.
- For the first main heading of the subject, draw a line out from the circle in any direction, and write the heading above or below the line.
- For sub-headings of the main heading, draw lines out from the first line for each subheading and label each one.
- For individual facts, draw lines out from the appropriate heading line.

Metacogs are one's own property. Once a person understands how to assemble the basic structure they can develop their own coding and conventions to take things further, for example to show linkages between facts. The following suggestions may assist educators and learners to enhance the effectiveness of their metacogs:

- Use single words or simple phrases for information. Excess words just clutter the metacog and take extra time to write down.
- Print words – joined up or indistinct writing can be more difficult to read and less attractive to look at.
- Use colour to separate different ideas – this will help your mind separate ideas where it is necessary, and helps visualisation of the metacog for easy recall. Colour also helps to show organisation.
- Use symbols and images where applicable. If a symbol means something to you, and conveys more information than words, use it. Pictures also help you to remember information.
- Use shapes, circles and boundaries to connect information – these are additional tools to help show the grouping of information.

Use the concept of analytical geometry as your topic and construct a mind map (or metacog) containing all the information (including terminology, definitions, formulae and examples) that you know about the topic of analytical geometry.

Possible questions to ask the learner on completion of their metacog:

- Briefly explain to me what the mathematics topic of analytical geometry entails.
- Identify and explain the distance formula, the derivation and use thereof for me on your metacog.
- How does the calculation of gradient in analytical geometry differ (or not) from the approach used to calculate gradient in working with functions?

Here is a suggested simple rubric for marking a metacog:

Task	Competent (2 Marks)	Still Developing (1 Mark)	Not Yet Developed (1 Mark)
Completion in Time Limit			
Main Headings			
Correct Theory (Formulae, Definitions, Terminology etc.)			
Explanation			
Readability			

10 marks for the questions, which are marked using the following scale:

0 - no attempt or a totally incorrect attempt has been made

1 - a correct attempt was made, but the learner did not get the correct answer

2 - a correct attempt was made and the answer is correct

Investigations

Investigations consist of open-ended questions that initiate and expand thought processes. Acquiring and developing problem-solving skills are an essential part of doing investigations.

It is suggested that 2 – 3 hours be allowed for this task. During the first 30 – 45 minutes learners could be encouraged to talk about the problem, clarify points of confusion, and discuss initial conjectures with others. The final written-up version should be done individually though and should be approximately four pages.

Assessing investigations may include feedback/ presentations from groups or individuals on the results keeping the following in mind:

- following of a logical sequence in solving the problems
- pre-knowledge required to solve the problem
- correct usage of mathematical language and notation
- purposefulness of solution
- quality of the written and oral presentation

Some examples of suggested marking rubrics are included on the next few pages, followed by a selection of topics for possible investigations.

The following guidelines should be provided to learners before they begin an investigation:

General Instructions Provided to Learners

- You may choose any one of the projects/investigations given (see model question on investigations)
- You should follow the instructions that accompany each task as these describe the way in which the final product must be presented.
- You may discuss the problem in groups to clarify issues, but each individual must write-up their own version.
- Copying from fellow learners will cause the task to be disqualified.
- Your educator is a resource to you, and though they will not provide you with answers / solutions, they may be approached for hints.

The investigation is to be handed in on the due date, indicated to you by your educator. It should have as a minimum:

- A description of the problem.
- A discussion of the way you set about dealing with the problem.
- A description of the final result with an appropriate justification of its validity.
- Some personal reflections that include mathematical or other lessons learnt, as well as the feelings experienced whilst engaging in the problem.
- The written-up version should be attractively and neatly presented on about four A4 pages.
- Whilst the use of technology is encouraged in the presentation, the mathematical content and processes must remain the major focus.

Below is an example of a possible rubric to use when marking investigations:

Level of Performance	Criteria
4	• Contains a complete response. • Clear, coherent, unambiguous and elegant explanation. • Includes clear and simple diagrams where appropriate. • Shows understanding of the question's mathematical ideas and processes. • Identifies all the important elements of the question. • Includes examples and counter examples. • Gives strong supporting arguments. • Goes beyond the requirements of the problem.
3	• Contains a complete response. • Explanation less elegant, less complete. • Shows understanding of the question's mathematical ideas and processes. • Identifies all the important elements of the question. • Does not go beyond the requirements of the problem.
2	• Contains an incomplete response. • Explanation is not logical and clear. • Shows some understanding of the question's mathematical ideas and processes. • Identifies some of the important elements of the question. • Presents arguments, but incomplete. • Includes diagrams, but inappropriate or unclear.
1	• Contains an incomplete response. • Omits significant parts or all of the question and response. • Contains major errors. • Uses inappropriate strategies.
0	• No visible response or attempt

Orals

An oral assessment involves the learner explaining to the class as a whole, a group or the educator his or her understanding of a concept, a problem or answering specific questions. The focus here is on the correct use of mathematical language by the learner and the conciseness and logical progression of their explanation as well as their communication skills.

Orals can be done in a number of ways:

- A learner explains the solution of a homework problem chosen by the educator.
- The educator asks the learner a specific question or set of questions to ascertain that the learner understands, and assesses the learner on their explanation.
- The educator observes a group of learners interacting and assesses the learners on their contributions and explanations within the group.
- A group is given a mark as a whole, according to the answer given to a question by any member of a group.

An example of a marking rubric for an oral:

1 - the learner has understood the question and attempts to answer it

2 - the learner uses correct mathematical language

2 - the explanation of the learner follows a logical progression

2 - the learner's explanation is concise and accurate

2 - the learner shows an understanding of the concept being explained

1 - the learner demonstrates good communication skills

Maximum mark = 10

An example of a peer-assessment rubric for an oral:

My name:

Name of person I am assessing:

Criteria	**Mark Awarded**	**Maximum Mark**
Correct Answer		2
Clarity of Explanation		3
Correctness of Explanation		3
Evidence of Understanding		2
Total		10

CHAPTER 1

Algebraic expressions

1.1 Introduction

- Content covered in this chapter includes understanding how numbers are classified as rational or irrational, estimating surds, rounding off, factorisation and simplification.
- This chapter provides a lot of core skills that learners will apply to the rest of mathematics. Ensure that learners are sufficiently proficient in the skills covered in this chapter.
- Rounding real numbers is an important skill that learners will use often. Ensure that learners are completely comfortable with this skill.
- Factorisation forms the groundwork for solving equations. Learners should be comfortable factorising trinomials and binomials.
- Factorisation should include types covered in grade 9 as well as trinomials, grouping in pairs and sum and difference of two cubes.

1.2 The real number system

1.3 Rational and irrational numbers

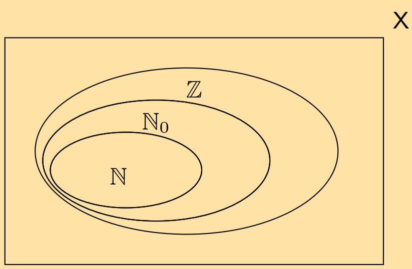
Decimal numbers

Converting terminating decimals into rational numbers

Converting recurring decimals into rational numbers

Exercise 1 – 1:

1. The figure here shows the Venn diagram for the special sets $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$.



   a) Where does the number $-\frac{12}{3}$ belong in the diagram?
   Solution:
   First simplify the fraction: $-\frac{12}{3} = -4$
   -4 is an integer, so it falls into the set $\mathbb{Z}$.
   b) In the following list, there are two false statements and one true statement. Which of the statements is **true**?
      i. Every integer is a natural number.
      ii. Every natural number is a whole number.
      iii. There are no decimals in the whole numbers.

Solution:

Consider each option carefully:

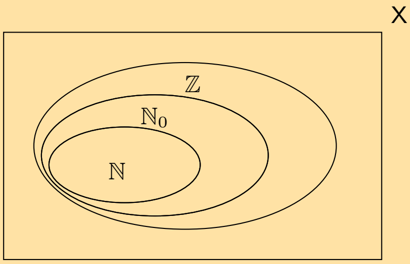
i. There are integers which do not fall into the natural numbers (all negative numbers), so this is false.

ii. The natural numbers are $\{1; 2; 3; \ldots\}$ and whole numbers are $\{0; 1; 2; 3; \ldots\}$ (the circle $\mathbb{N}$ is inside $\mathbb{N}_0$) so if a number is a natural number it must be a whole number. This is true.

iii. Whole numbers $\{0; 1; 2; 3; \ldots\}$ only go up in steps of 1, so there cannot be any decimal numbers in the whole numbers, making this false.

So only (ii) is true.

2. The figure here shows the Venn diagram for the special sets $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$.



a) Where does the number $-\frac{1}{2}$ belong in the diagram?

Solution:

$-\frac{1}{2}$ is in its simplest form, therefore it is not in $\mathbb{N}$, $\mathbb{N}_0$ or $\mathbb{Z}$. It is in the space between the rectangle and $\mathbb{Z}$.

b) In the following list, there are two false statements and one true statement. Which of the statements is **true**?

i. Every integer is a natural number.

ii. Every whole number is an integer.

iii. There are no decimals in the whole numbers.

Solution:

Consider each option carefully:

i. There are integers which do not fall into the natural numbers (all negative numbers), so this is false.

ii. The integers are $\{\ldots; -3; -2; -1; 0; 1; 2; 3; \ldots\}$ and the whole numbers are $\{0; 1; 2; 3; \ldots\}$ (the circle $\mathbb{Z}$ is inside $\mathbb{N}_0$) so if a number is an integer it must be a whole number. This is true.

iii. Whole numbers $\{0; 1; 2; 3; 4; \ldots\}$ only go up in steps of 1, so there cannot be any decimal numbers in the whole numbers, making this false.

So only (ii) is true.

3. State whether the following numbers are real, non-real or undefined.

a) $-\sqrt{3}$

Solution:

$-\sqrt{3}$ has no minus sign under the square root (the minus is outside the root) and is not divided by zero, so it is real.

b) $\frac{0}{\sqrt{2}}$

Solution:

$\frac{0}{\sqrt{2}}$ has no minus sign under the square root (the minus is outside the root) and is not divided by zero, so it is real.

c) $\sqrt{-9}$

Solution:

$\sqrt{-9}$ has a minus sign under the square root so it is non-real.

d) $\frac{-\sqrt{7}}{0}$

Solution:

$\frac{-\sqrt{7}}{0}$ has division by zero so it is undefined.

e) $-\sqrt{-16}$

Solution:

$-\sqrt{-16}$ has a negative number under the square root so it is non-real.

f) $\sqrt{2}$

Solution:

$\sqrt{2}$ has no minus under the square root (the minus is outside the root), is not divided by zero, so it is real.

4. State whether the following numbers are rational or irrational. If the number is rational, state whether it is a natural number, whole number or an integer.

a) $-\frac{1}{3}$

Solution:

$-\frac{1}{3}$ is rational. A fraction of integers is a rational number.

b) 0,651268962154862...

Solution:

0,651268962154862... is irrational. It cannot be simplified to a fraction of integers.

c) $\dfrac{\sqrt{9}}{3}$

Solution:

$\dfrac{\sqrt{9}}{3}$ is rational, an integer, a whole number and a natural number. An integer is a rational number.

d) π^2

Solution:

π^2 is irrational. It cannot be simplified to a fraction of integers.

e) π^4

Solution:

π^4 is irrational. It cannot be simplified to a fraction of integers.

f) $\sqrt[3]{19}$

Solution:

$\sqrt[3]{19}$ is irrational. It cannot be simplified to a fraction of integers.

g) $\left(\sqrt[3]{1}\right)^7$

Solution:

$\left(\sqrt[3]{1}\right)^7$ is rational, an integer, a whole number and a natural number. It can be written as an integer.

h) $\pi + 3$

Solution:

π is irrational. 3 is rational (it is an integer). Any rational number added to any irrational number is irrational. Therefore $\pi + 3$ is irrational.

i) $\pi +$ 0,858408346

Solution:

π is irrational. 0,858408346 is rational (it is a terminating decimal). Any rational number added to any irrational number is irrational.

Therefore $\pi +$ 0,858408346 is irrational.

5. If a is an integer, b is an integer and c is irrational, which of the following are rational numbers?

a) $\dfrac{5}{6}$

Solution:

$\frac{5}{6}$ is rational.

b) $\dfrac{a}{3}$

Solution:

Since a is an integer, $\frac{a}{3}$ is rational.

c) $\dfrac{-2}{b}$

Solution:

Since b is an integer, $\frac{-2}{b}$ is rational.

Note that b cannot be 0 as that makes the fraction undefined.

d) $\dfrac{1}{c}$

Solution:

Since c is irrational, $\frac{1}{c}$ is irrational.

6. For each of the following values of a state whether $\frac{a}{14}$ is rational or irrational.

   a) 1

      Solution:

      $\frac{a}{14} = \frac{1}{14}$ is rational.

   b) -10

      Solution:

      $\frac{a}{14} = \frac{-10}{14}$ is rational.

   c) $\sqrt{2}$

      Solution:

      $\frac{a}{14} = \frac{\sqrt{2}}{14}$ is irrational.

   d) 2,1

      Solution:

      $\frac{a}{14} = \frac{2{,}1}{14}$ is rational.

7. Consider the following list of numbers:

$$-3\ ;\ 0\ ;\ \sqrt{-1}\ ;\ -8\frac{4}{5}\ ;\ -\sqrt{8}\ ;\ \frac{22}{7}\ ;\ \frac{14}{0}\ ;\ 7\ ;\ 1{,}\overline{34}\ ;\ 3{,}3231089...\ ;\ 3+\sqrt{2}\ ;\ 9\frac{7}{10}\ ;\ \pi\ ;\ 11$$

   Which of the numbers are:

   a) natural numbers

      Solution:

      Check which of the numbers are in the set $\{1; 2; 3; 4; \ldots\}$. Therefore 7 and 11 are natural numbers.

   b) irrational numbers

      Solution:

      Remember that rational numbers can be written as $\frac{a}{b}$ where a and b are integers. Also remember that rational numbers include terminating decimal numbers. Therefore $-\sqrt{8}$; 3,3231089... ; $3+\sqrt{2}$; π are all irrational.

   c) non-real numbers

      Solution:

      Any number that is a square root of a negative number is non-real. Therefore only $\sqrt{-1}$ is non-real.

   d) rational numbers

      Solution:

      Remember that rational numbers can be written as $\frac{a}{b}$ where a and b are integers. Also remember that rational numbers include terminating decimal numbers. Therefore -3 ; 0 ; $-8\frac{4}{5}$; $\frac{22}{7}$; 7 ; $1{,}\overline{34}$; $9\frac{7}{10}$; 11 are all rational numbers.

   e) integers

      Solution:

      Check which of the numbers are in the set $\{\ldots; -3; -2; -1; 0; 1; 2; 3; \ldots\}$. Therefore -3 ; 7 ; 11 are integers.

   f) undefined

      Solution:

      Any fraction divided by 0 is undefined. Therefore only $\frac{14}{0}$ is undefined.

8. For each of the following numbers:

   - write the next three digits and
   - state whether the number is rational or irrational.

   a) $1{,}1\dot{5}$

      Solution:

      - Since there is a dot over the 5 we know that the 5 repeats. The next three digits are: 555
      - Rational, there is a repeating pattern of digits.

   b) 2,121314...

      Solution:

      - The number does not terminate (this is shown by the ...). There is also no indication of a repeating pattern of digits since there is not dot or bar over any of the numbers. The next three digits could be any numbers. Note that while it looks like there is a pattern in the digits we do not know if this pattern continues on.
      - Irrational, there is no repeating pattern.

c) 1,242244246...

Solution:

- The number does not terminate (this is shown by the . . .). There is also no indication of a repeating pattern of digits since there is not dot or bar over any of the numbers. The next three digits could be any numbers. Note that while it looks like there is a pattern in the digits we do not know if this pattern continues on.
- Irrational, there is no repeating pattern.

d) 3,324354...

Solution:

- The number does not terminate (this is shown by the . . .). There is also no indication of a repeating pattern of digits since there is not dot or bar over any of the numbers. The next three digits could be any numbers. Note that while it looks like there is a pattern in the digits we do not know if this pattern continues on.
- Irrational, there is no repeating pattern.

e) $3{,}3243\dot{5}\dot{4}$

Solution:

- Since there is a dot over both the 5 and the 4 we know that the pattern 54 repeats. The next three digits are: 545
- Rational, there is a repeating pattern.

9. Write the following as fractions:

a) 0,1

Solution:

$0{,}1 = \frac{1}{10}$

b) 0,12

Solution:

$$\begin{aligned} 0{,}12 &= \frac{1}{10} + \frac{2}{100} \\ &= \frac{10}{100} + \frac{2}{100} \\ &= \frac{12}{100} \\ &= \frac{3}{25} \end{aligned}$$

c) 0,58

Solution:

$$\begin{aligned} 0{,}58 &= \frac{5}{10} + \frac{8}{100} \\ &= \frac{50}{100} + \frac{8}{100} \\ &= \frac{58}{100} \\ &= \frac{29}{50} \end{aligned}$$

d) 0,2589

Solution:

$$\begin{aligned} 0{,}2589 &= \frac{2}{10} + \frac{5}{100} + \frac{8}{1000} + \frac{9}{10\,000} \\ &= \frac{2000}{10\,000} + \frac{500}{10\,000} + \frac{80}{10\,000} + \frac{9}{10\,000} \\ &= \frac{2589}{10\,000} \end{aligned}$$

10. Write the following using the recurring decimal notation:

a) 0,1111111...

Solution:

We see that only the digit 1 is repeated and so we can write this as: $0{,}\dot{1}$.

b) 0,1212121212...

Solution:

There is a repeating pattern of 12 and so we can write this number as: $0,\overline{12}$

c) 0,123123123123...

Solution:

There is a repeating pattern of 123 and so we can write this number as: $0,\overline{123}$

d) 0,11414541454145...

Solution:

The pattern 4145 repeats and so we can write this number as: $0,11\overline{4145}$.

11. Write the following in decimal form, using the recurring decimal notation:

a) $\frac{25}{45}$

Solution:

$$\begin{aligned}
45|\overline{25{,}0000} &= 0 \text{ remainder } 25 \\
45|\overline{25{,}^{25}0000} &= 5 \text{ remainder } 25 \\
45|\overline{25{,}^{25}0^{25}000} &= 5 \text{ remainder } 25 \\
45|\overline{25{,}^{25}0^{25}0^{25}00} &= 5 \text{ remainder } 25 \\
\frac{25}{45} &= 0{,}5555\ldots \\
&= 0{,}\dot{5}
\end{aligned}$$

b) $\frac{10}{18}$

Solution:

$$\begin{aligned}
18|\overline{10{,}0000} &= 0 \text{ remainder } 10 \\
18|\overline{10{,}^{10}0000} &= 5 \text{ remainder } 10 \\
18|\overline{10{,}^{10}0^{10}000} &= 5 \text{ remainder } 10 \\
18|\overline{10{,}^{10}0^{10}0^{10}00} &= 5 \text{ remainder } 10 \\
\frac{10}{18} &= 0{,}5555\ldots \\
&= 0{,}\dot{5}
\end{aligned}$$

c) $\frac{7}{33}$

Solution:

$$\begin{aligned}
33|\overline{7{,}0000} &= 0 \text{ remainder } 7 \\
33|\overline{7{,}^{7}0000} &= 2 \text{ remainder } 4 \\
33|\overline{7{,}^{4}0^{4}000} &= 1 \text{ remainder } 7 \\
33|\overline{7{,}^{7}0^{4}0^{7}00} &= 2 \text{ remainder } 4 \\
\frac{7}{33} &= 0{,}2121\ldots \\
&= 0{,}\dot{2}\dot{1}
\end{aligned}$$

d) $\frac{2}{3}$

Solution:

$$\begin{aligned}\frac{2}{3} &= 2\left(\frac{1}{3}\right)\\ &= 2(0{,}333333...)\\ &= 0{,}666666...\\ &= 0{,}\dot{6}\end{aligned}$$

e) $1\frac{3}{11}$
Solution:

$$\begin{aligned}1\frac{3}{11} &= 1 + 3\left(\frac{1}{11}\right)\\ &= 1 + 3(0{,}090909...)\\ &= 1 + 0{,}27272727...\\ &= 1{,}\overline{27}\end{aligned}$$

f) $4\frac{5}{6}$
Solution:

$$\begin{aligned}4\frac{5}{6} &= 4 + 5\left(\frac{1}{6}\right)\\ &= 4 + 5(0{,}1666666...)\\ &= 4 + 0{,}833333...\\ &= 4{,}8\dot{3}\end{aligned}$$

g) $2\frac{1}{9}$
Solution:

$$\begin{aligned}2\frac{1}{9} &= 2 + 0{,}1111111...\\ &= 2{,}\dot{1}\end{aligned}$$

12. Write the following decimals in fractional form:

a) $0{,}\dot{5}$
Solution:

$$\begin{aligned}x &= 0{,}55555... \text{ and}\\ 10x &= 5{,}55555...\\ 10x - x &= (5{,}55555...) - (0{,}55555...)\\ 9x &= 5\\ \therefore x &= \frac{5}{9}\end{aligned}$$

b) $0{,}6\dot{3}$
Solution:

$$\begin{aligned}10x &= 6{,}3333... \text{ and}\\ 100x &= 63{,}3333...\\ 100x - 10x &= (63{,}3333...) - (6{,}3333...)\\ 99x &= 57\\ \therefore x &= \frac{57}{90}\end{aligned}$$

c) $0,\dot{4}$

Solution:

$$\begin{aligned} x &= 0{,}4444\ldots \text{ and} \\ 10x &= 4{,}4444\ldots \\ 10x - x &= (4{,}4444\ldots) - (0{,}4444\ldots) \\ 9x &= 4 \\ \therefore x &= \frac{4}{9} \end{aligned}$$

d) $5,\overline{31}$

Solution:

$$\begin{aligned} x &= 5{,}313131\ldots \text{ and} \\ 100x &= 531{,}313131\ldots \\ 100x - x &= (531{,}313131\ldots) - (5{,}313131\ldots) \\ 99x &= 526 \\ \therefore x &= \frac{526}{99} \end{aligned}$$

e) $4,\overline{93}$

Solution:

$$\begin{aligned} x &= 4{,}939393\ldots \text{ and} \\ 100x &= 493{,}939393\ldots \\ 100x - x &= (493{,}939393\ldots) - (4{,}939393\ldots) \\ 99x &= 489 \\ \therefore x &= \frac{163}{33} \end{aligned}$$

f) $3,\overline{93}$

Solution:

$$\begin{aligned} x &= 3{,}939393\ldots \text{ and} \\ 100x &= 393{,}939393\ldots \\ 100x - x &= (393{,}939393\ldots) - (3{,}939393\ldots) \\ 99x &= 390 \\ \therefore x &= \frac{130}{33} \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2DBM	2. 2DBN	3a. 2DBP	3b. 2DBQ	3c. 2DBR	3d. 2DBS	3e. 2DBT	3f. 2DBV
4a. 2DBX	4b. 2DBY	4c. 2DC2	4d. 2DC3	4e. 2DC4	4f. 2DC5	4g. 2DC6	4h. 2DBZ
4i. 2DBW	5. 2DC7	6. 2DC8	7. 2DC9	8a. 2DCB	8b. 2DCC	8c. 2DCD	8d. 2DCF
8e. 2DCG	9a. 2DCH	9b. 2DCJ	9c. 2DCK	9d. 2DCM	10a. 2DCN	10b. 2DCP	10c. 2DCQ
10d. 2DCR	11a. 2DCS	11b. 2DCT	11c. 2DCV	11d. 2DCW	11e. 2DCX	11f. 2DCY	11g. 2DCZ
12a. 2DD2	12b. 2DD3	12c. 2DD4	12d. 2DD5	12e. 2DD6	12f. 2DD7		

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Exercise 1 – 2:

1. Round off the following to 3 decimal places:

   a) 12,56637061...
   Solution:
   Mark off the required number of decimal places: $12{,}566|37061\ldots$. The next digit is a 3 and so we round down: 12,566.

   b) 3,31662479...
   Solution:
   Mark off the required number of decimal places: $3{,}316|62479\ldots$. The next digit is a 6 and so we round up: 3,317.

   c) 0,2666666...
   Solution:
   Mark off the required number of decimal places: $0{,}266|6666\ldots$. The next digit is a 6 and so we round up: 0,267.

   d) 1,912931183...
   Solution:
   Mark off the required number of decimal places: $1{,}912|931183\ldots$. The next digit is a 9 and so we round up: 1,913.

   e) 6,32455532...
   Solution:
   Mark off the required number of decimal places: $6{,}324|55532\ldots$. The next digit is a 5 and so we round up: 6,325.

   f) 0,05555555...
   Solution:
   Mark off the required number of decimal places: $0{,}055|55555\ldots$. The next digit is a 5 and so we round up: 0,056.

2. Round off each of the following to the indicated number of decimal places:

   a) 345,04399906 to 4 decimal places.
   Solution:

   $$345{,}04399906 \approx 345{,}0440$$

   b) 1361,72980445 to 2 decimal places.
   Solution:

   $$1361{,}72980445 \approx 1361{,}73$$

   c) 728,00905239 to 6 decimal places.
   Solution:

   $$728{,}00905239 \approx 728{,}009052$$

   d) $\frac{1}{27}$ to 4 decimal places.
   Solution:
   We first write the fraction as a decimal and then we can round off.

   $$\begin{aligned}\frac{1}{27} &= 0{,}037037\ldots \\ &\approx 0{,}0370\end{aligned}$$

   e) $\frac{45}{99}$ to 5 decimal places.
   Solution:

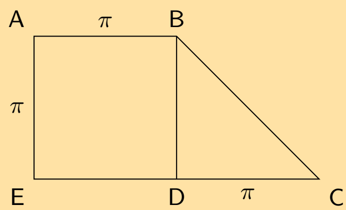
We first write the fraction as a decimal and then we can round off.

$$\begin{aligned}\frac{45}{99} &= 0{,}45454545...\\ &\approx 0{,}45455\end{aligned}$$

f) $\frac{1}{12}$ to 2 decimal places.
Solution:
We first write the fraction as a decimal and then we can round off.

$$\begin{aligned}\frac{1}{12} &= 0{,}08333...\\ &\approx 0{,}08\end{aligned}$$

3. Study the diagram below



a) Calculate the area of $ABDE$ to 2 decimal places.
Solution:
$ABDE$ is a square and so the area is just the length squared.

$$\begin{aligned}A &= l^2\\ &= \pi^2\\ &= 9{,}86904...\\ &\approx 9{,}87\end{aligned}$$

b) Calculate the area of BCD to 2 decimal places.
Solution:
BCD is a right-angled triangle and so we have the perpendicular height. The area is:

$$\begin{aligned}A &= \frac{1}{2}bh\\ &= \frac{1}{2}\pi^2\\ &= 4{,}934802...\\ &\approx 4{,}93\end{aligned}$$

c) Using you answers in (a) and (b) calculate the area of $ABCDE$.
Solution:
The area of $ABCDE$ is the sum of the areas of $ABDE$ and BCD.

$$\begin{aligned}A &= 9{,}87 + 4{,}93\\ &\approx 14{,}80\end{aligned}$$

d) Without rounding off, what is the area of $ABCDE$?
Solution:

$$
\begin{aligned}
A_{ABCDE} &= A_{ABDE} + A_{BCD} \\
&= l^2 + \frac{1}{2}bh \\
&= \pi^2 + \frac{1}{2}\pi^2 \\
&= 14{,}8044...
\end{aligned}
$$

4. Given $i = \frac{r}{600}$; $r = 7{,}4$; $n = 96$; $P = 200\ 000$.
   a) Calculate i correct to 2 decimal places.
   Solution:

$$
\begin{aligned}
i &= \frac{r}{600} \\
&= \frac{7{,}4}{600} \\
&= 0{,}01233 \\
&\approx 0{,}01
\end{aligned}
$$

   b) Using you answer from (a), calculate A in $A = P(1 + i)^n$.
   Solution:

$$
\begin{aligned}
A &= P(1 + i)^n \\
&= 200\ 000\,(1 + 0{,}01)^{96} \\
&= 519\ 854{,}59
\end{aligned}
$$

   c) Calculate A without rounding off your answer in (a), compare this answer with your answer in (b).
   Solution:

$$
\begin{aligned}
A &= P(1 + i)^n \\
A &= 200\ 000\left(1 + \frac{7{,}4}{600}\right)^{96} \\
&= 648\ 768{,}22
\end{aligned}
$$

   There is a 128 913,63 difference between the answer in (b) and the one calculated without rounding until the final step.

5. If it takes 1 person to carry 3 boxes, how many people are needed to carry 31 boxes?
   Solution:
   Each person can carry 3 boxes. So we need to divide 31 by 3 to find out how many people are needed to carry 31 boxes.

$$
\frac{31}{3} = 10{,}3333...
$$

   Therefore 11 people are needed to carry 31 boxes. We cannot have 0,333 of a person so we round up to the nearest whole number.

6. If 7 tickets cost R 35,20, how much does one ticket cost?
   Solution:
   Since 7 tickets cost R 35,20, 1 ticket must cost R 35,20 divided by 7.

$$
\frac{35{,}20}{7} = 5{,}028571429
$$

   Therefore one ticket costs R 5,03. Money should be rounded off to 2 decimal places.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2DD9 1b. 2DDB 1c. 2DDC 1d. 2DDD 1e. 2DDF 1f. 2DDG 2a. 2DDH 2b. 2DDJ
2c. 2DDK 2d. 2DDM 2e. 2DDN 2f. 2DDP 3. 2DDQ 4. 2DDR 5. 2DDS 6. 2DDT

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Exercise 1 – 3:

1. Determine between which two consecutive integers the following numbers lie, without using a calculator:

a) $\sqrt{18}$
Solution:
4 and 5 ($4^2 = 16$ and $5^2 = 25$)

b) $\sqrt{29}$
Solution:
5 and 6 ($5^2 = 25$ and $6^2 = 36$)

c) $\sqrt[3]{5}$
Solution:
1 and 2 ($1^3 = 1$ and $2^3 = 8$)

d) $\sqrt[3]{79}$
Solution:
4 and 5 ($4^3 = 64$ and $5^3 = 125$)

e) $\sqrt{155}$
Solution:
12 and 13 ($12^2 = 144$ and $13^2 = 169$)

f) $\sqrt{57}$
Solution:
7 and 8 ($7^2 = 49$ and $8^2 = 64$)

g) $\sqrt{71}$
Solution:
8 and 9 ($8^2 = 64$ and $9^2 = 81$)

h) $\sqrt[3]{123}$
Solution:
4 and 5 ($4^3 = 64$ and $5^3 = 125$)

i) $\sqrt[3]{90}$
Solution:
4 and 5 ($4^3 = 64$ and $5^3 = 125$)

j) $\sqrt[3]{81}$
Solution:
4 and 5 ($4^3 = 64$ and $5^3 = 125$)

2. Estimate the following surds to the nearest 1 decimal place, without using a calculator.

a) $\sqrt{10}$
Solution:
Since $3^2 = 9$ and $4^2 = 16$, $\sqrt{10}$ must lie between 3 and 4. But we note that 10 is closer to 9 than to 16 and so $\sqrt{10}$ will be closer to 3 than to 4.
3,1 or 3,2 are suitable estimates.

b) $\sqrt{82}$
Solution:
Since $9^2 = 81$ and $10^2 = 100$, $\sqrt{82}$ must lie between 9 and 10. But we note that 82 is closer to 81 than to 100 and so $\sqrt{82}$ will be closer to 9 than to 10.
9,1 is a suitable estimate.

c) $\sqrt{15}$
Solution:
Since $3^2 = 9$ and $4^2 = 16$, $\sqrt{15}$ must lie between 3 and 4. But we note that 15 is closer to 16 than to 9 and so $\sqrt{15}$ will be closer to 4 than to 3.
3,9 is a suitable estimate.

d) $\sqrt{90}$

Solution:
Since $9^2 = 81$ and $10^2 = 100$, $\sqrt{90}$ must lie between 9 and 10. But we note that 90 is about halfway between 81 and 100, so $\sqrt{90}$ will be halfway between 3 and 4.
3,5 is a suitable estimate.

3. Consider the following list of numbers:
$\frac{27}{7}$; $\sqrt{19}$; 2π ; $0{,}45$; $0{,}\overline{45}$; $-\sqrt{\frac{9}{4}}$; 6 ; $-\sqrt{8}$; $\sqrt{51}$
Without using a calculator, rank all the numbers in ascending order.
Solution:
Remember that negative numbers are smaller than positive numbers. It may also be helpful to write the fractions as decimals to help you estimate the number. For the surds you can estimate between which two numbers the surd lies and use that to help you rank these numbers.
   - $\frac{27}{7} \approx 3{,}857$
   - $\sqrt{19}$ lies between 4 and 5
   - $2\pi \approx 6{,}28$
   - $-\sqrt{\frac{9}{4}} = -\frac{3}{2} = -1{,}5$
   - $-\sqrt{8}$ lies between -2 and -3
   - $\sqrt{51}$ lies between 7 and 8

   Also note that $0{,}45 < 0{,}\overline{45}$.
   Therefore we get the following order: $-\sqrt{8}$; $-\sqrt{\frac{9}{4}}$; $0{,}45$; $0{,}\overline{45}$; $\frac{27}{7}$; $\sqrt{19}$; 6 ; 2π ; $\sqrt{51}$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1a. 2DDW 1b. 2DDX 1c. 2DDY 1d. 2DDZ 1e. 2DF2 1f. 2DF3
1g. 2DF4 1h. 2DF5 1i. 2DF6 1j. 2DF7 2a. 2DF8 2b. 2DF9
2c. 2DFB 2d. 2DFC 3. 2DFD

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1.6 Products

Multiplying a monomial and a binomial

Multiplying two binomials

Multiplying a binomial and a trinomial

Exercise 1 – 4:

1. Expand the following products:
   a) $2y(y+4)$
   Solution:
   $$2y(y+4) = 2y^2 + 8y$$
   b) $(y+5)(y+2)$
   Solution:

$$
\begin{aligned}
(y+5)(y+2) &= y^2 + 2y + 5y + 10 \\
&= y^2 + 7y + 10
\end{aligned}
$$

c) $(2-t)(1-2t)$
Solution:

$$
\begin{aligned}
(2-t)(1-2t) &= 2 - 4t - t + 2t^2 \\
&= 2t^2 - 5t + 2
\end{aligned}
$$

d) $(x-4)(x+4)$
Solution:

$$
\begin{aligned}
(x-4)(x+4) &= x^2 + 4x - 4x - 16 \\
&= x^2 - 16
\end{aligned}
$$

e) $-(4-x)(x+4)$
Solution:

$$
\begin{aligned}
-(4-x)(x+4) &= -(4x + 16 - x^2 - 4x) \\
&= -(16 - x^2) \\
&= -16 + x^2 \\
&= x^2 - 16
\end{aligned}
$$

f) $-(a+b)(b-a)$
Solution:

$$
\begin{aligned}
-(a+b)(b-a) &= (a+b)(a-b) \\
&= a^2 + ba - ba - 16 \\
&= a^2 - b^2
\end{aligned}
$$

g) $(2p+9)(3p+1)$
Solution:

$$
\begin{aligned}
(2p+9)(3p+1) &= 6p^2 + 2p + 27p + 9 \\
&= 6p^2 + 29p + 9
\end{aligned}
$$

h) $(3k-2)(k+6)$
Solution:

$$
\begin{aligned}
(3k-2)(k+6) &= 3k^2 + 18k - 2k - 12 \\
&= 3k^2 + 16k - 12
\end{aligned}
$$

i) $(s+6)^2$
Solution:

$$
\begin{aligned}
(s+6)^2 &= (s+6)(s+6) \\
&= s^2 + 6s + 6s + 36 \\
&= s^2 + 12s + 36
\end{aligned}
$$

j) $-(7-x)(7+x)$
Solution:

$$
\begin{aligned}
-(7-x)(7+x) &= -(49+7x-7x-x^2) \\
&= -(49-x^2) \\
&= x^2-49
\end{aligned}
$$

k) $(3x-1)(3x+1)$
Solution:

$$
\begin{aligned}
(3x-1)(3x+1) &= 9x^2+3x-3x-1 \\
&= 9x^2-1
\end{aligned}
$$

l) $(7k+2)(3-2k)$
Solution:

$$
\begin{aligned}
(7k+2)(3-2k) &= 21k-14k^2+6-4k \\
&= -14k^2+17k+6
\end{aligned}
$$

m) $(1-4x)^2$
Solution:

$$
\begin{aligned}
(1-4x)^2 &= (1-4x)(1-4x) \\
&= 1-4x-4x+16x^2 \\
&= 16x^2-8x+1
\end{aligned}
$$

n) $(-3-y)(5-y)$
Solution:

$$
\begin{aligned}
(-3-y)(5-y) &= -15+3y-5y+y^2 \\
&= y^2-2y-15
\end{aligned}
$$

o) $(8-x)(8+x)$
Solution:

$$
\begin{aligned}
(8-x)(8+x) &= 64+8x-8x-x^2 \\
&= -x^2+64
\end{aligned}
$$

p) $(9+x)^2$
Solution:

$$
\begin{aligned}
(9+x)^2 &= (9+x)(9+x) \\
& 81+9x+9x+x^2 \\
&= x^2+18x+81
\end{aligned}
$$

q) $(-7y+11)(-12y+3)$
Solution:

$$
\begin{aligned}
(-7y+11)(-12y+3) &= 84y^7-21y-132y+33 \\
&= 84y^2-153y+33
\end{aligned}
$$

r) $(g-5)^2$
Solution:

$$
\begin{aligned}
(g-5)^2 &= (g-5)(g-5) \\
&= g^2-5g-5g+25 \\
&= g^2-10g+25
\end{aligned}
$$

s) $(d+9)^2$
Solution:

$$\begin{aligned}(d+9)^2 &= (d+9)(d+9)\\ &= d^2+9d+9d+81\\ &= d^2+18d+81\end{aligned}$$

t) $(6d+7)(6d-7)$
Solution:

$$\begin{aligned}(6d+7)(6d-7) &= 36d^2-42d+42d-49\\ &= 36d^2-49\end{aligned}$$

u) $(5z+1)(5z-1)$
Solution:

$$\begin{aligned}(5z+1)(5z-1) &= 25z^2-5z+5z-1\\ &= 25z^2-1\end{aligned}$$

v) $(1-3h)(1+3h)$
Solution:

$$\begin{aligned}(1-3h)(1+3h) &= 1+3h-3h-9h^2\\ &= 1-9h^2\end{aligned}$$

w) $(2p+3)(2p+2)$
Solution:

$$\begin{aligned}(2p+3)(2p+2) &= 4p^2+4p+6p+6\\ &= 4p^2+10p+6\end{aligned}$$

x) $(8a+4)(a+7)$
Solution:

$$\begin{aligned}(8a+4)(a+7) &= 8a^2+56a+4a+28\\ &= 8a^2+60a+28\end{aligned}$$

y) $(5r+4)(2r+4)$
Solution:

$$\begin{aligned}(5r+4)(2r+4) &= 10r^2+20r+8r+16\\ &= 10r^2+28r+16\end{aligned}$$

z) $(w+1)(w-1)$
Solution:

$$\begin{aligned}(w+1)(w-1) &= w^2+w-w-1\\ &= w^2-1\end{aligned}$$

2. Expand the following products:

a) $(g + 11)(g - 11)$
Solution:

$$\begin{aligned}(g + 11)(g - 11) &= g^2 + 11g - 11g - 121\\ &= g^2 - 121\end{aligned}$$

b) $(4b - 2)(2b - 4)$
Solution:

$$\begin{aligned}(4b - 2)(2b - 4) &= 8b^2 - 16b - 4b + 8\\ &= 8b^2 - 20b + 8\end{aligned}$$

c) $(4b - 3)(2b - 1)$
Solution:

$$\begin{aligned}(4b - 3)(2b - 1) &= 8b^2 - 4b - 6b + 3\\ &= 8b^2 - 10b + 3\end{aligned}$$

d) $(6x - 4)(3x + 6)$
Solution:

$$\begin{aligned}(6x - 4)(3x + 6) &= 18x^2 + 36x - 12x - 24\\ &= 18x^2 + 24x - 24\end{aligned}$$

e) $(3w - 2)(2w + 7)$
Solution:

$$\begin{aligned}(3w - 2)(2w + 7) &= 6w^2 + 21w - 4w - 14\\ &= 6w^2 + 17w - 14\end{aligned}$$

f) $(2t - 3)^2$
Solution:

$$\begin{aligned}(2t - 3)^2 &= (2t - 3)(2t - 3)\\ &= 4t^2 - 6t - 6t + 9\\ &= 4t^2 - 12t + 9\end{aligned}$$

g) $(5p - 8)^2$
Solution:

$$\begin{aligned}(5p - 8)^2 &= (5p - 8)(5p - 8)\\ &= 25p^2 - 40p - 40p + 64\\ &= 25p^2 - 80p + 64\end{aligned}$$

h) $(4y + 5)^2$
Solution:

$$\begin{aligned}(4y + 5)^2 &= (4y + 5)(4y + 5)\\ &= 16y^2 + 20y + 20y + 25\\ &= 16y^2 + 40y + 25\end{aligned}$$

i) $(2y^6 + 3y^5)(-5y - 12)$
Solution:

$$\begin{aligned}(2y^6 + 3y^5)(-5y - 12) &= -10y^7 - 24y^6 - 15y^6 - 36y^5 \\ &= -10y^7 - 39y^6 - 36y^5\end{aligned}$$

j) $9(8y^2 - 2y + 3)$
Solution:

$$9(8y^2 - 2y + 3) = 72y^2 - 18y + 27$$

k) $(-2y^2 - 4y + 11)(5y - 12)$
Solution:

$$\begin{aligned}(-2y^2 - 4y + 11)(5y - 12) &= -10y^3 - 20y^2 + 55y + 24y^2 + 48y - 132 \\ &= -10y^3 + 4y^2 + 103y - 132\end{aligned}$$

l) $(7y^2 - 6y - 8)(-2y + 2)$
Solution:

$$\begin{aligned}(7y^2 - 6y - 8)(-2y + 2) &= -14y^3 + 12y^2 + 16y + 14y^2 - 12y - 16 \\ &= -14y^3 + 26y^2 + 4y - 16\end{aligned}$$

m) $(10y + 3)(-2y^2 - 11y + 2)$
Solution:

$$\begin{aligned}(10y + 3)(-2y^2 - 11y + 2) &= -20y^3 - 110y^2 + 20y - 6y^2 - 33y + 6 \\ &= -20y^3 - 116y^2 - 13y + 6\end{aligned}$$

n) $(-12y - 3)(2y^2 - 11y + 3)$
Solution:

$$\begin{aligned}(-12y - 3)(2y^2 - 11y + 3) &= -24y^3 + 132y^2 - 36y - 6y^2 + 33y - 9 \\ &= -24y^3 + 126y^2 - 3y - 9\end{aligned}$$

o) $(-10)(2y^2 + 8y + 3)$
Solution:

$$(-10)(2y^2 + 8y + 3) = -20y^2 - 80y - 30$$

p) $(7y + 3)(7y^2 + 3y + 10)$
Solution:

$$\begin{aligned}(7y + 3)(7y^2 + 3y + 10) &= 49y^3 + 21y^2 + 70y + 21y^2 + 9y + 30 \\ &= 49y^3 + 42y^2 + 79y + 30\end{aligned}$$

q) $(a + 2b)(a^2 + b^2 + 2ab)$
Solution:

$$\begin{aligned}(a + 2b)(a^2 + b^2 + 2ab) &= a^3 + ab^2 + 2a^2b + 2a^2b + 2b^3 + 4ab^2 \\ &= a^3 + 4a^2b + 5ab^2 + 2b^3\end{aligned}$$

r) $(x+y)(x^2-xy+y^2)$
Solution:

$$\begin{aligned}(x+y)(x^2-xy+y^2) &= x^3-x^2y+xy^2+x^2y-xy^2+y^3\\ &= x^3+y^3\end{aligned}$$

s) $3m(9m^2+2)+5m^2(5m+6)$
Solution:

$$\begin{aligned}3m(9m^2+2)+5m^2(5m+6) &= 27m^3+6m+25m^3+30m^2\\ &= 52m^3+6m+30m^2\end{aligned}$$

t) $4x^2(10x^3+4)+4x^3(2x^2+6)$
Solution:

$$\begin{aligned}4x^2(10x^3+4)+4x^3(2x^2+6) &= 40x^5+16x^2+8x^5+24x^3\\ &= 48x^5+16x^2+24x^3\end{aligned}$$

u) $3k^3(k^2+3)+2k^2(6k^3+7)$
Solution:

$$\begin{aligned}3k^3(k^2+3)+2k^2(6k^3+7) &= 3k^5+9k^3+12k^5+14k^2\\ &= 15k^5+9k^3+14k^2\end{aligned}$$

v) $(3x+2)(3x-2)(9x^2-4)$
Solution:

$$\begin{aligned}(3x+2)(3x-2)(9x^2-4) &= (9x^2-4)(9x^2-4)\\ &= 81x^4-36x-36x+16\\ &= 81x^4-72x+16\end{aligned}$$

w) $(-6y^4+11y^2+3y)(y+4)(y-4)$
Solution:

$$\begin{aligned}(-6y^4+11y^2+3y)(y+4)(y-4) &= (-6y^4+11y^2+3y)(y^2-16)\\ &= -6y^6+96y^4+11y^4-176y^2+3y^3-48y\\ &= -6y^6+107y^4+3y^3-176y^2-48y\end{aligned}$$

x) $(x+2)(x-3)(x^2+2x-3)$
Solution:

$$\begin{aligned}(x+2)(x-3)(x^2+2x-3) &= (x^2-x-6)(x^2+2x-3)\\ &= x^4+2x^3-3x^2-x^3-2x^2+3x-6x^2-12x+18\\ &= x^4+x^3-11x^2-9x+18\end{aligned}$$

y) $(a+2)^2-(2a-4)^2$
Solution:

$$\begin{aligned}(a+2)^2-(2a-4)^2 &= a^2+4a+4-(4a^2-16a+16)\\ &= a^2+4a+4-4a^2+16a-16\\ &= -3a^2+20a-12\end{aligned}$$

3. Expand the following products:

a) $(2x+3)^2-(x-2)^2$
Solution:

$$\begin{aligned}(2x+3)^2-(x-2)^2 &= 4x^2+12x+9-(x^2-4x+4)\\ &= 4x^2+12x+9-x^2+4x-4\\ &= 3x^2+16x+5\end{aligned}$$

b) $(2a^2-a-1)(a^2+3a+2)$
Solution:

$$\begin{aligned}(2a^2-a-1)(a^2+3a+2) &= 2a^4+6a^3+4a^2-a^3-3a^2-2a-a^2-3a-2\\ &= 2a^4+5a^3-5a-2\end{aligned}$$

c) $(y^2+4y-1)(1-4y-y^2)$
Solution:

$$\begin{aligned}(y^2+4y-1)(1-4y-y^2) &= y^2-4y^3-y^4+4y-16y^2-4y^3-1+4y+y^2\\ &= -y^4-8y^3-14y^2+8y-1\end{aligned}$$

d) $2(x-2y)(x^2+xy+y^2)$
Solution:

$$\begin{aligned}2(x-2y)(x^2+xy+y^2) &= 2(x^3+x^2y+xy^2-2x^2y-2xy^2-y^3)\\ &= 2(x^3-x^2y-xy^2-y^3)\\ &= 2x^3-2x^2y-2xy^2-2y^3\end{aligned}$$

e) $3(a-3b)(a^2+3ab-b^2)$
Solution:

$$\begin{aligned}3(a-3b)(a^2+3ab-b^2) &= 3(a^3+3a^2b-ab^2-3a^2b-9ab^2+3b^3)\\ &= 3(a^3-10ab^2+3b^3)\\ &= 3a^3-30ab^2+9b^3\end{aligned}$$

f) $(2a-b)(2a+b)(2a^2-3ab+b^2)$
Solution:

$$\begin{aligned}(2a-b)(2a+b)(2a^2-3ab+b^2) &= (4a^2-b^2)(2a^2-3ab+b^2)\\ &= 8a^4-12a^3b+4a^2b^2-2a^2b^2+3ab^3-b^4\\ &= 8a^4-12a^3b+2a^2b^2+3ab^3-b^4\end{aligned}$$

g) $2(3x+y)(3x-y)-(3x-y)^2$
Solution:

$$\begin{aligned}2(3x+y)(3x-y)-(3x-y)^2 &= 2(9x^2-y^2)-9x^2+6xy-y^2\\ &= 18x^2-2y^2-9x^2+6xy-y^2\\ &= 9x^2+6xy-3y^2\end{aligned}$$

h) $(x+y)(x-3y)+(2x-y)^2$
Solution:

$$\begin{aligned}(x+y)(x-3y)+(2x-y)^2 &= x^2-3xy+xy-3y^2+4x^2-4xy+y^2\\ &= 5x^2-6xy-2y^2\end{aligned}$$

i) $\left(\frac{x}{3}-\frac{3}{x}\right)\left(\frac{x}{4}+\frac{4}{x}\right)$
Solution:

$$\begin{aligned}\left(\frac{x}{3}-\frac{3}{x}\right)\left(\frac{x}{4}+\frac{4}{x}\right)&=\frac{x^2}{12}+\frac{4}{3}-\frac{3}{4}+\frac{12}{x^2}\\&=\frac{x^2}{12}+\frac{16}{12}-\frac{9}{12}+\frac{12}{x^2}\\&=\frac{x^2}{12}+\frac{7}{12}+\frac{3}{x^2}\end{aligned}$$

j) $\left(x-\frac{2}{x}\right)\left(\frac{x}{3}+\frac{4}{x}\right)$
Solution:

$$\begin{aligned}\left(x-\frac{2}{x}\right)\left(\frac{x}{3}+\frac{4}{x}\right)&=\frac{x^2}{3}+4-\frac{2}{3}-\frac{8}{x^2}\\&=\frac{x^2}{3}+\frac{12}{3}-\frac{2}{3}-\frac{8}{x^2}\\&=\frac{x^2}{3}+\frac{10}{3}-\frac{8}{x^2}\end{aligned}$$

k) $\frac{1}{2}(10x-12y)+\frac{1}{3}(15x-18y)$
Solution:

$$\begin{aligned}\frac{1}{2}(10x-12y)+\frac{1}{3}(15x-18y)&=5x-6y+5x-6y\\&=10x-12y\end{aligned}$$

l) $\frac{1}{2}a(4a+6b)+\frac{1}{4}(8a+12b)$
Solution:

$$\frac{1}{2}a(4a+6b)+\frac{1}{4}(8a+12b)=2a^2+3ab+2a+3b$$

4. What is the value of b, in $(x+b)(x-1)=x^2+3x-4$
Solution:

$$(x+b)(x-1)=x^2-x+bx-b$$

From the constant term we see that $b=4$. We can check the x term: $-x+4x=3x$.

5. What is the value of g, in $(x-2)(x+g)=x^2-6x+8$
Solution:

$$(x-2)(x+g)=x^2+gx-2x-2g$$

From the constant term we see that $-2g=8$, therefore $g=-4$. We can check the x term: $-4x-2x=-6x$.

6. In $(x-4)(x+k)=x^2+bx+c$:

a) For which of these values of k will b be positive?
$-3;-1;0;3;5$
Solution:

$$(x-4)(x+k)=x^2+kx-4x-4k$$

The x term is $kx-4x$ so for b to be positive $k>4$. Therefore $k=5$.

b) For which of these values of k will c be positive?
$-3;-1;0;3;5$
Solution:

$$(x-4)(x+k) = x^2 + kx - 4x - 4k$$

The constant term is $-4k$ so for c to be positive $k < 0$. Therefore $k = -3$ or $k = -1$.

c) For what real values of k will c be positive?

Solution:

From the previous question we see that $k < 0$ will make c positive.

d) For what real values of k will b be positive?

Solution:

From earlier we see that $k > 4$ will make b positive.

7. Answer the following:

a) Expand $\left(x + \frac{4}{x}\right)^2$.

Solution:

$$\begin{aligned}\left(x + \frac{4}{x}\right)^2 &= \left(x + \frac{4}{x}\right)\left(x + \frac{4}{x}\right) \\ &= x^2 + 8 + \frac{16}{x^2}\end{aligned}$$

b) Given that $\left(x + \frac{4}{x}\right)^2 = 14$, determine the value of $x^2 + \frac{16}{x^2}$ without solving for x.

Solution:

$$\left(x + \frac{4}{x}\right)^2 = x^2 + 8 + \frac{16}{x^2}$$

Now we note that the above expression can also be written as $x^2 + \frac{16}{x^2} + 8$. Since $\left(x + \frac{4}{x}\right)^2 = 14$ we get:

$$\begin{aligned}14 &= x^2 + 8 + \frac{16}{x^2} \\ 14 - 8 &= x^2 + \frac{16}{x^2} \\ 6 &= x^2 + \frac{16}{x^2}\end{aligned}$$

8. Answer the following:

a) Expand: $\left(a + \frac{1}{a}\right)^2$

Solution:

$$\left(a + \frac{1}{a}\right)^2 = a^2 + 2 + \frac{1}{a^2}$$

b) Given that $\left(a + \frac{1}{a}\right) = 3$, determine the value of $\left(a + \frac{1}{a}\right)^2$ without solving for a.

Solution:

$$\begin{aligned}\left(a + \frac{1}{a}\right)^2 &= 3^2 \\ &= 9\end{aligned}$$

c) Given that $\left(a - \frac{1}{a}\right) = 3$, determine the value of $\left(a + \frac{1}{a}\right)^2$ without solving for a.

Solution:

We note that:

$$\left(a+\frac{1}{a}\right)^2 = a^2+2+\frac{1}{a^2}$$

$$\left(a-\frac{1}{a}\right)^2 = a^2-2+\frac{1}{a^2}$$

Next we note that if we add 4 to $\left(a-\frac{1}{a}\right)^2$ we get $\left(a+\frac{1}{a}\right)^2$. Therefore:

$$\begin{aligned}\left(a+\frac{1}{a}\right)^2 &= a^2-2+\frac{1}{a^2}+4\\ &= 3^2+4\\ &= 9+4\\ &= 13\end{aligned}$$

9. Answer the following:

   a) Expand: $\left(3y+\frac{1}{2y}\right)^2$

   Solution:

$$\left(3y+\frac{1}{2y}\right)^2 = 9y^2+3+\frac{1}{4y^2}$$

   b) Given that $3y+\frac{1}{2y}=4$, determine the value of $\left(3y+\frac{1}{2y}\right)^2$ without solving for y.

   Solution:

$$\begin{aligned}\left(3y+\frac{1}{2y}\right)^2 &= 4^2\\ &= 16\end{aligned}$$

10. Answer the following:

   a) Expand: $\left(a+\frac{1}{3a}\right)^2$

   Solution:

$$\left(a+\frac{1}{3a}\right)^2 = a^2+\frac{2}{3}+\frac{1}{9a^2}$$

   b) Expand: $\left(a+\frac{1}{3a}\right)\left(a^2-\frac{1}{3}+\frac{1}{9a^2}\right)$

   Solution:

$$\begin{aligned}\left(a+\frac{1}{3a}\right)\left(a^2-\frac{1}{3}+\frac{1}{9a^2}\right) &= a^3-\frac{1}{3}a+\frac{1}{9a}+\frac{1}{3}a-\frac{1}{9a}+\frac{1}{27a^3}\\ &= a^3+\frac{1}{27a^3}\end{aligned}$$

   c) Given that $a+\frac{1}{3a}=2$, determine the value of $a^3+\frac{1}{27a^3}$ without solving for a.

   Solution:

$$a^3 + \frac{1}{27a^3} = \left(a + \frac{1}{3a}\right)\left(a^2 - \frac{1}{3} + \frac{1}{9a^2}\right)$$
$$= 2\left(a^2 - \frac{1}{3} + \frac{1}{9a^2}\right)$$

$$a^2 - \frac{1}{3} + \frac{1}{9a^2} = \left(a + \frac{1}{3a}\right)^2 - 1$$
$$= 4 - 1$$
$$= 3$$

$$a^3 + \frac{1}{27a^3} = 2(3)$$
$$= 6$$

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1a. 2DFG	1b. 2DFH	1c. 2DFJ	1d. 2DFK	1e. 2DFM	1f. 2DFN	1g. 2DFP	1h. 2DFQ
1i. 2DFR	1j. 2DFS	1k. 2DFT	1l. 2DFV	1m. 2DFW	1n. 2DFX	1o. 2DFY	1p. 2DFZ
1q. 2DG3	1r. 2DG4	1s. 2DG5	1t. 2DG6	1u. 2DG7	1v. 2DG8	1w. 2DG9	1x. 2DGB
1y. 2DGC	1z. 2DGD	2a. 2DGF	2b. 2DGH	2c. 2DGJ	2d. 2DGK	2e. 2DGM	2f. 2DGN
2g. 2DGP	2h. 2DGQ	2i. 2DG2	2j. 2DGR	2k. 2DGS	2l. 2DGT	2m. 2DGV	2n. 2DGW
2o. 2DGX	2p. 2DGY	2q. 2DGZ	2r. 2DH2	2s. 2DH3	2t. 2DH4	2u. 2DH5	2v. 2DH6
2w. 2DH7	2x. 2DH8	2y. 2DH9	3a. 2DHB	3b. 2DHC	3c. 2DHD	3d. 2DHF	3e. 2DHG
3f. 2DHH	3g. 2DHJ	3h. 2DHK	3i. 2DHM	3j. 2DHN	3k. 2DHP	3l. 2DHQ	4. 2DHR
5. 2DHS	6. 2DHT	7. 2DHV	8. 2DHW	9. 2DHX	10. 2DHY		

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1.7 Factorisation

Common factors

Exercise 1 – 5:

Factorise:

1. $12x + 32y$
   Solution:
   $$12x + 32y = 4(3x + 8y)$$
2. $-2ab^2 - 4a^2b$
   Solution:
   $$-2ab^2 - 4a^2b = -2ab(b + 2a)$$
3. $18ab - 3bc$
   Solution:
   $$18ab - 3bc = 3b(6a - c)$$
4. $12kj + 18kq$
   Solution:
   $$12kj + 18kq = 6k(2j + 3q)$$

5. $-12a + 24a^3$
   Solution:

$$-12a + 24a^3 = 12a(-1 + 2a^2)$$

6. $-2ab - 8a$
   Solution:

$$-2ab - 8a = -2a(b + 4)$$

7. $24kj - 16k^2j$
   Solution:

$$24kj - 16k^2j = 8kj(3 - 2k)$$

8. $-a^2b - b^2a$
   Solution:

$$-a^2b - b^2a = -ab(a + b)$$

9. $72b^2q - 18b^3q^2$
   Solution:

$$72b^2q - 18b^3q^2 = 18b^2q(4 - bq)$$

10. $125x^6 - 5y^2$
    Solution:

$$\begin{aligned} 125x^6 - 5y^2 &= 5(25x^6 - y^2) \\ &= 5(5x^3 - y)(5x^3 + y) \end{aligned}$$

11. $6x^2 + 2x + 10x^3$
    Solution:

$$6x^2 + 2x + 10x^3 = 2x(3x + 1 + 5x^2)$$

12. $2xy^2 + xy^2z + 3xy$
    Solution:

$$2xy^2 + xy^2z + 3xy = xy(2y + yz + 3)$$

13. $12k^2j + 24k^2j^2$
    Solution:

$$12k^2j + 24k^2j^2 = 12k^2j(1 + 2j)$$

14. $3a^2 + 6a - 18$
    Solution:

$$3a^2 + 6a - 18 = 3(a^2 + 2a - 6)$$

15. $7a + 4$
    Solution:

$$7a + 4$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2DJ2 2. 2DJ3 3. 2DJ4 4. 2DJ5 5. 2DJ6 6. 2DJ7 7. 2DJ8 8. 2DJ9
9. 2DJB 10. 2DJC 11. 2DJD 12. 2DJF 13. 2DJG 14. 2DJH 15. 2DJJ

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Exercise 1 – 6:

Factorise:

1. $4(y-3)+k(3-y)$
   Solution:
$$\begin{aligned} 4(y-3)+k(3-y) &= 4(y-3)-k(y-3) \\ &= (y-3)(4-k) \end{aligned}$$

2. $a^2(a-1)-25(a-1)$
   Solution:
$$\begin{aligned} a^2(a-1)-25(a-1) &= (a-1)(a^2-25) \\ &= (a-1)(a-5)(a+5) \end{aligned}$$

3. $bm(b+4)-6m(b+4)$
   Solution:
$$\begin{aligned} bm(b+4)-6m(b+4) &= (b+4)(bm-6m) \\ &= (b+4)(m)(b-6) \end{aligned}$$

4. $a^2(a+7)+9(a+7)$
   Solution:
$$a^2(a+7)+9(a+7) = (a+7)(a^2+9)$$

5. $3b(b-4)-7(4-b)$
   Solution:
$$\begin{aligned} 3b(b-4)-7(4-b) &= 3b(b-4)+7(b-4) \\ &= (b-4)(3b+7) \end{aligned}$$

6. $3g(z+6)+2(6+z)$
   Solution:
$$\begin{aligned} 3g(z+6)+2(6+z) &= 3g(z+6)+2(z+6) \\ &= (z+6)(3g+2) \end{aligned}$$

7. $4b(y+2)+5(2+y)$
   Solution:
$$\begin{aligned} 4b(y+2)+5(2+y) &= 4b(y+2)+5(y+2) \\ &= (y+2)(4b+5) \end{aligned}$$

8. $3d(r+5)+14(5+r)$
   Solution:
$$\begin{aligned} 3d(r+5)+14(5+r) &= 3d(r+5)+14(r+5) \\ &= (r+5)(3d+14) \end{aligned}$$

9. $(6x+y)^2 - 9$
Solution:

$$(6x+y)^2 - 9 = (6x+y-3)(6x+y+3)$$

10. $4x^2 - (4x-3y)^2$
Solution:

$$\begin{aligned} 4x^2 - (4x-3y)^2 &= (2x+4x-3y)(2x-(4x-3y)) \\ &= (6x-3y)(3y-2x) \\ &= 3(2x-y)(3y-2x) \end{aligned}$$

11. $16a^2 - (3b+4c)^2$
Solution:

$$\begin{aligned} 16a^2 - (3b+4c)^2 &= (4a+3b+4c)(4a-(3b+4c)) \\ &= (4a+3b+4c)(4a-3b-4c) \end{aligned}$$

12. $(b-4)^2 - 9(b-5)^2$
Solution:

$$\begin{aligned} (b-4)^2 - 9(b-5)^2 &= (b-4-3(b-5))(b-4+3(b-5)) \\ &= (-2b+11)(4b-19) \end{aligned}$$

13. $4(a-3)^2 - 49(4a-5)$
Solution:

$$\begin{aligned} 4(a-3)^2 - 49(4a-5)^2 &= (2(a-3)-7(4a-5))(2(a-3)+7(4a-5)) \\ &= (2a-6-28a+35)(2a-6+28a-35) \\ &= (29-26a)(30a-41) \end{aligned}$$

14. $16k^2 - 4$
Solution:

$$16k^2 - 4 = (4k-2)(4k+2)$$

15. $a^2b^2c^2 - 1$
Solution:

$$a^2b^2c^2 - 1 = (abc-1)(abc+1)$$

16. $\frac{1}{9}a^2 - 4b^2$
Solution:

$$\frac{1}{9}a^2 - 4b^2 = \left(\frac{1}{3}a - 2b\right)\left(\frac{1}{3}a + 2b\right)$$

17. $\frac{1}{2}x^2 - 2$
Solution:

$$\begin{aligned} \frac{1}{2}x^2 - 2 &= 2\left(\frac{1}{4}x^2 - 1\right) \\ &= 2\left(\frac{1}{2}x + 1\right)\left(\frac{1}{2}x - 1\right) \end{aligned}$$

18. $y^2 - 8$
Solution:
Note that $\left(\sqrt{8}\right)^2 = 8$

$$y^2 - 8 = (y - \sqrt{8})(y + \sqrt{8})$$

19. $y^2 - 13$
Solution:
Note that $\left(\sqrt{13}\right)^2 = 13$

$$y^2 - 13 = (y - \sqrt{13})(y + \sqrt{13})$$

20. $a^2(a - 2ab - 15b^2) - 9b^2(a^2 - 2ab - 15b^2)$
Solution:

$$\begin{aligned} a^2(a - 2ab - 15b^2) - 9b^2(a^2 - 2ab - 15b^2) &= (a^2 - 2ab - 15b^2)(a^2 - 9b^2) \\ &= (a - 5b)(a + 3b)(a - 3b)(a + 3b) \\ &= (a - 3b)(a - 5b)(a + 3b)^2 \end{aligned}$$

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1. 2DJM	2. 2DJN	3. 2DJP	4. 2DJQ	5. 2DJR	6. 2DJS	7. 2DJT	8. 2DJV
9. 2DJW	10. 2DJX	11. 2DJY	12. 2DJZ	13. 2DK2	14. 2DK3	15. 2DK4	16. 2DK5
17. 2DK6	18. 2DK7	19. 2DK8	20. 2DK9				

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Factorising by grouping in pairs

Exercise 1 – 7:

Factorise the following:

1. $6d - 9r + 2t^5d - 3t^5r$
Solution:

$$\begin{aligned} 6d - 9r + 2t^5d - 3t^5r &= 3(2d - 3r) + t^5(2d - 3r) \\ &= (2d - 3r)(3 + t^5) \end{aligned}$$

2. $9z - 18m + b^3z - 2b^3m$
Solution:

$$\begin{aligned} 9z - 18m + b^3z - 2b^3m &= 9(z - 2m) + b^3(z - 2m) \\ &= (z - 2m)(9 + b^3) \end{aligned}$$

3. $35z - 10y + 7c^5z - 2c^5y$
Solution:

$$\begin{aligned} 35z - 10y + 7c^5z - 2c^5y &= 5(7z - 2y) + c^5(7z - 2y) \\ &= (7z - 2y)(5 + c^5) \end{aligned}$$

4. $6x + a + 2ax + 3$

Solution:

$$\begin{aligned}6x + a + 2ax + 3 &= 6x + 3 + a + 2ax\\&= 3(2x + 1) + a(2x + 1)\\&= (3 + a)(2x + 1)\end{aligned}$$

5. $x^2 - 6x + 5x - 30$

Solution:

$$\begin{aligned}x^2 - 6x + 5x - 30 &= x(x - 6) + 5(x - 6)\\&= (x + 5)(x - 6)\end{aligned}$$

6. $5x + 10y - ax - 2ay$

Solution:

$$\begin{aligned}5x + 10y - ax - 2ay &= 5(x + 2y) - a(x + 2y)\\&= (5 - a)(x + 2y)\end{aligned}$$

7. $a^2 - 2a - ax + 2x$

Solution:

$$\begin{aligned}a^2 - 2a - ax + 2x &= a(a - 2) - x(a - 2)\\&= (a - x)(a - 2)\end{aligned}$$

8. $5xy - 3y + 10x - 6$

Solution:

$$\begin{aligned}5xy - 3y + 10x - 6 &= y(5x - 3) + 2(5x - 3)\\&= (y + 2)(5x - 3)\end{aligned}$$

9. $ab - a^2 - a + b$

Solution:

$$\begin{aligned}ab - a^2 - a + b &= -a^2 - a + ab + b\\&= -a(a + 1) + b(a + 1)\\&= (-a + b)(a + 1)\end{aligned}$$

10. $14m - 4n + 7jm - 2jn$

Solution:

$$\begin{aligned}14m - 4n + 7jm - 2jn &= 2(7m - 2n) + j(7m - 2n)\\&= (7m - 2n)(2 + j)\end{aligned}$$

11. $28r - 20x + 7gr - 5gx$

Solution:

$$\begin{aligned}28r - 20x + 7gr - 5gx &= 4(7r - 5x) + g(7r - 5x)\\&= (7r - 5x)(4 + g)\end{aligned}$$

12. $25d - 15m + 5yd - 3ym$

Solution:

$$\begin{aligned}25d - 15m + 5yd - 3ym &= 5(5d - 3m) + y(5d - 3m)\\&= (5d - 3m)(5 + y)\end{aligned}$$

13. $45q - 18z + 5cq - 2cz$
Solution:

$$\begin{aligned} 45q - 18z + 5cq - 2cz &= 9(5q - 2z) + c(5q - 2z) \\ &= (5q - 2z)(9 + c) \end{aligned}$$

14. $6j - 15v + 2yj - 5yv$
Solution:

$$\begin{aligned} 6j - 15v + 2yj - 5yv &= 3(2j - 5v) + y(2j - 5v) \\ &= (2j - 5v)(3 + y) \end{aligned}$$

15. $16a - 40k + 2za - 5zk$
Solution:

$$\begin{aligned} 16a - 40k + 2za - 5zk &= 8(2a - 5k) + z(2a - 5k) \\ &= (2a - 5k)(8 + z) \end{aligned}$$

16. $ax - bx + ay - by + 2a - 2b$
Solution:

$$\begin{aligned} ax - bx + ay - by + 2a - 2b &= x(a - b) + y(a - b) + 2(a - b) \\ &= (a - b)(x + y + 2) \end{aligned}$$

17. $3ax + bx - 3ay - by - 9a - 3b$
Solution:

$$\begin{aligned} 3ax + bx - 3ay - by - 9a - 3b &= x(3a + b) - y(3a + b) - 3(3a + b) \\ &= (3a + b)(x - y - 3) \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2DKB 2. 2DKC 3. 2DKD 4. 2DKF 5. 2DKG 6. 2DKH 7. 2DKJ
8. 2DKK 9. 2DKM 10. 2DKN 11. 2DKP 12. 2DKQ 13. 2DKR 14. 2DKS
15. 2DKT 16. 2DKV 17. 2DKW

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Factorising a quadratic trinomial

General procedure for factorising a trinomial

Exercise 1 – 8:

Factorise the following:

1. $x^2 + 8x + 15$
Solution:

$$x^2 + 8x + 15 = (x + 5)(x + 3)$$

2. $x^2 + 9x + 8$
   Solution:

$$x^2 + 9x + 8 = (x + 8)(x + 1)$$

3. $x^2 + 12x + 36$
   Solution:

$$\begin{aligned} x^2 + 12x + 36 &= (x + 6)(x + 6) \\ &= (x + 6)^2 \end{aligned}$$

4. $2h^2 + 5h - 3$
   Solution:

$$2h^2 + 5h - 3 = (h + 3)(2h - 1)$$

5. $3x^2 + 4x + 1$
   Solution:

$$3x^2 + 4x + 1 = (x + 1)(3x + 1)$$

6. $3s^2 + s - 10$
   Solution:

$$3s^2 + s - 10 = (s + 2)(3s - 5)$$

7. $x^2 - 2x - 15$
   Solution:

$$x^2 - 2x - 15 = (x + 3)(x - 5)$$

8. $x^2 + 2x - 3$
   Solution:

$$x^2 + 2x - 3 = (x + 3)(x - 1)$$

9. $x^2 + x - 20$
   Solution:

$$x^2 + x - 20 = (x + 5)(x - 4)$$

10. $x^2 - x - 20$
    Solution:

$$x^2 - x - 20 = (x - 5)(x + 4)$$

11. $2x^2 - 22x + 20$
    Solution:

$$\begin{aligned} 2x^2 + 22x + 20 &= 2(x^2 + 11x + 10) \\ &= 2(x + 1)(x + 10) \end{aligned}$$

12. $6a^2 + 14a + 8$
    Solution:

$$\begin{aligned} 6a^2 + 14a + 8 &= 2(3a^2 + 7a + 4) \\ &= 2\,(a + 1)\,(3a + 4) \end{aligned}$$

13. $6v^2 - 27v + 27$
    Solution:

$$\begin{aligned} 6v^2 - 27v + 27 &= 3(2v^2 - 9v + 9) \\ &= 3\,(2v - 3)\,(v - 3) \end{aligned}$$

14. $6g^2 - 15g - 9$
Solution:

$$\begin{aligned}6g^2 - 15g - 9 &= 3(2g^2 - 5g - 3)\\ &= 3\,(g-3)\,(2g+1)\end{aligned}$$

15. $3x^2 + 19x + 6$
Solution:

$$3x^2 + 19x + 6 = (3x+1)(x+6)$$

16. $3x^2 + 17x - 6$
Solution:

$$3x^2 + 17x - 6 = (3x-1)(x+6)$$

17. $7x^2 - 6x - 1$
Solution:

$$7x^2 - 6x - 1 = (7x+1)(x-1)$$

18. $6x^2 - 15x - 9$
Solution:

$$\begin{aligned}6x^2 - 15x - 9 &= 3(2x^2 - 5x - 3)\\ &= 3(2x+1)(x-3)\end{aligned}$$

19. $a^2 - 7ab + 12b$
Solution:

$$a^2 - 7ab + 12b^2 = (a-4b)(a-3b)$$

20. $3a^2 + 5ab - 12b^2$
Solution:

$$3a^2 + 5ab - 12b^2 = (3a-4b)(a+3b)$$

21. $98x^4 + 14x^2 - 4$
Solution:

$$\begin{aligned}98x^4 + 14x^2 - 4 &= 2(49x^4 - 7x^2 - 2)\\ &= 2((7x+2)(7x-1))\end{aligned}$$

22. $(x-2)^2 - 7(x-2) + 12$
Solution:

$$\begin{aligned}(x-2)^2 - 7(x-2) + 12 &= ((x-2)-4)((x-2)-3)\\ &= (x-6)(x-5)\end{aligned}$$

23. $(a-2)^2 - 4(a-2) - 5$
Solution:

$$\begin{aligned}(a-2)^2 - 4(a-2) - 5 &= ((a-2)-5)((a-2)+1)\\ &=(a-7)(a-1)\end{aligned}$$

24. $(y+3)^2 - 3(y+3) - 18$
Solution:

$$\begin{aligned}(y+3)^2 - 3(y+3) - 18 &= ((y+3)-6)((y+3)+3)\\ &= (y-3)(y+6)\end{aligned}$$

25. $3(b^2 + 5b) + 12$
Solution:

$$\begin{aligned} 3(b^2 + 5b) + 12 &= 3(b^2 + 5b) + 3(4) \\ &= 3(b^2 + 5b + 4) \\ &= 3(b + 4)(b + 1) \end{aligned}$$

26. $6(a^2 + 3a) - 168$
Solution:

$$\begin{aligned} 6(a^2 + 3a) - 168 &= 6(a^2 + 3a) - 6(28) \\ &= 6(a^2 + 3a - 28) \\ &= 6(a + 7)(a - 4) \end{aligned}$$

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1. 2DKY 2. 2DKZ 3. 2DM2 4. 2DM3 5. 2DM4 6. 2DM5 7. 2DM6 8. 2DM7
9. 2DM8 10. 2DM9 11. 2DMB 12. 2DMC 13. 2DMD 14. 2DMF 15. 2DMG 16. 2DMH
17. 2DMJ 18. 2DMK 19. 2DMM 20. 2DMN 21. 2DMP 22. 2DMQ 23. 2DMR 24. 2DMS
25. 2DMT 26. 2DMV

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Sum and difference of two cubes

Exercise 1 – 9:

Factorise:

1. $w^3 - 8$
Solution:

$$w^3 - 8 = (w - 2)(w^2 + 2w + 4)$$

2. $g^3 + 64$
Solution:

$$g^3 + 64 = (g + 4)(g^2 - 4g + 16)$$

3. $h^3 + 1$
Solution:

$$h^3 + 1 = (h + 1)(h^2 - h + 1)$$

4. $x^3 + 8$
Solution:

$$\begin{aligned} x^3 + 8 &= (x + 2)[(x)^2 - (x)(2) + (2)^2] \\ &= (x + 2)(x^2 - 2x + 4) \end{aligned}$$

5. $27 - m^3$
Solution:

$$
\begin{aligned}
27 - m^3 &= (3 - m)[(3)^2 + (3)(m) + (m)^2] \\
&= (3 - m)(9 + 3m + m^2)
\end{aligned}
$$

6. $2x^3 - 2y^3$
   Solution:

$$
\begin{aligned}
2x^3 - 2y^3 &= 2(x^3 - y^3) \\
&= 2(x - y)[(x)^2 + (x)(y) + y^2] \\
&= 2(x - y)(x^2 + xy + y^2)
\end{aligned}
$$

7. $3k^3 + 81q^3$
   Solution:

$$
\begin{aligned}
3k^3 + 81q^3 &= 3(k^3 + 27q^3) \\
&= 3(k + 3q)[(k)^2 - (k)(3q) + (3q)^2] \\
&= 3(k + 3q)(k^2 - 3kq + 9q^2)
\end{aligned}
$$

8. $64t^3 - 1$
   Solution:

$$
\begin{aligned}
64t^3 - 1 &= (4t - 1)[(4t)^2 + (4t)(1) + (1)^2] \\
&= (4t - 1)(16t^2 + 4t + 1)
\end{aligned}
$$

9. $64x^2 - 1$
   Solution:

$$
64x^2 - 1 = (8x - 1)(8x + 1)
$$

10. $125x^3 + 1$
    Solution:

$$
\begin{aligned}
125x^3 + 1 &= (5x + 1)[(5x)^2 - (5x)(1) + (1)^2] \\
&= (5x + 1)(25x^2 - 5x + 1)
\end{aligned}
$$

11. $25x^3 + 1$
    Solution:
    Note that $\left(\sqrt[3]{25}\right)^3 = 25$.

$$
\begin{aligned}
25x^3 + 1 &= (\sqrt[3]{25}x + 1)[(\sqrt[3]{25}x)^2 - (\sqrt[3]{25}x)(1) + (1)^2] \\
&= (\sqrt[3]{25}x + 1)((\sqrt[3]{25})^2x^2 - \sqrt[3]{25}x + 1)
\end{aligned}
$$

12. $z - 125z^4$
    Solution:

$$
\begin{aligned}
z - 125z^4 &= (z)(1 - 125z^3) \\
&= (z)(1 - 5z)[(1)^2 + (1)(5z) + (5z)^2] \\
&= (z)(1 - 5z)(1 + 5z + 25z^2)
\end{aligned}
$$

13. $8m^6 + n^9$
    Solution:

$$
\begin{aligned}
8m^6 + n^9 &= (2m^2)^3 + (n^3)^3 \\
&= (2m^2 + n^3)[(2m^2)^2 - (2m^2)(n^3) + (n^3)^2] \\
&= (2m^2 + n^3)(4m^4 - 2m^2n^3 + n^6)
\end{aligned}
$$

14. $216n^3 - k^3$
Solution:

$$216n^3 - k^3 = (6n - k)(36n^2 + 6nk + k^2)$$

15. $125s^3 + d^3$
Solution:

$$125s^3 + d^3 = (5s + d)(25s^2 - 5sd + d^2)$$

16. $8k^3 + r^3$
Solution:

$$8k^3 + r^3 = (2k + r)(4k^2 - 2kr + r^2)$$

17. $8j^3k^3l^3 - b^3$
Solution:

$$8j^3k^3l^3 - b^3 = (2jkl - b)(4j^2k^2l^2 + 2jklabc + b^2)$$

18. $27x^3y^3 + w^3$
Solution:

$$27x^3y^3 + w^3 = (3xy + w)(9x^2y^2 - 3xyw + w^2)$$

19. $128m^3 + 2f^3$
Solution:

$$
\begin{aligned}
128m^3 + 2f^3 &= 2(64m^3 + f^3) \\
&= 2(4m + f)(16m^2 - 4mf + f^2)
\end{aligned}
$$

20. $p^{15} - \frac{1}{8}y^{12}$
Solution:

$$
\begin{aligned}
p^{15} - \frac{1}{8}y^{12} &= (p^5)^3 - (\frac{1}{2}y^4)^3 \\
&= \left(p^5 - \frac{1}{2}y^4\right)\left[\left(p^5\right)^2 + \left(p^5\right)\left(\frac{1}{2}y^4\right) + \left(\frac{1}{2}y^4\right)\right] \\
&= \left(p^5 - \frac{1}{2}y^4\right)\left(p^{10} + \frac{1}{2}p^5y^4 + \frac{1}{4}y^8\right)
\end{aligned}
$$

21. $\frac{27}{t^3} - s^3$
Solution:

$$\frac{27}{t^3} - s^3 = (\frac{3}{t} - s)(\frac{9}{t^2} + \frac{3s}{t} + s^2)$$

22. $\frac{1}{64q^3} - h^3$
Solution:

$$\frac{1}{64q^3} - h^3 = (\frac{1}{4q} - h)(\frac{1}{16q^2} + \frac{h}{4q} + h^2)$$

23. $72g^3 + \frac{1}{3}v^3$
Solution:

$$\begin{aligned} 72g^3 + \frac{1}{3}v^3 &= \frac{1}{3}(216g^3 + v^3) \\ &= \frac{1}{3}(6g + v)(36g^2 - 6gv + v^2) \end{aligned}$$

24. $1 - (x - y)^3$
Solution:

$$\begin{aligned} 1 - (x - y)^3 &= (1 - (x - y))[(1)^2 - (1)(x - y) + (x - y)^2] \\ &= (1 - x + y)(1 - x + y + x^2 - 2xy + y^2) \end{aligned}$$

25. $h^4(8g^6 + h^3) - (8g^6 + h^3)$
Solution:

$$\begin{aligned} h^4(8g^6 + h^3) - (8g^6 + h^3) &= (h^4 - 1)(8g^6 + h^3) \\ &= (h^2 - 1)(h^2 + 1)(2g^2 + h)(4g^4 - 2g^2h + h^2) \\ &= (h - 1)(h + 1)(h^2 + 1)(2g^2 + h)(4g^4 - 2g^2h + h^2) \end{aligned}$$

26. $x(125w^3 - h^3) + y(125w^3 - h^3)$
Solution:

$$\begin{aligned} x(125w^3 - h^3) + y(125w^3 - h^3) &= (x + y)(125w^3 - h^3) \\ &= (x + y)(5w - h)(25w^2 + 5wh + h^2) \end{aligned}$$

27. $x^2(27p^3 + w^3) - 5x(27p^3 + w^3) - 6(27p^3 + w^3)$
Solution:

$$\begin{aligned} x^2(27p^3 + w^3) - 5x(27p^3 + w^3) - 6(27p^3 + w^3) &= (x^2 - 5x - 6)(27p^3 + w^3) \\ &= (x - 6)(x + 1)(3p + w)(9p^2 - 3pw + w^2) \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2DMW	2. 2DMX	3. 2DMY	4. 2DMZ	5. 2DN2	6. 2DN3	7. 2DN4	8. 2DN5
9. 2DN6	10. 2DN7	11. 2DN8	12. 2DN9	13. 2DNB	14. 2DNC	15. 2DND	16. 2DNF
17. 2DNG	18. 2DNH	19. 2DNJ	20. 2DNK	21. 2DNM	22. 2DNN	23. 2DNP	24. 2DNQ
25. 2DNR	26. 2DNS	27. 2DNT					

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1.8 Simplification of fractions

Exercise 1 – 10:

1. Simplify (assume all denominators are non-zero):

a) $\frac{3a}{15}$
Solution:

$$\frac{3a}{15} = \frac{a}{5}$$

b) $\frac{2a+10}{4}$

Solution:

$$\begin{aligned}\frac{2a+10}{4} &= \frac{2(a+5)}{4}\\ &= \frac{a+5}{2}\end{aligned}$$

c) $\frac{5a+20}{a+4}$

Solution:

$$\begin{aligned}\frac{5a+20}{a+4} &= \frac{5(a+4)}{a+4}\\ &= 5\end{aligned}$$

d) $\frac{a^2-4a}{a-4}$

Solution:

$$\begin{aligned}\frac{a^2-4a}{a-4} &= \frac{a(a-4)}{a-4}\\ &= a\end{aligned}$$

e) $\frac{3a^2-9a}{2a-6}$

Solution:

$$\begin{aligned}\frac{3a^2-9a}{2a-6} &= \frac{3a(a-3)}{2(a-3)}\\ &= \frac{3a}{2}\end{aligned}$$

f) $\frac{9a+27}{9a+18}$

Solution:

$$\begin{aligned}\frac{9a+27}{9a+18} &= \frac{9(a+3)}{9(a+2)}\\ &= \frac{a+3}{a+2}\end{aligned}$$

Note restriction: $a \neq -2$.

g) $\frac{6ab+2a}{2b}$

Solution:

$$\begin{aligned}\frac{6ab+2a}{2b} &= \frac{2a(3b+1)}{2b}\\ &= \frac{a(3b+1)}{b}\end{aligned}$$

Note restriction: $b \neq 0$.

h) $\dfrac{16x^2y - 8xy}{12x - 6}$

Solution:

$$\begin{aligned}\frac{16x^2y - 8xy}{12x - 6} &= \frac{8xy(2x-1)}{6(2x-1)}\\ &= \frac{8xy}{6}\\ &= \frac{4xy}{3}\end{aligned}$$

i) $\dfrac{4xyp - 8xp}{12xy}$

Solution:

$$\begin{aligned}\frac{4xyp - 8xp}{12xy} &= \frac{4xp(y-2)}{12xy}\\ &= \frac{p(y-2)}{3y}\end{aligned}$$

Note restriction: $y \neq 0$.

j) $\dfrac{9x^2 - 16}{6x - 8}$

Solution:

$$\begin{aligned}\frac{9x^2 - 16}{6x - 8} &= \frac{(3x-4)(3x+4)}{2(3x-4)}\\ &= \frac{3x+4}{2}\end{aligned}$$

k) $\dfrac{b^2 - 81a^2}{18a - 2b}$

Solution:

$$\begin{aligned}\frac{b^2 - 81a^2}{18a - 2b} &= \frac{(b-9)(b+9)}{2(9-b)}\\ &= -\frac{b+9}{2}\end{aligned}$$

l) $\dfrac{t^2 - s^2}{s^2 - 2st + t^2}$

Solution:

$$\begin{aligned}\frac{t^2 - s^2}{s^2 - 2st + t^2} &= \frac{(t-s)(t+s)}{(s-t)^2}\\ &= \frac{-(s-t)(t+s)}{(s-t)^2}\\ &= \frac{-(t+s)}{s-t}\end{aligned}$$

Note restriction: $s \neq t$

m) $\dfrac{x^2 - 2x - 15}{5x - 25}$

Solution:

$$\begin{aligned}\frac{x^2 - 2x - 15}{5x - 25} &= \frac{(x-5)(x+3)}{5(x-5)}\\ &= \frac{x+3}{5}\end{aligned}$$

n) $\dfrac{x^2+2x-15}{x^2+8x+15}$

Solution:

$$\begin{aligned}\frac{x^2+2x-15}{x^2+8x+15} &= \frac{(x+5)(x-3)}{(x+3)(x+5)}\\ &= \frac{x-3}{x+3}\end{aligned}$$

Note restriction: $x \neq -3$.

o) $\dfrac{x^2-x-6}{x^3-27}$

Solution:

$$\begin{aligned}\frac{x^2-x-6}{x^3-27} &= \frac{(x-3)(x+2)}{(x-3)(x^2+3x+9)}\\ &= \frac{x+2}{x^2+3x+9}\end{aligned}$$

p) $\dfrac{a^2+6a-16}{a^3-8}$

Solution:

$$\begin{aligned}\frac{a^2+6a-16}{a^3-8} &= \frac{(a+8)(a-2)}{(a-2)(a^2+2a+4)}\\ &= \frac{a+8}{a^2+2a+4}\end{aligned}$$

q) $\dfrac{a^2-4ab-12b^2}{a^2+4ab+4b^2}$

Solution:

$$\begin{aligned}\frac{a^2-4ab-12b^2}{a^2+4ab+4b^2} &= \frac{(a-6b)(a+2b)}{(a+2b)^2}\\ &= \frac{a-6b}{a+2b}\end{aligned}$$

Note restriction: $a \neq -2b$.

r) $\dfrac{6a^2-7a-3}{3ab+b}$

Solution:

$$\begin{aligned}\frac{6a^2-7a-3}{3ab+b} &= \frac{(2a-3)(3a+1)}{b(3a+1)}\\ &= \frac{2a-3}{b}\end{aligned}$$

Note restriction: $b \neq 0$.

s) $\dfrac{2x^2-x-1}{x^3-x}$

Solution:

$$\begin{aligned}\frac{2x^2-x-1}{x^3-x} &= \frac{(2x+1)(x-1)}{x(x-1)(x+1)}\\ &= \frac{2x+1}{x(x+1)}\end{aligned}$$

Note restrictions: $x \neq -1$ and $x \neq 0$.

t) $\dfrac{qz + qr + 16z + 16r}{z + r}$

Solution:

$$\begin{aligned}\frac{qz + qr + 16z + 16r}{(z + r)} &= \frac{q(z + r) + 16(z + r)}{(z + r)}\\ &= \frac{(z + r)(q + 16)}{(z + r)}\\ &= q + 16\end{aligned}$$

u) $\dfrac{pz - pq + 5z - 5q}{z - q}$

Solution:

$$\begin{aligned}\frac{pz - pq + 5z - 5q}{(z - q)} &= \frac{p(z - q) + 5(z - q)}{(z - q)}\\ &= \frac{(z - q)(p + 5)}{(z - q)}\\ &= p + 5\end{aligned}$$

v) $\dfrac{hx - hg + 13x - 13g}{x - g}$

Solution:

$$\begin{aligned}\frac{hx - hg + 13x - 13g}{(x - g)} &= \frac{h(x - g) + 13(x - g)}{(x - g)}\\ &= \frac{(x - g)(h + 13)}{(x - g)}\\ &= h + 13\end{aligned}$$

w) $\dfrac{f^2a - fa^2}{f - a}$

Solution:

$$\begin{aligned}\frac{f^2a - fa^2}{f - a} &= \frac{af(f - a)}{(f - a)}\\ &= af\end{aligned}$$

2. Simplify (assume all denominators are non-zero):

a) $\dfrac{b^2 + 10b + 21}{3(b^2 - 9)} \div \dfrac{2b^2 + 14b}{30b^2 - 90b}$

Solution:

$$\begin{aligned}\frac{b^2 + 10b + 21}{3(b^2 - 9)} \div \frac{2b^2 + 14b}{30b^2 - 90b} &= \frac{b^2 + 10b + 21}{3(b^2 - 9)} \times \frac{30b^2 - 90b}{2b^2 + 14b}\\ &= \frac{(b + 7)(b + 3)}{3(b - 3)(b + 3)} \times \frac{30b(b - 3)}{2b(b + 7)}\\ &= \frac{1}{3} \times \frac{30}{2}\\ &= 5\end{aligned}$$

b) $\dfrac{x^2 + 17x + 70}{5(x^2 - 100)} \div \dfrac{3x^2 + 21x}{45x^2 - 450x}$

Solution:

$$\begin{aligned}\frac{x^2+17x+70}{5(x^2-100)} \div \frac{3x^2+21x}{45x^2-450x} &= \frac{x^2+17x+70}{5(x^2-100)} \times \frac{45x^2-450x}{3x^2+21x} \\ &= \frac{(x+7)(x+10)}{5(x-10)(x+10)} \times \frac{45x(x-10)}{3x(x+7)} \\ &= \frac{1}{5} \times \frac{45}{3} \\ &= 3\end{aligned}$$

c) $\dfrac{z^2+17z+66}{3(z^2-121)} \div \dfrac{2z^2+12z}{24z^2-264z}$

Solution:

$$\begin{aligned}\frac{z^2+17z+66}{3(z^2-121)} \div \frac{2z^2+12z}{24z^2-264z} &= \frac{z^2+17z+66}{3(z^2-121)} \times \frac{24z^2-264z}{2z^2+12z} \\ &= \frac{(z+6)(z+11)}{3(z-11)(z+11)} \times \frac{24z(z-11)}{2z(z+6)} \\ &= \frac{1}{3} \times \frac{24}{2} \\ &= 4\end{aligned}$$

d) $\dfrac{3a+9}{14} \div \dfrac{7a+21}{a+3}$

Solution:

$$\begin{aligned}\frac{3a+9}{14} \div \frac{7a+21}{a+3} &= \frac{3(a+3)}{14} \div \frac{7(a+3)}{a+3} \\ &= \frac{3(a+3)}{14} \div 7 \\ &= \frac{3(a+3)}{14} \times \frac{1}{7} \\ &= \frac{3(a+3)}{98}\end{aligned}$$

e) $\dfrac{a^2-5a}{2a+10} \times \dfrac{4a}{3a+15}$

Solution:

$$\begin{aligned}\frac{a^2-5a}{2a+10} \times \frac{4a}{3a+15} &= \frac{a(a-5)}{2(a+5)} \times \frac{4a}{3(a+5)} \\ &= \frac{[a(a-5)][4a]}{[2(a+5)][3(a+5)]} \\ &= \frac{4a^2(a-5)}{6(a+5)^2}\end{aligned}$$

Note restriction: $a \neq -5$.

f) $\dfrac{3xp+4p}{8p} \div \dfrac{12p^2}{3x+4}$

Solution:

$$\begin{aligned}\frac{3xp+4p}{8p} \div \frac{12p^2}{3x+4} &= \frac{p(3x+4)}{8p} \div \frac{12p^2}{3x+4} \\ &= \frac{3x+4}{8} \times \frac{3x+4}{12p^2} \\ &= \frac{[3x+4][3x+4]}{[8][12p^2]} \\ &= \frac{(3x+4)^2}{96p^2}\end{aligned}$$

Note restriction: $p \neq 0$.

g) $\dfrac{24a-8}{12} \div \dfrac{9a-3}{6}$

Solution:

$$\begin{aligned}
\frac{24a-8}{12} \div \frac{9a-3}{6} &= \frac{8(3a-1)}{12} \div \frac{3(a-1)}{6} \\
&= \frac{2(3a-1)}{3} \times \frac{2}{a-1} \\
&= \frac{[2(3x-1)][2]}{[3][a-1]} \\
&= \frac{4(3a-1)}{3(a-1)}
\end{aligned}$$

Note restriction: $a \neq 1$.

h) $\dfrac{a^2+2a}{5} \div \dfrac{2a+4}{20}$

Solution:

$$\begin{aligned}
\frac{a^2+2a}{5} \div \frac{2a+4}{20} &= \frac{a(a+2)}{5} \div \frac{2(a+2)}{20} \\
&= \frac{a(a+2)}{5} \times \frac{10}{a+2} \\
&= \frac{[a(a+2)][10]}{[5][a+2]} \\
&= \frac{10a}{5} \\
&= 2a
\end{aligned}$$

i) $\dfrac{p^2+pq}{7p} \times \dfrac{21q}{8p+8q}$

Solution:

$$\begin{aligned}
\frac{p^2+pq}{7p} \times \frac{21q}{8p+8q} &= \frac{p(p+q)}{7p} \times \frac{21q}{8(p+q)} \\
&= \frac{[p(p+q)][21q]}{[7p][8(p+q)]} \\
&= \frac{21pq}{56p} \\
&= \frac{3q}{8}
\end{aligned}$$

j) $\dfrac{5ab-15b}{4a-12} \div \dfrac{6b^2}{a+b}$

Solution:

$$\begin{aligned}
\frac{5ab-15b}{4a-12} \div \frac{6b^2}{a+b} &= \frac{5b(a-3)}{4(a-3)} \div \frac{6b^2}{a+b} \\
&= \frac{5b}{4} \times \frac{a+b}{6b^2} \\
&= \frac{[5b][a+b]}{[4][6b^2]} \\
&= \frac{30b^3}{4(a+b)}
\end{aligned}$$

Note restriction: $a \neq -b$.

k) $\dfrac{16-x^2}{x^2-x-12} \times \dfrac{x+3}{x+4}$

Solution:

$$\begin{aligned}\frac{16-x^2}{x^2-x-12} \times \frac{x+3}{x+4} &= \frac{(4-x)(4+x)}{(x-4)(x+3)} \times \frac{x+3}{x+4} \\ &= -1\end{aligned}$$

l) $\dfrac{a^3+b^3}{a^3} \times \dfrac{5a+5b}{a^2+2ab+b^2}$

Solution:

$$\begin{aligned}\frac{a^3+b^3}{a^3} \times \frac{5a+5b}{a^2+2ab+b^2} &= \frac{(a+b)(a^2-ab+b^2)}{a^3} \times \frac{5(a+b)}{(a+b)^2} \\ &= \frac{a^2-ab+b^2}{a^3} \times 5 \\ &= \frac{5(a^2-ab+b^2)}{a^3}\end{aligned}$$

Note restrictions: $a \neq \pm 0$.

m) $\dfrac{a-4}{a+5a+4} \times \dfrac{a^2+2a+1}{a^2-3a-4}$

Solution:

$$\begin{aligned}\frac{a-4}{a+5a+4} \times \frac{a^2+2a+1}{a^2-3a-4} &= \frac{a-4}{(a+4)(a+1)} \times \frac{(a+1)^2}{(a-4)(a+1)} \\ &= \frac{1}{a+4}\end{aligned}$$

Note restrictions: $a \neq -4$.

n) $\dfrac{3x+2}{x^2-6x+8} \times \dfrac{x-2}{3x^2+8x+4}$

Solution:

$$\begin{aligned}\frac{3x+2}{x^2-6x+8} \times \frac{x-2}{3x^2+8x+4} &= \frac{3x+2}{(x-4)(x-2)} \times \frac{x-2}{(3x+2)(x+2)} \\ &= \frac{1}{(x-4)(x+2)}\end{aligned}$$

Note restrictions: $x \neq 4$ and $x \neq -2$.

o) $\dfrac{a^2-2a+8}{a^2+6a+8} \times \dfrac{a^2+a-12}{3} - \dfrac{3}{2}$

Solution:

$$\begin{aligned}\frac{a^2-2a+8}{a^2+6a+8} \times \frac{a^2+a-12}{3} - \frac{3}{2} &= \frac{(a-4)(a+2)}{(a+2)(a+4)} \times \frac{(a+4)(a-3)}{3} - \frac{3}{2} \\ &= \frac{(a-4)(a-3)}{3} - \frac{3}{2} \\ &= \frac{2(a-4)(a-3)-9}{6} \\ &= \frac{2(a^2-7a+12)-9}{6} \\ &= \frac{2a^2-14a+15}{6}\end{aligned}$$

p) $\dfrac{4x^2-1}{3x^2+10x+3} \div \dfrac{6x^2+5x+1}{4x^2+7x-3} \times \dfrac{9x^2+6x+1}{8x^2-6x+1}$

Solution:

$$\begin{aligned}
&\frac{4x^2-1}{3x^2+10x+3} \div \frac{6x^2+5x+1}{4x^2+7x-3} \times \frac{9x^2+6x+1}{8x^2-6x+1}\\
&= \frac{(2x-1)(2x+1)}{(x+3)(3x+1)} \times \frac{(x+3)(4x-1)}{(2x+1)(3x+1)} \times \frac{(3x+1)^2}{(2x-1)(4x-1)}\\
&= 1
\end{aligned}$$

q) $\dfrac{x+4}{3} - \dfrac{x-2}{2}$

Solution:

$$\begin{aligned}
\frac{x+4}{3} - \frac{x-2}{2} &= \frac{2(x+4)-3(x-2)}{6}\\
&= \frac{2x+8-3x+6}{6}\\
&= \frac{14-x}{6}
\end{aligned}$$

r) $\dfrac{p^3+q^3}{p^2} \times \dfrac{3p-3q}{p^2-q^2}$

Solution:

$$\begin{aligned}
\frac{p^3+q^3}{p^2} \times \frac{3p-3q}{p^2-q^2} &= \frac{(p+q)(p^2-pq+q^2)}{p^2} \times \frac{3(p-q)}{(p-q)(p+q)}\\
&= \frac{(p+q)(p^2-pq+q^2)}{p^2} \times \frac{3}{p+q}\\
&= \frac{3(p^2-pq+q^2)}{p^2}
\end{aligned}$$

Note restriction: $p \neq 0$.

3. Simplify (assume all denominators are non-zero):

a) $\dfrac{x-3}{3} - \dfrac{x+5}{4}$

Solution:

$$\begin{aligned}
\frac{x-3}{3} - \frac{x+5}{4} &= \frac{4(x-3)-3(x+5)}{12}\\
&= \frac{4x-12-3x-15}{12}\\
&= \frac{x-27}{12}
\end{aligned}$$

b) $\dfrac{2x-4}{9} - \dfrac{x-3}{4} + 1$

Solution:

$$\begin{aligned}
\frac{2x-4}{9} - \frac{x-3}{4} + 1 &= \frac{4(2x-4)-9(x-3)+36}{36}\\
&= \frac{8x-16-9x+27+36}{36}\\
&= \frac{47-x}{36}
\end{aligned}$$

c) $1 + \dfrac{3x-4}{4} - \dfrac{x+2}{3}$

Solution:

$$\begin{aligned}1 + \frac{3x-4}{4} - \frac{x+2}{3} &= \frac{12 + 3(3x-4) - 4(x+2)}{12} \\ &= \frac{12 + 9x - 12 - 4x - 8}{12} \\ &= \frac{5x-8}{12}\end{aligned}$$

d) $\frac{11}{a+11} + \frac{8}{a-8}$

Solution:

$$\begin{aligned}\frac{11}{a+11} + \frac{8}{a-8} &= \frac{11(a-8) + 8(a+11)}{(a+11)(a-8)} \\ &= \frac{11a - 88 + 8a + 88}{(a+11)(a-8)} \\ &= \frac{19a}{(a+11)(a-8)}\end{aligned}$$

Note restrictions: $a \neq -11$ and $a \neq 8$.

e) $\frac{12}{x-12} - \frac{6}{x-6}$

Solution:

$$\begin{aligned}\frac{12}{x-12} - \frac{6}{x-6} &= \frac{12(x-6) - 6(x-12)}{(x-12)(x-6)} \\ &= \frac{12x - 72 - 6x + 72}{(x-12)(x-6)} \\ &= \frac{6x}{(x-12)(x-6)}\end{aligned}$$

Note restriction: $x \neq 12$ and $x \neq 6$.

f) $\frac{12}{r+12} + \frac{8}{r-8}$

Solution:

$$\begin{aligned}\frac{12}{r+12} + \frac{8}{r-8} &= \frac{12(r-8) + 8(r+12)}{(r+12)(r-8)} \\ &= \frac{12r - 96 + 8r + 96}{(r+12)(r-8)} \\ &= \frac{20r}{(r+12)(r-8)}\end{aligned}$$

Note restriction: $r \neq -12$ and $r \neq 8$.

g) $\frac{2}{xy} + \frac{4}{xz} + \frac{3}{yz}$

Solution:

$$\begin{aligned}\frac{2}{xy} + \frac{4}{xz} + \frac{3}{yz} &= \frac{2z}{xyz} + \frac{4y}{xyz} + \frac{3x}{xyz} \\ &= \frac{2z + 4y + 3x}{xyz}\end{aligned}$$

Note restrictions: $x \neq 0$; $y \neq 0$ and $z \neq 0$.

h) $\frac{5}{t-2} - \frac{1}{t-3}$

Solution:

$$\begin{aligned}
\frac{5}{t-2}-\frac{1}{t-3} &= \frac{(5)(t-3)}{(t-3)(t-2)}-\frac{1(t-2)}{(t-2)(t-3)} \\
&= \frac{5(t-3)-(t-3)}{(t-2)(t-3)} \\
&= \frac{5t-15-t+3}{(t-2)(t-3)} \\
&= \frac{4t-12}{(t-2)(t-3)}
\end{aligned}$$

Note restrictions: $t \neq 2$ and $t \neq 3$.

i) $\dfrac{k+2}{k^2+2}-\dfrac{1}{k+2}$

Solution:

$$\begin{aligned}
\frac{k+2}{k^2+2}-\frac{1}{k+2} &= \frac{(k+2)(k+2)}{(k^2+2)(k+2)}-\frac{1(k^2+2)}{(k^2+2)(k+2)} \\
&= \frac{(k+2)^2-(k^2+2)}{(k^2+2)(k+2)} \\
&= \frac{k^2+4k+4-k^2-2}{(k^2+2)(k+2)} \\
&= \frac{4k+2}{(k^2+2)(k+2)} \\
&= \frac{2(k+2)}{(k^2+2)(k+2)}
\end{aligned}$$

Note restrictions: $k \neq -2$ and $k^2 \neq \pm\sqrt{2}$.

j) $\dfrac{t+2}{3q}+\dfrac{t+1}{2q}$

Solution:

$$\begin{aligned}
\frac{t+2}{3q}+\frac{t+1}{2q} &= \frac{(t+2)(2q)}{(3q)(2q)}+\frac{(t+1)(3q)}{(3q)(2q)} \\
&= \frac{(2tq+4q)+(3tq+3q)}{6q^2} \\
&= \frac{q(5t+7)}{6q^2} \\
&= \frac{5t+7}{6q}
\end{aligned}$$

Note restriction: $q \neq 0$.

k) $\dfrac{3}{p^2-4}+\dfrac{2}{(p-2)^2}$

Solution:

$$\begin{aligned}
\frac{3}{p^2-4}+\frac{2}{(p-2)^2} &= \frac{3(p-2)^2}{(p^2-4)(p-2)^2}+\frac{2(p^2-4)}{(p^2-4)(p-2)^2} \\
&= \frac{3(p-2)(p-2)+2(p-2)(p+2)}{(p+2)(p-2)^3} \\
&= \frac{[p-2][3(p-2)+2(p+2)]}{(p+2)(p-2)^3} \\
&= \frac{3p-6+2p+4}{(p+2)(p-2)^2} \\
&= \frac{5p-2}{(p+2)(p-2)^2}
\end{aligned}$$

Note restriction: $p \neq \pm 2$.

l) $\dfrac{x}{x+y} + \dfrac{x^2}{y^2-x^2}$

Solution:

$$\begin{aligned}\frac{x}{x+y} + \frac{x^2}{y^2-x^2} &= \frac{x}{x+y} + \frac{x^2}{(x+y)(x-y)} \\ &= \frac{x(x-y)+x^2}{(x+y)(x-y)} \\ &= \frac{x^2-xy+x^2}{(x+y)(x-y)} \\ &= \frac{2x^2-xy}{(x+y)(x-y)}\end{aligned}$$

Note restriction: $x \neq \pm y$.

m) $\dfrac{1}{m+n} + \dfrac{3mn}{m^3+n^3}$

Solution:

$$\begin{aligned}\frac{1}{m+n} + \frac{3mn}{m^3+n^3} &= \frac{1}{m+n} + \frac{3mn}{(m+n)(m^2-mn+n^2)} \\ &= \frac{1(m^2-mn+n^2)+3mn}{(m+n)(m^2-mn+n^2)} \\ &= \frac{m^2+2mn+n^2}{(m+n)(m^2-mn+n^2)} \\ &= \frac{m+n}{m^2-mn+n^2}\end{aligned}$$

n) $\dfrac{h}{h^3-f^3} - \dfrac{1}{h^2+hf+f^2}$

Solution:

$$\begin{aligned}\frac{h}{h^3-f^3} - \frac{1}{h^2+hf+f^2} &= \frac{h}{(h-f)(h^2+hf+f^2)} - \frac{1}{h^2+hf+f^2} \\ &= \frac{h-h+f}{(h+f)(h^2+hf+f^2)} \\ &= \frac{f}{(h+f)(h^2+hf+f^2)}\end{aligned}$$

o) $\dfrac{x^2-1}{3} \times \dfrac{1}{x-1} - \dfrac{1}{2}$

Solution:

$$\begin{aligned}\frac{x^2-1}{3} \times \frac{1}{x-1} - \frac{1}{2} &= \frac{(x^2-1)(1)}{(3)(x-1)} - \frac{1}{2} \\ &= \frac{x^2-1}{3x-3} - \frac{1}{2} \\ &= \frac{(x^2-1)(2)}{2(3x-3)} - \frac{3x-3}{2(3x-3)} \\ &= \frac{2x^2-2-3x+3}{6x-6} \\ &= \frac{(x-1)(2x-1)}{6(x-1)} \\ &= \frac{2x-1}{6}\end{aligned}$$

p) $\dfrac{x^2-2x+1}{(x-1)^3} - \dfrac{x^2+x+1}{x^3-1}$

Solution:

$$\begin{aligned}
\frac{x^2-2x+1}{(x-1)^3}-\frac{x^2+x+1}{x^3-1} &= \frac{(x-1)^2}{(x-1)^3}-\frac{x^2+x+1}{x^3-1}\\
&= \frac{1}{(x-1)}-\frac{x^2+x+1}{(x-1)(x^2+x+1)}\\
&= \frac{1}{(x-1)}-\frac{1}{(x-1)}\\
&= 0
\end{aligned}$$

q) $\dfrac{1}{(x-1)^2}-\dfrac{2x}{x^3-1}$

Solution:

$$\begin{aligned}
\frac{1}{(x-1)^2}-\frac{2x}{x^3-1} &= \frac{1}{(x-1)^2}-\frac{2x}{(x-1)(x^2+x+1)}\\
&= \frac{x^2+x+1-2x(x-1)}{(x-1)^2(x^2+x+1)}\\
&= \frac{x^2+x+1-2x^2+2x}{(x-1)^2(x^2+x+1)}\\
&= \frac{-x^2+3x+1}{(x-1)^2(x^2+x+1)}
\end{aligned}$$

r) $\dfrac{t^2+2t-8}{t^2+t-6}+\dfrac{1}{t^2-9}+\dfrac{t+1}{t-3}$

Solution:

$$\begin{aligned}
\frac{t^2+2t-8}{t^2+t-6}+\frac{1}{t^2-9}+\frac{t+1}{t-3} &= \frac{(t+4)(t-2)}{(t+3)(t-2)}+\frac{1}{(t-3)(t+3)}+\frac{t+1}{t-3}\\
&= \frac{t+4}{t+3}+\frac{1}{(t-3)(t+3)}+\frac{t+1}{t-3}\\
&= \frac{(t-3)(t+4)+1+(t+1)(t+3)}{(t-3)(t+3)}\\
&= \frac{t^2+t-12+1+t^2+4t+3}{(t-3)(t+3)}\\
&= \frac{2t^2+5t-8}{(t-3)(t+3)}\\
&= \frac{2t^2+5t-8}{t^2-9}
\end{aligned}$$

Note restriction: $t \neq \pm 3$.

s) $\dfrac{x^2-3x+9}{x^3+27}+\dfrac{x-2}{x^2+4x+3}-\dfrac{1}{x-2}$

Solution:

$$\begin{aligned}
\frac{x^2-3x+9}{x^3+27}+\frac{x-2}{x^2+4x+3}-\frac{1}{x-2} &= \frac{x^2-3x+9}{(x+3)(x^2-3x+9)}+\frac{x-2}{(x+3)(x+1)}-\frac{1}{x-2}\\
&= \frac{(x+1)(x-2)+(x-2)^2-(x+3)(x+1)}{(x+3)(x+1)(x-2)}\\
&= \frac{x^2-x-2+x^2-4x+4-x^2-4x-3}{(x+3)(x+1)(x-2)}\\
&= \frac{x^2-9x-1}{(x+3)(x+1)(x-2)}
\end{aligned}$$

Note restrictions: $x \neq -3$; $x \neq -1$ and $x \neq 2$.

t) $\dfrac{1}{a^2 - 4ab + 4b^2} + \dfrac{a^2 + 2ab + b^2}{a^3 - 8b^3} - \dfrac{1}{a^2 - 4b^2}$

Solution:

$$\begin{aligned}
&\frac{1}{a^2 - 4ab + 4b^2} + \frac{a^2 + 2ab + b^2}{a^3 - 8b^3} - \frac{1}{a^2 - 4b^2} \\
&= \frac{1}{(a - 2b)(a - 2b)} + \frac{a^2 + 2ab + 4b^2}{(a - 2b)(a^2 + 2ab + 4b^2} - \frac{1}{(a - 2b)(a + 2b)} \\
&= \frac{(a + 2b) + (a - 2b)(a + 2b) - (a - 2b)}{(a - 2b)^2(a + 2b)} \\
&= \frac{a + 2b + a^2 - 4b^2 - a + 2b}{(a - 2b)^2(a + 2b)} \\
&= \frac{a^2 + 4b - 4b^2}{(a - 2b)^2(a + 2b)}
\end{aligned}$$

Note restriction: $a \neq \pm 2b$.

4. What are the restrictions in the following:

a) $\dfrac{1}{x - 2}$

Solution:

We need to find the value of x that will make the denominator equal to 0. Therefore:

$$\begin{aligned}
x - 2 &\neq 0 \\
x &\neq 2
\end{aligned}$$

b) $\dfrac{3x - 9}{4x + 4}$

Solution:

First simplify the fraction:

$$\frac{3x - 9}{4x + 4} = \frac{3(x - 1)}{4(x + 1)}$$

Now we can determine the restriction:

$$\begin{aligned}
4(x + 1) &\neq 0 \\
x + 1 &\neq 0 \\
x &\neq -1
\end{aligned}$$

c) $\dfrac{3}{x} - \dfrac{1}{x^2 - 1}$

Solution:

First simplify the fraction:

$$\frac{3}{x} - \frac{1}{x^2 - 1} = \frac{3}{x} - \frac{1}{(x - 1)(x + 1)}$$

Now we can determine the restrictions. There are three restrictions in this case:

$$\begin{aligned}
x &\neq 0 \\
x - 1 &\neq 0 \\
x + 1 &\neq 0
\end{aligned}$$

Therefore: $x \neq 0$ and $x \neq \pm 1$

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1a. 2DNW	1b. 2DNX	1c. 2DNY	1d. 2DNZ	1e. 2DP2	1f. 2DP3	1g. 2DP4	1h. 2DP5
1i. 2DP6	1j. 2DP7	1k. 2DP8	1l. 2DP9	1m. 2DPB	1n. 2DPC	1o. 2DPD	1p. 2DPF
1q. 2DPG	1r. 2DPH	1s. 2DPJ	1t. 2DPK	1u. 2DPM	1v. 2DPN	1w. 2DPP	2a. 2DPQ
2b. 2DPR	2c. 2DPS	2d. 2DPT	2e. 2DPV	2f. 2DPW	2g. 2DPX	2h. 2DPY	2i. 2DPZ
2j. 2DQ2	2k. 2DQ3	2l. 2DQ4	2m. 2DQ5	2n. 2DQ6	2o. 2DQ7	2p. 2DQ8	2q. 2DQ9
2r. 2DQB	3a. 2DQC	3b. 2DQD	3c. 2DQF	3d. 2DQG	3e. 2DQH	3f. 2DQJ	3g. 2DQK
3h. 2DQM	3i. 2DQN	3j. 2DQP	3k. 2DQQ	3l. 2DQR	3m. 2DQS	3n. 2DQT	3o. 2DQV
3p. 2DQW	3q. 2DQX	3r. 2DQY	3s. 2DQZ	3t. 2DR2	4a. 2DR3	4b. 2DR4	4c. 2DR5

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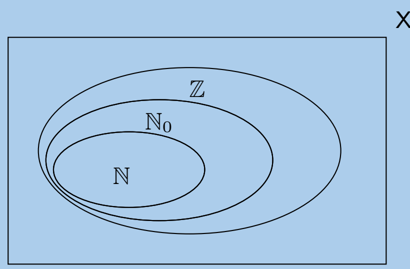
1.9 Chapter summary

End of chapter Exercise 1 – 11:

1. The figure here shows the Venn diagram for the special sets $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$.



   a) Where does the number 2,13 belong in the diagram?
   Solution:
   2,13 is in its simplest form, therefore it is not in $\mathbb{N}$, $\mathbb{N}_0$ or $\mathbb{Z}$). It is in the space between the rectangle and $\mathbb{Z}$

   b) In the following list, there are two false statements and one true statement. Which of the statements is **true**?
   - Every natural number is an integer.
   - Every whole number is a natural number.
   - There are fractions in the integers.

   Solution:
   Consider each statement:
   - Integers are natural numbers and negative natural numbers. Therefore this statement is true.
   - 0 is not a natural number, therefore this statement is false.
   - Integers are natural numbers and negative natural numbers, no fractions. Therefore this is false.

   The only true statement is (i).

2. State whether the following numbers are real, non-real or undefined.

   a) $-\sqrt{-5}$
   Solution:
   This is the square root of a negative number and so is non-real.

   b) $\frac{\sqrt{8}}{0}$
   Solution:
   We are dividing by 0 and so this is undefined.

   c) $-\sqrt{15}$
   Solution:
   This is the square root of a positive number and so is real.

d) $-\sqrt{7}$

Solution:

This is the square root of a positive number and so is real.

e) $\sqrt{-1}$

Solution:

This is the square root of a negative number and so is non-real.

f) $\sqrt{2}$

Solution:

This is the square root of a positive number and so is real.

3. State whether each of the following numbers are rational or irrational.

a) $\sqrt[3]{4}$

Solution:

Irrational. It cannot be simplified to a fraction of integers.

b) 45π

Solution:

Irrational. It cannot be simplified to a fraction of integers

c) $\sqrt{9}$

Solution:

$\sqrt{9} = 3$

Rational. Can be simplified to an integer

d) $\sqrt[3]{8}$

Solution:

$\sqrt[3]{8} = 2$

Rational. Can be simplified to an integers.

4. If a is an integer, b is an integer and c is irrational, which of the following are rational numbers?

a) $\frac{-b}{a}$

Solution:

We have a fraction of integers and so this is rational.

b) $c \div c$

Solution:

When we divide a number by itself we get 1 and so this is rational.

c) $\frac{a}{c}$

Solution:

We are dividing an integer by an irrational number and so this is irrational. However if $a = 0$ then the fraction is equal to 0 and the number is rational.

d) $\frac{1}{c}$

Solution:

We are dividing an integer by an irrational number and so this is irrational.

5. Consider the following list of numbers:

$$\sqrt[3]{26}\ ;\ \frac{3}{2}\ ;\ \sqrt{-24}\ ;\ \sqrt{39}\ ;\ 7{,}\dot{1}\dot{1}\ ;\ \pi^2\ ;\ \frac{\pi}{2}\ ;\ 7{,}12\ ;\ -\sqrt{24}\ ;\ \frac{\sqrt{2}}{0}\ ;\ 3\pi\ ;\ \sqrt{78}\ ;\ 9\ ;\ \pi$$

a) Which of the numbers are non-real numbers?

Solution:

Only $\sqrt{-24}$ is non-real as it is the square root of a negative number.

b) Without using a calculator, rank all the real numbers in ascending order.

Solution:

We exclude $\sqrt{-24}$ from the list as it is non-real. We also exclude $\frac{\sqrt{2}}{0}$ as it is undefined. Then we note that:

- $\sqrt[3]{26}$ lies between 2 and 8
- $\frac{3}{2} = 1{,}5$
- $\sqrt{39}$ lies between 6 and 7

- $\pi^2 \approx 9{,}8696$
- $\frac{\pi}{2} \approx 1{,}5708$
- $-\sqrt{24}$ lies between -4 and -5
- $3\pi \approx 9{,}4248$
- $\sqrt{78}$ lies between 8 and 9
- $\pi \approx 3{,}1416$

Therefore the ordering is: $-\sqrt{24}$; $\frac{3}{2}$; $\frac{\pi}{2}$; $\sqrt[3]{26}$; π ; $\sqrt{39}$; $7{,}\dot{1}\dot{1}$; $7{,}12$; $\sqrt{78}$; 9 ; 3π ; π^2

c) Which of the numbers are irrational numbers?

Solution:

Any number that cannot be written as a fraction of integers is irrational. Therefore $-\sqrt{24}$; $\frac{\pi}{2}$; $\sqrt[3]{26}$; π ; $\sqrt{39}$; $\sqrt{78}$; 3π ; π^2 are all irrational.

d) Which of the numbers are rational numbers?

Solution:

All numbers that can be written as a fraction of integers are rational numbers. Therefore $\frac{3}{2}$; $7{,}\dot{1}\dot{1}$; $7{,}12$; 9 are all rational numbers.

e) Which of the numbers are integers?

Solution:

Only 9 is an integer.

f) Which of the numbers are undefined?

Solution:

Any fraction that has a denominator of 0 is undefined, therefore only $\frac{\sqrt{2}}{0}$ is undefined.

6. Write each decimal as a simple fraction.

a) 0,12

Solution:

$$\begin{aligned} 0{,}12 &= \frac{1}{10} + \frac{2}{100} \\ &= \frac{12}{100} \\ &= \frac{3}{25} \end{aligned}$$

b) 0,006

Solution:

$$\begin{aligned} 0{,}006 &= \frac{6}{1000} \\ &= \frac{3}{500} \end{aligned}$$

c) $4{,}\overline{14}$

Solution:

$$\begin{aligned} x &= 4{,}141414\ldots \\ 100x &= 414{,}141414\ldots \\ 100x - x &= (414{,}141414\ldots) - (4{,}141414\ldots) \\ 99x &= 410 \\ \therefore x &= \frac{410}{99} \end{aligned}$$

d) 1,59

Solution:

$$\begin{aligned} 1{,}59 &= 1 + \frac{5}{10} + \frac{9}{100} \\ &= 1\frac{59}{100} \end{aligned}$$

e) $12{,}27\dot{7}$
Solution:

$$\begin{aligned} x &= 12{,}27\dot{7} \\ 10x &= 122{,}\dot{7} \\ 100x &= 1227{,}\dot{7} \\ \therefore 100x - 10x &= 90x = 1105 \\ \therefore x &= \frac{1105}{90} \\ &= \frac{221}{18} \end{aligned}$$

f) $0{,}8\dot{2}$
Solution:

$$\begin{aligned} 0{,}8\dot{2} &= 0{,}82222{,}.. \\ x &= 0{,}8222... \\ 10x &= 8{,}222... \\ 100x &= 82{,}222... \\ 100x - 10x &= 82{,}222 - 8{,}222... \\ 90x &= 74{,}000 \\ 90x &= 74 \\ \therefore x &= \frac{37}{45} \end{aligned}$$

g) $7{,}\overline{36}$
Solution:

$$\begin{aligned} x &= 7{,}363636... \\ 100x &= 736{,}363636... \\ 100x - x &= (736{,}363636...) - (7{,}363636...) \\ 99x &= 729 \\ \therefore x &= \frac{81}{11} \end{aligned}$$

7. Show that the decimal $3{,}21\dot{1}\dot{8}$ is a rational number.
Solution:

$$\begin{aligned} x &= 3{,}21\dot{1}\dot{8} \\ 1000x &= 32\ 118{,}\overline{18} \\ \therefore 10\ 000x - x &= 9999x = 32\ 115 \\ \therefore x &= \frac{32\ 115}{9999} \end{aligned}$$

This is a rational number because both the numerator and denominator are integers.

8. Write the following fractions as decimal numbers:

a) $\frac{1}{18}$
Solution:

$$
\begin{aligned}
18|\overline{1{,}0000} &= 0 \text{ remainder } 0\\
18|\overline{1{,}^{1}0000} &= 0 \text{ remainder } 0\\
18|\overline{1{,}^{10}0^{1}000} &= 5 \text{ remainder } 10\\
18|\overline{1{,}^{10}0^{1}0^{1}00} &= 5 \text{ remainder } 10\\
\frac{1}{18} &= 0{,}05555...\\
&= 0{,}0\dot{5}
\end{aligned}
$$

b) $1\frac{1}{2}$

Solution:

$$
\begin{aligned}
1\frac{1}{2} &= \frac{3}{2}\\
2|\overline{3{,}0000} &= 1 \text{ remainder } 1\\
2|\overline{3{,}^{1}0000} &= 5 \text{ remainder } 0\\
&= 1{,}5
\end{aligned}
$$

9. Express $0{,}\overline{78}$ as a fraction $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ (show all working).

Solution:

$$
\begin{aligned}
x &= 0{,}\overline{78}\\
100x &= 78{,}\overline{78}\\
\therefore 100x - x &= 99\\
\therefore x &= \frac{78}{99}
\end{aligned}
$$

10. For each of the following numbers:
   - write the next three digits;
   - state whether the number is rational or irrational.

a) 1,11235...

Solution:

- The number does not terminate (this is shown by the . . .). There is also no indication of a repeating pattern of digits since there is not dot or bar over any of the numbers. The next three digits could be any numbers.
- Irrational, there is no repeating pattern.

b) $1{,}\dot{1}$

Solution:

- Since there is a dot over the 1 we know that the 1 repeats. The next three digits are: 111
- Rational, there is a repeating pattern of digits.

11. Write the following rational numbers to 2 decimal places.

a) $\frac{1}{2}$

Solution:

To write to two decimal places we must convert to decimal: $\frac{1}{2} = 0{,}50$.

b) 1

Solution:

To write to two decimal places just add a comma and two 0's: 1,00.

c) $0{,}11111\bar{1}$

Solution:

We mark where the cut off point is, determine if it has to be rounded up or not and then write the answer. In this case there is a 1 after the cut off point so we do not round up. The final answer is: $0{,}11111\bar{1} \approx 0{,}11$.

d) $0{,}99999\bar{1}$

Solution:

We mark where the cut off point is, determine if it has to be rounded up or not and then write the answer. In this case there is a 9 after the cut off point so we round up. The final answer is: $0{,}99999\bar{1} \approx 1{,}00$.

12. Round off the following irrational numbers to 3 decimal places.

   a) 3,141592654...
      Solution:
      3,142 (round up as there is a 5 after the cut off point).
   b) 1,618033989...
      Solution:
      1,618 (no rounding as there is a 0 after the cut off point).
   c) 1,41421356...
      Solution:
      1,414 (no rounding as there is a 2 after the cut off point).
   d) 2,71828182845904523536...
      Solution:
      2,718 (no rounding as there is a 2 after the cut off point).

13. Round off the number 1523,00195593 to 4 decimal places.
    Solution:
    $1523{,}00195593 \approx 1523{,}0020$
14. Round off the number 1982,94028996 to 6 decimal places.
    Solution:
    $1982{,}94028996 \approx 1982{,}940290$
15. Round off the number 101,52378984 to 4 decimal places.
    Solution:
    $101{,}52378984 \approx 101{,}5238$
16. Use your calculator and write the following irrational numbers to 3 decimal places.

   a) $\sqrt{2}$
      Solution:
      $\sqrt{2} \approx 1{,}414213562... \approx 1{,}414$
   b) $\sqrt{3}$
      Solution:
      $\sqrt{3} \approx 1{,}732050808... \approx 1{,}732$
   c) $\sqrt{5}$
      Solution:
      $\sqrt{5} \approx 2{,}236067977... \approx 2{,}236$
   d) $\sqrt{6}$
      Solution:
      $\sqrt{6} \approx 2{,}449489743... \approx 2{,}449$

17. Use your calculator (where necessary) and write the following numbers to 5 decimal places. State whether the numbers are irrational or rational.

   a) $\sqrt{8}$
      Solution:
      $\sqrt{8} \approx 2{,}828427125... \approx 2{,}82843$
      Irrational number.
   b) $\sqrt{768}$
      Solution:
      $\sqrt{768} \approx 27{,}71281292... \approx 27{,}71281$
      Irrational number.
   c) $\sqrt{0{,}49}$
      Solution:
      $\sqrt{0{,}49} = 0{,}70000$
      Rational number.
   d) $\sqrt{0{,}0016}$
      Solution:
      $\sqrt{0{,}0016} = 0{,}04000$
      Rational number.
   e) $\sqrt{0{,}25}$

Solution:

$\sqrt{0,25} = 0,50000$

Rational number.

f) $\sqrt{36}$

Solution:

$\sqrt{36} = 6,00000$

Rational number.

g) $\sqrt{1960}$

Solution:

$\sqrt{1960} \approx 44,27188724... \approx 44,27189$

Irrational number.

h) $\sqrt{0,0036}$

Solution:

$\sqrt{0,0036} = 0,06000$

Rational number.

i) $-8\sqrt{0,04}$

Solution:

$-8\sqrt{0,04} = -8(0,20000) = -1,60000$

Rational number.

j) $5\sqrt{80}$

Solution:

$5\sqrt{80} \approx 5(8,94427191...) \approx 44,72136$

Irrational number.

18. Round off:

a) $\frac{\sqrt{2}}{2}$ to the nearest 2 decimal places.

Solution:

$$\begin{aligned} \frac{\sqrt{2}}{2} &\approx 0,7071... \\ &\approx 0,71 \end{aligned}$$

b) $\sqrt{14}$ to the nearest 3 decimal places.

Solution:

$$\begin{aligned} \sqrt{14} &\approx 3,741657... \\ &\approx 3,742 \end{aligned}$$

19. Write the following irrational numbers to 3 decimal places and then write each one as a rational number to get an approximation of the irrational number.

a) 3,141592654...

Solution:

$$\begin{aligned} 3,141592654... &\approx 3,142 \\ &\approx 3\frac{142}{1000} \\ &\approx \frac{1571}{500} \end{aligned}$$

b) 1,618033989...

Solution:

$$\begin{aligned} 1,618033989... &\approx 1,618 \\ &\approx 1\frac{618}{1000} \\ &\approx \frac{809}{500} \end{aligned}$$

c) 1,41421356...

Solution:

$$\begin{aligned}
1{,}41421356... &\approx 1{,}414 \\
&\approx 1\frac{414}{1000} \\
&\approx \frac{707}{500}
\end{aligned}$$

d) 2,71828182845904523536...

Solution:

$$\begin{aligned}
2{,}71828182845904523536... &\approx 2{,}718 \\
&\approx 2\frac{718}{1000} \\
&\approx \frac{1359}{500}
\end{aligned}$$

20. Determine between which two consecutive integers the following irrational numbers lie, without using a calculator.

a) $\sqrt{5}$

Solution:

2 and 3 ($2^2 = 4$ and $3^2 = 9$)

b) $\sqrt{10}$

Solution:

3 and 4 ($3^2 = 9$ and $4^2 = 16$)

c) $\sqrt{20}$

Solution:

4 and 5 ($4^2 = 16$ and $5^2 = 25$)

d) $\sqrt{30}$

Solution:

5 and 6 ($5^2 = 25$ and $6^2 = 36$)

e) $\sqrt[3]{5}$

Solution:

1 and 2 ($1^3 = 1$ and $2^3 = 8$)

f) $\sqrt[3]{10}$

Solution:

2 and 3 ($2^3 = 8$ and $3^3 = 27$)

g) $\sqrt[3]{20}$

Solution:

2 and 3 ($2^3 = 8$ and $3^3 = 27$)

h) $\sqrt[3]{30}$

Solution:

3 and 4 ($3^3 = 27$ and $4^3 = 64$)

i) $\sqrt{90}$

Solution:

9 and 10 ($9^2 = 81$ and $10^2 = 100$)

j) $\sqrt{72}$

Solution:

8 and 9 ($8^2 = 64$ and $9^2 = 81$)

k) $\sqrt[3]{58}$

Solution:

3 and 4 ($3^3 = 27$ and $4^3 = 64$)

l) $\sqrt[3]{118}$

Solution:

4 and 5 ($4^3 = 64$ and $5^3 = 125$)

21. Estimate the following surds to the nearest 1 decimal place, without using a calculator.

a) $\sqrt{14}$

Solution:

$\sqrt{14}$ lies between 3 and 4. Since $3^2 = 9$ and $4^2 = 16$ it lies closer to 4 than to 3.
Therefore 3,7 or 3,8 are suitable estimates.

b) $\sqrt{110}$

Solution:

$\sqrt{110}$ lies between 10 and 11. Since $10^2 = 100$ and $11^2 = 121$ it lies almost exactly between 10 and 11.
Therefore 10,5 is a suitable estimate.

c) $\sqrt{48}$

Solution:

$\sqrt{48}$ lies between 6 and 7. Since $6^2 = 36$ and $7^2 = 49$ it lies closer to 7 than to 6.
Therefore 6,9 is a suitable estimate.

d) $\sqrt{57}$

Solution:

$\sqrt{57}$ lies between 7 and 8. Since $7^2 = 49$ and $8^2 = 64$ it lies almost exactly between the two numbers.
Therefore 4,5 or 4,6 are suitable estimates.

22. Expand the following products:

a) $(a+5)^2$

Solution:

$$\begin{aligned}(a+5)^2 &= (a+5)(a+5)\\ &= a^2+5a+5a+25\\ &= a^2+10a+25\end{aligned}$$

b) $(n+12)^2$

Solution:

$$\begin{aligned}(n+12)^2 &= (n+12)(n+12)\\ &= n^2+12n+12n+144\\ &= n^2+24n+144\end{aligned}$$

c) $(d-4)^2$

Solution:

$$\begin{aligned}(d-4)^2 &= (d-4)(d-4)\\ &= d^2-4d-4d+16\\ &= d^2-8d+16\end{aligned}$$

d) $(7w+2)(7w-2)$

Solution:

$$\begin{aligned}(7w+2)(7w-2) &= 49w^2-14w+14w-4\\ &= 49w^2-4\end{aligned}$$

e) $(12q+1)(12q-1)$

Solution:

$$\begin{aligned}(12q+1)(12q-1) &= 144q^2-12q+12q-1\\ &= 144q^2-1\end{aligned}$$

f) $-(-x-2)(x+2)$
Solution:

$$\begin{aligned}-(-x-2)(x+2) &= (x+2)(x+2)\\ &= x^2+2x+2x+4\\ &= x^2+4x+4\end{aligned}$$

g) $(5k-4)(5k+4)$
Solution:

$$\begin{aligned}(5k-4)(5k+4) &= 25k^2+20k-20k-16\\ &= 25k^2-16\end{aligned}$$

h) $(5f+4)(2f+2)$
Solution:

$$\begin{aligned}(5f+4)(2f+2) &= 10f^2+10f+8f+8\\ &= 10f^2+18f+8\end{aligned}$$

i) $(3n+6)(6n+5)$
Solution:

$$\begin{aligned}(3n+6)(6n+5) &= 18n^2+15n+36n+30\\ &= 18n^2+51n+30\end{aligned}$$

j) $(2g+6)(g+6)$
Solution:

$$\begin{aligned}(2g+6)(g+6) &= 2g^2+12g+6g+36\\ &= 2g^2+18g+36\end{aligned}$$

k) $(4y+1)(4y+8)$
Solution:

$$\begin{aligned}(4y+1)(4y+8) &= 16y^2+32y+4y+8\\ &= 16y^2+36y+8\end{aligned}$$

l) $(d-3)(7d+2)$
Solution:

$$\begin{aligned}(d-3)(7d+2) &= 7d^2+2d-21d-6\\ &= 7d^2-19d-6\end{aligned}$$

m) $(6z-4)(z-2)$
Solution:

$$\begin{aligned}(6z-4)(z-2) &= 6z^2-12z-4z+8\\ &= 6z^2-16z+8\end{aligned}$$

n) $(5w-11)^2$
Solution:

$$\begin{aligned}(5w-11)^2 &= (5w-11)(5w-11)\\ &= 25w^2-55w-55w+121\\ &= 25w^2-110w+121\end{aligned}$$

o) $(5s-1)^2$
Solution:

$$\begin{aligned}(5s-1)^2 &= (5s-1)(5s-1)\\ &= 25s^2-5s-5s+1\\ &= 25s^2-10s+1\end{aligned}$$

p) $(3d-8)^2$
Solution:

$$\begin{aligned}(3d-8)^2 &= (3d-8)(3d-8)\\ &= 9d^2-24d-24d+64\\ &= 9d^2-48d+64\end{aligned}$$

q) $5f^2(3f+5)+7f(3f^2+7)$
Solution:

$$\begin{aligned}5f^2(3f+5)+7f(3f^2+7) &= 15f^3+25f^2+21f^3+49f\\ &= 36f^3+25f^2+49f\end{aligned}$$

r) $8d(4d^3+2)+6d^2(7d^2+4)$
Solution:

$$\begin{aligned}8d(4d^3+2)+6d^2(7d^2+4) &= 32d^4+16d+42d^4+24d^2\\ &= 74d^4+16d+24d^2\end{aligned}$$

s) $5x^2(2x+2)+7x(7x^2+7)$
Solution:

$$\begin{aligned}5x^2(2x+2)+7x(7x^2+7) &= 10x^3+10x^2+49x^3+49x\\ &= 59x^3+10x^2+49x\end{aligned}$$

23. Expand the following:

a) $(y^4+3y^2+y)(y+1)(y-2)$
Solution:

$$\begin{aligned}(y^4+3y^2+y)(y+1)(y-2) &= (y^4+3y^2+y)(y^2-y-2)\\ &= y^6-y^5-2y^4+3y^4-3y^3-6y^2+y^3-y^2-2y\\ &= y^6-y^5+y^4-2y^3-7y^2-2y\end{aligned}$$

b) $(x+1)^2-(x-1)^2$
Solution:

$$\begin{aligned}(x+1)^2-(x-1)^2 &= x^2+2x+1-(x^2-2x+1)\\ &= x^2+2x+1-x^2+2x-1\\ &= 4x\end{aligned}$$

c) $(x^2+2x+1)(x^2-2x+1)$
Solution:

$$\begin{aligned}(x^2+2x+1)(x^2-2x+1) &= x^4-2x^3+x^2+2x^3-4x^2+2x+x^2-2x+1\\ &= x^4-2x^2+1\end{aligned}$$

d) $(4a-3b)(16a^2+12ab+9b^2)$
Solution:

$$\begin{aligned}(4a-3b)(16a^2+12ab+9b^2) &= 64a^3+48a^2b+36ab^2-48a^2b-36ab^2-27b^3\\ &= 64a^3-27b^3\end{aligned}$$

e) $2(x+3y)(x^2-xy-y^2)$
Solution:

$$\begin{aligned}2(x+3y)(x^2-xy-y^2) &= 2(x^3-x^2y-xy^2+3x^2y-3xy^2-3y^3)\\ &= 2x^3+4x^2y-8xy^2-6y^3\end{aligned}$$

f) $(3a-5b)(3a+5b)(a^2+ab-b^2)$
Solution:

$$\begin{aligned}(3a-5b)(3a+5b)(a^2+ab-b^2) &= (9a^2-25b^2)(a^2+ab-b^2)\\ &= 9a^4+9a^3-9a^2b^2-25a^2b^2+25ab^3-25b^4\\ &= 9a^4+9a^3-34a^2b^2+25ab^3-25b^4\end{aligned}$$

g) $\left(y-\frac{1}{y}\right)\left(y+\frac{1}{y}\right)$
Solution:

$$\begin{aligned}\left(y-\frac{1}{y}\right)\left(y+\frac{1}{y}\right) &= y^2+1-1+\frac{1}{y^2}\\ &= y^2-\frac{1}{y^2}\end{aligned}$$

h) $\left(\frac{a}{3}-\frac{3}{a}\right)\left(\frac{a}{3}+\frac{3}{a}\right)$
Solution:

$$\begin{aligned}\left(\frac{a}{3}-\frac{3}{a}\right)\left(\frac{a}{3}+\frac{3}{a}\right) &= \frac{a^2}{9}+1-1+\frac{3}{a^2}\\ &= \frac{a^2}{9}-\frac{3}{a^2}\end{aligned}$$

i) $\frac{1}{3}(12x-9y)+\frac{1}{6}(12x+18y)$
Solution:

$$\begin{aligned}\frac{1}{3}(12x-9y)+\frac{1}{6}(12x+18y) &= 4x-3y+2x+3y\\ &= 6x\end{aligned}$$

j) $(x+2)(x-2)-(x+2)^2$
Solution:

$$\begin{aligned}(x+2)(x-2)-(x+2)^2 &= x^2-4-(x^2+4x+4)\\ &= -4x-8\end{aligned}$$

24. What is the value of e in $(x-4)(x+e)=x^2-16$?
Solution:

$$(x-4)(x+e) = x^2+ex-4x-4e$$

From the constant term we see that $4e=16$, therefore $e=4$.

25. In $(x+2)(x+k) = x^2 + bx + c$:

a) For which of these values of k will b be positive?
$-6\ ;\ -1\ ;\ 0\ ;\ 1\ ;\ 6$
Solution:

$$\begin{aligned}(x+2)(x+k) &= x^2 + kx + 2x + 2k \\ &= x^2 + (k+2)x + 2k\end{aligned}$$

The b term is $k+2$ and so any value greater than -2 will make the b term positive.
Therefore $-1\ ;\ 0\ ;\ 1\ ;\ 6$

b) For which of these values of k will c be positive?
$-6\ ;\ -1\ ;\ 0\ ;\ 1\ ;\ 6$
Solution:
From above we see that the c term is $2k$. Therefore any positive value of k will make c positive.
Therefore $0\ ;\ 1\ ;\ 6$

c) For what values of k will c be positive?
Solution:
From above we see that the c term is $2k$. Therefore any positive value of k will make c positive.
Therefore $k > 0$

d) For what values of k will b be positive?
Solution:
From above we see that any value greater than -2 will make the b term positive.
Therefore $k > -2$.

26. Answer the following:

a) Expand: $\left(3a - \frac{1}{2a}\right)^2$
Solution:

$$\left(3a - \frac{1}{2a}\right)^2 = 9a^2 + 3 + \frac{1}{4a^2}$$

b) Expand: $\left(3a - \frac{1}{2a}\right)\left(9a^2 + \frac{3}{2} + \frac{1}{4a^2}\right)$
Solution:

$$\begin{aligned}\left(3a - \frac{1}{2a}\right)\left(9a^2 + \frac{3}{2} + \frac{1}{4a^2}\right) &= 27a^3 + \frac{9}{2}a + \frac{3}{4a} - \frac{9}{2}a - \frac{3}{4a} - \frac{1}{8a^3} \\ &= 27a^3 - \frac{1}{8a^3}\end{aligned}$$

c) Given that $3a - \frac{1}{2a} = 7$, determine the value of $27a^3 - \frac{1}{8a^3}$ without solving for a.
Solution:

$$\begin{aligned}27a^3 - \frac{1}{8a^3} &= \left(3a - \frac{1}{2a}\right)\left(9a^2 + \frac{3}{2} + \frac{1}{4a^2}\right) \\ &= 7\left(9a^2 + \frac{3}{2} + \frac{1}{4a^2}\right)\end{aligned}$$

$$\begin{aligned}9a^2 + \frac{3}{2} + \frac{1}{4a^2} &= \left(3a - \frac{1}{2a}\right)^2 + \frac{9}{2} \\ &= 7^2 + \frac{9}{2}\end{aligned}$$

$$27a^3 - \frac{1}{8a^3} = 374\frac{1}{2}$$

27. Solve by factorising:

a) $17^2 - 15^2$
Solution:

$$\begin{aligned}17^2 - 15^2 &= (17 - 15)(17 + 15) \\ &= 2(32) \\ &= 64\end{aligned}$$

b) $13^2 - 12^2$
Solution:

$$\begin{aligned}13^2 - 12^2 &= (13 - 12)(13 + 12) \\ &= 25\end{aligned}$$

c) $120045^2 - 120035^2$
Solution:

$$\begin{aligned}120045^2 - 120035^2 &= (120045 - 120035)(120045 + 120035) \\ &= 10(240080) \\ &= 2400800\end{aligned}$$

d) $26^2 - 24^2$
Solution:

$$\begin{aligned}26^2 - 24^2 &= (26 - 24)(26 + 24) \\ &= 2(50) \\ &= 100\end{aligned}$$

28. Represent the following as a product of its prime factors:

a) 143
Solution:

$$\begin{aligned}143 &= 144 - 1 \\ &= (12 - 1)(12 + 1) \\ &= 11 \times 13\end{aligned}$$

b) 168
Solution:

$$\begin{aligned}168 &= 169 - 1 \\ &= (13 - 1)(13 + 1) \\ &= 12(14) \\ &= 3 \times 2^2 \times 2 \times 7 \\ &= 2^3 \times 3 \times 7\end{aligned}$$

c) 899
Solution:

$$\begin{aligned}899 &= 900 - 1 \\ &= (30 - 1)(30 + 1) \\ &= 29 \times 31\end{aligned}$$

d) 99
Solution:

$$\begin{aligned} 99 &= 100 - 1 \\ &= (10-1)(10+1) \\ &= 3^2 \times 11 \end{aligned}$$

e) 1599
Solution:

$$\begin{aligned} 1599 &= 1600 - 1 \\ &= (40-1)(40+1) \\ &= 39(41) \\ &= 3 \times 13 \times 41 \end{aligned}$$

29. Factorise:

a) $a^2 - 9$
Solution:

$$a^2 - 9 = (a-3)(a+3)$$

b) $9b^2 - 81$
Solution:

$$\begin{aligned} 9b^2 - 81 &= 9(b^2 - 9) \\ &= 9(b-3)(b+3) \end{aligned}$$

c) $m^2 - \frac{1}{9}$
Solution:

$$m^2 - \frac{1}{9} = \left(m - \frac{1}{3}\right)\left(m + \frac{1}{3}\right)$$

d) $5 - 5a^2b^6$
Solution:

$$\begin{aligned} 5 - 5a^2b^6 &= 5(1 - a^2b^6) \\ &= 5(1 - ab^3)(1 + ab^3) \end{aligned}$$

e) $16ba^4 - 81b$
Solution:

$$\begin{aligned} 16ba^4 - 81b &= b(16a^4 - 81) \\ &= b(4a^2 - 9)(4a^2 + 9) \\ &= b(2a-3)(2a+3)(4a^2+9) \end{aligned}$$

f) $a^2 - 10a + 25$
Solution:

$$a^2 - 10a + 25 = (a-5)(a-5)$$

g) $16b^2 + 56b + 49$
Solution:

$$16b^2 + 56b + 49 = (4b+7)(4b+7)$$

h) $-4b^2 - 144b^8 + 48b^5$
Solution:

$$\begin{aligned} -4b^2 - 144b^8 + 48b^5 &= -4b^2(1 + 36b^6 - 12b^3) \\ &= -4b^2(6b^3 - 1)(6b^3 - 1) \\ &= -4b^2(6b^3 - 1)^2 \end{aligned}$$

i) $16 - x^4$
Solution:

$$\begin{aligned} 16 - x^4 &= (4 - x^2)(4 + x^2) \\ &= (4 + x^2)(2 + x)(2 - x) \end{aligned}$$

j) $7x^2 - 14x + 7xy - 14y$
Solution:

$$\begin{aligned} 7x^2 - 14x + 7xy - 14y &= 7(x^2 - 2x + xy - 2y) \\ &= 7(x(x - 2) + y(x - 2)) \\ &= 7(x - 2)(x + y) \end{aligned}$$

k) $y^2 - 7y - 30$
Solution:

$$y^2 - 7y - 30 = (y - 10)(y + 3)$$

l) $1 - x - x^2 + x^3$
Solution:

$$\begin{aligned} 1 - x - x^2 + x^3 &= (1 - x) - x^2(1 - x) \\ &= (1 - x)(1 - x^2) \\ &= (1 - x)(1 - x)(1 + x) \\ &= (1 - x)^2(1 + x) \end{aligned}$$

m) $-3(1 - p^2) + p + 1$
Solution:

$$\begin{aligned} -3(1 - p^2) + p + 1 &= -3(1 - p)(1 + p) + (1 + p) \\ &= (1 + p)[-3(1 - p) + 1] \\ &= (1 + p)(-2 + 3p) \end{aligned}$$

n) $x^2 - 2x + 1 - y^4$
Solution:

$$\begin{aligned} x^2 - 2x + 1 - y^4 &= x(x - 2) + (1 - y^2)(1 + y^2) \\ &= x(x - 2) + (1 + y)(1 - y)(1 + y^2) \end{aligned}$$

o) $4b(x^3 - 1) + x(1 - x^3)$
Solution:

$$\begin{aligned} 4b(x^3 - 1) + x(1 - x^3) &= (x^3 - 1)(4b - x) \\ &= (x - 1)(x^2 + x + 1)(4b - x) \end{aligned}$$

p) $3m(v-7)+19(-7+v)$
Solution:

$$\begin{aligned}3m(v-7)+19(-7+v) &= 3m(v-7)+19(v-7)\\ &= (v-7)(3m+19)\end{aligned}$$

q) $3f(z+3)+19(3+z)$
Solution:

$$\begin{aligned}3f(z+3)+19(3+z) &= 3f(z+3)+19(z+3)\\ &= (3f+19)(z+3)\end{aligned}$$

r) $3p^3-\frac{1}{9}$
Solution:

$$3p^3-\frac{1}{9}=3(p-\frac{1}{3})(p^2+\frac{p}{3}+\frac{1}{9})$$

s) $8x^6-125y^9$
Solution:

$$8x^6-125y^9=(2x^2-5y^3)(4x^4+10x^2y^3+25y^6)$$

t) $(2+p)^3-8(p+1)^3$
Solution:

$$\begin{aligned}(2+p)^3-8(p+1)^3 &= [(p+2)-2(p+1)][(p+2)^2+2(p+2)(p+1)+4(p+1)^2]\\ &= [p+2-2p-2][p^2+4p+4+2p^2+6p+4+4p^2+8p+4]\\ &= (-p)(12+18p+7p^2)\end{aligned}$$

u) $\frac{1}{3}a^3-a^2b+2a^2b-6ab^2+3ab^2-9b^3$
Solution:

$$\begin{aligned}\frac{1}{3}a^3-a^2b+2a^2b-6ab^2+3ab^2-9b^3 &= \frac{1}{3}a^2(a-3b)+2ab(a-3b)+3b^2(a-3b)\\ &= (\frac{1}{3}a^2+2ab+3b^2)(a-3b)\\ &= \frac{(a^2+6ab+9b^2)(a-3b)}{3}\\ &= \frac{(a+3b)^2(a-3b)}{3}\end{aligned}$$

v) $6a^2-17a+5$
Solution:

$$6a^2-17a+5=(2a-5)(3a-1)$$

w) $s^2+2s-15$
Solution:

$$s^2+2s-15=(s-3)(s+5)$$

x) $16v+24h+2j^5v+3j^5h$
Solution:

$$\begin{aligned}16v+24h+2j^5v+3j^5h &= 8(2v+3h)+j^5(2v+3h)\\ &= (2v+3h)(8+j^5)\end{aligned}$$

y) $18h-45g+2m^3h-5m^3g$
Solution:

$$\begin{aligned}18h-45g+2m^3h-5m^3g &= 9(2h-5g)+m^3(2h-5g)\\ &= (2h-5g)(9+m^3)\end{aligned}$$

z) $63d-18s+7u^2d-2u^2s$
Solution:

$$\begin{aligned}63d-18s+7u^2d-2u^2s &= 9(7d-2s)+u^2(7d-2s)\\ &= (7d-2s)(9+u^2)\end{aligned}$$

30. Factorise the following:

a) $6a^2+14a+8$
Solution:

$$\begin{aligned}6a^2+14a+8 &= 2(3a^2+7a+4)\\ &= 2\left(a+1\right)\left(3a+4\right)\end{aligned}$$

b) $6g^2-15g-9$
Solution:

$$\begin{aligned}6g^2-15g-9 &= 3(2g^2-5g-3)\\ &= 3\left(g-3\right)\left(2g+1\right)\end{aligned}$$

c) $125g^3-r^3$
Solution:

$$125g^3-r^3=(5g-r)(25g^2+5gr+r^2)$$

d) $8r^3+z^3$
Solution:

$$8r^3+z^3=(2r+z)(4r^2-2rz+z^2)$$

e) $14m-4n+7jm-2jn$
Solution:

$$\begin{aligned}14m-4n+7jm-2jn &= 2(7m-2n)+j(7m-2n)\\ &= (7m-2n)(2+j)\end{aligned}$$

f) $25d-15m+5yd-3ym$
Solution:

$$\begin{aligned}25d-15m+5yd-3ym &= 5(5d-3m)+y(5d-3m)\\ &= (5d-3m)(5+y)\end{aligned}$$

g) g^3-27
Solution:

$$g^3-27=(g-3)(g^2+3g+9)$$

h) z^3+125
Solution:

$$z^3+125=(z+5)(z^2-5z+25)$$

i) $b^2 - (3a - 2b)^2$

Solution:

$$\begin{aligned} b^2 - (3a - 2b)^2 &= (b - (3a - 2b))(b + 3a - 2b) \\ &= (3b - 3a)(3a - b) \\ &= 3(b - a)(3a - b) \end{aligned}$$

j) $9y^2 - (4x + 2y)^2$

Solution:

$$\begin{aligned} 9y^2 - (4x + 2y)^2 &= (3y + 4x + 2y)(3y - (4x + 2y)) \\ &= (4x + 5y)(y - 4x) \end{aligned}$$

k) $16x^6 - 3y^8$

Solution:

$$\begin{aligned} 16x^6 - 3y^8 &= 4(4x^6 - 9y^8) \\ &= 4(4x^6 - 9y^8) \\ &= 4(4x^3 - 3y^4)(4x^3 + 3y^4) \end{aligned}$$

l) $\frac{1}{6}a^2 - 24b^4$

Solution:

$$\begin{aligned} \frac{1}{6}a^2 - 24b^4 &= \frac{1}{6}\left(a^2 - 144b^4\right) \\ &= \frac{1}{6}\left(a - 12b^2\right)\left(a + 12b^2\right) \end{aligned}$$

m) $4(a - 3) - 81x^2(a - 3)$

Solution:

$$\begin{aligned} 4(a - 3) - 81x^2(a - 3) &= (a - 3)(4 - 81x^2) \\ &= (a - 3)(2 - 9x)(2 + 9x) \end{aligned}$$

n) $(2 + b)^2 - 11(2 + b) - 12$

Solution:

$$\begin{aligned} (2 + b)^2 - 11(2 + b) - 12 &= ((2 + b) + 1)((2 + b) - 12) \\ &= (b + 3)(b - 10) \end{aligned}$$

o) $2x^2 + 7xy + 5y^2$

Solution:

$$2x^2 + 7xy + 5y^2 = (2x + 5y)(x + y)$$

p) $x^2 - 2xy - 15y^2$

Solution:

$$x^2 - 2xy - 15y^2 = (x - 5y)(x + 3y)$$

q) $4x^4 + 11x^2 + 6$

Solution:

$$4x^4 + 11x^2 + 6 = (4x^2 + 3)(x^2 + 2)$$

r) $6x^4 - 38x^2 + 40$
Solution:

$$\begin{aligned} 6x^4 - 38x^2 + 40 &= 2(3x^4 - 19x^2 + 20) \\ &= 2(3x^2 - 4)(x^2 - 5) \end{aligned}$$

s) $9a^2x + 9a^2y + 27a^2 - b^2x - b^2y - 3b^2$
Solution:

$$\begin{aligned} 9a^2x + 9a^2y + 27a^2 - b^2x - b^2y - 3b^2 &= (9a^2 - b^2)(x + y + 3) \\ &= (3a + b)(3a - b)(x + y + 3) \end{aligned}$$

t) $2(2y^2 - 5y) - 24$
Solution:

$$\begin{aligned} 2(2y^2 - 5y) - 24 &= 2(2y^2 - 5y) - 2(12) \\ &= 2(2y^2 - 5y - 12) \\ &= 2(2y + 3)(y - 4) \end{aligned}$$

u) $\dfrac{1}{2}x^3 - \frac{9}{2}x - 2x^2 + 18$
Solution:

$$\begin{aligned} \frac{1}{2}x^3 - \frac{9}{2}x - 2x^2 + 18 &= \frac{x^3 - 9x - 4x^2 + 36}{2} \\ &= \frac{x^2(x - 4) - 9(x - 4)}{2} \\ &= \frac{(x - 4)(x^2 - 9)}{2} \\ &= \frac{(x - 4)(x - 3)(x + 3)}{2} \end{aligned}$$

v) $27r^3s^3 - 1$
Solution:

$$27r^3s^3 - 1 = (3rs - 1)(9r^2s^2 + 3rs + 1)$$

w) $\dfrac{1}{125h^3} + r^3$
Solution:

$$\frac{1}{125h^3} + r^3 = \left(\frac{1}{5h} + r\right)\left(\frac{1}{25h^2} - \frac{r}{5h} + r^2\right)$$

x) $j(64n^3 - b^3) + k(64n^3 - b^3)$
Solution:

$$\begin{aligned} j(64n^3 - b^3) + k(64n^3 - b^3) &= (j + k)(64n^3 - b^3) \\ &= (j + k)(4n - b)(16n^2 + 4nb + b^2) \end{aligned}$$

31. Simplify the following:

a) $(a - 2)^2 - a(a + 4)$
Solution:

$$\begin{aligned} (a - 2)^2 - a(a + 4) &= a^2 - 4a + 4 - a^2 - 4a \\ &= -8a + 4 \end{aligned}$$

b) $(5a-4b)(25a^2+20ab+16b^2)$

Solution:

$$\begin{aligned}(5a-4b)(25a^2+20ab+16b^2) &= 125a^3+100a^2b+80ab^2-100a^2b-80ab^2-64b^3\\ &= 125a^3-64b^3\end{aligned}$$

c) $(2m-3)(4m^2+9)(2m+3)$

Solution:

$$\begin{aligned}(2m-3)(4m^2+9)(2m+3) &= (4m^2-9)(4m^2+9)\\ &= 16m^4-81\end{aligned}$$

d) $(a+2b-c)(a+2b+c)$

Solution:

$$\begin{aligned}(a+2b-c)(a+2b+c) &= a^2+2ab+ac+2ab+4b^2+2bc-ac-2bc-c^2\\ &= a^2+4ab+4b^2-c^2\end{aligned}$$

e) $\dfrac{m^2+11m+18}{4(m^2-4)} \div \dfrac{3m^2+27m}{24m^2-48m}$

Solution:

$$\begin{aligned}\frac{m^2+11m+18}{4(m^2-4)} \div \frac{3m^2+27m}{24m^2-48m} &= \frac{m^2+11m+18}{4(m^2-4)} \times \frac{24m^2-48m}{3m^2+27m}\\ &= \frac{(m+9)(m+2)}{4(m-2)(m+2)} \times \frac{24m(m-2)}{3m(m+9)}\\ &= \frac{1}{4} \times \frac{24}{3}\\ &= 2\end{aligned}$$

f) $\dfrac{t^2+9t+18}{5(t^2-9)} \div \dfrac{4t^2+24t}{100t^2-300t}$

Solution:

$$\begin{aligned}\frac{t^2+9t+18}{5(t^2-9)} \div \frac{4t^2+24t}{100t^2-300t} &= \frac{t^2+9t+18}{5(t^2-9)} \times \frac{100t^2-300t}{4t^2+24t}\\ &= \frac{(t+6)(t+3)}{5(t-3)(t+3)} \times \frac{100t(t-3)}{4t(t+6)}\\ &= \frac{1}{5} \times \frac{100}{4}\\ &= 5\end{aligned}$$

g) $\dfrac{4-b^2}{3b-6}$

Solution:

$$\begin{aligned}\frac{4-b^2}{3b-6} &= \frac{(2-b)(2+b)}{3(b-2)}\\ &= -\frac{2+b}{3}\end{aligned}$$

h) $\dfrac{x^2+2x+4}{x^3-8}$

Solution:

$$\begin{aligned}\frac{x^2+2x+4}{x^3-8} &= \frac{x^2+2x+4}{(x-2)(x^2+2x+4)} \\ &= \frac{1}{x-2}\end{aligned}$$

i) $\dfrac{x^2-5x-14}{3x+6}$

Solution:

$$\begin{aligned}\frac{x^2-5x-14}{3x+6} &= \frac{(x-7)(x+2)}{3(x+2)} \\ &= \frac{x-7}{3}\end{aligned}$$

j) $\dfrac{d^2+23d+132}{5(d^2-121)} \div \dfrac{4d^2+48d}{100d^2-1100d}$

Solution:

$$\begin{aligned}\frac{d^2+23d+132}{5(d^2-121)} \div \frac{4d^2+48d}{100d^2-1100d} &= \frac{d^2+23d+132}{5(d^2-121)} \times \frac{100d^2-1100d}{4d^2+48d} \\ &= \frac{(d+12)(d+11)}{5(d-11)(d+11)} \times \frac{100d(d-11)}{4d(d+12)} \\ &= \frac{1}{5} \times \frac{100}{4} \\ &= 5\end{aligned}$$

k) $\dfrac{a-2}{a^2+4a+3} \div \dfrac{(a-1)(a+1)}{a-1} \times \dfrac{a^2-2a-15}{a-2}$

Solution:

$$\begin{aligned}&\frac{a-2}{a^2+4a+3} \div \frac{(a-1)(a+1)}{a-1} \times \frac{a^2-2a-15}{a-2} \\ &= \frac{a-2}{(a+1)(a+3)} \div \frac{(a-1)(a+1)}{a-1} \times \frac{(a+3)(a-5)}{a-2} \\ &= \frac{a-2}{(a+1)(a+3)} \times \frac{a-1}{(a-1)(a+1)} \times \frac{(a+3)(a-5)}{a-2} \\ &= \frac{a-5}{(a+2)^2}\end{aligned}$$

l) $\dfrac{a+6}{a^2+12a+11} \times \dfrac{a^2+14a+33}{a+3} \div \dfrac{a^3+216}{a+1}$

Solution:

$$\begin{aligned}&\frac{a+6}{a^2+12a+11} \times \frac{a^2+14a+33}{a+3} \div \frac{a^3+216}{a+1} \\ &= \frac{a+6}{(a+11)(a+1)} \times \frac{(a+11)(a+3)}{a+3} \times \frac{a+1}{(a+6)(a^2+6a+36)} \\ &= \frac{1}{a^2+6a+36}\end{aligned}$$

m) $2 \div \dfrac{a+b}{a+2b} \times \dfrac{b^2-ba-6a^2}{a^2-4b^2} \times \dfrac{a^2-b-2b^2}{3a-b}$

Solution:

$$\begin{aligned}&2 \div \frac{a+b}{a+2b} \times \frac{b^2-ba-6a^2}{a^2-4b^2} \times \frac{a^2-b-2b^2}{3a-b} \\ &= 2 \times \frac{a+2b}{a+b} \times \frac{(b-3a)(b+2a)}{(a-2b)(a+2b)} \times \frac{(a-2b)(a+b)}{3a-b} \\ &= -2(2a+b)\end{aligned}$$

n) $\dfrac{st + sb + 31t + 31b}{t + b}$
Solution:

$$\begin{aligned}\frac{st + sb + 31t + 31b}{(t + b)} &= \frac{s(t + b) + 31(t + b)}{(t + b)} \\ &= \frac{(t + b)(s + 31)}{(t + b)} \\ &= s + 31\end{aligned}$$

o) $\dfrac{ny + nq + 8y + 8q}{y + q}$
Solution:

$$\begin{aligned}\frac{ny + nq + 8y + 8q}{(y + q)} &= \frac{n(y + q) + 8(y + q)}{(y + q)} \\ &= \frac{(y + q)(n + 8)}{(y + q)} \\ &= n + 8\end{aligned}$$

p) $\dfrac{p^2 - q^2}{p} \div \dfrac{p + q}{p^2 - pq}$
Solution:

$$\begin{aligned}\frac{p^2 - q^2}{p} \div \frac{p + q}{p^2 - pq} &= \frac{(p - q)(p + q)}{p} \times \frac{p(p - q)}{p + q} \\ &= (p - q)^2 \\ &= p^2 - 2pq + q^2\end{aligned}$$

q) $\dfrac{2}{x} + \dfrac{x}{2} - \dfrac{2x}{3}$
Solution:

$$\begin{aligned}\frac{2}{x} + \frac{x}{2} - \frac{2x}{3} &= \frac{12 + 3x^2 - 4x^2}{6x} \\ &= \frac{12 - x^2}{6x}\end{aligned}$$

r) $\dfrac{1}{a + 7} - \dfrac{a + 7}{a^2 - 49}$
Solution:

$$\begin{aligned}\frac{1}{a + 7} - \frac{a + 7}{a^2 - 49} &= \frac{1}{a + 7} - \frac{a + 7}{(a + 7)(a - 7)} \\ &= \frac{-14}{(a + 7)(a - 7)}\end{aligned}$$

s) $\dfrac{x + 2}{2x^3} + 16$
Solution:

$$\begin{aligned}\frac{x + 2}{2x^3} + 16 &= \frac{(x + 2) + 16(2x^3)}{2x^3} \\ &= \frac{32x^3 + x + 2}{2x^3}\end{aligned}$$

t) $\dfrac{1-2a}{4a^2-1} - \dfrac{a-1}{2a^2-3a+1} - \dfrac{1}{1-a}$

Solution:

$$\begin{aligned}\frac{1-2a}{4a^2-1} - \frac{a-1}{2a^2-3a+1} - \frac{1}{1-a} &= \frac{1-2a}{(2a-1)(2a+1)} - \frac{a-1}{(2a-1)(a-1)} + \frac{1}{a-1} \\ &= -\frac{(2a-1)}{(2a-1)(2a+1)} - \frac{1}{2a-1} + \frac{1}{a-1} \\ &= \frac{4a-1}{(2a+1)(2a-1)(a-1)}\end{aligned}$$

u) $\dfrac{1}{2}x + \dfrac{x-2}{3} + 4$

Solution:

$$\begin{aligned}\frac{1}{2}x + \frac{x-2}{3} + 4 &= \frac{3x + 2(x-2) + (2)(3)(4)}{6} \\ &= \frac{3x + 2x - 4 + 24}{6} \\ &= \frac{5x + 20}{6}\end{aligned}$$

v) $\dfrac{1}{x^2+2x} + \dfrac{4x^2-x-3}{x^2+2x-3}$

Solution:

$$\begin{aligned}\frac{1}{x^2+2x} + \frac{4x^2-x-3}{x^2+2x-3} &= \frac{1}{x(x+2)} + \frac{(4x+3)(x-1)}{(x-1)(x+3)} \\ &= \frac{1}{x(x+2)} + \frac{4x+3}{x+3} \\ &= \frac{x+3+x(4x+3)(x+2)}{x(x+2)(x+3)} \\ &= \frac{x+3+x(4x^2+11x+6)}{x(x+2)(x+3)} \\ &= \frac{4x^3+11x^2+7x+3}{x(x+2)(x+3)}\end{aligned}$$

w) $\dfrac{b^2+6b+9}{b^2-9} + \dfrac{b^2-6b+8}{(b-2)(b+3)} + \dfrac{1}{b+3}$

Solution:

$$\begin{aligned}\frac{b^2+6b+9}{b^2-9} + \frac{b^2-6b+8}{(b-2)(b+3)} + \frac{1}{b+3} &= \frac{(b+3)^2}{(b+3)(b-3)} + \frac{(b-4)(b-2)}{(b-2)(b+3)} + \frac{1}{b+3} \\ &= \frac{b+3}{b-3} + \frac{b-4}{b+3} + \frac{1}{b+3} \\ &= \frac{(b+3)^2 + (b-3)(b-4) + b - 3}{(b-3)(b+3)} \\ &= \frac{b^2+6b+9+b^2-7b+12+b-3}{(b-3)(b+3)} \\ &= \frac{2b^2+18}{(b-3)(b+3)} \\ &= \frac{2(b^2+9)}{(b-3)(b+3)}\end{aligned}$$

x) $\dfrac{x^2+2x}{x^2+x+6} \times \dfrac{x^2+2x+1}{x^2+3x+2}$

Solution:

$$\frac{x^2+2x}{x^2+x+6} \times \frac{x^2+2x+1}{x^2+3x+2} = \frac{x(x+2)}{x^2+x+6} \times \frac{(x+1)(x+1)}{(x+2)(x+1)}$$
$$= \frac{x(x+1)}{x^2+x+6}$$

y) $\frac{12}{z+12} + \frac{5}{z-5}$
Solution:

$$\frac{12}{z+12} + \frac{5}{z-5} = \frac{12(z-5)+5(z+12)}{(z+12)(z-5)}$$
$$= \frac{12z-60+5z+60}{(z+12)(z-5)}$$
$$= \frac{17z}{(z+12)(z-5)}$$

z) $\frac{11}{w-11} - \frac{4}{w-4}$
Solution:

$$\frac{11}{w-11} - \frac{4}{w-4}$$
$$= \frac{11(w-4)-4(w-11)}{(w-11)(w-4)}$$
$$= \frac{11w-44-4w+44}{(w-11)(w-4)}$$
$$= \frac{7w}{(w-11)(w-4)}$$

32. Show that $(2x-1)^2-(x-3)^2$ can be simplified to $(x+2)(3x-4)$.
Solution:

$$(2x-1)^2-(x-3)^2 = (2x-1)(2x-1)-(x-3)(x-3)$$
$$= 4x^2-2x-2x+1-(x^2-3x-3x-9)$$
$$= 3x^2+2x-8$$
$$= (3x-4)(x+2)$$

33. What must be added to x^2-x+4 to make it equal to $(x+2)^2$?
Solution:
Suppose A must be added to the expression to get the desired result.

$$\therefore (x^2-x+4)+A = (x+2)^2$$
$$\therefore A = (x+2)(x+2)-(x^2-x+4)$$
$$= x^2+2x+2x+4-x^2+x-4$$
$$= 5x$$

Therefore $5x$ must be added.

34. Evaluate $\frac{x^3+1}{x^2-x+1}$ if $x = 7{,}85$ without using a calculator. Show your work.
Solution:
First simplify the expression:

$$\frac{x^3+1}{x^2-x+1} = \frac{(x+1)(x^2-x+1)}{x^2-x+1}$$
$$= x+1$$

Now substitute the value of x: $7{,}85+1=8{,}85$.

35. With what expression must $(a - 2b)$ be multiplied to get a product of $(a^3 - 8b^3)$?
Solution:

$$(a - 2b)(a^2 + 2ab + 4b^2) = a^3 - 8b^3$$

So, the expression is $a^2 + 2ab + 4b^2$.

36. With what expression must $27x^3 + 1$ be divided to get a quotient of $3x + 1$?
Solution:

$$\begin{aligned} 27x^3 + 1 &= (3x + 1)(9x^2 - 3x + 1) \\ \frac{(3x + 1)(9x^2 - 3x + 1)}{9x^2 - 3x + 1} &= 3x + 1 \end{aligned}$$

Therefore the expression is $9x^2 - 3x + 1$.

37. What are the restrictions on the following?

a) $\dfrac{4}{3x^2 + 2x - 1}$
Solution:

$$\frac{4}{3x^2 + 2x - 1} = \frac{4}{(3x - 1)(x + 1)}$$

$$x \neq \frac{1}{3} \text{ and } x \neq -1$$

b) $\dfrac{a}{3(b - a) + ab - a^2}$
Solution:

$$\begin{aligned} \frac{a}{3(b - a) + ab - a^2} &= \frac{a}{3(b - a) + a(b - a)} \\ &= \frac{a}{(b - a)(a + 3)} \end{aligned}$$

$$a \neq b \text{ and } a \neq -3$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2DR7	2a. 2DR8	2b. 2DR9	2c. 2DRB	2d. 2DRC	2e. 2DRD	2f. 2DRF	3a. 2DRG
3b. 2DRH	3c. 2DRJ	3d. 2DRK	4. 2DRM	5. 2DRN	6a. 2DRP	6b. 2DRQ	6c. 2DRR
6d. 2DRS	6e. 2DRT	6f. 2DRV	6g. 2DRW	7. 2DRX	8a. 2DRY	8b. 2DRZ	9. 2DS2
10a. 2DS3	10b. 2DS4	11a. 2DS5	11b. 2DS6	11c. 2DS7	11d. 2DS8	12a. 2DS9	12b. 2DSB
12c. 2DSC	12d. 2DSD	13. 2DSF	14. 2DSG	15. 2DSH	16a. 2DSJ	16b. 2DSK	16c. 2DSM
16d. 2DSN	17a. 2DSP	17b. 2DSQ	17c. 2DSR	17d. 2DSS	17e. 2DST	17f. 2DSV	17g. 2DSW
17h. 2DSX	17i. 2DSY	17j. 2DSZ	18a. 2DT2	18b. 2DT3	19a. 2DT4	19b. 2DT5	19c. 2DT6
19d. 2DT7	20a. 2DT8	20b. 2DT9	20c. 2DTB	20d. 2DTC	20e. 2DTD	20f. 2DTF	20g. 2DTG
20h. 2DTH	20i. 2DTJ	20j. 2DTK	20k. 2DTM	20l. 2DTN	21a. 2DTP	21b. 2DTQ	21c. 2DTR
21d. 2DTS	22a. 2DTT	22b. 2DTW	22c. 2DTX	22d. 2DTY	22e. 2DTZ	22f. 2DV2	22g. 2DV3
22h. 2DV4	22i. 2DV5	22j. 2DV6	22k. 2DV7	22l. 2DV8	22m. 2DV9	22n. 2DVB	22o. 2DVC
22p. 2DVD	22q. 2DVF	22r. 2DVG	22s. 2DVH	23a. 2DVJ	23b. 2DVK	23c. 2DVM	23d. 2DVN
23e. 2DVP	23f. 2DVQ	23g. 2DVR	23h. 2DVS	23i. 2DVT	23j. 2DVV	24. 2DVW	25. 2DVX
26. 2DVY	27a. 2DVZ	27b. 2DW2	27c. 2DW3	27d. 2DW4	28a. 2DW5	28b. 2DW6	28c. 2DW7
28d. 2DW8	28e. 2DW9	29a. 2DWB	29b. 2DWC	29c. 2DWD	29d. 2DWF	29e. 2DWG	29f. 2DWH
29g. 2DWJ	29h. 2DWK	29i. 2DWM	29j. 2DWN	29k. 2DWP	29l. 2DWQ	29m. 2DWR	29n. 2DWS
29o. 2DWT	29p. 2DWV	29q. 2DWW	29r. 2DWX	29s. 2DWY	29t. 2DWZ	29u. 2DX2	29v. 2DX3
29w. 2DX4	29x. 2DX5	29y. 2DX6	29z. 2DX7	30a. 2DX8	30b. 2DX9	30c. 2DXB	30d. 2DXC
30e. 2DXD	30f. 2DXF	30g. 2DXG	30h. 2DXH	30i. 2DXJ	30j. 2DXK	30k. 2DXM	30l. 2DXN
30m. 2DXP	30n. 2DXQ	30o. 2DXR	30p. 2DXS	30q. 2DXT	30r. 2DXV	30s. 2DXW	30t. 2DXX
30u. 2DXY	30v. 2DXZ	30w. 2DY2	30x. 2DY3	31a. 2DY4	31b. 2DY5	31c. 2DY6	31d. 2DY7
31e. 2DY8	31f. 2DY9	31g. 2DYB	31h. 2DYC	31i. 2DYD	31j. 2DYF	31k. 2DYG	31l. 2DYH
31m. 2DYJ	31n. 2DYK	31o. 2DYM	31p. 2DYN	31q. 2DYP	31r. 2DYQ	31s. 2DYR	31t. 2DYS
31u. 2DYT	31v. 2DYV	31w. 2DYW	31x. 2DYX	31y. 2DYY	31z. 2DYZ	32. 2DZ2	33. 2DZ3
34. 2DZ4	35. 2DZ5	36. 2DZ6	37a. 2DZ7	37b. 2DZ8			

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CHAPTER 2

Exponents

2.1 Introduction

- Content covered in this chapter includes the laws of exponents from grade 9 and simplifying expressions with exponents as well as solving simple exponential equations.
- The content in this chapter will be used in exponential equations later on as well as in grade 11 for financial calculations.
- Note that the rational exponent law is not covered in this chapter, this is only introduced in grade 11.

2.2 Revision of exponent laws

Exercise 2 – 1:

Simplify without using a calculator:

1. 16^0
   Solution:
$$16^0 = 1$$
2. $16a^0$
   Solution:
$$\begin{aligned} 16a^0 &= 16(1) \\ &= 16 \end{aligned}$$
3. $11^{9x} \times 11^{2x}$
   Solution:
$$\begin{aligned} 11^{9x} \times 11^{2x} &= 11^{9x+2x} \\ &= 11^{11x} \end{aligned}$$
4. $10^{6x} \times 10^{2x}$
   Solution:
$$\begin{aligned} 10^{6x} \times 10^{2x} &= 10^{6x+2x} \\ &= 10^{8x} \end{aligned}$$
5. $(6c)^3$
   Solution:
$$\begin{aligned} (6c)^3 &= 6^3c^3 \\ &= 216c^3 \end{aligned}$$
6. $(5n)^3$
   Solution:
$$\begin{aligned} (5n)^3 &= 5^3n^3 \\ &= 125n^3 \end{aligned}$$

7. $\dfrac{2^{-2}}{3^2}$
Solution:

$$\begin{aligned}\frac{2^{-2}}{3^2} &= \frac{\frac{1}{2^2}}{9} \\ &= \frac{1}{4} \times \frac{1}{9} \\ &= \frac{1}{36}\end{aligned}$$

8. $\dfrac{5}{2^{-3}}$
Solution:

$$\begin{aligned}\frac{5}{2^{-3}} &= \frac{5}{\frac{1}{2^3}} \\ &= \frac{5}{1} \times \frac{8}{1} \\ &= 40\end{aligned}$$

9. $\left(\dfrac{2}{3}\right)^{-3}$
Solution:

$$\begin{aligned}\left(\frac{2}{3}\right)^{-3} &= \frac{2^{-3}}{3^{-3}} \\ &= \frac{1}{8} \times \frac{27}{1} \\ &= \frac{27}{8}\end{aligned}$$

10. $\dfrac{a^2}{a^{-1}}$
Solution:

$$\frac{a^2}{a^{-1}} = a^3$$

11. $\dfrac{xy^{-3}}{x^4y}$
Solution:

$$\frac{xy^{-3}}{x^4y} = \frac{1}{x^3y^4}$$

12. x^2x^{3t+1}
Solution:

$$\begin{aligned}x^2x^{3t+1} &= x^2x^{3t}x^1 \\ &= x^{2+1}x^{3t} \\ &= x^3x^{3t} \\ &= x^{3t+3}\end{aligned}$$

13. $3 \times 3^{2a} \times 3^2$
Solution:

$$\begin{aligned}3 \times 3^{2a} \times 3^2 &= 3^{1+2a+2} \\ &= 3^{2a+3}\end{aligned}$$

14. $\dfrac{2^{m+20}}{2^{m+20}}$

Solution:

$$\begin{aligned}\frac{2^{m+20}}{2^{m+20}} &= 2^{(m+20)-(m+20)}\\ &= 2^{m+20-m-20}\\ &= 2^{0}\\ &= 1\end{aligned}$$

15. $\dfrac{2^{x+4}}{2^{x+3}}$

Solution:

$$\begin{aligned}\frac{2^{x+4}}{2^{x+3}} &= 2^{(x+4)-(x+3)}\\ &= 2^{x+4-x-3}\\ &= 2^{1}\\ &= 2\end{aligned}$$

16. $(2a^4)(3ab^2)$

Solution:

$$(2a^4)(3ab^2) = 6a^5b^2$$

17. $(7m^4n)(8m^6n^8)$

Solution:

$$(7m^4n)(8m^6n^8) = 56m^{10}n^9$$

18. $2(-a^7b^8)(-4a^3b^6)(-9a^6b^2)$

Solution:

$$\begin{aligned}2(-a^7b^8)(-4a^3b^6)(-9a^6b^2) &= -72a^{7+3+6}b^{8+6+2}\\ &= -72a^{16}b^{16}\end{aligned}$$

19. $(-9x^3y^6)\left(\frac{1}{9}x^8y^7\right)\left(\frac{1}{5}x^3y^6\right)$

Solution:

$$(-9x^3y^6)\left(\frac{1}{9}x^8y^7\right)\left(\frac{1}{5}x^3y^6\right) = -\frac{1}{5}x^{14}y^{19}$$

20. $\dfrac{a^{3x}}{a^{x}}$

Solution:

$$\begin{aligned}\frac{a^{3x}}{a^{x}} &= a^{3x}\times a^{-x}\\ &= a^{3x-x}\\ &= a^{2x}\end{aligned}$$

21. $\dfrac{20x^{10}a^4}{4x^9a^3}$

Solution:

$$\frac{20x^{10}a^4}{4x^9a^3}$$
$$= 5x^{(10-9)}a^{(4-3)}$$
$$= 5ax$$

22. $\dfrac{18c^{10}p^8}{9c^6p^5}$

Solution:

$$\frac{18c^{10}p^8}{9c^6p^5}$$
$$= 2c^{(10-6)}p^{(8-5)}$$
$$= 2c^4p^3$$

23. $\dfrac{6m^8a^{10}}{2m^3a^5}$

Solution:

$$\frac{6m^8a^{10}}{2m^3a^5}$$
$$= 3m^{(8-3)}a^{(10-5)}$$
$$= 3a^5m^5$$

24. $3^{12} \div 3^9$

Solution:

$$\begin{aligned}3^{12} \div 3^9 &= 3^{12-9}\\ &= 3^3\\ &= 27\end{aligned}$$

25. $\dfrac{7(a^3)^3}{a^7}$

Solution:

$$\begin{aligned}\frac{7(a^3)^3}{a^7} &= \frac{7a^9}{a^7}\\ &= 7a^2\end{aligned}$$

26. $\dfrac{9(ab^4)^8}{a^3b^5}$

Solution:

$$\begin{aligned}\frac{9(ab^4)^8}{a^3b^5} &= \frac{9a^8b^{32}}{a^3b^5}\\ &= 9a^5b^{27}\end{aligned}$$

27. $\dfrac{2^2}{6^2}$

Solution:

$$\begin{aligned}\frac{2^2}{6^2} &= \left(\frac{2}{6}\right)^2\\ &= \frac{1}{3^2}\\ &= \frac{1}{9}\end{aligned}$$

28. $\left(\frac{a^6}{b^7}\right)^5$

Solution:

$$\left(\frac{a^6}{b^7}\right)^5 = \frac{a^{30}}{b^{35}}$$

29. $\left(2t^4\right)^3$

Solution:

$$\begin{aligned}\left(2t^4\right)^3 &= 2^3 t^{(4)(3)} \\ &= 8t^{12}\end{aligned}$$

30. $\left(3^{n+3}\right)^2$

Solution:

$$\begin{aligned}\left(3^{n+3}\right)^2 &= 3^{(n+3)(2)} \\ &= 3^{2n+6}\end{aligned}$$

31. $\frac{3^n 9^{n-3}}{27^{n-1}}$

Solution:

$$\begin{aligned}\frac{3^n 9^{n-3}}{27^{n-1}} &= \frac{3^n 3^{2(n-3)}}{3^{3(n-1)}} \\ &= \frac{3^n 3^{2n-6}}{3^{3n-3}} \\ &= \frac{3^{n+2n-6}}{3^{3n-3}} \\ &= \frac{3^{3n-6}}{3^{3n-3}} \\ &= 3^{(3n-6)-(3n-3)} \\ &= 3^{3n-6-3n+3} \\ &= 3^{-3} \\ &= \frac{1}{27}\end{aligned}$$

32. $\frac{13^c + 13^{c+2}}{3 \times 13^c - 13^c}$

Solution:

$$\begin{aligned}\frac{13^c + 13^{c+2}}{3 \times 13^c - 13^c} &= \frac{13^c(1 + 13^2)}{13^c(3 - 1)} \\ &= \frac{(1 + 13^2)}{(3 - 1)} \\ &= \frac{1 + 169}{3 - 1} \\ &= \frac{170}{2} \\ &= \frac{85}{1} \\ &= 85\end{aligned}$$

33. $\frac{3^{5x} \times 81^{5x} \times 3^3}{9^{8x}}$

Solution:

$$\begin{aligned}\frac{3^{5x}\times 81^{5x}\times 3^3}{9^{8x}} &= \frac{3^{5x}\times\left(3^4\right)^{5x}\times 3^3}{\left(3^2\right)^{8x}}\\ &= \frac{3^{5x}\times 3^{20x}\times 3^3}{3^{16x}}\\ &= \frac{3^{5x+20x+3}}{3^{16x}}\\ &= \frac{3^{25x+3}}{3^{16x}}\\ &= 3^{25x+3-16x}\\ &= 3^{9x+3}\end{aligned}$$

34. $\dfrac{16^x - 144^b}{4^x - 12^b}$

Solution:

$$\begin{aligned}\frac{16^x - 144^b}{4^x - 12^b} &= \frac{\left(4^2\right)^x - \left(12^2\right)^b}{4^x - 12^b}\\ &= \frac{\left(4^x\right)^2 - \left(12^b\right)^2}{4^x - 12^b}\\ &= \frac{\left(4^x - 12^b\right)\left(4^x + 12^b\right)}{4^x - 12^b}\\ &= 4^x + 12^b\end{aligned}$$

35. $\dfrac{5^{2y-3}2^{4y+4}}{10^{-5y+5}}$

Solution:

$$\begin{aligned}\frac{5^{2y-3}2^{4y+4}}{10^{-5y+5}} &= \frac{5^{2y-3}\,.\,2^{4y+4}}{(5\times 2)^{-5y+5}}\\ &= \frac{5^{2y-3}2^{4y+4}}{5^{-5y+5}2^{-5y+5}}\\ &= 5^{(2y-3)-(-5y+5)}\times 2^{(4y+4)-(-5y+5)}\\ &= 5^{7y-8}\times 2^{9y-1}\end{aligned}$$

36. $\dfrac{6^4\times 12^3\times 4^5}{30^3\times 3^6}$

Solution:

$$\begin{aligned}\frac{6^4\times 12^3\times 4^5}{30^3\times 3^6} &= \frac{\left(3^4\times 2^4\right)\times\left(3^3\times 4^3\right)\times 4^5}{\left(3^3\times 10^3\right)\times 3^6}\\ &= \frac{3^4\times 2^4\times 3^3\times 2^6\times 2^{10}}{3^3\times 2^3\times 5^3\times 3^6}\\ &3^{4+3-3-6}\,.\,2^{4+6+10-3}\,.\,5^{-3}\\ &= \frac{2^{17}}{3^2 5^3}\end{aligned}$$

37. $\dfrac{9^3\times 20^2}{4\times 5^2\times 3^5}$

Solution:

$$
\begin{aligned}
\frac{9^3 \times 20^2}{4 \times 5^2 \times 3^5} &= \frac{3^6 4^2 5^2}{4 \times 5^2 3^5} \\
&= \frac{3^6 2^4 5^2}{2^2 5^2 3^5} \\
&= 3^{6-5} 2^{4-2} 5^{2-2} \\
&= 3 \times 2^2 \\
&= 12
\end{aligned}
$$

38. $\dfrac{7^b + 7^{b-2}}{4 \times 7^b + 3 \times 7^b}$

Solution:

$$
\begin{aligned}
\frac{7^b + 7^{b-2}}{4 \times 7^b - 3 \times 7^b} &= \frac{7^b(1 + 7^{-2})}{7^b(4 - 3)} \\
&= \frac{(1 + 7^{-2})}{1} \\
&= \frac{1 + \frac{1}{49}}{1} \\
&= \frac{51}{49}
\end{aligned}
$$

39. $\dfrac{12^y - 96^y}{3^y + 6^y}$

Solution:

$$
\begin{aligned}
\frac{12^y - 96^y}{3^y + 6^y} &= \frac{(4\,.\,3)^y - \left(2^5\,.\,3\right)^y}{3^y + (2\,.\,3)^y} \\
&= \frac{3^y\left(4^y - 2^{5y}\right)}{3^y\,(2 + 1)} \\
&= \frac{4^y - 2^{5y}}{3}
\end{aligned}
$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2DZG	2. 2DZH	3. 2DZJ	4. 2DZK	5. 2DZM	6. 2DZN	7. 2DZP	8. 2DZQ
9. 2DZR	10. 2DZS	11. 2DZT	12. 2DZV	13. 2DZW	14. 2DZX	15. 2DZY	16. 2DZZ
17. 2F22	18. 2F23	19. 2F24	20. 2F25	21. 2F26	22. 2F27	23. 2F28	24. 2F29
25. 2F2B	26. 2F2C	27. 2F2D	28. 2F2F	29. 2F2G	30. 2F2H	31. 2F2J	32. 2F2K
33. 2F2M	34. 2F2N	35. 2F2P	36. 2F2Q	37. 2F2R	38. 2F2S	39. 2F2T	

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2.3 Rational exponents

According to CAPS, the rational exponent law is introduced in Grade 11 but you may choose to introduce learners to the rational exponent law $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ at this stage.

Exercise 2 – 2:

Simplify without using a calculator:

1. $t^{\frac{1}{4}} \times 3t^{\frac{7}{4}}$

Solution:

$$
\begin{aligned}
t^{\frac{1}{4}} \times 3t^{\frac{7}{4}} &= 3t^{\frac{1}{4}+\frac{7}{4}} \\
&= 3t^{\frac{8}{4}} \\
&= 3t^{2}
\end{aligned}
$$

2. $\dfrac{16x^2}{(4x^2)^{\frac{1}{2}}}$

Solution:

$$
\begin{aligned}
\frac{16x^2}{(4x^2)^{\frac{1}{2}}} &= \frac{4^2x^2}{4^{\frac{1}{2}}x^{(2)\left(\frac{1}{2}\right)}} \\
&= \frac{4^2x^2}{4^{\frac{1}{2}}x} \\
&= 4^{2-\frac{1}{2}} \,.\, x^{2-1} \\
&= \left(2^2\right)^{\frac{3}{2}} x \\
&= 2^3x \\
&= 8x
\end{aligned}
$$

3. $(0{,}25)^{\frac{1}{2}}$

Solution:

$$
\begin{aligned}
(0{,}25)^{\frac{1}{2}} &= \left(\frac{1}{4}\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2^2}\right)^{\frac{1}{2}} \\
&= \left(2^{-2}\right)^{\frac{1}{2}} \\
&= 2^{-1} \\
&= \frac{1}{2}
\end{aligned}
$$

4. $(27)^{-\frac{1}{3}}$

Solution:

$$
\begin{aligned}
(27)^{-\frac{1}{3}} &= \left(3^3\right)^{-\frac{1}{3}} \\
&= 3^{-1} \\
&= \frac{1}{3}
\end{aligned}
$$

5. $\left(3p^2\right)^{\frac{1}{2}} \times \left(3p^4\right)^{\frac{1}{2}}$

Solution:

$$
\begin{aligned}
\left(3p^2\right)^{\frac{1}{2}} \times \left(3p^4\right)^{\frac{1}{2}} &= 3^{\frac{1}{2}}p \times 3^{\frac{1}{2}}p^2 \\
&= 3^{\frac{1}{2}+\frac{1}{2}} \times p^{1+2} \\
&= 3p^3
\end{aligned}
$$

6. $12\left(a^4b^8\right)^{\frac{1}{2}} \times \left(512a^3b^3\right)^{\frac{1}{3}}$

Solution:

$$
\begin{aligned}
12\left(a^4b^8\right)^{\frac{1}{2}} \times \left(512a^3b^3\right)^{\frac{1}{3}} &= 12a^{\frac{4}{2}}b^{\frac{8}{2}} \times (512)^{\frac{1}{3}}a^{\frac{3}{3}}b^{\frac{3}{3}} \\
&= 12a^2b^4 \times \left(8^3\right)^{\frac{1}{3}}a^1b^1 \\
&= 12a^2b^4 \times 8a^1b^1 \\
&= 96a^3b^5
\end{aligned}
$$

7. $\left((-2)^4a^6b^2\right)^{\frac{1}{2}}$

   Solution:

$$
\begin{aligned}
\left((-2)^4a^6b^2\right)^{\frac{1}{2}} &= (-2)^2(a^3b) \\
&= 4a^3b
\end{aligned}
$$

8. $\left(a^{-2}b^6\right)^{\frac{1}{2}}$

   Solution:

$$
\begin{aligned}
\left(a^{-2}b^6\right)^{\frac{1}{2}} &= a^{-1}b^3 \\
&= \frac{b^3}{a}
\end{aligned}
$$

9. $\left(16x^{12}b^6\right)^{\frac{1}{3}}$

   Solution:

$$
\begin{aligned}
\left(16x^{12}b^6\right)^{\frac{1}{3}} &= \left((8 \times 2)x^{12}b^6\right)^{\frac{1}{3}} \\
&= 2 \,.\, 2^{\frac{1}{3}}a^4b^2
\end{aligned}
$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2F2W 2. 2F2X 3. 2F2Y 4. 2F2Z 5. 2F32 6. 2F33 7. 2F34 8. 2F35 9. 2F36

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2.4 Exponential equations

Learners may find Worked Example 13 much easier using the k-substitution method. You may choose to return to this example once the k-substitution has been taught.

The solution using k-substitution is as follows:

$$
\begin{aligned}
2^x - 2^{4-x} &= 0 \\
2^x - 2^4 \,.\, 2^{-x} &= 0 \\
2^x - \frac{2^4}{2^x} &= 0 \\
\text{Let } 2^x &= k \\
k - \frac{2^4}{k} &= 0 \\
\times k \qquad k^2 - 16 &= 0
\end{aligned}
$$

$$\begin{aligned}(k-4)(k+4) &= 0 \\ k &= -4 \qquad \text{or } k = 4 \\ 2^x &\neq -4 \qquad 2^x = 4 \\ &\qquad\quad\;\; 2^x = 2^2 = 4 \\ x &= 2\end{aligned}$$

Exercise 2 – 3:

1. Solve for the variable:

a) $2^{x+5} = 32$

Solution:

$$\begin{aligned}2^{x+5} &= 32 \\ 2^{x+5} &= 2^5 \\ \therefore x+5 &= 5 \\ x &= 0\end{aligned}$$

b) $5^{2x+2} = \frac{1}{125}$

Solution:

$$\begin{aligned}5^{2x+2} &= \frac{1}{125} \\ 5^{2x+2} &= \frac{1}{5^3} \\ 5^{2x+2} &= 5^{-3} \\ \therefore 2x+2 &= -3 \\ 2x &= -5 \\ x &= -\frac{5}{2}\end{aligned}$$

c) $64^{y+1} = 16^{2y+5}$

Solution:

$$\begin{aligned}64^{y+1} &= 16^{2y+5} \\ 2^{6(y+1)} &= 2^{4(2y+5)} \\ 2^{6y+6} &= 2^{8y+20} \\ \therefore 6y+6 &= 8y+20 \\ 2y &= -14 \\ y &= -7\end{aligned}$$

d) $3^{9x-2} = 27$

Solution:

$$\begin{aligned}3^{9x-2} &= 27 \\ 3^{9x-2} &= 3^3 \\ \therefore 9x-2 &= 3 \\ 9x &= 5 \\ x &= \frac{5}{9}\end{aligned}$$

e) $25 = 5^{z-4}$

Solution:

$$\begin{aligned} 25 &= 5^{z-4} \\ 5^2 &= 5^{z-4} \\ 2 &= z-4 \\ 2+4 &= z \\ 6 &= z \end{aligned}$$

f) $-\frac{1}{2} \,.\, 6^{\frac{m}{2}+3} = -18$

Solution:

$$\begin{aligned} (-2)\left(-\frac{1}{2}6^{\frac{m}{2}+3}\right) &= (-18)\,(-2) \\ 6^{\frac{m}{2}+3} &= 36 \\ 6^{\frac{m}{2}+3} &= 6^2 \\ \frac{m}{2}+3 &= 2 \\ \frac{m}{2} &= -1 \\ m &= -2 \end{aligned}$$

g) $81^{k+2} = 27^{k+4}$

Solution:

$$\begin{aligned} 81^{k+2} &= 27^{k+4} \\ 3^{4(k+2)} &= 3^{3(k+4)} \\ \therefore 4k+8 &= 3k+12 \\ k &= 4 \end{aligned}$$

h) $25^{1-2x} - 5^4 = 0$

Solution:

$$\begin{aligned} 25^{1-2x} - 5^4 &= 0 \\ 5^{2(1-2x)} &= 5^4 \\ 5^{2-4x} &= 5^4 \\ \therefore 2-4x &= 4 \\ 4x &= -2 \\ x &= -\frac{1}{2} \end{aligned}$$

i) $27^x \times 9^{x-2} = 1$

Solution:

$$\begin{aligned} 27^x \times 9^{x-2} &= 1 \\ 3^{3x} \times 3^{2(x-2)} &= 1 \\ 3^{3x+2x-4} &= 3^0 \\ \therefore 5x-4 &= 0 \\ 5x &= 4 \\ x &= \frac{4}{5} \end{aligned}$$

j) $2^t + 2^{t+2} = 40$

Solution:

$$
\begin{aligned}
2^t + 2^{t+2} &= 40 \\
2^t(1 + 2^2) &= 40 \\
2^t(5) &= 40 \\
2^t &= 8 \\
2^t &= 2^3 \\
\therefore t &= 3
\end{aligned}
$$

k) $(7^x - 49)(3^x - 27) = 0$
Solution:

$$
\begin{aligned}
(7^x - 49)(3^x - 27) &= 0 \\
(7^x - 7^2)(3^x - 3^3) &= 0 \\
\therefore 7^x - 7^2 &= 0 \text{ or } 3^x - 3^3 = 0 \\
\therefore 7^x &= 7^2 \text{ or } 3^x = 3^3 \\
\therefore x &= 2 \text{ or } x = 3
\end{aligned}
$$

l) $(2 \,.\, 2^x - 16)(3^{x+1} - 9) = 0$
Solution:

$$
\begin{aligned}
(2 \,.\, 2^x - 16)(3^{x+1} - 9) &= 0 \\
(2^{x+1} - 2^4)(3^{x+1} - 3^2) &= 0 \\
\therefore 2^{x+1} - 2^4 &= 0 \text{ or } 3^{x+1} - 3^2 = 0 \\
x + 1 &= 4 \text{ or } x + 1 = 2 \\
\therefore x &= 3 \text{ or } x = 1
\end{aligned}
$$

m) $(10^x - 1)(3^x - 81) = 0$
Solution:

$$
\begin{aligned}
(10^x - 1)(3^x - 81) &= 0 \\
(10^x - 10^0)(3^x - 3^4) &= 0 \\
\therefore 10^x - 10^0 &= 0 \text{ or } 3^x - 3^4 = 0 \\
\therefore x &= 0 \text{ or } x = 4
\end{aligned}
$$

n) $2 \times 5^{2-x} = 5 + 5^x$
Solution:

$$
\begin{aligned}
2 \times 5^{2-x} &= 5 + 5^x \\
2(5^2)(5^{-x}) &= 5 + 5^x \\
\frac{2(5^2)}{5^x} - 5 - 5^x &= 0 \\
\left(\frac{50}{5^x}\right) \times 5^x - 5 \times 5^x - 5^x \times 5^x &= 0 \\
50 - 5(5^x) - (5^x)^2 &= 0 \\
(5^x - 5)(5^x + 10) &= 0 \\
5^x - 5 &= 0 \text{ or } 5^x + 10 = 0 \\
5^x &= 5 \text{ or } 5^x = -10 \\
x &= 1 \text{ or } \text{ undefined} \\
\therefore x &= 1
\end{aligned}
$$

o) $9^m + 3^{3-2m} = 28$

Solution:

$$
\begin{aligned}
9^m + 3^{3-2m} &= 28 \\
3^{2m} + 3^3.3^{-2m} &= 28 \\
3^{2m} + \frac{27}{3^{2m}} - 28 &= 0 \\
\left(3^{2m}\right)^2 - 28\left(3^{2m}\right) + 27 &= 0 \\
\left(3^{2m} - 27\right)\left(3^{2m} - 1\right) &= 0 \\
3^{2m} - 27 = 0 \text{ or } 3^{2m} - 1 &= 0 \\
3^{2m} = 3^3 \text{ or } 3^{2m} &= 3^0 \\
2m = 3 \text{ or } 2m &= 0 \\
\therefore m &= \frac{3}{2} \text{ or } 0
\end{aligned}
$$

p) $y - 2y^{\frac{1}{2}} + 1 = 0$
Solution:

$$
\begin{aligned}
y - 2y^{\frac{1}{2}} + 1 &= 0 \\
\left(y^{\frac{1}{2}} - 1\right)\left(y^{\frac{1}{2}} - 1\right) &= 0 \\
y^{\frac{1}{2}} - 1 &= 0 \\
y^{\frac{1}{2}} &= 1 \\
y^{\frac{1}{2}\times 2} &= 1^{1\times 2} \\
y &= 1^2 \\
\therefore y &= 1
\end{aligned}
$$

q) $4^{x+3} = 0{,}5$
Solution:

$$
\begin{aligned}
4^{x+3} &= 0{,}5 \\
2^{2x+6} &= \frac{1}{2} \\
2^{2x+6} &= 2^{-1} \\
\therefore 2x + 6 &= -1 \\
2x &= -7 \\
x &= -\frac{7}{2}
\end{aligned}
$$

r) $2^a = 0{,}125$
Solution:

$$
\begin{aligned}
2^a &= 0{,}125 \\
2^a &= \frac{1}{8} \\
2^a &= 2^{-3} \\
\therefore a &= -3
\end{aligned}
$$

s) $10^x = 0{,}001$
Solution:

$$\begin{aligned}
10^x &= 0{,}001\\
10^x &= \frac{1}{1000}\\
10^x &= 10^{-3}\\
\therefore x &= -3
\end{aligned}$$

t) $2^{x^2-2x-3} = 1$

Solution:

$$\begin{aligned}
2^{x^2-2x-3} &= 1\\
2^{x^2-2x-3} &= 2^0\\
\therefore x^2 - 2x - 3 &= 0\\
(x-3)(x+1) &= 0\\
\therefore x &= 3 \text{ or } -1
\end{aligned}$$

u) $\dfrac{8^x - 1}{2^x - 1} = 8.2^x + 9$

Solution:

$$\begin{aligned}
\frac{8^x - 1}{2^x - 1} &= 8.2^x + 9\\
\frac{2^{3x} - 1}{2^x - 1} &= 8.2^x + 9\\
\frac{(2^x - 1)(2^{2x} + 2^x + 1)}{2^x - 1} &= 8.2^x + 9\\
2^{2x} + 2^x + 1 &= 8.2^x + 9\\
2^{2x} + 2^x &= 8.2^x + 8\\
2^x.2^x + 2^x &= 8(2^x + 1)\\
2^x(2^x + 1) &= 8(2^x + 1)\\
2^x &= 2^3\\
\therefore x &= 3
\end{aligned}$$

v) $\dfrac{27^x - 1}{9^x + 3^x + 1} = -\dfrac{8}{9}$

Solution:

$$\begin{aligned}
\frac{27^x - 1}{9^x + 3^x + 1} &= -\frac{8}{9}\\
\frac{3^{3x} - 1}{9^x + 3^x + 1} &= -\frac{8}{9}\\
\frac{(3^x - 1)(9^x + 3^x + 1)}{9^x + 3^x + 1} &= -\frac{8}{9}\\
3^x - 1 &= -\frac{8}{9}\\
3^x &= \frac{1}{9}\\
3^x &= \frac{1}{3^2}\\
3^x &= 3^{-2}\\
\therefore x &= -2
\end{aligned}$$

2. The growth of algae can be modelled by the function $f(t) = 2^t$. Find the value of t such that $f(t) = 128$.

Solution:

$$f(t) = 2^t = 128$$
$$2^t = 2^7$$
$$\therefore t = 7$$

3. Use trial and error to find the value of x correct to 2 decimal places
$2^x = 7$
Solution:

$$2^2 = 4 \text{ and } 2^3 = 8$$
$$\text{so } 2 < x < 3 \text{ but closer to } 3$$
$$\text{Test}$$
$$2^{2{,}9} = 7{,}464$$
$$2^{2{,}8} = 6{,}964$$
$$2^{2{,}81} = 7{,}01$$
$$2^{2{,}805} = 6{,}989$$
$$2^{2{,}809} = 7{,}007$$
$$\therefore x \approx 2{,}81$$

4. Use trial and error to find the value of x correct to 2 decimal places
$5^x = 11$
Solution:

$$5^1 = 5 \text{ and } 5^2 = 25$$
$$\text{so } 1 < x < 2$$
$$\text{Test}$$
$$5^{1{,}5} = 11{,}180$$
$$5^{1{,}4} = 9{,}51$$
$$5^{1{,}45} = 10{,}31$$
$$5^{1{,}49} = 11{,}001$$
$$\therefore x \approx 1{,}49$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2F37	1b. 2F38	1c. 2F39	1d. 2F3B	1e. 2F3C	1f. 2F3D
1g. 2F3F	1h. 2F3G	1i. 2F3H	1j. 2F3J	1k. 2F3K	1l. 2F3M
1m. 2F3N	1n. 2F3P	1o. 2F3Q	1p. 2F3R	1q. 2F3S	1r. 2F3T
1s. 2F3V	1t. 2F3W	1u. 2F3X	1v. 2F3Y	2. 2F3Z	3. 2F42
4. 2F43					

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End of chapter Exercise 2 – 4:

1. Simplify:
   a) $(8x)^3$
   Solution:

$$(8x)^3 = 8^3x^3$$
$$= 512x^3$$

b) $t^3 \times 2t^0$
Solution:

$$t^3 \times 2t^0 = t^3 \times 2(1)$$
$$= 2t^3$$

c) $5^{2x+y} \times 5^{3(x+z)}$
Solution:

$$5^{2x+y} \times 5^{3(x+z)} = 5^{2x+y+3x+3z}$$
$$= 5^{5x+y+3z}$$

d) $15^{3x} \times 15^{12x}$
Solution:

$$15^{3x} \times 15^{12x} = 15^{3x+12x}$$
$$= 15^{15x}$$

e) $\frac{7^{y+7}}{7^{y+6}}$
Solution:

$$\frac{7^{y+7}}{7^{y+6}} = 7^{(y+7)-(y+6)}$$

$$7^{(y+7)-(y+6)} = 7^{y+7-y-6}$$
$$= 7^1$$
$$= 7$$

f) $3(d^4)(7d^3)$
Solution:

$$3(d^4)(7d^3) = 21d^7$$

g) $(\frac{1}{7}a^2b^9)(6a^6b^2)(-3a^7b)$
Solution:

$$\left(\frac{1}{7}a^2b^9\right)(6a^6b^2)(-3a^7b) = -\frac{18}{7}a^{15}b^{12}$$

h) $\left(b^{k+1}\right)^k$
Solution:

$$\left(b^{k+1}\right)^k = b^{k^2+k}$$

i) $\frac{24c^8m^7}{6c^2m^5}$
Solution:

$$\frac{24c^8m^7}{6c^2m^5}$$
$$= 4c^{(8-2)}m^{(7-5)}$$
$$= 4c^6m^2$$

j) $\dfrac{2(x^4)^3}{x^{12}}$

Solution:

$$\begin{aligned}\frac{2(x^4)^3}{x^{12}} &= \frac{2x^{12}}{x^{12}}\\ &= 2\end{aligned}$$

k) $\dfrac{a^6b^5}{7(a^8b^3)^2}$

Solution:

$$\begin{aligned}\frac{a^6b^5}{7(a^8b^3)^2} &= \frac{a^6b^5}{7a^{16}b^6}\\ &= \frac{1}{7a^{10}b}\end{aligned}$$

l) $\left(\frac{a^7}{b^4}\right)^2$

Solution:

$$\left(\frac{a^7}{b^4}\right)^2 = \frac{a^{14}}{b^8}$$

m) $\dfrac{6^{5p}}{9^p}$

Solution:

$$\begin{aligned}\frac{6^{5p}}{9^p} &= \frac{2^{5p}.3^{5p}}{3^{2p}}\\ &= 2^{5p}.3^{5p-2p}\\ &= 2^{5p}.3^{3p}\end{aligned}$$

n) $m^{-2t} \times \left(3m^t\right)^3$

Solution:

$$\begin{aligned}m^{-2t} \times \left(3m^t\right)^3 &= m^{-2t} \times 3^3m^{3t}\\ &= m^{-2t+3t}.27\\ &= 27m^t\end{aligned}$$

o) $\dfrac{3x^{-3}}{(3x)^2}$

Solution:

$$\begin{aligned}\frac{3x^{-3}}{(3x)^2} &= 3^{1-2}.x^{-3-2}\\ &= 3^{-1}.x^{-5}\\ &= \frac{1}{3x^5}\end{aligned}$$

p) $\dfrac{5^{b-3}}{5^{b+1}}$

Solution:

$$\begin{aligned}\frac{5^{b-3}}{5^{b+1}} &= 5^{b-3-b-1}\\ &= 5^{-4}\\ &= \frac{1}{625}\end{aligned}$$

q) $\dfrac{2^{a-2}3^{a+3}}{6^a}$

Solution:

$$\begin{aligned}\frac{2^{a-2}3^{a+3}}{6^a} &= \frac{2^{a-2}3^{a+3}}{(2.3)^a}\\ &= \frac{2^{a-2}3^{a+3}}{2^a.3^a}\\ &= 2^{a-2-a}.3^{a+3-a}\\ &= 2^{-2}.3^3\\ &= \frac{27}{4}\end{aligned}$$

r) $\dfrac{3^n 9^{n-3}}{27^{n-1}}$

Solution:

$$\begin{aligned}\frac{3^n 9^{n-3}}{27^{n-1}} &= \frac{3^n.\left(3^2\right)^{n-3}}{(3^3)^{n-1}}\\ &= \frac{3^n.3^{2n-6}}{3^{3n-3}}\\ &= 3^{n+2n-6-3n+3}\\ &= 3^{-3}\\ &= \frac{1}{27}\end{aligned}$$

s) $\dfrac{3^3}{9^3}$

Solution:

$$\begin{aligned}\frac{3^3}{9^3} &= \left(\frac{3}{9}\right)^3\\ &= \frac{1}{3^3}\\ &= \frac{1}{27}\end{aligned}$$

t) $\dfrac{x^{-1}}{x^4y^{-2}}$

Solution:

$$\frac{x^{-1}}{x^4y^{-2}} = \frac{y^2}{x^5}$$

u) $\dfrac{(-1)^4}{(-2)^{-3}}$

Solution:

$$\begin{aligned}\frac{(-1)^4}{(-2)^{-3}} &= \frac{1}{(-2)^{-3}}\\ &= (-2)^3\\ &= -8\end{aligned}$$

v) $\left(\dfrac{2x^{2a}}{y^{-b}}\right)^3$

Solution:

$$\left(\frac{2x^{2a}}{y^{-b}}\right)^3 = \frac{2^3\left(x^{2a}\right)^3}{\left(y^{-b}\right)^3}$$
$$= \frac{2^3x^{6a}}{y^{-3b}}$$
$$= 2^3x^{6a}y^{3b}$$
$$= 8x^{6a}y^{3b}$$

w) $\frac{2^{3x-1}8^{x+1}}{4^{2x-2}}$

Solution:

$$\frac{2^{3x-1}8^{x+1}}{4^{2x-2}} = \frac{2^{3x-1}.2^{3(x+1)}}{2^{2(2x-2)}}$$
$$= 2^{3x-1+3x+3-4x+4}$$
$$= 2^{2x+6}$$
$$= 4^{x+3}$$

x) $\frac{6^{2x}11^{2x}}{22^{2x-1}3^{2x}}$

Solution:

$$\frac{6^{2x}11^{2x}}{22^{2x-1}3^{2x}} = \frac{(3\ .\ 2)^{2x}\ .\ 11^{2x}}{(2\ .\ 11)^{2x-1}\ .\ 3^{2x}}$$
$$= \frac{3^{2x}\ .\ 2^{2x}\ .\ 11^{2x}}{2^{2x-1}\ .\ 11^{2x-1}\ .\ 3^{2x}}$$
$$= 3^{2x-2x}\ .\ 2^{2x-2x+1}\ .\ 11^{2x-2x+1}$$
$$= 3^0\ .\ 2^1\ .\ 11^1$$
$$= 22$$

y) $\frac{(-3)^{-3}(-3)^2}{(-3)^{-4}}$

Solution:

$$\frac{(-3)^{-3}(-3)^2}{(-3)^{-4}} = (-3)^{-3+2+4}$$
$$= (-3)^3$$
$$= -27$$

z) $\left(3^{-1}+2^{-1}\right)^{-1}$

Solution:

$$\left(3^{-1}+2^{-1}\right)^{-1} = \left(\frac{1}{3}+\frac{1}{2}\right)^{-1}$$
$$= \left(\frac{2}{6}+\frac{3}{6}\right)^{-1}$$
$$= \left(\frac{5}{6}\right)^{-1}$$
$$= \frac{5^{-1}}{6^{-1}}$$
$$= \frac{6}{5}$$

2. Simplify:

a) $\dfrac{9^{n-1}.27^{3-2n}}{81^{2-n}}$

Solution:

$$\begin{aligned}\frac{9^{n-1}.27^{3-2n}}{81^{2-n}} &= \frac{3^{2(n-1)}.3^{3(3-2n)}}{3^{4(2-n)}}\\ &= 3^{2(n-1)+3(3-2n)-4(2-n)}\\ &= 3^{2n-2+9-6n-8+4n}\\ &= \frac{1}{3}\end{aligned}$$

b) $\dfrac{2^{3n+2}\ .\ 8^{n-3}}{4^{3n-2}}$

Solution:

$$\begin{aligned}\frac{2^{3n+2}\ .\ 8^{n-3}}{4^{3n-2}} &= \frac{2^{3n+2}\ .\ 2^{3(n-3)}}{2^{2(3n-2)}}\\ &= 2^{3n+2+3(n-3)-2(3n-2)}\\ &= 2^{3n+2+3n-9-6n+4}\\ &= \frac{1}{8}\end{aligned}$$

c) $\dfrac{3^{t+3}+3^t}{2\times 3^t}$

Solution:

$$\begin{aligned}\frac{3^{t+3}+3^t}{2\times 3^t} &= \frac{3^t.3^3+3^t}{2.3^t}\\ &= \frac{3^t(3^3+1)}{2.3^t}\\ &= \frac{3^3+1}{2}\\ &= \frac{28}{2}\\ &= 14\end{aligned}$$

d) $\dfrac{2^{3p}+1}{2^p+1}$

Solution:

$$\begin{aligned}\frac{2^{3p}+1}{2^p+1} &= \frac{(2^p+1)\left(2^{2p}-2^p+1\right)}{(2^p+1)}\\ &= 2^{2p}-2^p+1\end{aligned}$$

e) $\left(a^{10}b^6\right)^{\frac{1}{2}}$

Solution:

$$\left(a^{10}b^6\right)^{\frac{1}{2}} = a^5b^3$$

f) $\left(9x^8y^4\right)^{\frac{1}{2}}$

Solution:

$$\left(9x^8y^4\right)^{\frac{1}{2}} = 3x^4y^2$$

g) $\dfrac{13^a+13^{a+2}}{6\times 13^a-13^a}$

Solution:

$$\begin{aligned}\frac{13^{a}+13^{a+2}}{6\times 13^{a}-13^{a}} &= \frac{13^{a}(1+13^{2})}{13^{a}(6-1)} \\ &= \frac{(1+13^{2})}{(6-1)} \\ &= \frac{1+169}{6-1} \\ &= \frac{170}{5} \\ &= \frac{34}{1} \\ &= 34\end{aligned}$$

h) $\dfrac{3^{8z}\times 27^{8z}\times 3^{2}}{9^{6z}}$

Solution:

$$\begin{aligned}\frac{3^{8z}\times 27^{8z}\times 3^{2}}{9^{6z}} &= \frac{3^{8z}\times \left(3^{3}\right)^{8z}\times 3^{2}}{\left(3^{2}\right)^{6z}} \\ &= \frac{3^{8z}\times 3^{24z}\times 3^{2}}{3^{12z}} \\ &= \frac{3^{8z+24z+2}}{3^{12z}} \\ &= \frac{3^{32z+2}}{3^{12z}} \\ &= 3^{32z+2-12z} \\ &= 3^{20z+2}\end{aligned}$$

i) $\dfrac{121^{b}-16^{p}}{11^{b}+4^{p}}$

Solution:

$$\begin{aligned}\frac{121^{b}-16^{p}}{11^{b}+4^{p}} &= \frac{\left(11^{2}\right)^{b}-\left(4^{2}\right)^{p}}{11^{b}+4^{p}} \\ &= \frac{\left(11^{b}\right)^{2}-(4^{p})^{2}}{11^{b}+4^{p}} \\ &= \frac{\left(11^{b}-4^{p}\right)\left(11^{b}+4^{p}\right)}{11^{b}+4^{p}} \\ &= \frac{\left(11^{b}-4^{p}\right)\left(11^{b}+4^{p}\right)}{11^{b}+4^{p}} \\ &= 11^{b}-4^{p}\end{aligned}$$

j) $\dfrac{11^{-4c-4}4^{4c-3}}{22^{-6c-2}}$

Solution:

$$\begin{aligned}\frac{11^{-4c-4}4^{4c-3}}{22^{-6c-2}} &= \frac{11^{-4c-4}\left(2^{2}\right)^{4c-3}}{(11\times 2)^{-6c-2}} \\ &= \frac{11^{-4c-4}2^{8c-6}}{11^{-6c-2}2^{-6c-2}} \\ &= 11^{(-4c-4)-(-6c-2)}\times 2^{(8c-6)-(-6c-2)} \\ &= 11^{2c-2}\times 2^{14c-4}\end{aligned}$$

k) $\dfrac{12^4 \times 2^4}{16^6 \times 10}$

Solution:

$$\begin{aligned}\frac{12^4 \times 2^4}{16^6 \times 10} &= \frac{(3 \times 2^2)^4 \times 2^4}{(2^4)^6 \times (2 \times 5)} \\ &\frac{3^4 \times 2^8 \times 2^4}{2^{24} \times 2 \times 5} \\ &= \frac{3^4}{2^{13} \times 5}\end{aligned}$$

l) $\dfrac{5^6 \times 3^{16} \times 2^7}{10^8 \times 9^6}$

Solution:

$$\begin{aligned}\frac{5^6 \times 3^{16} \times 2^7}{10^8 \times 9^6} &= \frac{5^6 3^{16} 2^7}{2^8 5^8 3^{12}} \\ &= \frac{3^4}{2 \times 5^2} \\ &= \frac{81}{50}\end{aligned}$$

m) $(0{,}81)^{\frac{1}{2}}$

Solution:

$$\begin{aligned}(0{,}81)^{\frac{1}{2}} &= \left(\frac{81}{100}\right)^{\frac{1}{2}} \\ &= \left(\frac{9^2}{10^2}\right)^{\frac{1}{2}} \\ &= \left[\left(\frac{9}{10}\right)^2\right]^{\frac{1}{2}} \\ &= \frac{9}{10}\end{aligned}$$

n) $12\left(a^{10}b^{20}\right)^{\frac{1}{5}} \times \left(729a^{12}b^{15}\right)^{\frac{1}{3}}$

Solution:

$$\begin{aligned}12\left(a^{10}b^{20}\right)^{\frac{1}{5}} \times \left(729a^{12}b^{15}\right)^{\frac{1}{3}} &= 12a^{\frac{10}{5}}b^{\frac{20}{5}} \times (729)^{\frac{1}{3}}a^{\frac{12}{3}}b^{\frac{15}{3}} \\ &= 12a^2b^4 \times \left(9^3\right)^{\frac{1}{3}}a^4b^5 \\ &= 12a^2b^4 \times 9a^4b^5 \\ &= 108a^6b^9\end{aligned}$$

o) $2\left(p^{30}q^{20}\right)^{\frac{1}{5}} \times \left(1331p^{12}q^6\right)^{\frac{1}{3}}$

Solution:

$$\begin{aligned}2\left(p^{30}q^{20}\right)^{\frac{1}{5}} \times \left(1331p^{12}q^6\right)^{\frac{1}{3}} &= 2p^{\frac{30}{5}}q^{\frac{20}{5}} \times (1331)^{\frac{1}{3}}p^{\frac{12}{3}}q^{\frac{6}{3}} \\ &= 2p^6q^4 \times \left(11^3\right)^{\frac{1}{3}}p^4q^2 \\ &= 2p^6q^4 \times 11p^4q^2 \\ &= 22p^{10}q^6\end{aligned}$$

p) $\dfrac{a^{-1}-b^{-1}}{a-b}$

Solution:

$$\begin{aligned}\frac{a^{-1}-b^{-1}}{a-b} &= \frac{\frac{1}{a}-\frac{1}{b}}{a-b}\\ &= \frac{\frac{b-a}{ab}}{a-b}\\ &= \frac{-(a-b)}{ab(a-b)}\\ &= -\frac{1}{ab}\\ &= -(ab)^{-1}\end{aligned}$$

q) $\left((x^{36})^{\frac{1}{2}}\right)^{\frac{1}{3}}$

Solution:

$$\begin{aligned}\left((x^{36})^{\frac{1}{2}}\right)^{\frac{1}{3}} &= (x^{18})^{\frac{1}{3}}\\ &= x^{6}\end{aligned}$$

r) $\left(\dfrac{2}{3}\right)^{x+y} \cdot \left(\dfrac{3}{2}\right)^{x-y}$

Solution:

$$\begin{aligned}\left(\frac{2}{3}\right)^{x+y} \cdot \left(\frac{3}{2}\right)^{x-y} &= \left(\frac{2}{3}\right)^{x+y} \cdot \left(\frac{2}{3}\right)^{-(x-y)}\\ &= \left(\frac{2}{3}\right)^{x+y-(x-y)}\\ &= \left(\frac{2}{3}\right)^{2y}\end{aligned}$$

s) $(a^{\frac{1}{2}}+a^{-\frac{1}{2}})^2-(a^{\frac{1}{2}}-a^{-\frac{1}{2}})^2$

Solution:

$$\begin{aligned}(a^{\frac{1}{2}}+a^{-\frac{1}{2}})^2-(a^{\frac{1}{2}}-a^{-\frac{1}{2}})^2 &= (a^{\frac{1}{2}}+a^{-\frac{1}{2}}-(a^{\frac{1}{2}}-a^{-\frac{1}{2}}))(a^{\frac{1}{2}}+a^{-\frac{1}{2}}+(a^{\frac{1}{2}}-a^{-\frac{1}{2}}))\\ &= (2a^{-\frac{1}{2}})(2a^{\frac{1}{2}})\\ &= 4a^{\frac{1}{2}-\frac{1}{2}}\\ &= 4a^{0}\\ &= 4\end{aligned}$$

3. Solve:

a) $3^x = \dfrac{1}{27}$

Solution:

$$\begin{aligned}3^x &= \frac{1}{27}\\ 3^x &= \frac{1}{3^3}\\ 3^x &= 3^{-3}\\ \therefore x &= -3\end{aligned}$$

b) $121 = 11^{m-1}$

Solution:

$$\begin{aligned} 121 &= 11^{m-1} \\ 11^2 &= 11^{m-1} \\ \therefore 2 &= m - 1 \\ 2 + 1 &= m \\ 3 &= m \end{aligned}$$

c) $5^{t-1} = 1$

Solution:

$$\begin{aligned} 5^{t-1} &= 1 \\ 5^{t-1} &= 5^0 \\ \therefore t - 1 &= 0 \\ t &= 1 \end{aligned}$$

d) $2 \times 7^{3x} = 98$

Solution:

$$\begin{aligned} 2 \times 7^{3x} &= 98 \\ 7^{3x} &= 49 \\ 7^{3x} &= 7^2 \\ \therefore 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

e) $-\frac{64}{3} = -\frac{4}{3}2^{-\frac{c}{3}+1}$

Solution:

$$\begin{aligned} \left(-\frac{3}{4}\right)\left(-\frac{64}{3}\right) &= \left(-\frac{4}{3} \, . \, 2^{-\frac{c}{3}+1}\right)\left(-\frac{3}{4}\right) \\ 16 &= 2^{-\frac{c}{3}+1} \\ \therefore 2^4 &= 2^{-\frac{c}{3}+1} \\ 4 &= -\frac{c}{3} + 1 \\ -9 &= c \end{aligned}$$

f) $-\frac{1}{2}6^{-n-3} = -18$

Solution:

$$\begin{aligned} (-2)\left(-\frac{1}{2} \, . \, 6^{-n-3}\right) &= (-18)(-2) \\ 6^{-n-3} &= 36 \\ 6^{-n-3} &= 6^2 \\ \therefore -n - 3 &= 2 \\ n &= -5 \end{aligned}$$

g) $2^{m+1} = (0{,}5)^{m-2}$

Solution:

$$
\begin{aligned}
2^{m+1} &= (0{,}5)^{m-2} \\
2^{m+1} &= \left(\frac{1}{2}\right)^{m-2} \\
2^{m+1} &= \left(2^{-1}\right)^{m-2} \\
2^{m+1} &= 2^{2-m} \\
\therefore m+1 &= 2-m \\
m &= \frac{1}{2}
\end{aligned}
$$

h) $3^{y+1} = 5^{y+1}$
Solution:

$$
\begin{aligned}
3^{y+1} &= 5^{y+1} \\
\therefore y+1 &= 0 \\
y &= -1
\end{aligned}
$$

i) $z^{\frac{3}{2}} = 64$
Solution:

$$
\begin{aligned}
z^{\frac{3}{2}} &= 64 \\
z^{\frac{3}{2}} &= 4^3 \\
\left(z^{\frac{3}{2}}\right)^{\frac{2}{3}} &= \left(4^3\right)^{\frac{2}{3}} \\
z &= 4^2 \\
z &= 16
\end{aligned}
$$

j) $16x^{\frac{1}{2}} - 4 = 0$
Solution:

$$
\begin{aligned}
16x^{\frac{1}{2}} - 4 &= 0 \\
16x^{\frac{1}{2}} &= 4 \\
x^{\frac{1}{2}} &= \frac{4}{16} \\
x^{\frac{1}{2}} &= \frac{1}{4} \\
\left(x^{\frac{1}{2}}\right)^2 &= \left(\frac{1}{4}\right)^2 \\
x &= \frac{1}{16}
\end{aligned}
$$

k) $m^0 + m^{-1} = 0$
Solution:

$$
\begin{aligned}
m^0 + m^{-1} &= 0 \\
1 + m^{-1} &= 0 \\
m^{-1} &= -1 \\
\left(m^{-1}\right)^{-1} &= (-1)^{-1} \\
m &= -1
\end{aligned}
$$

l) $t^{\frac{1}{2}} - 3t^{\frac{1}{4}} + 2 = 0$
Solution:

$$
\begin{aligned}
t^{\frac{1}{2}} - 3t^{\frac{1}{4}} + 2 &= 0 \\
\left(t^{\frac{1}{4}} - 1\right)\left(t^{\frac{1}{4}} - 2\right) &= 0 \\
t^{\frac{1}{4}} - 1 = 0 \text{ or } t^{\frac{1}{4}} - 2 &= 0 \\
t^{\frac{1}{4}} = 1 \text{ or } t^{\frac{1}{4}} &= 2 \\
\left(t^{\frac{1}{4}}\right)^4 = (1)^4 \text{ or } \left(t^{\frac{1}{4}}\right)^4 &= (2)^4 \\
t &= 1 \text{ or } 16
\end{aligned}
$$

m) $3^p + 3^p + 3^p = 27$

Solution:

$$
\begin{aligned}
3^p + 3^p + 3^p &= 27 \\
3.3^p &= 27 \\
3^{p+1} &= 3^3 \\
\therefore p + 1 &= 3 \\
p &= 2
\end{aligned}
$$

n) $k^{-1} - 7k^{-\frac{1}{2}} - 18 = 0$

Solution:

$$
\begin{aligned}
k^{-1} - 7k^{-\frac{1}{2}} - 18 &= 0 \\
\left(k^{-\frac{1}{2}} - 9\right)\left(k^{-\frac{1}{2}} + 2\right) &= 0 \\
k^{-\frac{1}{2}} - 9 = 0 \text{ or } k^{-\frac{1}{2}} + 2 &= 0 \\
k^{-\frac{1}{2}} = 9 \text{ or } k^{-\frac{1}{2}} &= -2 \\
\left(k^{-\frac{1}{2}}\right)^{-2} = (9)^{-2} \text{ or } \left(k^{-\frac{1}{2}}\right)^{-2} &= (-2)^{-2} \\
k &= \frac{1}{81} \text{ or } \frac{1}{4}
\end{aligned}
$$

We check both answers and find that $k = \frac{1}{81}$ is the only solution.

o) $x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 18 = 0$

Solution:

$$
\begin{aligned}
\left(x^{\frac{1}{4}} + 6\right)\left(x^{\frac{1}{4}} - 3\right) &= 0 \\
x^{\frac{1}{4}} + 6 = 0 \text{ or } x^{\frac{1}{4}} - 3 &= 0 \\
x^{\frac{1}{4}} = -6 \text{ or } x^{\frac{1}{4}} &= 3 \\
\left(x^{\frac{1}{4}}\right)^4 = (-6)^4 \text{ or } \left(x^{\frac{1}{4}}\right)^4 &= (3)^4 \\
x &= 1296 \text{ or } 81
\end{aligned}
$$

We check both answers and find that $x = 81$ is the only solution.

p) $\dfrac{16^x - 1}{4^2x + 1} = 3$

Solution:

$$\frac{16^x - 1}{4^2x + 1} = 3$$
$$\frac{(4^{2x} - 1)(4^{2x} + 1)}{4^{2x} + 1} = 3$$
$$4^{2x} - 1 = 3$$
$$4^{2x} = 4^1$$
$$\therefore 2x = 1$$
$$x = \frac{1}{2}$$

q) $(2^x - 8)(3^x - 9) = 0$
Solution:

$$(2^x - 8)(3^x - 9) = 0$$
$$(2^x - 2^3)(3^x - 3^2) = 0$$
$$\therefore 2^x - 2^3 = 0 \text{ or } 3^x - 3^2 = 0$$
$$\therefore x = 3 \text{ or } x = 2$$

r) $(6^x - 36)(16 - 4^x) = 0$
Solution:

$$(6^x - 36)(16 - 4^x) = 0$$
$$(6^x - 6^2)(4^2 - 4^x) = 0$$
$$\therefore 6^x - 6^2 = 0 \text{ or } 4^2 - 4^x = 0$$
$$\therefore x = 2$$

s) $5.2^{x^2+1} = 20$
Solution:

$$5.2^{x^2+1} = 20$$
$$2^{x^2+1} = 4$$
$$2^{x^2+1} = 2^2$$
$$\therefore x^2 + 1 = 2$$
$$x^2 - 1 = 0$$
$$(x + 1)(x - 1) = 0$$

$$\therefore x = 1 \text{ or } x = -1$$

t) $27^{x-2} = 9^{2x+1}$
Solution:

$$27^{x-2} = 9^{2x+1}$$
$$(3^3)^{x-2} = (3^2)^{2x+1}$$
$$3^{3x-6} = 3^{4x+2}$$
$$\therefore 3x - 6 = 4x + 2$$
$$x = -8$$

u) $\frac{8^x - 1}{2^x - 1} = 7$
Solution:

$$\begin{aligned}
\frac{8^x-1}{2^x-1} &= 7 \\
\frac{(2^3)^x-1}{2^x-1} &= 7 \\
\frac{(2^x)^3-1}{2^x-1} &= 7 \\
\frac{(2^x-1)((2^x)^2+2^x+1)}{(2^x-1)} &= 7 \\
(2^{2x}+2^x+1) &= 7 \\
2^{2x}+2^x-6 &= 0 \\
(2^x+3)(2^x-2) &= 0 \\
\therefore 2^x+3=0 \text{ or } 2^x-2 &= 0 \\
2^x \neq -3 \text{ or } 2^x-2 &= 0 \\
2^x &= 2 \\
x &= 1
\end{aligned}$$

v) $\frac{35^x}{5^x}=\frac{1}{7}$

Solution:

$$\begin{aligned}
\frac{35^x}{5^x} &= \frac{1}{7} \\
\frac{7^x5^x}{5^x} &= \frac{1}{7} \\
7^x &= 7^{-1} \\
\therefore x &= -1
\end{aligned}$$

w) $\frac{a^{3x}\ .\ a^{\frac{1}{x}}}{a^{-4}}=1$

Solution:

$$\begin{aligned}
\frac{a^{3x}.a^{\frac{1}{x}}}{a^{-4}} &= 1 \\
a^{3x+\frac{1}{x}+4} &= a^0 \\
\therefore 3x+\frac{1}{x}+4 &= 0 \\
3x^2+1+4x &= 0 \\
(3x+1)(x+1) &= 0 \\
\therefore x &= -\frac{1}{3} \text{ or } x=-1
\end{aligned}$$

x) $2x^{\frac{1}{2}}+1=-x$

Solution:

$$\begin{aligned}
2x^{\frac{1}{2}}+1 &= -x \\
x+2x^{\frac{1}{2}}+1 &= 0 \\
(x^{\frac{1}{2}})^2+2x^{\frac{1}{2}}+1^2 &= 0 \\
(x^{\frac{1}{2}}+1)^2 &= 0 \\
x^{\frac{1}{2}} &= -1 \\
x &= 1
\end{aligned}$$

However $2(1)^{\frac{1}{2}}+1=2\neq-(1)$ $\therefore$ no solution exists

4. Use trial and error to find the value of x correct to 2 decimal places
   $4^x = 44$
   Solution:

$$\begin{aligned} 4^2 = 16 \text{ and } 4^3 &= 64 \\ \text{so } 2 < x &< 3 \\ \text{Test} & \\ 4^{2{,}5} &= 32 \\ 4^{2{,}75} &= 45{,}255 \\ 4^{2{,}70} &= 42{,}224 \\ 4^{2{,}73} &= 44{,}017 \\ 4^{2{,}725} &= 43{,}713 \\ \therefore x &\approx 2{,}73 \end{aligned}$$

5. Use trial and error to find the value of x correct to 2 decimal places
   $3^x = 30$
   Solution:

$$\begin{aligned} 3^3 = 27 \text{ and } 3^4 &= 81 \\ \text{so } 3 < x &< 4 \\ \text{Test} & \\ 3^{3{,}1} &= 30{,}014 \\ 3^{3{,}05} &= 28{,}525 \\ 3^{3{,}08} &= 29{,}480 \\ 3^{3{,}09} &= 29{,}806 \\ 3^{3{,}095} &= 29{,}970 \\ 3^{3{,}096} &= 30{,}003 \\ \therefore x &\approx 3{,}10 \end{aligned}$$

6. Explain why the following statements are false:

   a) $\dfrac{1}{a^{-1}+b^{-1}} = a+b$
   Solution:
   The sum of two powers of the same degree is not the power of the sum of the bases

$$a+b = \frac{1}{(a+b)^{-1}} \neq \frac{1}{a^{-1}+b^{-1}}$$

   b) $(a+b)^2 = a^2+b^2$
   Solution:
   The sum of two powers of the same degree is not the power of the sum of the bases

$$(a+b)^2 = a^2+2ab+b^2 \neq a^2+b^2$$

   c) $\left(\frac{1}{a^2}\right)^{\frac{1}{3}} = a^{\frac{2}{3}}$
   Solution:
   A negative sign is missing, when a power is moved from the denominator to the numerator, the sign of the exponent changes.
   From the question we must note that $a \neq 0$

$$\begin{aligned} \left(\frac{1}{a^2}\right)^{\frac{1}{3}} &= \left(a^{-2}\right)^{\frac{1}{3}} \\ &= a^{-\frac{2}{3}} \end{aligned}$$

d) $2.3^x = 6^x$
Solution:
We cannot multiply bases unless they are raised to the same power

$$6^x = (2 \times 3)^x = 2^x.3^x \neq 2.3^x$$

e) $x^{-\frac{1}{2}} = \dfrac{1}{-x^{\frac{1}{2}}}$
Solution:
The sign of a base is not changed when an exponent is moved from the denominator to the numerator in a fraction

$$x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}} \neq \frac{1}{-x^{\frac{1}{2}}}$$

f) $(3x^4y^2)^3 = 3x^{12}y^6$
Solution:
The power of a product is the product of all the bases raised to the same power

$$\begin{aligned}(3x^4y^2)^3 &= (3)^3(x^4)^3(y^2)^3 \\ &= 27x^{12}y^6 \neq 3x^{12}y^6\end{aligned}$$

7. If $2^{2013}.5^{2015}$ is written out in full how many digits will there be?
Solution:

$$\begin{aligned}2^{2013}.5^{2015} &= 2^{2013}.5^{2013+2} \\ &= 2^{2013}.5^{2013}.5^2 \\ &= 25(2^{2013}.5^{2013}) \\ &= 25(10^{2013}) \\ &= 25 \times 10^{2013}\end{aligned}$$

10^{2013} has 2014 digits therefore 25×10^{2013} 2015 digits.

8. Prove that $\dfrac{2^{n+1} + 2^n}{2^n - 2^{n-1}} = \dfrac{3^{n+1} + 3^n}{3^n - 3^{n-1}}$
Solution:

$$\frac{2^{n+1}+2^n}{2^n-2^{n-1}} = \frac{3^{n+1}+3^n}{3^n-3^{n-1}}$$

$$\begin{aligned}
\text{R.H.S} &= \frac{3^{n+1}+3^n}{3^n-3^{n-1}} \\
&= \frac{3^n(3^1+3^0)}{3^n(3^0-3^{-1})} \\
&= \frac{4}{1-\frac{1}{3}} \\
&= \frac{4}{\frac{2}{3}} \\
&= \frac{12}{2} \\
&= 6
\end{aligned}$$

$$\begin{aligned}
\text{L.H.S} &= \frac{2^{n+1}+2^n}{2^n-2^{n-1}} \\
&= \frac{2^n(2^1+2^0)}{2^n(2^0-2^{-1})} \\
&= \frac{3}{1-\frac{1}{2}} \\
&= \frac{3}{\frac{1}{2}} \\
&= 6
\end{aligned}$$

$$\therefore \text{R.H.S} = \text{L.H.S}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2F45	1b. 2F46	1c. 2F47	1d. 2F48	1e. 2F49	1f. 2F4B
1g. 2F4C	1h. 2F4D	1i. 2F4F	1j. 2F4G	1k. 2F4H	1l. 2F4J
1m. 2F4K	1n. 2F4M	1o. 2F4N	1p. 2F4P	1q. 2F4Q	1r. 2F4R
1s. 2F4S	1t. 2F4T	1u. 2F4V	1v. 2F4W	1w. 2F4X	1x. 2F4Y
1y. 2F4Z	1z. 2F52	2a. 2F53	2b. 2F54	2c. 2F55	2d. 2F56
2e. 2F57	2f. 2F58	2g. 2F59	2h. 2F5B	2i. 2F5C	2j. 2F5D
2k. 2F5F	2l. 2F5G	2m. 2F5H	2n. 2F5J	2o. 2F5K	2p. 2F5M
2q. 2F5N	2r. 2F5P	2s. 2F5Q	3a. 2F5R	3b. 2F5S	3c. 2F5T
3d. 2F5V	3e. 2F5W	3f. 2F5X	3g. 2F5Y	3h. 2F5Z	3i. 2F62
3j. 2F63	3k. 2F64	3l. 2F65	3m. 2F66	3n. 2F67	3o. 2F68
3p. 2F69	3q. 2F6B	3r. 2F6C	3s. 2F6D	3t. 2F6F	3u. 2F6G
3v. 2F6H	3w. 2F6J	3x. 2F6K	4. 2F6M	5. 2F6N	6a. 2F6P
6b. 2F6Q	6c. 2F6R	6d. 2F6S	6e. 2F6T	6f. 2F6V	7. 2F6W
8. 2F6X					

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CHAPTER 3

Number patterns

- This chapter covers investigating number patterns that involve a common difference and the general term is linear.
- Arithmetic sequences are only covered in grade 12 so do not use $T_n = a + (n-1)d$ here.
- The focus of this chapter is more about investigating patterns in numbers and diagrams rather than on formulae.

3.1 Introduction

3.2 Describing sequences

Some learners may see example 3 as $2^1; 2^2; 2^3; \ldots$ and see a pattern with the powers. You may choose to discuss this in class as a precursor to geometric series which will be introduced in Grade 12.

Common difference

Exercise 3 – 1:

1. Use the given pattern to complete the table below.

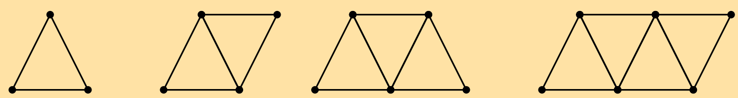


Figure number	1	2	3	4	n
Number of dots					
Number of lines					
Total					

Solution:

Figure number	1	2	3	4	n
Number of dots	3	4	5	6	$n+2$
Number of lines	3	5	7	9	$2n+1$
Total	6	9	12	15	$3(n+1)$

2. Consider the sequence shown here: $-4\ ;\ -1\ ;\ 2\ ;\ 5\ ;\ 8\ ;\ 11\ ;\ 14\ ;\ 17\ ;\ \ldots$
If $T_n = 2$ what is the value of T_{n-1}?
Solution:

$$T_3 = 2$$
$$\therefore T_{n-1} = -1$$

3. Consider the sequence shown here: $C\ ;\ D\ ;\ E\ ;\ F\ ;\ G\ ;\ H\ ;\ I\ ;\ J\ ;\ \ldots$
If $T_n = G$ what is the value of T_{n-4}?
Solution:

$$T_5 = G$$
$$\therefore T_{n-4} = C$$

4. For each of the following sequences determine the common difference. If the sequence is not linear, write "no common difference".

a) $9\ ;\ -7\ ;\ -8\ ;\ -25\ ;\ -34\ ;\ \ldots$
Solution:

$$
\begin{aligned}
d &= T_2 - T_1 = (-7) - (9) = -16 \\
d &= T_3 - T_2 = (-8) - (-7) = -1 \\
d &= T_4 - T_3 = (-25) - (-8) = -17
\end{aligned}
$$

You can see that the results are not the same - the difference is not 'common.' That means that this sequence of numbers in not linear, and it has no common difference.

b) $5 \ ; \ 12 \ ; \ 19 \ ; \ 26 \ ; \ 33 \ ; \ \ldots$
Solution:

$$
\begin{aligned}
d &= T_2 - T_1 = (12) - (5) = 7 \\
d &= T_3 - T_2 = (19) - (12) = 7 \\
d &= T_4 - T_3 = (26) - (19) = 7
\end{aligned}
$$

All of the results are the same, which means we have found the **common** difference for these numbers: $d = 7$.

c) 2,93 ; 1,99 ; 1,14 ; 0,35 ; ...
Solution:

$$
\begin{aligned}
d &= T_2 - T_1 = (1{,}99) - (2{,}93) = -0{,}94 \\
d &= T_3 - T_2 = (1{,}14) - (1{,}99) = -0{,}85
\end{aligned}
$$

In this case the sequence is not linear. Therefore the final answer is that there is no common difference.

d) 2,53 ; 1,88 ; 1,23 ; 0,58 ; ...
Solution:

$$
\begin{aligned}
d &= T_2 - T_1 = (1{,}88) - (2{,}53) = -0{,}65 \\
d &= T_3 - T_2 = (1{,}23) - (1{,}88) = -0{,}65
\end{aligned}
$$

The common difference is $d = -0{,}65$.

5. Write down the next three terms in each of the following sequences:

a) $5 \ ; \ 15 \ ; \ 25 \ ; \ \ldots$
Solution:
The common difference is:

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= 15 - 5 \\
&= 10
\end{aligned}
$$

Therefore we add 10 each time to get the next term in the sequence. The next three numbers are:
35, 45 and 55
and the sequence becomes:
$5 \ ; \ 15 \ ; \ 25 \ ; \ 35 \ ; \ 45 \ ; \ 55 \ ; \ \ldots$

b) $-8 \ ; \ -3 \ ; \ 2 \ ; \ \ldots$
Solution:
The common difference is:

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= -3 - (-8) \\
&= 5
\end{aligned}
$$

Therefore we add 5 each time to get the next term in the sequence. The next three numbers are:
7, 12 and 17
and the sequence becomes:
$-8 \ ; \ -3 \ ; \ 2 \ ; \ 7 \ ; \ 12 \ ; \ 17 \ ; \ \ldots$

c) $30\ ;\ 27\ ;\ 24\ ;\ \ldots$

Solution:

The common difference is:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 27 - 30 \\ &= -3 \end{aligned}$$

Therefore we subtract 3 each time to get the next term in the sequence. The next three numbers are: 21, 18 and 15

and the sequence becomes:

$30\ ;\ 27\ ;\ 24\ ;\ 21\ ;\ 18\ ;\ 15\ ;\ \ldots$

d) $-13{,}1\ ;\ -18{,}1\ ;\ -23{,}1\ ;\ldots$

Solution:

$$\begin{aligned} d &= T_2 - T_1 \text{ or } T_3 - T_2 \\ &= (-18{,}1) - (-13{,}1) \text{ or } (-23{,}1) - (-18{,}1) \\ &= -5 \\ \text{Therefore } T_4 &= -28{,}1 \\ T_5 &= -33{,}1 \\ T_6 &= -38{,}1 \end{aligned}$$

e) $-9x\ ; -19x\ ; -29x\ ;\ldots$

Solution:

$$\begin{aligned} d &= T_2 - T_1 \text{ or } T_3 - T_2 \\ &= (-19x) - (-9x) \text{ or } (-29x) - (-19x) \\ &= -10x \\ \text{Therefore } T_4 &= -39x \\ T_5 &= -49x \\ T_6 &= -59x \end{aligned}$$

f) $-15{,}8\ ;\ 4{,}2\ ;\ 24{,}2\ ;\ldots$

Solution:

$$\begin{aligned} d &= T_2 - T_1 \text{ or } T_3 - T_2 \\ &= (4{,}2) - (-15{,}8) \text{ or } (24{,}2) - (4{,}2) \\ &= 20 \\ \text{Therefore } T_4 &= 44{,}2 \\ T_5 &= 64{,}2 \\ T_6 &= 84{,}2 \end{aligned}$$

g) $30b\ ; 34b\ ; 38b\ ;\ldots$

Solution:

$$\begin{aligned} d &= T_2 - T_1 \text{ or } T_3 - T_2 \\ &= (34b) - (30b) \text{ or } (38b) - (34b) \\ &= 4b \\ \text{Therefore } T_4 &= 42b \\ T_5 &= 46b \\ T_6 &= 50b \end{aligned}$$

6. Given a pattern which starts with the numbers: $3\ ;\ 8\ ;\ 13\ ;\ 18\ ;\ \ldots$ determine the values of T_6 and T_9.

Solution:

$$3\ ;\ 8\ ;\ 13\ ;\ 18\ ;\ 23\ ;\ \underline{28}\ ;\ 33\ ;\ 38\ ;\ \underline{43}\ ;\ \ldots$$

$T_6 = 28$ and $T_9 = 43$

7. Given a sequence which starts with the letters: $C\ ;\ D\ ;\ E\ ;\ F\ ;\ \ldots$ determine the values of T_5 and T_8.
Solution:

$$C\ ;\ D\ ;\ E\ ;\ F\ ;\ \underline{G}\ ;\ H\ ;\ I\ ;\ \underline{J}\ ;\ \ldots$$

$T_5 = G$ and $T_8 = J$

8. Given a pattern which starts with the numbers: $7\ ;\ 11\ ;\ 15\ ;\ 19\ ;\ \ldots$ determine the values of T_5 and T_8.
Solution:

$$7\ ;\ 11\ ;\ 15\ ;\ 19\ ;\ \underline{23}\ ;\ 27\ ;\ 31\ ;\ \underline{35}\ ;\ \ldots$$

$T_5 = 23$ and $T_8 = 35$

9. The general term is given for each sequence below. Calculate the missing terms (each missing term is represented by $\ldots$).

a) $0\ ;\ 3\ ;\ \ldots\ ;\ 15\ ;\ 24 \qquad T_n = n^2 - 1$
Solution:
The third term is:

$$\begin{aligned}
T_n &= n^2 - 1 \\
T_3 &= (3)^2 - 1 \\
&= 9 - 1 \\
&= 8
\end{aligned}$$

The fourth term is:

$$\begin{aligned}
T_n &= n^2 - 1 \\
T_4 &= (4)^2 - 1 \\
&= 16 - 1 \\
&= 15
\end{aligned}$$

Therefore the only missing term is the third one, which is 8. The full sequence is:
$0\ ;\ 8\ ;\ 15\ ;\ 24$

b) $3\ ;\ 2\ ;\ 1\ ;\ 0\ ;\ \ldots ; -2 \qquad T_n = -n + 4$
Solution:
The fifth term is:

$$\begin{aligned}
T_n &= -n + 4 \\
T_5 &= -(5) + 4 \\
&= -1
\end{aligned}$$

The sixth term is:

$$\begin{aligned}
T_n &= -n + 4 \\
T_6 &= -(6) + 4 \\
&= -2
\end{aligned}$$

Therefore the only missing term is the fifth one, which is -1. The full sequence is:
$3\ ;\ 2\ ;\ 1\ ;\ 0\ ;\ -1\ ;\ -2$

c) $-11\ ;\ \ldots\ ;\ -7\ ;\ \ldots\ ;\ -3 \qquad T_n = -13 + 2n$
Solution:
The second term is:

$$
\begin{aligned}
T_n &= -13 + 2n \\
T_2 &= -13 + 2(2) \\
&= -13 + 4 \\
&= -9
\end{aligned}
$$

The third term is:

$$
\begin{aligned}
T_n &= -13 + 2n \\
T_3 &= -13 + 2(3) \\
&= -13 + 6 \\
&= -7
\end{aligned}
$$

The fourth term is:

$$
\begin{aligned}
T_n &= -13 + 2n \\
T_4 &= -13 + 2(4) \\
&= -13 + 8 \\
&= -5
\end{aligned}
$$

The fifth term is:

$$
\begin{aligned}
T_n &= -13 + 2n \\
T_5 &= -13 + 2(5) \\
&= -13 + 10 \\
&= -3
\end{aligned}
$$

Therefore the two missing terms are the second and fourth ones, which are -9 and -5. The full sequence is: -11 ; -9 ; -7 ; -5 ; -3

d) 1 ; 10 ; 19 ; ... ; 37 $\quad T_n = 9n - 8$
Solution:

$$
\begin{aligned}
T_n &= 9n - 8 \\
T_4 &= 9(4) - 8 \\
&= 28
\end{aligned}
$$

e) 9 ; ... ; 21 ; ... ; 33 $\quad T_n = 6n + 3$
Solution:
To find the two missing terms, we use the equation for the general term:

$$
\begin{aligned}
T_n &= 6n + 3 \\
T_2 &= 6(2) + 3 \\
&= 15 \\
T_4 &= 6(4) + 3 \\
&= 27
\end{aligned}
$$

10. Find the general formula for the following sequences and then find T_{10}, T_{50} and T_{100}

a) $2; 5; 8; 11; 14; \ldots$
Solution:
We first need to find d:

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= 5 - 2 \\
&= 3
\end{aligned}
$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 2 \\ T_2 &= a + d = 2 + 3 \\ &= 2 + 1(3) \\ T_3 &= T_2 + d = 2 + 3 + 3 \\ &= 2 + 2(3) \\ T_4 &= T_3 + d = 2 + 3 + 3 + 3 \\ &= 2 + 3(3) \\ T_n &= T_{n-1} + d = 2 + 3(n-1) \\ &= 3n - 1 \end{aligned}$$

The general formula is $T_n = 3n - 1$.
T_{10}, T_{50} and T_{100} are:

$$\begin{aligned} T_{10} &= 3(10) - 1 \\ &= 29 \\ T_{50} &= 3(50) - 1 \\ &= 149 \\ T_{100} &= 3(100) - 1 \\ &= 299 \end{aligned}$$

b) $0; 4; 8; 12; 16; \ldots$

Solution:

We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 0 \\ T_2 &= a + d = 0 + 4 \\ &= 4(1) \\ T_3 &= T_2 + d = 0 + 4 + 4 \\ &= 4(2) \\ T_4 &= T_3 + d = 0 + 4 + 4 + 4 \\ &= 4(3) \\ T_n &= T_{n-1} + d = 0 + 4(n-1) \\ &= 4n - 4 \end{aligned}$$

The general formula is $T_n = 4n - 4$.
T_{10}, T_{50} and T_{100} are:

$$\begin{aligned} T_{10} &= 4(10) - 4 \\ &= 36 \\ T_{50} &= 4(50) - 4 \\ &= 196 \\ T_{100} &= 4(100) - 4 \\ &= 396 \end{aligned}$$

c) $2; -1; -4; -7; -10; \ldots$

Solution:
We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= -1 - 2 \\ &= -3 \end{aligned}$$

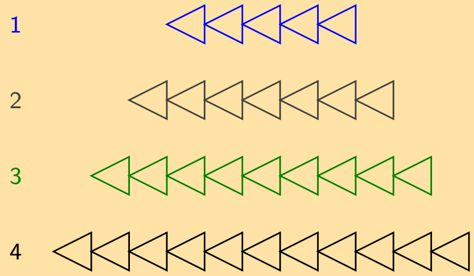
Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 2 \\ T_2 &= a + d = 2 + (-3) \\ &= 2 + (-3)(1) \\ T_3 &= T_2 + d = 2 + (-3) + (-3) \\ &= 2 + (-3)(2) \\ T_4 &= T_3 + d = 2 + (-3) + (-3) + (-3) \\ &= 2 + (-3)(3) \\ T_n &= T_{n-1} + d = 2 + (-3)(n-1) \\ &= 5 - 3n \end{aligned}$$

The general formula is $T_n = 5 - 3n$.
T_{10}, T_{50} and T_{100} are:

$$\begin{aligned} T_{10} &= 5 - 3(10) \\ &= -25 \\ T_{50} &= 5 - 3(50) \\ &= -145 \\ T_{100} &= 5 - 3(100) \\ &= -295 \end{aligned}$$

11. The diagram below shows pictures which follow a pattern.



a) How many triangles will there be in the 5^{th} picture?
Solution:
$5\ ;\ 7\ ;\ 9\ ;\ 11\ ;\ \ldots$
Therefore two triangles are added each time and the fifth picture will have 13 triangles.

b) Determine the formula for the n^{th} term.
Solution:
The general term of the pattern is:

$$\begin{aligned} T_n &= T_1 + d = 5 + (2)(n-1) \\ &= 2n + 3 \end{aligned}$$

c) Use the formula to find how many triangles are in the 25^{th} picture of the diagram.
Solution:

$$\begin{aligned} T_n &= 2n + 3 \\ T_{25} &= 2(25) + 3 \longleftarrow \text{substitute n } = 25 \\ &= 53 \end{aligned}$$

12. Study the following sequence: $15\ ;\ 23\ ;\ 31\ ;\ 39\ ;\ \ldots$

    a) Write down the next 3 terms.
    Solution:
    We note that we add 8 to each term to get the next term. Therefore the next three terms are $47\ ;\ 55\ ;\ 63$.

    b) Find the general formula for the sequence
    Solution:

$$\begin{aligned} T_n &= T_1 + d(n-1) \\ &= 15 + 8(n-1) \\ &= 8n + 7 \end{aligned}$$

    c) Find the value of n if T_n is 191.
    Solution:

$$\begin{aligned} 191 &= 8n + 7 \\ 184 &= 8n \\ n &= 23 \end{aligned}$$

13. Study the following sequence: $-44\ ;\ -14\ ;\ 16\ ;\ 46\ ;\ \ldots$

    a) Write down the next 3 terms.
    Solution:
    We note that we add 30 to each term to get the next term. Therefore the next three terms are $76\ ;\ 106\ ;\ 136$.

    b) Find the general formula for the sequence
    Solution:

$$\begin{aligned} T_n &= T_1 + d(n-1) \\ &= -44 + 30(n-1) \\ &= 30n - 74 \end{aligned}$$

    c) Find the value of n if T_n is 406.
    Solution:

$$\begin{aligned} 406 &= 30n - 74 \\ 480 &= 30n \\ n &= 16 \end{aligned}$$

14. Consider the following list:

$$-z-5\ ;\ -4z-5\ ;\ -6z-2\ ;\ -8z-5\ ;\ -10z-5\ ;\ \ldots$$

    a) Find the common difference for the terms of the list. If the sequence is not linear (if it does not have a common difference), write "no common difference".
    Solution:

$$\begin{aligned} d &= T_2 - T_1 = (-4z-5) - (-z-5) = -3z \\ &= T_3 - T_2 = (-6z-2) - (-4z-5) = -2z+3 \\ &= T_4 - T_3 = (-8z-5) - (-6z-2) = -2z-3 \end{aligned}$$

    No common difference.

b) If you are now told that $z = -2$, determine the values of T_1 and T_2.
Solution:

$$\begin{aligned} T_1 &= -z - 5 \\ &= -(-2) - 5 \\ &= -3 \\ T_2 &= -4z - 5 \\ &= -4(-2) - 5 \\ &= 3 \end{aligned}$$

15. Consider the following pattern:

$$2n + 4 \ ; \ 1 \ ; \ -2n - 2 \ ; \ -4n - 5 \ ; \ -6n - 8 \ ; \ \ldots$$

a) Find the common difference for the terms of the pattern. If the sequence is not linear (if it does not have a common difference), write "no common difference".
Solution:

$$\begin{aligned} d &= T_2 - T_1 = (1) - (2n + 4) = -2n - 3 \\ &= T_3 - T_2 = (-2n - 2) - (1) = -2n - 3 \\ &= T_4 - T_3 = (-4n - 5) - (-2n - 2) = -2n - 3 \end{aligned}$$

The common difference for these numbers: $d = -2n - 3$.

b) If you are now told that $n = -1$, determine the values of T_1 and T_3.
Solution:

$$\begin{aligned} T_1 &= 2n + 4 \\ &= 2(-1) + 4 \\ &= 2 \\ T_3 &= -2n - 2 \\ &= -2(-1) - 2 \\ &= 0 \end{aligned}$$

16. a) If the following terms make a linear sequence:

$$\frac{k}{3} - 1 \ ; \ -\frac{5k}{3} + 2 \ ; \ -\frac{2k}{3} + 10 \ ; \ \ldots$$

Determine the value of k. If the answer is a non-integer, write the answer as a simplified fraction.
Solution:

$$\begin{aligned} T_2 - T_1 &= T_3 - T_2 \\ \left(-\frac{5k}{3} + 2\right) - \left(\frac{k}{3} - 1\right) &= \left(-\frac{2k}{3} + 10\right) - \left(-\frac{5k}{3} + 2\right) \\ 3\left(-\frac{5k}{3} + 2\right) - 3\left(\frac{k}{3} - 1\right) &= 3\left(-\frac{2k}{3} + 10\right) - 3\left(-\frac{5k}{3} + 2\right) \\ -5k + 6 - (k - 3) &= -2k + 30 - (-5k + 6) \\ -6k + 9 &= 3k + 24 \\ -15 &= 9k \\ k &= -\frac{5}{3} \end{aligned}$$

b) Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.

Solution:

$$\begin{aligned}
\text{First term: } T_1 &= \frac{k}{3} - 1 \\
&= \frac{\left(-\frac{5}{3}\right)}{3} - 1 \\
&= -\frac{14}{9} \\
\text{Second term: } T_2 &= -\frac{5k}{3} + 2 \\
&= -\frac{5\left(-\frac{5}{3}\right)}{3} + 2 \\
&= \frac{43}{9} \\
\text{Third term: } T_3 &= -\frac{2k}{3} + 10 \\
&= -\frac{2\left(-\frac{5}{3}\right)}{3} + 10 \\
&= \frac{100}{9}
\end{aligned}$$

The first three terms of this sequence are: $-\frac{14}{9}$, $\frac{43}{9}$ and $\frac{100}{9}$.

17. a) If the following terms make a linear sequence:

$$y - \frac{3}{2}\ ;\ -y - \frac{7}{2}\ ;\ -7y - \frac{15}{2}\ ;\ \ldots$$

find y. If the answer is a non-integer, write the answer as a simplified fraction.

Solution:

$$\begin{aligned}
T_2 - T_1 &= T_3 - T_2 \\
\left(-y - \frac{7}{2}\right) - \left(y - \frac{3}{2}\right) &= \left(-7y - \frac{15}{2}\right) - \left(-y - \frac{7}{2}\right) \\
2\left(-y - \frac{7}{2}\right) - 2\left(y - \frac{3}{2}\right) &= 2\left(-7y - \frac{15}{2}\right) - 2\left(-y - \frac{7}{2}\right) \\
-2y - 7 - (2y - 3) &= -14y - 15 - (-2y - 7) \\
-4y - 4 &= -12y - 8 \\
8y &= -4 \\
y &= -\frac{1}{2}
\end{aligned}$$

b) Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.

Solution:

$$\begin{aligned}
\text{First term: } T_1 &= y - \frac{3}{2} \\
&= \left(-\frac{1}{2}\right) - \frac{3}{2} \\
&= -2 \\
\text{Second term: } T_2 &= -y - \frac{7}{2} \\
&= -\left(-\frac{1}{2}\right) - \frac{7}{2} \\
&= -3 \\
\text{Third term: } T_3 &= -7y - \frac{15}{2} \\
&= -7\left(-\frac{1}{2}\right) - \frac{15}{2} \\
&= -4
\end{aligned}$$

The first three terms of this sequence are: $-2, -3$ and -4.

18. What is the 649^{th} letter of the sequence:
PATTERNPATTERNPATTERNPATTERNPATTERNPATTERNPATTE.............?
Solution:
The word "PATTERN" is 7 letters long, so:

$$\frac{649}{7} = 92 \text{ r } 5$$

The remainder of 5 shows us that the 649^{th} letter is the 5^{th} letter in the word, which is E

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2F73	2. 2F74	3. 2F75	4. 2F76	5a. 2F77	5b. 2F78
5c. 2F79	5d. 2F7B	5e. 2F7C	5f. 2F7D	5g. 2F7F	6. 2F7G
7. 2F7H	8. 2F7J	9a. 2F7K	9b. 2F7M	9c. 2F7N	9d. 2F7P
9e. 2F7Q	10a. 2F7R	10b. 2F7S	10c. 2F7T	11. 2F7V	12. 2F7W
13. 2F7X	14. 2F7Y	15. 2F7Z	16. 2F82	17. 2F83	18. 2F84

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3.3 Chapter summary

End of chapter Exercise 3 – 2:

1. Analyse the diagram and complete the table.

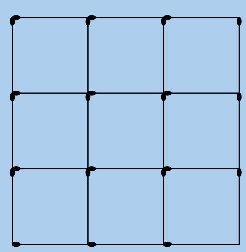


Figure number $(n \times n)$	1×1	2×2	3×3	4×4	$n \times n$
Number of horizontal matches					
Number of vertical matches					
Total number of matches					

Solution:

Figure number ($n \times n$)	1×1	2×2	3×3	4×4	$n \times n$
Number of horizontal matches	2	6	12	20	$n(n+1)$
Number of vertical matches	2	6	12	20	$n(n+1)$
Total number of matches	4	12	24	40	$2n(n+1)$

2. Given a list of numbers: $7\ ;\ 4\ ;\ 1\ ;\ -2\ ;\ -5\ ;\ \ldots$ determine the common difference for the list (if there is one).
Solution:

$$\begin{aligned} d &= T_2 - T_1 = (4) - (7) = -3 \\ &= T_3 - T_2 = (1) - (4) = -3 \\ &= T_4 - T_3 = (-2) - (1) = -3 \end{aligned}$$

All of the results are the same, which means we have found the **common** difference for these numbers: $d = -3$.

3. For the pattern here: $-0{,}55\ ;\ 0{,}99\ ;\ 2{,}49\ ;\ 3{,}91\ ;\ \ldots$ calculate the common difference.
If the pattern is not linear, write "no common difference". Otherwise, give your answer as a decimal.
Solution:

$$\begin{aligned} d &= T_2 - T_1 = (0{,}99) - (-0{,}55) = 1{,}54 \\ d &= T_3 - T_2 = (2{,}49) - (0{,}99) = 1{,}5 \end{aligned}$$

In this case the sequence is not linear. Therefore the final answer is that there is no common difference.

4. Consider the list shown here: $2\ ;\ 7\ ;\ 12\ ;\ 17\ ;\ 22\ ;\ 27\ ;\ 32\ ;\ 37\ ;\ \ldots$
If $T_5 = 22$ what is the value of T_{n-3}?
Solution:

$$\begin{aligned} T_5 &= 22 \\ \therefore T_{n-3} &= 7 \end{aligned}$$

5. Write down the next three terms in each of the following linear sequences:

a) $-10{,}2\ ;\ -29{,}2\ ;\ -48{,}2\ ;\ \ldots$
Solution:

$$\begin{aligned} d &= T_2 - T_1 \text{ or } T_3 - T_2 \\ &= (-29{,}2) - (-10{,}2) \text{ or } (-48{,}2) - (-29{,}2) \\ &= -19 \\ \text{Therefore } T_4 &= -67{,}2 \\ T_5 &= -86{,}2 \\ T_6 &= -105{,}2 \end{aligned}$$

b) $50r\ ;\ 46r\ ;\ 42r\ ;\ \ldots$
Solution:

$$\begin{aligned} d &= T_2 - T_1 \text{ or } T_3 - T_2 \\ &= (46r) - (50r) \text{ or } (42r) - (46r) \\ &= -4r \\ \text{Therefore } T_4 &= 38r \\ T_5 &= 34r \\ T_6 &= 30r \end{aligned}$$

6. Given a sequence which starts with the numbers: $6\ ;\ 11\ ;\ 16\ ;\ 21\ ;\ \ldots$ determine the values of T_6 and T_8.
Solution:

$$6;\ 11;\ 16;\ 21;\ 26;\ \underline{31};\ 36;\ \underline{41};\ \ldots$$

$T_6 = 31$ and $T_8 = 41$

7. Given a list which starts with the letters: $A\ ;\ B\ ;\ C\ ;\ D\ ;\ \ldots$ determine the values of T_6 and T_{10}.
   Solution:

$$A\ ;\ B\ ;\ C\ ;\ D\ ;\ E\ ;\ \underline{F}\ ;\ G\ ;\ H\ ;\ I\ ;\ \underline{J}\ ;\ \ldots$$

   $T_6 = F$ and $T_{10} = J$

8. Find the sixth term in each of the following sequences:
   a) $4\ ;\ 13\ ;\ 22\ ;\ 31\ ;\ \ldots$
      Solution:
      We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 13 - 4 \\ &= 9 \end{aligned}$$

      Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 4 \\ T_2 &= a + d = 4 + 9 \\ &= 4 + 9(1) \\ T_3 &= T_2 + d = 4 + 9 + 9 \\ &= 4 + 9(2) \\ T_n &= T_{n-1} + d = 4 + 9(n-1) \\ &= 9n - 5 \end{aligned}$$

      The general formula is $T_n = 9n - 5$.
      T_6 is:

$$\begin{aligned} T_6 &= 9(6) - 5 \\ &= 49 \end{aligned}$$

      $T_6 = 49$
   b) $5\ ;\ 2\ ;\ -1\ ;\ -4\ ;\ \ldots$
      Solution:
      We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 2 - 5 \\ &= -3 \end{aligned}$$

      Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 5 \\ T_2 &= a + d = 5 + (-3) \\ &= 5 + (-3)(1) \\ T_3 &= T_2 + d = 5 + (-3) + (-3) \\ &= 5 + (-3)(2) \\ T_n &= T_{n-1} + d = 5 + (-3)(n-1) \\ &= 7 - 3n \end{aligned}$$

      The general formula is $T_n = 7 - 3n$.
      T_6 is:

$$\begin{aligned} T_6 &= 7 - 3(6) \\ &= -11 \end{aligned}$$

      $T_6 = -11$

c) 7,4 ; 9,7 ; 12 ; 14,3 ; ...

Solution:

We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 9{,}7 - 7{,}4 \\ &= 2{,}3 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 7{,}4 \\ T_2 &= a + d = 7{,}4 + 2{,}3 \\ &= 7{,}4 + 2{,}3(1) \\ T_3 &= T_2 + d = 7{,}4 + 2{,}3 + 2{,}3 \\ &= 7{,}4 + 2{,}3(2) \\ T_n &= T_{n-1} + d = 7{,}4 + 2{,}3(n-1) \\ &= 7{,}4 + 2{,}3n - 2{,}3 = 2{,}3n + 5{,}1 \end{aligned}$$

The general formula is $T_n = 2{,}3n + 5{,}1$.

T_6 is:

$$\begin{aligned} T_6 &= 2{,}3(6) + 5{,}1 \\ &= 18{,}9 \end{aligned}$$

$T_6 = 18{,}9$

9. Find the general formula for the following sequences and then find T_{10}, T_{15} and T_{30}

a) -18 ; -22 ; -26 ; -30 ; -34 ; ...

Solution:

$$\begin{aligned} d &= T_2 - T_1 \\ &= (-22) - (-18) \\ &= -4 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = -18 \\ T_2 &= a + d = -18 + (-4) \\ &= -18 + (-4)(1) \\ T_3 &= T_2 + d = -18 + (-4) + (-4) \\ &= -18 + (-4)(2) \\ T_n &= T_{n-1} + d = -18 + (-4)(n-1) \\ &= -4n - 14 \end{aligned}$$

The general formula is $T_n = -4n - 14$.

$$\begin{aligned} T_{10} &= -4(10) - 14 \\ &= -54 \\ T_{15} &= -4(15) - 14 \\ &= -74 \\ T_{30} &= -4(30) - 14 \\ &= -134 \end{aligned}$$

b) $1; -6; -13; -20; -27; \ldots$

Solution:

$$\begin{aligned} d &= T_2 - T_1 \\ &= (-6) - (1) \\ &= -7 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 1 \\ T_2 &= a + d = 1 + (-7) \\ &= 1 + (-7)(1) \\ T_3 &= T_2 + d = 1 + (-7) + (-7) \\ &= 1 + (-7)(2) \\ T_n &= T_{n-1} + d = 1 + (-7)(n-1) \\ &= -7n + 8 \end{aligned}$$

The general formula is $T_n = -7n + 8$.

$$\begin{aligned} T_{10} &= -7(10) + 8 \\ &= -62 \\ T_{15} &= -7(15) + 8 \\ &= -97 \\ T_{30} &= -7(30) + 8 \\ &= -202 \end{aligned}$$

10. The general term is given for each sequence below. Calculate the missing terms (each missing term is represented by $\ldots$).

a) $10 \; ; \; \ldots \; ; \; 14 \; ; \; \ldots \; ; \; 18 \qquad T_n = 2n + 8$

Solution:

$$\begin{aligned} T_n &= 2n + 8 \\ T_2 &= 2(2) + 8 \\ &= 12 \\ T_4 &= 2(4) + 8 \\ &= 16 \end{aligned}$$

The missing terms are 12 and 16

b) $2 \; ; \; -2 \; ; \; -6 \; ; \; \ldots \; ; \; -14 \qquad T_n = -4n + 6$

Solution:

$$\begin{aligned} T_n &= -4n + 6 \\ T_4 &= -4(4) + 6 \\ &= -10 \end{aligned}$$

c) $8 \; ; \; \ldots \; ; \; 38 \; ; \; \ldots \; ; \; 68 \qquad T_n = 15n - 7$

Solution:

$$\begin{aligned} T_n &= 15n - 7 \\ T_2 &= 15(2) - 7 \\ &= 23 \\ T_4 &= 15(4) - 7 \\ &= 53 \end{aligned}$$

11. Find the general term in each of the following sequences:

a) $3\ ;\ 7\ ;\ 11\ ;\ 15\ ;\ \ldots$

Solution:

We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 7 - 3 \\ &= 4 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 3 \\ T_2 &= a + d = 3 + 4 \\ &= 3 + 4(1) \\ T_3 &= T_2 + d = 3 + 4 + 4 \\ &= 3 + 4(2) \\ T_n &= T_{n-1} + d = 3 + 4(n-1) \\ &= 4n - 1 \end{aligned}$$

The general formula is $T_n = 4n - 1$.

b) $-2\ ;\ 1\ ;\ 4\ ;\ 7\ ;\ \ldots$

Solution:

We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 1 - (-2) \\ &= 3 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = -2 \\ T_2 &= a + d = -2 + 3 \\ &= -2 + 3(1) \\ T_3 &= T_2 + d = -2 + 3 + 3 \\ &= -2 + 3(2) \\ T_n &= T_{n-1} + d = -2 + 3(n-1) \\ &= 3n - 5 \end{aligned}$$

The general formula is $T_n = 3n - 5$.

c) $11\ ;\ 15\ ;\ 19\ ;\ 23\ ;\ \ldots$

Solution:

We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 15 - 11 \\ &= 4 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 11 \\ T_2 &= a + d = 11 + 4 \\ &= 11 + 4(1) \\ T_3 &= T_2 + d = 11 + 4 + 4 \\ &= 11 + 4(2) \\ T_n &= T_{n-1} + d = 11 + 4(n-1) \\ &= 4n + 7 \end{aligned}$$

The general formula is $T_n = 4n + 7$.

d) $\frac{1}{3}$; $\frac{2}{3}$; 1 ; $1\frac{1}{3}$; $\ldots$

Solution:

We first need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = \frac{1}{3} \\ T_2 &= a + d = \frac{1}{3} + \frac{1}{3} \\ &= \frac{1}{3} + \frac{1}{3}(1) \\ T_3 &= T_2 + d = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \\ &= \frac{1}{3} + \frac{1}{3}(2) \\ T_n &= T_{n-1} + d = \frac{1}{3} + \frac{1}{3}(n-1) \\ &= \frac{1}{3} + \frac{1}{3}n - \frac{1}{3} \\ &= \frac{1}{3}n \end{aligned}$$

The general formula is $T_n = \frac{1}{3}n$.

12. Study the following sequence -7 ; -21 ; -35 ; $\ldots$

a) Write down the next 3 terms:

Solution:

-49 ; -63 ; 77

b) Find the general formula for the sequence.

Solution:

$$\begin{aligned} T_n &= -7 - 14(n-1) \\ T_n &= -7 - 14n + 14 \\ T_n &= -14n + 7 \end{aligned}$$

c) Find the value of n if T_n is -917.

Solution:

$$\begin{aligned} -917 &= 7 - 14n \\ -924 &= -14n \\ n &= 66 \end{aligned}$$

13. What is the 346^{th} letter of the sequence:
COMMONCOMMON............?

Solution:

The word "COMMON" is 6 letters long, so:

$$\frac{346}{6} = 57 \text{ r } 4$$

The remainder of 4 shows us that the 346^{th} letter is the $4th^{\text{th}}$ letter in the word, which is M

14. What is the 1000^{th} letter of the sequence:
MATHEMATICSMATHEMATICSMATHEMATICSMATHE ....?
Solution:
The word "MATHEMATICS" is 11 letters long, so:

$$\frac{1000}{11} = 90 \text{ r } 10$$

The remainder of 10 shows us that the 1000^{th} letter is the tenth letter in the word, which is C

15. The seating of a sports stadium is arranged so that the first row has 15 seats, the second row has 19 seats, the third row has 23 seats and so on. Calculate how many seats are in the 25^{th} row.
Solution:
We start by writing the given information as a sequence:

$$15; 19; 23; \ldots$$

Now we can calculate d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 19 - 15 \\ &= 4 \end{aligned}$$

Next we note that for each successive term we add d to the last term. We can express this as:

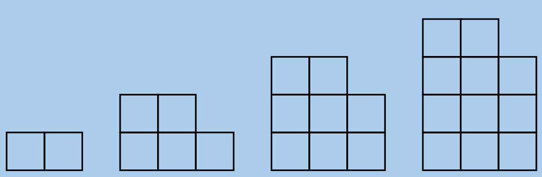
$$\begin{aligned} T_1 &= a = 15 \\ T_2 &= a + d = 15 + 4 \\ &= 15 + 4(1) \\ T_3 &= T_2 + d = 15 + 4 + 4 \\ &= 15 + 4(2) \\ T_n &= T_{n-1} + d = 15 + 4(n-1) \\ &= 4n + 11 \end{aligned}$$

The general formula is $T_n = 4n + 11$.
The 25^{th} row is represented by T_{25}. The number of seats in this row is:

$$\begin{aligned} T_{25} &= 4(25) + 11 \\ &= 111 \end{aligned}$$

There are 111 seats in the 25^{th} row.

16. The diagram below shows pictures which follow a pattern.



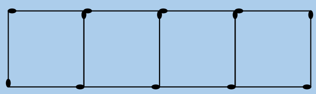
a) How many boxes will there be in the sixth picture?
Solution:
$2\ ;\ 5\ ;\ 8\ ;\ 11\ ;\ \ldots$
Therefore three boxes are added each time and the sixth picture will have 17 boxes

b) Determine the formula for the n^{th} term.
Solution:
The general term of the pattern is: $T_n = 3n - 1$.

c) Use the formula to find how many boxes are in the 30^{th} picture of the diagram.
Solution:

$$
\begin{aligned}
T_n &= 3n - 1 \\
T_{30} &= 3(30) - 1 \longleftarrow \text{ substitute n } = 30 \\
&= 89
\end{aligned}
$$

17. A single square is made from 4 matchsticks. Two squares in a row need 7 matchsticks and three squares need 10 matchsticks.



Answer the following questions for this sequence.

a) Determine the first term.
Solution:
We begin by writing a sequence to represent this:

$$4\ ;\ 7\ ;\ 10\ ;\ \ldots$$

We see from this that the first term is 4.
$T_1 = 4$

b) Determine the common difference.
Solution:
The common difference (d) is:

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= 7 - 4 \\
&= 3
\end{aligned}
$$

c) Determine the general formula.
Solution:
To determine the general formula we note that for each successive term we add d to the last term. We can express this as:

$$
\begin{aligned}
T_1 &= a = 4 \\
T_2 &= a + d = 4 + 3 \\
&= 4 + 3(1) \\
T_3 &= T_2 + d = 4 + 3 + 3 \\
&= 4 + 3(2) \\
T_n &= T_{n-1} + d = 4 + 3(n - 1) \\
&= 3n + 1
\end{aligned}
$$

The general formula is $T_n = 3n + 1$.

d) A row has twenty-five squares. How many matchsticks are there in this row?
Solution:
We note that a row with twenty-five squares is represented by T_{25}. The number of matchsticks in this row is:

$$
\begin{aligned}
T_{25} &= 3(25) + 1 \\
&= 76
\end{aligned}
$$

There are 76 matchsticks in the row with twenty-five squares.

18. You would like to start saving some money, but because you have never tried to save money before, you decide to start slowly. At the end of the first week you deposit R 5 into your bank account. Then at the end of the second week you deposit R 10 and at the end of the third week, R 15. After how many weeks will you deposit R 50 into your bank account?
Solution:
We begin by writing down a sequence to represent this:

$$5 \; ; \; 10 \; ; \; 15 \; ; \; \ldots$$

Next we need to find d:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 10 - 5 \\ &= 5 \end{aligned}$$

Now we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 5 \\ T_2 &= a + d = 5 + 5 \\ &= 5 + 5(1) \\ T_3 &= T_2 + d = 5 + 5 + 5 \\ &= 5 + 5(2) \\ T_n &= T_{n-1} + d = 5 + 5(n-1) \\ &= 5n \end{aligned}$$

The general formula is $T_n = 5n$.
Now we need to find n such that $T_n = 50$:

$$\begin{aligned} T_n &= 5n \\ 50 &= 5n \\ \therefore n &= 10 \end{aligned}$$

After the 10$^{\text{th}}$ week you will deposit R 50 into your bank account.

19. Consider the following list:

$$-4y - 3 \; ; \; -y \; ; \; 2y + 3 \; ; \; 5y + 6 \; ; \; 8y + 9 \; ; \; \ldots$$

a) Find the common difference for the terms of the list. If the sequence is not linear (if it does not have a common difference), write "no common difference".
Solution:

$$\begin{aligned} d &= T_2 - T_1 = (-y) - (-4y - 3) = 3y + 3 \\ d &= T_3 - T_2 = (2y + 3) - (-y) = 3y + 3 \\ d &= T_4 - T_3 = (5y + 6) - (2y + 3) = 3y + 3 \end{aligned}$$

The common difference for these numbers: $d = 3y + 3$.

b) If you are now told that $y = 1$, determine the values of T_1 and T_2.
Solution:

$$\begin{aligned} T_1 &= -4y - 3 \\ &= -4(1) - 3 \\ &= -7 \\ T_2 &= -y \\ &= -(1) \\ &= -1 \end{aligned}$$

20. a) If the following terms make a linear sequence:

$$2n + \frac{1}{2} \; ; \; 3n + \frac{5}{2} \; ; \; 7n + \frac{11}{2} \; ; \; \ldots$$

Determine the value of n. If the answer is a non-integer, write the answer as a simplified fraction.
Solution:

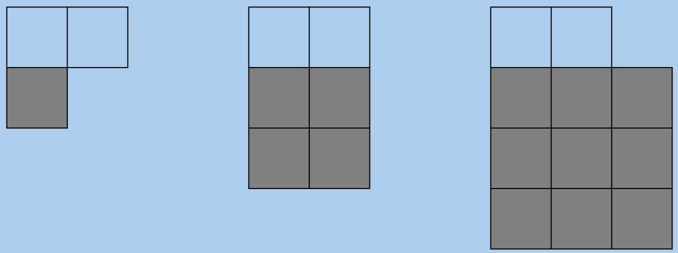
$$\begin{aligned}
T_2 - T_1 &= T_3 - T_2 \\
\left(3n + \frac{5}{2}\right) - \left(2n + \frac{1}{2}\right) &= \left(7n + \frac{11}{2}\right) - \left(3n + \frac{5}{2}\right) \\
2\left(3n + \frac{5}{2}\right) - 2\left(2n + \frac{1}{2}\right) &= 2\left(7n + \frac{11}{2}\right) - 2\left(3n + \frac{5}{2}\right) \\
6n + 5 - (4n + 1) &= 14n + 11 - (6n + 5) \\
2n + 4 &= 8n + 6 \\
-2 &= 6n \\
n &= -\frac{1}{3}
\end{aligned}$$

b) Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.
Solution:

$$\begin{aligned}
\text{First term: } T_1 &= 2n + \frac{1}{2} \\
&= 2\left(-\frac{1}{3}\right) + \frac{1}{2} \\
&= -\frac{1}{6} \\
\text{Second term: } T_2 &= 3n + \frac{5}{2} \\
&= 3\left(-\frac{1}{3}\right) + \frac{5}{2} \\
&= \frac{3}{2} \\
\text{Third term: } T_3 &= 7n + \frac{11}{2} \\
&= 7\left(-\frac{1}{3}\right) + \frac{11}{2} \\
&= \frac{19}{6}
\end{aligned}$$

The first three terms of this sequence are: $-\frac{1}{6}$, $\frac{3}{2}$ and $\frac{19}{6}$.

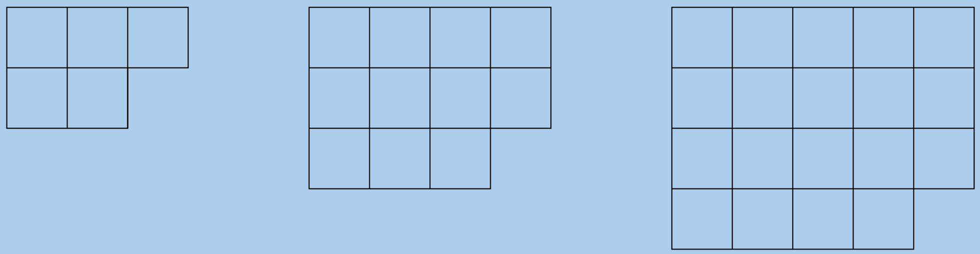
21. How many blocks will there be in the 85^{th} picture?
Hint: Use the grey blocks to help.



Solution:
The grey blocks can be represented by n^2 and there are always 2 white blocks.

$$\begin{aligned}
T_n &= n^2 + 2 \\
T_{85} &= 85^2 + 2 \\
T_{85} &= 7227 \text{ blocks}
\end{aligned}$$

22. Analyse the picture below:



a) How many blocks are there in the next picture?
Solution:
Picture 1: $2^2 + 1$
Picture 2: $3^2 + 2$
Picture 3: $4^2 + 3$
Picture 4: $5^2 + 4 = 29$ blocks

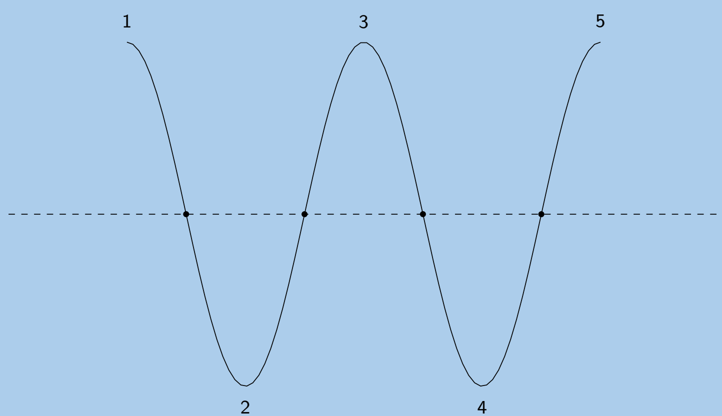
b) Write down the general formula for this pattern.
Solution:
Look at:
Picture 1: $2^2 + 1 \qquad (n = 1)$

$$T_n = (n+1)^2 + n$$

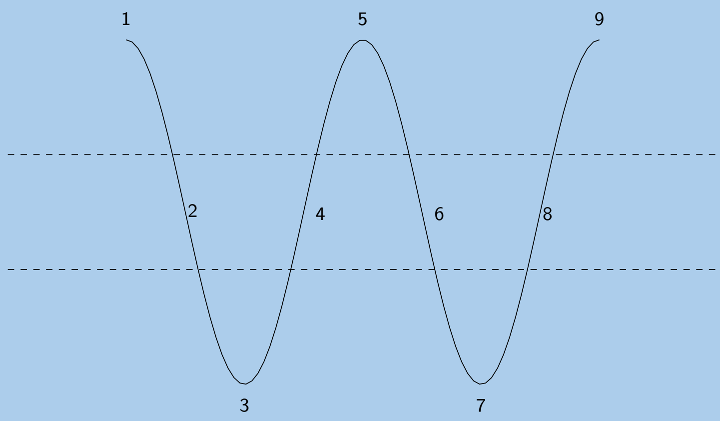
c) How many blocks will there be in the 14th picture?
Solution:

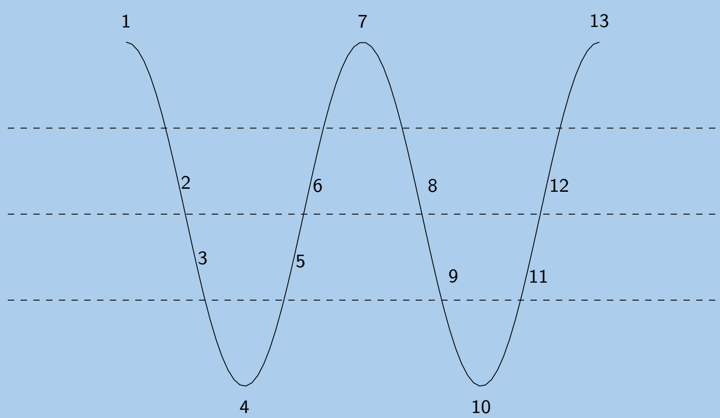
$$\begin{aligned} T_n &= (n+1)^2 + n \\ T_{14} &= (14+1)^2 + 14 \\ T_{14} &= 239 \text{ blocks} \end{aligned}$$

23. A horizontal line intersects a piece of string at 4 points and divides it into five parts, as shown below.



If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at 4 points, determine the number of parts into which the string will be divided.
Solution:
We need to determine a pattern for this scenario.
The first line divides the string into five parts. We can redraw the diagram to show the string with 2 and 3 lines:





From this we see that the two lines cut the string into 9 pieces. Three lines cut the string into 13 pieces. So for each line added we cut the line into 4 more pieces.
So we can write the following sequence:

$$5\ ;\ 9\ ;\ 13\ ;\ \ldots$$

The common difference is 4.
Next we note that for each successive term we add d to the last term. We can express this as:

$$\begin{aligned} T_1 &= a = 5 \\ T_2 &= a + d = 5 + 4 \\ &= 5 + 4(1) \\ T_3 &= T_2 + d = 5 + 4 + 4 \\ &= 5 + 4(2) \\ T_n &= T_{n-1} + d = 5 + 4(n-1) \\ &= 4n + 1 \end{aligned}$$

The general formula is $T_n = 4n + 1$.
When there are 19 lines we are working with T_{19}:

$$\begin{aligned} T_{19} &= 4(19) + 1 \\ &= 77 \end{aligned}$$

Therefore the string will be cut into 77 parts.

24. Use a calculator to explore and then generalise your findings to determine the:
    a) units digit of 3^{2007}
    Solution:

$$\begin{array}{|l|l|l|}
3^1 = 3 & 3^5 = 243 & 3^9 = 19683 \\
3^2 = 9 & 3^6 = 729 & 3^{10} = 59049 \\
3^3 = 27 & 3^7 = 2187 & 3^{11} = 177147 \\
3^4 = 81 & 3^8 = 6561 & 3^{12} = 531441
\end{array}$$

$\frac{2007}{4} = 501$ r 3
Therefore 3^{2007} will follow the same pattern as the third row
therefore the units digit is 7

b) tens digit of 7^{2008}

Solution:

$$\begin{array}{|l|l|l|}
7^1 = 07 & 7^5 = 16807 & 7^9 = 40353607 \\
7^2 = 49 & 7^6 = 117649 & 7^{10} = 282475249 \\
7^3 = 343 & 7^7 = 823543 & 7^{11} = 1977326743 \\
7^4 = 2401 & 7^8 = 576801 &
\end{array}$$

$\frac{2008}{4} = 502$ r 0
Therefore 7^{2008} will follow the same pattern as the fourth row
therefore the tens digit is 0

c) remainder when 7^{250} is divided by 5

Solution:

$$\begin{array}{|l|l|}
\frac{7^1}{5} : \text{Remainder} = 2 & \frac{7^5}{5} : \text{Remainder} = 2 \\
\frac{7^2}{5} : \text{Remainder} = 4 & \frac{7^6}{5} : \text{Remainder} = 4 \\
\frac{7^3}{5} : \text{Remainder} = 3 & \frac{7^7}{5} : \text{Remainder} = 3 \\
\frac{7^4}{5} : \text{Remainder} = 1 & \frac{7^8}{5} : \text{Remainder} = 1
\end{array}$$

$\frac{250}{4} = 62$ r 0
Therefore 2^{250} will follow the same pattern as the second row
therefore the remainder is 4

25. Analyse the diagram and complete the table.
The dots follow a triangular pattern and the formula is $T_n = \frac{n(n+1)}{2}$.
The general formula for the lines is $T_n = \frac{3n(n-1)}{2}$.

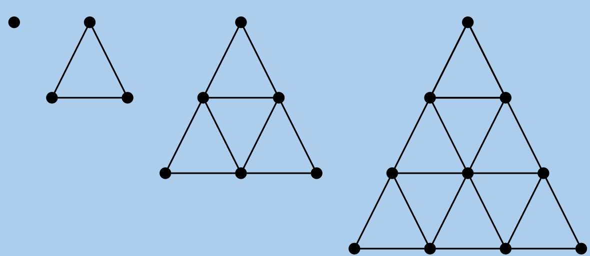


Figure number	1	2	3	4	5	20	n
Number of dots							
Number of lines							
Total							

Solution:
We are given the general formula for both the lines and the dots. We can determine the general formula for the sum of the lines and dots by adding the general formula for the lines to the general formula for the dots.

$$
\begin{aligned}
T_n &= \frac{n(n+1)}{2} + \frac{3n(n-1)}{2} \\
&= \frac{n^2 + n + 3n^2 - 3n}{2} \\
&= \frac{4n^2 - 2n}{2} \\
&= 2n^2 - n
\end{aligned}
$$

Figure number	1	2	3	4	5	20	n
Number of dots	1	3	6	10	15	210	$\frac{n(n+1)}{2}$
Number of lines	0	3	9	18	30	570	$\frac{3n(n-1)}{2}$
Total	1	6	15	28	45	780	$2n^2 - n$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2F86	2. 2F88	3. 2F89	4. 2F8B	5. 2F8C	6. 2F8D
7. 2F8F	8a. 2F8G	8b. 2F8H	8c. 2F8J	9. 2F8K	10. 2F8M
11a. 2F8N	11b. 2F8P	11c. 2F8Q	11d. 2F8R	12. 2F8S	13. 2F8T
14. 2F8V	15. 2F8W	16. 2F8X	17. 2F8Y	18. 2F8Z	19. 2F93
20. 2F94	21. 2F95	22. 2F96	23. 2F92	24a. 2F97	24b. 2F98
24c. 2F99	25. 2F87				

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CHAPTER 4

Equations and inequalities

4.1 Introduction

- This chapter covers linear, quadratic and simultaneous linear equations as well as word problems, literal equations and linear inequalities.
- Linear equations were covered in earlier grades and are revised here.
- Word problems can include any of linear, quadratic and simultaneous equations.
- For linear inequalities learners must know interval notation and be able to represent the solution graphically.

4.2 Solving linear equations

Method for solving linear equations

Exercise 4 – 1:

Solve the following equations (assume all denominators are non-zero):

1. $2y - 3 = 7$
   Solution:
   $$\begin{aligned} 2y - 3 &= 7 \\ 2y &= 10 \\ y &= 5 \end{aligned}$$
2. $2c = c - 8$
   Solution:
   $$\begin{aligned} 2c &= c - 8 \\ c &= -8 \end{aligned}$$
3. $3 = 1 - 2c$
   Solution:
   $$\begin{aligned} 3 &= 1 - 2c \\ 2c &= 1 - (3) \\ 2c &= -2 \\ c &= \frac{-2}{2} \\ &= -1 \end{aligned}$$
4. $4b + 5 = -7$
   Solution:
   $$\begin{aligned} 4b + 5 &= -7 \\ 4b &= -7 - (5) \\ 4b &= -12 \\ b &= \frac{-12}{4} \\ &= -3 \end{aligned}$$

5. $-3y = 0$
Solution:

$$\begin{aligned} -3y &= 0 \\ y &= 0 \end{aligned}$$

6. $16y + 4 = -10$
Solution:

$$\begin{aligned} 16y + 4 &= -10 \\ 16y &= -14 \\ y &= -\frac{14}{16} \\ &= -\frac{7}{8} \end{aligned}$$

7. $12y + 0 = 144$
Solution:

$$\begin{aligned} 12y + 0 &= 144 \\ 12y &= 144 \\ y &= 12 \end{aligned}$$

8. $7 + 5y = 62$
Solution:

$$\begin{aligned} 7 + 5y &= 62 \\ 5y &= 55 \\ y &= 11 \end{aligned}$$

9. $55 = 5x + \frac{3}{4}$
Solution:

$$\begin{aligned} 55 &= 5x + \frac{3}{4} \\ 220 &= 20x + 3 \\ 20x &= 217 \\ x &= \frac{217}{20} \end{aligned}$$

10. $5x = 2x + 45$
Solution:

$$\begin{aligned} 5x &= 2x + 45 \\ 3x &= 45 \\ x &= 15 \end{aligned}$$

11. $23x - 12 = 6 + 3x$
Solution:

$$\begin{aligned} 23x - 12 &= 6 + 3x \\ 20x &= 18 \\ x &= \frac{18}{20} \\ &= \frac{9}{10} \end{aligned}$$

12. $12 - 6x + 34x = 2x - 24 - 64$
Solution:

$$\begin{aligned} 12 - 6x + 34x &= 2x - 24 - 64 \\ 12 + 28x &= 2x - 88 \\ 26x &= -100 \\ x &= -\frac{100}{26} \\ &= -\frac{50}{13} \end{aligned}$$

13. $6x + 3x = 4 - 5(2x - 3)$
Solution:

$$\begin{aligned} 6x + 3x &= 4 - 5(2x - 3) \\ 9x &= 4 - 10x + 15 \\ 19x &= 19 \\ x &= 1 \end{aligned}$$

14. $18 - 2p = p + 9$
Solution:

$$\begin{aligned} 18 - 2p &= p + 9 \\ 9 &= 3p \\ p &= 3 \end{aligned}$$

15. $\frac{4}{p} = \frac{16}{24}$
Solution:

$$\begin{aligned} \frac{4}{p} &= \frac{16}{24} \\ (4)(24) &= (16)(p) \\ 16p &= 96 \\ p &= 6 \end{aligned}$$

16. $-(-16 - p) = 13p - 1$
Solution:

$$\begin{aligned} -(-16 - p) &= 13p - 1 \\ 16 + p &= 13p - 1 \\ 17 &= 12p \\ p &= \frac{17}{12} \end{aligned}$$

17. $3f - 10 = 10$
Solution:

$$\begin{aligned} 3f - 10 &= 10 \\ 3f &= 20 \\ f &= \frac{20}{3} \end{aligned}$$

18. $3f + 16 = 4f - 10$
Solution:

$$\begin{aligned} 3f + 16 &= 4f - 10 \\ f &= 26 \end{aligned}$$

19. $10f + 5 = -2f - 3f + 80$
Solution:

$$\begin{aligned} 10f + 5 &= -2f - 3f + 80 \\ 10f + 5 &= -5f + 80 \\ 15f &= 75 \\ f &= 5 \end{aligned}$$

20. $8(f - 4) = 5(f - 4)$
Solution:

$$\begin{aligned} 8(f - 4) &= 5(f - 4) \\ 8f - 32 &= 5f - 20 \\ 3f &= 12 \\ f &= 4 \end{aligned}$$

21. $6 = 6(f + 7) + 5f$
Solution:

$$\begin{aligned} 6 &= 6(f + 7) + 5f \\ 6 &= 6f + 42 + 5f \\ -36 &= 11f \\ f &= -\frac{36}{11} \end{aligned}$$

22. $-7x = 8(1 - x)$
Solution:

$$\begin{aligned} -7x &= 8(1 - x) \\ -7x &= 8 - 8x \\ x &= 8 \end{aligned}$$

23. $5 - \frac{7}{b} = \frac{2(b + 4)}{b}$
Solution:

$$\begin{aligned} 5 - \frac{7}{b} &= \frac{2(b + 4)}{b} \\ \frac{5b - 7}{b} &= \frac{2b + 8}{b} \\ 5b - 7 &= 2b + 8 \\ 3b &= 15 \\ b &= 5 \end{aligned}$$

24. $\frac{x + 2}{4} - \frac{x - 6}{3} = \frac{1}{2}$
Solution:

$$\begin{aligned} \frac{x + 2}{4} - \frac{x - 6}{3} &= \frac{1}{2} \\ \frac{3(x + 2) - 4(x - 6)}{12} &= \frac{1}{2} \\ \frac{3x + 6 - 4x + 24}{12} &= \frac{1}{2} \\ (-x + 30)(2) &= 12 \\ -2x + 60 &= 12 \\ -2x &= -48 \\ x &= 24 \end{aligned}$$

25. $1 = \frac{3a-4}{2a+6}$

Solution:

Note that $a \neq --3$

$$\begin{aligned} 1 &= \frac{3a-4}{2a+6} \\ 2a+6 &= 3a-4 \\ a &= 10 \end{aligned}$$

26. $\frac{2-5a}{3} - 6 = \frac{4a}{3} + 2 - a$

Solution:

$$\begin{aligned} \frac{2-5a}{3} - 6 &= \frac{4a}{3} + 2 - a \\ \frac{2-5a}{3} - \frac{4a}{3} + a &= 8 \\ \frac{2-5a-4a+3a}{3} &= 8 \\ 2-6a &= 24 \\ 6a &= -22 \\ a &= -\frac{22}{6} \end{aligned}$$

27. $2 - \frac{4}{b+5} = \frac{3b}{b+5}$

Solution:

Note $b \neq -5$

$$\begin{aligned} 2 - \frac{4}{b+5} &= \frac{3b}{b+5} \\ 2 &= \frac{3b+4}{b+5} \\ 2b+10 &= 3b+4 \\ b &= 6 \end{aligned}$$

28. $3 - \frac{y-2}{4} = 4$

Solution:

$$\begin{aligned} 3 - \frac{y-2}{4} &= 4 \\ -\frac{y-2}{4} &= 1 \\ -y+2 &= 4 \\ y &= -2 \end{aligned}$$

29. $1{,}5x + 3{,}125 = 1{,}25x$

Solution:

$$\begin{aligned} 1{,}5x + 3{,}125 &= 1{,}25x \\ 1{,}5x - 1{,}25x &= -3{,}125 \\ 0{,}25x &= -3{,}125 \\ x &= -12{,}5 \end{aligned}$$

30. $1{,}3(2{,}7x+1) = 4{,}1 - x$

Solution:

$$
\begin{aligned}
1{,}3(2{,}7x+1) &= 4{,}1-x \\
3{,}51x+1{,}3 &= 4{,}1-x \\
4{,}51x &= 2{,}8 \\
x &= \frac{2{,}8}{4{,}51} \\
&= \frac{280}{451}
\end{aligned}
$$

31. $6{,}5x - 4{,}15 = 7 + 4{,}25x$
Solution:

$$
\begin{aligned}
6{,}5x-4{,}15 &= 7+4{,}25x \\
2{,}25x &= 11{,}15 \\
x &= \frac{11{,}15}{2{,}25} \\
&= \frac{1115}{225} \\
&= \frac{223}{45}
\end{aligned}
$$

32. $\frac{1}{3}P + \frac{1}{2}P - 10 = 0$
Solution:

$$
\begin{aligned}
\frac{1}{3}P+\frac{1}{2}P-10 &= 0 \\
\frac{2+3}{6}P &= 10 \\
5P &= 60 \\
P &= 12
\end{aligned}
$$

33. $1\frac{1}{4}(x-1) - 1\frac{1}{2}(3x+2) = 0$
Solution:

$$
\begin{aligned}
1\frac{1}{4}(x-1)-1\frac{1}{2}(3x+2) &= 0 \\
\frac{5}{4}x-\frac{5}{4}-\frac{3}{2}(3x)-\frac{3}{2}(2) &= 0 \\
\frac{5}{4}x-\frac{5}{4}-\frac{9}{2}x-\frac{6}{2} &= 0 \\
\frac{5-18}{4}x+\frac{-5-12}{4} &= 0 \\
\frac{-13}{4}x &= \frac{17}{4} \\
-13x &= 17 \\
x &= -\frac{17}{13}
\end{aligned}
$$

34. $\frac{1}{5}(x-1) = \frac{1}{3}(x-2) + 3$
Solution:

$$\begin{aligned}
\frac{1}{5}(x-1) &= \frac{1}{3}(x-2)+3 \\
\frac{1}{5}x - \frac{1}{5} &= \frac{1}{3}x - \frac{2}{3} + 3 \\
-\frac{1}{5} + \frac{2}{3} - 3 &= \frac{2}{15}x \\
-\frac{38}{15} &= \frac{2}{15}x \\
x &= -\frac{38}{2} \\
x &= -19
\end{aligned}$$

35. $\frac{5}{2a} + \frac{1}{6a} - \frac{3}{a} = 2$
Solution:

$$\begin{aligned}
\frac{5}{2a} + \frac{1}{6a} - \frac{3}{a} &= 2 \\
\frac{5(3)+1-3(6)}{6a} &= 2 \\
\frac{15+1-18}{6a} &= 2 \\
\frac{-2}{6a} &= 2 \\
-2 &= 12a \\
a &= -\frac{1}{6}
\end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2F9C	2. 2F9D	3. 2F9F	4. 2F9G	5. 2F9H	6. 2F9J	7. 2F9K	8. 2F9M	9. 2F9N
10. 2F9P	11. 2F9Q	12. 2F9R	13. 2F9S	14. 2F9T	15. 2F9V	16. 2F9W	17. 2F9X	18. 2F9Y
19. 2F9Z	20. 2FB2	21. 2FB3	22. 2FB4	23. 2FB5	24. 2FB6	25. 2FB7	26. 2FB8	27. 2FB9
28. 2FBB	29. 2FBC	30. 2FBD	31. 2FBF	32. 2FBG	33. 2FBH	34. 2FBJ	35. 2FBK	

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4.3 Solving quadratic equations

Method for solving quadratic equations

Exercise 4 – 2:

1. Write the following in standard form

a) $(r+4)(5r-4) = -16$
Solution:

$$\begin{aligned}
(r+4)(5r-4) &= -16 \\
5r^2 - 4r + 20r - 16 + 16 &= 0 \\
5r^2 - 4r + 20r - 16 + 16 &= 0 \\
5r^2 + 16r &= 0
\end{aligned}$$

b) $(3r-8)(2r-3) = -15$

Solution:

$$\begin{aligned}(3r-8)(2r-3)&=-15\\ 6r^2-9r-16r+24+15&=0\\ 6r^2-9r-16r+24+15&=0\\ 6r^2-25r+39&=0\end{aligned}$$

c) $(d+5)(2d+5)=8$
Solution:

$$\begin{aligned}(d+5)(2d+5)&=8\\ 2d^2+5d+10d+25-8&=0\\ 2d^2+5d+10d+25-8&=0\\ 2d^2+15d+17&=0\end{aligned}$$

2. Solve the following equations:

a) $x^2+2x-15=0$
Solution:

$$\begin{aligned}x^2+2x-15&=0\\ (x-3)(x+5)&=0\\ \therefore x&=-5 \text{ or } x=3\end{aligned}$$

b) $p^2-7p-18=0$
Solution:

$$\begin{aligned}p^2-7p-18&=0\\ (p-9)(p+2)&=0\\ \therefore p&=-2 \text{ or } p=9\end{aligned}$$

c) $9x^2-6x-8=0$
Solution:

$$\begin{aligned}9x^2-6x-8&=0\\ (3x+2)(3x-4)&=0\\ 3x+2&=0\\ x&=-\frac{2}{3}\\ &\text{or}\\ 3x-4&=0\\ x&=\frac{4}{3}\\ \therefore x&=-\frac{2}{3} \text{ or } x=\frac{4}{3}\end{aligned}$$

d) $5x^2+21x-54=0$
Solution:

$$
\begin{aligned}
5x^2 + 21x - 54 &= 0 \\
(5x - 9)(x + 6) &= 0 \\
5x - 9 &= 0 \\
x &= \frac{9}{5} \\
&\text{or} \\
x + 6 &= 0 \\
x &= -6 \\
\therefore x &= \frac{9}{5} \text{ or } x = -6
\end{aligned}
$$

e) $4z^2 + 12z + 8 = 0$
Solution:

$$
\begin{aligned}
4z^2 + 12z + 8 &= 0 \\
z^2 + 3z + 2 &= 0 \\
(z + 1)(z + 2) &= 0 \\
z &= -2 \text{ or } z = -1
\end{aligned}
$$

f) $-b^2 + 7b - 12 = 0$
Solution:

$$
\begin{aligned}
-b^2 + 7b - 12 &= 0 \\
b^2 - 7b + 12 &= 0 \\
(b - 4)(b - 3) &= 0 \\
b &= 3 \text{ or } b = 4
\end{aligned}
$$

g) $-3a^2 + 27a - 54 = 0$
Solution:

$$
\begin{aligned}
-3a^2 + 27a - 54 &= 0 \\
a^2 - 9a + 18 &= 0 \\
(a - 6)(a - 3) &= 0 \\
a &= 3 \text{ or } a = 6.
\end{aligned}
$$

h) $4y^2 - 9 = 0$
Solution:

$$
\begin{aligned}
4y^2 - 9 &= 0 \\
(2y - 3)(2y + 3) &= 0 \\
2y - 3 &= 0 \\
y &= \frac{3}{2} \\
&\text{or} \\
2y + 3 &= 0 \\
y &= -\frac{3}{2} \\
\therefore y &= \frac{3}{2} \text{ or } y = -\frac{3}{2}
\end{aligned}
$$

i) $4x^2 + 16x - 9 = 0$
Solution:

$$
\begin{aligned}
4x^2+16x-9&=0\\
(2x-1)(2x+9)&=0\\
2x-1&=0\\
x&=\frac{1}{2}\\
\text{or}&\\
2x+9&=0\\
y&=-\frac{9}{2}\\
\therefore x=\frac{1}{2}&\text{ or } x=-\frac{9}{2}
\end{aligned}
$$

j) $4x^2-12x=-9$
Solution:

$$
\begin{aligned}
4x^2-12x&=-9\\
4x^2-12x+9&=0\\
(2x-3)(2x-3)&=0\\
2x-3&=0\\
x&=\frac{3}{2}
\end{aligned}
$$

k) $20m+25m^2=0$
Solution:

$$
\begin{aligned}
20m+25m^2&=0\\
5m(4+5m)&=0\\
5m&=0\\
m&=0\\
\text{or}&\\
4+5m&=0\\
m&=-\frac{4}{5}\\
\therefore m=0&\text{ or } m=-\frac{4}{5}
\end{aligned}
$$

l) $2x^2-5x-12=0$
Solution:

$$
\begin{aligned}
2x^2-5x-12&=0\\
(2x+3)(x-4)&=0\\
2x+3&=0\\
x&=-\frac{3}{2}\\
\text{or}&\\
x-4&=0\\
x&=4\\
\therefore x=-\frac{3}{2}&\text{ or } x=4
\end{aligned}
$$

m) $-75x^2+290x=240$
Solution:

$$-75x^2 + 290x = 240$$
$$-75x^2 + 290x - 240 = 0$$
$$-15x^2 + 58x - 48 = 0$$
$$(5x - 6)(3x - 8) = 0$$
$$5x - 6 = 0$$
$$x = \frac{6}{5}$$
or
$$3x - 8 = 0$$
$$x = \frac{8}{3}$$
$$\therefore x = \frac{6}{5} \text{ or } x = \frac{8}{3}$$

n) $2x = \frac{1}{3}x^2 - 3x + 14\frac{2}{3}$
Solution:

$$2x = \frac{1}{3}x^2 - 3x + 14\frac{2}{3}$$
$$6x = x^2 - 9x + 44$$
$$x^2 - 15x + 44 = 0$$
$$(x - 4)(x - 11) = 0$$
$$x - 4 = 0$$
$$x = 4$$
or
$$x - 11 = 0$$
$$x = 11$$
$$\therefore x = 4 \text{ or } x = 11$$

o) $x^2 - 4x = -4$
Solution:

$$x^2 - 4x = -4$$
$$x^2 - 4x + 4 = 0$$
$$(x - 2)(x - 2) = 0$$
$$x - 2 = 0$$
$$x = 2$$

p) $-x^2 + 4x - 6 = 4x^2 - 14x + 3$
Solution:

$$-x^2 + 4x - 6 = 4x^2 - 14x + 3$$
$$5x^2 - 18x + 9 = 0$$
$$(5x - 3)(x - 3) = 0$$
$$5x - 3 = 0$$
$$x = \frac{3}{5}$$
or
$$x - 3 = 0$$
$$x = 3$$
$$\therefore x = \frac{3}{5} \text{ or } x = 3$$

q) $t^2 = 3t$
Solution:

$$\begin{aligned} t^2 &= 3t \\ t^2 - 3t &= 0 \\ t(t-3) &= 0 \\ t &= 0 \\ &\text{or} \\ t - 3 &= 0 \\ t &= 3 \\ \therefore t &= 0 \text{ or } t = 3 \end{aligned}$$

r) $x^2 - 10x = -25$
Solution:

$$\begin{aligned} x^2 - 10x &= -25 \\ x^2 - 10x + 25 &= 0 \\ (x-5)(x-5) &= 0 \\ x - 5 &= 0 \\ x &= 5 \end{aligned}$$

s) $x^2 = 18$
Solution:

$$\begin{aligned} x^2 &= 18 \\ \therefore x &= \sqrt{18} \text{ or } x = -\sqrt{18} \end{aligned}$$

t) $p^2 - 6p = 7$
Solution:

$$\begin{aligned} p^2 - 6p &= 7 \\ p^2 - 6p - 7 &= 0 \\ (p-7)(p+1) &= 0 \\ p - 7 &= 0 \\ p &= 7 \\ &\text{or} \\ p + 1 &= 0 \\ p &= -1 \\ \therefore p &= 7 \text{ or } p = -1 \end{aligned}$$

u) $4x^2 - 17x - 77 = 0$
Solution:

$$\begin{aligned} 4x^2 - 17x - 77 &= 0 \\ (4x+11)(x-7) &= 0 \\ 4x + 11 &= 0 \\ x &= -\frac{11}{4} \\ &\text{or} \\ x - 7 &= 0 \\ x &= 7 \\ \therefore x &= -\frac{11}{4} \text{ or } x = 7 \end{aligned}$$

v) $14x^2 + 5x = 6$
Solution:

$$\begin{aligned} 14x^2 + 5x &= 6 \\ 14x^2 + 5x - 6 &= 0 \\ (7x + 6)(2x - 1) &= 0 \\ 7x + 6 &= 0 \\ x &= -\frac{6}{7} \\ &\text{or} \\ 2x - 1 &= 0 \\ x &= \frac{1}{2} \\ \therefore x &= -\frac{6}{7} \text{ or } x = \frac{1}{2} \end{aligned}$$

w) $2x^2 - 2x = 12$
Solution:

$$\begin{aligned} 2x^2 - 2x &= 12 \\ x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \\ x - 3 &= 0 \\ x &= 3 \\ &\text{or} \\ x + 2 &= 0 \\ x &= -2 \\ \therefore x &= 3 \text{ or } x = -2 \end{aligned}$$

x) $(2a - 3)^2 - 16 = 0$
Solution:

$$\begin{aligned} (2a - 3)^2 - 16 &= 0 \\ (2a - 3 + 4)(2a - 3 - 4) &= 0 \\ (2a + 1)(2a - 7) &= 0 \\ \therefore a &= -\frac{1}{2} \text{ or } a = 3{,}5 \end{aligned}$$

y) $(x - 6)^2 - 24 = 1$
Solution:

$$\begin{aligned} (x - 6)^2 - 24 &= 1 \\ (x - 6)^2 - 25 &= 0 \\ (x - 6 - 5)(x - 6 + 5) &= 0 \\ (x - 11)(x - 1) &= 0 \\ \therefore x &= 11 \text{ or } x = 1 \end{aligned}$$

3. Solve the following equations (note the restrictions that apply):

a) $3y = \frac{54}{2y}$
Solution:
Note $y \neq 0$

$$\begin{aligned}
3y &= \frac{54}{2y} \\
3y^2 &= 27 \\
y^2 &= 9 \\
y^2 - 9 &= 0 \\
(y-3)(y+3) &= 0 \\
\therefore y &= 3 \text{ or } y = -3
\end{aligned}$$

b) $\frac{10z}{3} = 1 - \frac{1}{3z}$

Solution:

Note $z \neq 0$

$$\begin{aligned}
\frac{10z}{3} &= 1 - \frac{1}{3z} \\
10z^2 &= 3z - 1 \\
10z^2 - 3z + 1 &= 0 \\
(5z+1)(2z-1) &= 0 \\
\therefore z &= -\frac{1}{5} \text{ or } z = \frac{1}{2}
\end{aligned}$$

c) $x + 2 = \frac{18}{x} - 1$

Solution:

Note $x \neq 0$

$$\begin{aligned}
x + 2 &= \frac{18}{x} - 1 \\
x^2 + 2x &= 18 - x \\
x^2 + 3x - 18 &= 0 \\
(x-3)(x+6) &= 0 \\
\therefore x &= 3 \text{ or } x = -6
\end{aligned}$$

d) $y - 3 = \frac{5}{4} - \frac{1}{y}$

Solution:

Note $y \neq 0$

$$\begin{aligned}
y - 3 &= \frac{5}{4} - \frac{1}{y} \\
4y^2 - 12y &= 5y - 4 \\
4y^2 - 17y + 4 &= 0 \\
(4y-1)(y-4) &= 0 \\
\therefore y &= \frac{1}{4} \text{ or } y = 4
\end{aligned}$$

e) $\frac{1}{2}(b-1) = \frac{1}{3}\left(\frac{2}{b} + 4\right)$

Solution:

Note $b \neq 0$

$$\frac{1}{2}(b-1) = \frac{1}{3}\left(\frac{2}{b}+4\right)$$
$$3(b-1) = 2\left(\frac{2}{b}+4\right)$$
$$3b-3 = \frac{4}{b}+8$$
$$3b^2-3b = 4+8b$$
$$3b^2-11b-4 = 0$$
$$(3b+1)(b-4) =$$
$$\therefore b = -\frac{1}{3}b \text{ or } b = 4$$

f) $3(y+1) = \frac{4}{y}+2$

Solution:

Note $y \neq 0$

$$3(y+1) = \frac{4}{y}+2$$
$$3y+3 = \frac{4}{y}+2$$
$$3y^2+3y = 4+2y$$
$$3y^2+y-4 = 0$$
$$(3y+4)(y-1) = 0$$
$$\therefore y = -\frac{4}{3} \text{ or } y = 1$$

g) $(x+1)^2-2(x+1)-15 = 0$

Solution:

$$(x+1)^2-2(x+1)-15 = 0$$
$$((x+1)-5)((x+1)+3) = 0$$
$$(x-4)(x+4) = 0$$
$$\therefore x = 4 \text{ or } x = -4$$

h) $z^4-1 = 0$

Solution:

$$z^4-1 = 0$$
$$(z^2-1)(z^2+1) = 0$$
$$(z-1)(z+1)(z^2+1) = 0$$
$$\therefore z = 1 \text{ or } z = -1$$

Note that z^2+1 has no real solutions.

i) $b^4-13b^2+36 = 0$

Solution:

$$b^4-13b^2+36 = 0$$
$$(b^2-4)(b^2-9) = 0$$
$$(b-2)(b+2)(b-3)(b+3) = 0$$
$$\therefore b = \pm 2 \text{ or } b = \pm 3$$

j) $\frac{a+1}{3a-4}+\frac{9}{2a+5}+\frac{2a+3}{2a+5} = 0$

Solution:

$$\begin{aligned}
\frac{a+1}{3a-4}+\frac{9}{2a+5}+\frac{2a+3}{2a+5}&=0\\
\frac{(a+1)(2a+5)+9(3a-4)+(2a+3)(3a-4)}{(3a-4)(2a+5)}&=0\\
2a^2+7a+5+27a-36+6a^2+a-12&=0\\
8a^2+35a-43&=0\\
(8a+43)(a-1)&=0\\
8a+43&=0\\
a&=-\frac{43}{8}\\
&\text{or}\\
a-1&=0\\
a&=1\\
\therefore a&=-\frac{43}{8}\text{ or } a=1
\end{aligned}$$

k) $\frac{x^2-2x-3}{x+1}=0$

Solution:

Note $x \neq -1$

$$\begin{aligned}
\frac{x^2-2x-3}{x+1}&=0\\
\frac{(x+1)(x-3)}{x+1}&=0\\
\therefore x&=3
\end{aligned}$$

l) $x+2=\frac{6x-12}{x-2}$

Solution:

Note $x \neq 2$

$$\begin{aligned}
x+2&=\frac{6x-12}{x-2}\\
(x+2)(x-2)&=6x-12\\
x^2-4&=6x-12\\
x^2-6x+8&=0\\
(x-2)(x-4)&=0\\
\therefore x&=4
\end{aligned}$$

m) $\frac{3(a^2+1)+10a}{3a+1}=1$

Solution:

Note $a \neq -\frac{1}{3}$

$$\begin{aligned}
\frac{3(a^2+1)+10a}{3a+1}&=1\\
3(a^2+1)+10a&=3a+1\\
3a^2+3+10a-3a-1&=0\\
3a^2+7a+2&=0\\
(3a+1)(a+2)&=0\\
\therefore a&=-2
\end{aligned}$$

n) $\dfrac{3}{9a^2-3a+1} - \dfrac{3a+4}{27a^3+1} = \dfrac{1}{9a^2-1}$

Solution:

$$\begin{aligned}
\frac{3}{9a^2-3a+1} - \frac{3a+4}{27a^3+1} &= \frac{1}{9a^2-1} \\
\frac{3}{9a^2-3a+1} - \frac{3a+4}{(3a+1)(9a^2-3a+1)} &= \frac{1}{(3a-1)(3a+1)} \\
\frac{3(9a^2-1)-(3a-1)(3a+4)}{(3a+1)(3a-1)(9a^2-3a+1)} &= \frac{9a^2-3a+1}{(3a-1)(3a+1)(9a^2-3a+1)} \\
27a^2-3-9a^2-9a+4 &= 9a^2-3a+1 \\
9a^2-6a &= 0 \\
3a(3a-2) &= 0 \\
3a &= 0 \\
a &= 0 \\
&\text{or} \\
3a-2 &= 0 \\
a &= \frac{2}{3} \\
\therefore a = 0 \text{ or } a &= \frac{2}{3}
\end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FBN	1b. 2FBP	1c. 2FBQ	2a. 2FBR	2b. 2FBS	2c. 2FBT
2d. 2FBV	2e. 2FBW	2f. 2FBX	2g. 2FBY	2h. 2FBZ	2i. 2FC2
2j. 2FC3	2k. 2FC4	2l. 2FC5	2m. 2FC6	2n. 2FC7	2o. 2FC8
2p. 2FC9	2q. 2FCB	2r. 2FCC	2s. 2FCD	2t. 2FCF	2u. 2FCG
2v. 2FCH	2w. 2FCJ	2x. 2FCK	2y. 2FCM	3a. 2FCN	3b. 2FCP
3c. 2FCQ	3d. 2FCR	3e. 2FCS	3f. 2FCT	3g. 2FCV	3h. 2FCW
3i. 2FCX	3j. 2FCY	3k. 2FCZ	3l. 2FD2	3m. 2FD3	3n. 2FD4

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4.4 Solving simultaneous equations

Solving by substitution

Solving by elimination

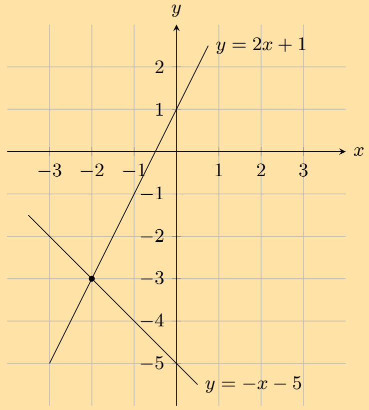
Solving graphically

This section can be included in the chapter on functions and graphs with graphs of linear equations. Before beginning this section it may be necessary to revise plotting graphs of linear equations with your learners.

It is also important that learners are either given the graphs or are encouraged to draw accurate graphs on graph paper to help them solve simultaneous equations graphically. Graph sketching software can be used in this section to ensure that graphs are accurate.

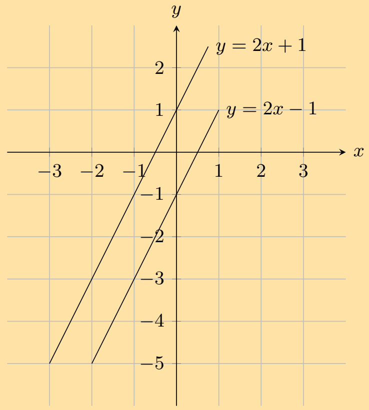
Exercise 4 – 3:

1. Look at the graph below



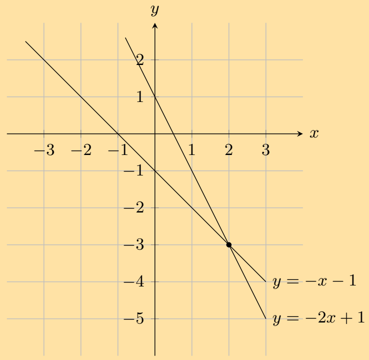
Solve the equations $y = 2x + 1$ and $y = -x - 5$ simultaneously
Solution:
From the graph we can see that the lines intersect at $x = -2$ and $y = -3$

2. Look at the graph below



Solve the equations $y = 2x - 1$ and $y = 2x + 1$ simultaneously
Solution:
The lines are parallel. Therefore there is no solution to x and y.

3. Look at the graph below



Solve the equations $y = -2x + 1$ and $y = -x - 1$ simultaneously
Solution:
From the graph we can see that the lines intersect at $x = 2$ and $y = -3$

4. Solve for x and y:

a) $-10x = -1$ and $-4x + 10y = -9$.
Solution:
Solve for x:

$$\begin{aligned} -10x &= -1 \\ \therefore x &= \frac{1}{10} \end{aligned}$$

Substitute the value of x into the second equation and solve for y:

$$\begin{aligned} -4x + 10y &= -9 \\ -4\left(\frac{1}{10}\right) + 10y &= -9 \\ \frac{-4}{10} + 10y &= -9 \\ 100y &= -90 + 4 \\ y &= \frac{-86}{100} \\ &= \frac{-43}{50} \end{aligned}$$

Therefore $x = \frac{1}{10}$ and $y = -\frac{43}{50}$.

b) $3x - 14y = 0$ and $x - 4y + 1 = 0$
Solution:
Write x in terms of y:

$$\begin{aligned} 3x - 14y &= 0 \\ 3x &= 14y \\ x &= \frac{14}{3}y \end{aligned}$$

Substitute value of x into second equation:

$$\begin{aligned} x - 4y + 1 &= 0 \\ \frac{14}{3}y - 4y + 1 &= 0 \\ 14y - 12y + 3 &= 0 \\ 2y &= -3 \\ y &= -\frac{3}{2} \end{aligned}$$

Substitute value of y back into first equation:

$$\begin{aligned} x &= \frac{14\left(-\frac{3}{2}\right)}{3} \\ &= -7 \end{aligned}$$

Therefore $x = -7$ and $y = -\frac{3}{2}$.

c) $x + y = 8$ and $3x + 2y = 21$
Solution:
Write x in terms of y:

$$\begin{aligned} x + y &= 8 \\ x &= 8 - y \end{aligned}$$

Substitute value of x into second equation:

$$\begin{aligned} 3x + 2y &= 21 \\ 3(8 - y) + 2y &= 21 \\ 24 - 3y + 2y &= 21 \\ y &= 3 \end{aligned}$$

Substitute value of y back into first equation:

$$x = 5$$

Therefore $x = 5$ and $y = 3$.

d) $y = 2x + 1$ and $x + 2y + 3 = 0$

Solution:

Write y in terms of x:

$$y = 2x + 1$$

Substitute value of y into second equation:

$$\begin{aligned} x + 2y + 3 &= 0 \\ x + 2(2x + 1) + 3 &= 0 \\ x + 4x + 2 + 3 &= 0 \\ 5x &= -5 \\ x &= -1 \end{aligned}$$

Substitute value of x back into first equation:

$$\begin{aligned} y &= 2(-1) + 1 \\ &= -1 \end{aligned}$$

Therefore $x = -1$ and $y = -1$.

e) $5x - 4y = 69$ and $2x + 3y = 23$

Solution:

Make x the subject of the first equation:

$$\begin{aligned} 5x - 4y &= 69 \\ 5x &= 69 + 4y \\ x &= \frac{69 + 4y}{5} \end{aligned}$$

Substitute value of x into second equation:

$$\begin{aligned} 2x + 3y &= 23 \\ 2\left(\frac{69 + 4y}{5}\right) + 3y &= 23 \\ 2(69 + 4y) + 3(5)y &= 23(5) \\ 138 + 8y + 15y &= 115 \\ 23y &= -23 \\ \therefore y &= -1 \end{aligned}$$

Substitute value of y back into first equation:

$$\begin{aligned} x &= \frac{69 + 4y}{5} \\ &= \frac{69 + 4(-1)}{5} \\ &= 13 \end{aligned}$$

Therefore $x = 13$ and $y = -1$.

f) $x + 3y = 26$ and $5x + 4y = 75$

Solution:

Make x the subject of the first equation:

$$\begin{aligned} x + 3y &= 26 \\ x &= 26 - 3y \end{aligned}$$

Substitute value of x into second equation:

$$\begin{aligned} 5x + 4y &= 75 \\ 5(26 - 3y) + 4y &= 75 \\ 130 - 15y + 4y &= 75 \\ -11y &= -55 \\ \therefore y &= 5 \end{aligned}$$

Substitute value of y back into first equation:

$$\begin{aligned} x &= 26 - 3y \\ &= 26 - 3(5) \\ &= 11 \end{aligned}$$

Therefore $x = 11$ and $y = 5$.

g) $3x - 4y = 19$ and $2x - 8y = 2$

Solution:

If we multiply the first equation by 2 then the coefficient of y will be the same in both equations:

$$\begin{aligned} 3x - 4y &= 19 \\ 3(2)x - 4(2)y &= 19(2) \\ 6x - 8y &= 38 \end{aligned}$$

Now we can subtract the second equation from the first:

$$\begin{array}{rrcl} & 6x - 8y & = & 38 \\ - & (2x - 8y & = & 2) \\ \hline & 4x + 0 & = & 36 \end{array}$$

Solve for x:

$$\begin{aligned} \therefore x &= \frac{36}{4} \\ &= 9 \end{aligned}$$

Substitute the value of x into the first equation and solve for y:

$$\begin{aligned} 3x - 4y &= 19 \\ 3(9) - 4y &= 19 \\ \therefore y &= \frac{19 - 3(9)}{-4} \\ &= 2 \end{aligned}$$

Therefore $x = 9$ and $y = 2$.

h) $\frac{a}{2} + b = 4$ and $\frac{a}{4} - \frac{b}{4} = 1$

Solution:

Make a the subject of the first equation:

$$\begin{aligned}\frac{a}{2}+b&=4\\a+2b&=8\\a&=8-2b\end{aligned}$$

Substitute value of a into second equation:

$$\begin{aligned}\frac{a}{4}-\frac{b}{4}&=1\\a-b&=4\\8-2b-b&=4\\3b&=4\\b&=\frac{4}{3}\end{aligned}$$

Substitute value of b back into first equation:

$$\begin{aligned}a&=8-2\left(\frac{4}{3}\right)\\&=\frac{16}{3}\end{aligned}$$

Therefore $a=\frac{16}{3}$ and $b=\frac{4}{3}$.

i) $-10x+y=-1$ and $-10x-2y=5$

Solution:

If we subtract the second equation from the first then we can solve for y:

$$\begin{array}{rrcr} & -10x+y & = & -1\\ - & (-10x-2y & = & 5)\\ \hline & 0+3y & = & -6\end{array}$$

Solve for y:

$$\begin{aligned}3y&=-6\\\therefore y&=-2\end{aligned}$$

Substitute the value of y into the first equation and solve for x:

$$\begin{aligned}-10x+y&=-1\\-10x-2&=-1\\-10x&=1\\x&=\frac{1}{-10}\end{aligned}$$

Therefore $x=\frac{-1}{10}$ and $y=-2$.

j) $-10x-10y=-2$ and $2x+3y=2$

Solution:

Make x the subject of the first equation:

$$\begin{aligned}-10x-10y&=-2\\5x+5y&=1\\5x&=1-5y\\\therefore x&=-y+\frac{1}{5}\end{aligned}$$

Substitute the value of x into the second equation and solve for y:

$$
\begin{aligned}
2x + 3y &= 2 \\
2\left(-y + \frac{1}{5}\right) + 3y &= 2 \\
-2y + \frac{2}{5} + 3y &= 2 \\
y &= \frac{8}{5}
\end{aligned}
$$

Substitute the value of y in the first equation:

$$
\begin{aligned}
5x + 5y &= 1 \\
5x + 5\left(\frac{8}{5}\right) &= 1 \\
5x + 8 &= 1 \\
5x &= -7 \\
x &= \frac{-7}{5}
\end{aligned}
$$

Therefore $x = -\frac{7}{5}$ and $y = \frac{8}{5}$.

k) $\dfrac{1}{x} + \dfrac{1}{y} = 3$ and $\dfrac{1}{x} - \dfrac{1}{y} = 11$

Solution:

Rearrange both equations by multiplying by xy:

$$
\begin{aligned}
\frac{1}{x} + \frac{1}{y} &= 3 \\
y + x &= 3xy
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{x} - \frac{1}{y} &= 11 \\
y - x &= 11xy
\end{aligned}
$$

Add the two equations together:

$$
\begin{array}{rccc}
 & y + x & = & 3xy \\
+ & (y - x & = & 11xy) \\
\hline
 & 2y + 0 & = & 14xy
\end{array}
$$

Solve for x:

$$
\begin{aligned}
2y &= 14xy \\
y &= 7xy \\
1 &= 7x \\
x &= \frac{1}{7}
\end{aligned}
$$

Substitute value of x back into first equation:

$$
\begin{aligned}
y + \frac{1}{7} &= 3\left(\frac{1}{7}\right) y \\
7y + 1 &= 3y \\
4y &= -1 \\
y &= -\frac{1}{4}
\end{aligned}
$$

Therefore $x = \frac{1}{7}$ and $y = -\frac{1}{4}$.

l) $y = \frac{2(x^2+2)-3}{x^2+2}$ and $y = 2 - \frac{3}{x^2+2}$

Solution:

Let

$$\begin{aligned} \frac{2(x^2+2)-3}{x^2+2} &= 2 - \frac{3}{x^2+2} \\ 2x^2+4-3 &= 2(x^2+2)-3 \\ 2x^2+1 &= 2x^2+1 \\ 0 &= 0 \end{aligned}$$

Since this is true for all x in the real numbers, x can be any real number.

Look at what happens to y when x is very small or very large:

The smallest x can be is 0. When $x = 0$, $y = 2 - \frac{3}{2} = \frac{1}{2}$.

If x gets very large, then the fraction $\frac{3}{x^2+2}$ becomes very small (think about what happens when you divide a small number by a very large number). Then $y = 2 - 0 = 2$.

From this we can see that $\frac{1}{2} \leq y \leq 2$.

Therefore x can be any real number, $\frac{1}{2} \leq y < 2$.

m) $3a + b = \frac{6}{2a}$ and $3a^2 = 3 - ab$

Solution:

Note $a \neq 0$

Look at the first equation

$$\begin{aligned} 3a + b &= \frac{6}{2a} \\ 6a^2 + 2ab &= 6 \\ 6a^2 &= 6 - 2ab \\ 3a^2 &= 3 - ab \end{aligned}$$

Note that this is the same as the second equation

a and b can be any real number except for 0.

5. Solve graphically and check your answer algebraically:

a) $y + 2x = 0$ and $y - 2x - 4 = 0$

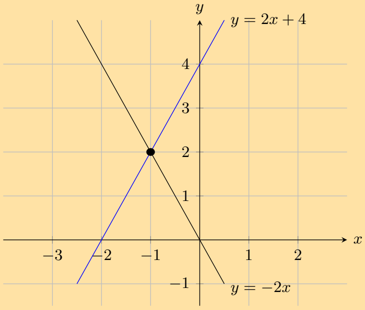
Solution:

First write the equations in standard form:

$$\begin{aligned} y + 2x &= 0 \\ y &= -2x \end{aligned}$$

$$\begin{aligned} y - 2x - 4 &= 0 \\ y &= 2x + 4 \end{aligned}$$

Draw the graph:



The graphs intersect at $(-1;2)$ so $x=-1$ and $y=2$.
Checking algebraically we get:

$$y=-2x$$

Substitute value of y into second equation:

$$\begin{aligned} y-2x-4&=0\\ -2x-2x-4&=0\\ -4x&=4\\ x&=-1 \end{aligned}$$

Substitute the value of x back into the first equation:

$$\begin{aligned} y&=-2(-1)\\ y&=2 \end{aligned}$$

b) $x+2y=1$ and $\frac{x}{3}+\frac{y}{2}=1$

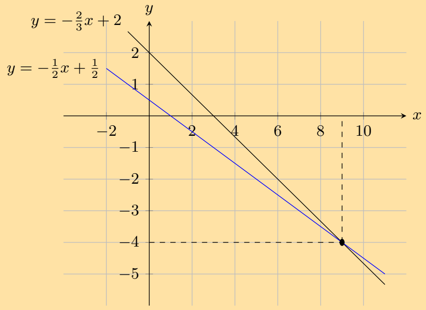
Solution:

First write the equations in standard form:

$$\begin{aligned} x+2y&=1\\ 2y&=-x+1\\ y&=-\frac{1}{2}x+\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \frac{x}{3}+\frac{y}{2}&=1\\ y&=-\frac{2}{3}x+2 \end{aligned}$$

Draw the graph:



The graphs intersect at $(9;-4)$ so $x=9$ and $y=-4$.
Checking algebraically we get:

$$x=-2y+1$$

Substitute value of x into first equation:

$$\begin{aligned} \frac{-2y+1}{3}+\frac{y}{2}&=1\\ -4y+2+3y&=6\\ y&=-4 \end{aligned}$$

Substitute the value of y back into the first equation:

$$
\begin{aligned}
x + 2(-4) &= 1 \\
x - 8 &= 1 \\
x &= 9
\end{aligned}
$$

c) $y - 2 = 6x$ and $y - x = -3$

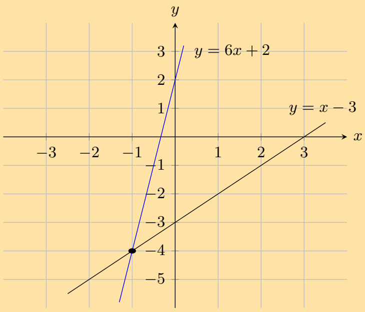
Solution:

First write the equations in standard form:

$$
\begin{aligned}
y - 2 &= 6x \\
y &= 6x + 2
\end{aligned}
$$

$$
\begin{aligned}
y - x &= -3 \\
y &= x - 3
\end{aligned}
$$

Draw the graph:



The graphs intersect at $(-1; -4)$ so $x = -1$ and $y = -4$.

Checking algebraically we get:

$$
y = 6x + 2
$$

Substitute value of y into first equation:

$$
\begin{aligned}
6x + 2 &= x - 3 \\
5x &= -5 \\
x &= -1
\end{aligned}
$$

Substitute the value of x back into the first equation:

$$
\begin{aligned}
y &= 6(-1) + 2 \\
y &= -4
\end{aligned}
$$

d) $2x + y = 5$ and $3x - 2y = 4$

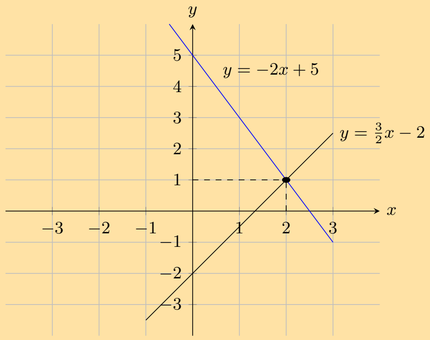
Solution:

First write the equations in standard form:

$$
\begin{aligned}
2x + y &= 5 \\
y &= -2x + 5
\end{aligned}
$$

$$
\begin{aligned}
3x - 2y &= 4 \\
2y &= 3x - 4 \\
y &= \frac{3}{2}x - 2
\end{aligned}
$$

Draw the graph:



The graphs intersect at $(2; 1)$ so $x = 2$ and $y = 1$.
Checking algebraically we get:

$$y = -2x + 5$$

Substitute value of y into first equation:

$$\begin{aligned} -2x + 5 &= \frac{3}{2}x - 2 \\ -4x + 10 &= 3x - 4 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

Substitute the value of x back into the first equation:

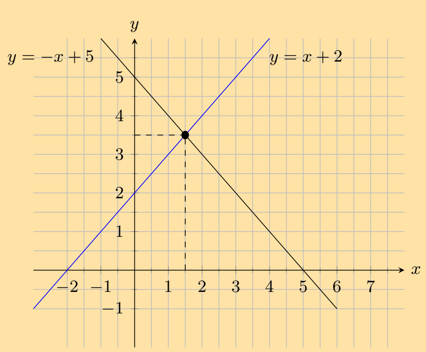
$$\begin{aligned} x &= -2(2) + 5 \\ y &= 1 \end{aligned}$$

e) $5 = x + y$ and $x = y - 2$
Solution:
First write the equations in standard form:

$$\begin{aligned} 5 &= x + y \\ y &= -x + 5 \end{aligned}$$

$$\begin{aligned} x &= y - 2 \\ y &= x + 2 \end{aligned}$$

Draw the graph:



The graphs intersect at $(1{,}5; 3{,}5)$ so $x = 1{,}5$ and $y = 3{,}5$.
Checking algebraically we get:

$$y = -x + 5$$

Substitute value of y into second equation:

$$\begin{aligned} x &= -x + 5 - 2 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

Substitute the value of x back into the first equation:

$$\begin{aligned} 5 &= \frac{3}{2} + y \\ y &= \frac{7}{2} \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FD6	2. 2FD7	3. 2FD8	4a. 2FDM	4b. 2FD9	4c. 2FDB	4d. 2FDC
4e. 2FDD	4f. 2FDF	4g. 2FDG	4h. 2FDH	4i. 2FDJ	4j. 2FDK	4k. 2FDN
4l. 2FDP	4m. 2FDQ	5a. 2FDT	5b. 2FDR	5c. 2FDV	5d. 2FDW	5e. 2FDS

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4.5 Word problems

Problem solving strategy

Exercise 4 – 4:

1. Two jets are flying towards each other from airports that are 1200 km apart. One jet is flying at 250 km·h^{-1} and the other jet at 350 km·h^{-1}. If they took off at the same time, how long will it take for the jets to pass each other?
   Solution:
   Let distance $d_1 = 1200 - x$ km and distance $d_2 = x$ km.
   Speed $s_1 = 250$ km·h^{-1} and speed $s_2 = 350$ km·h^{-1}.
   Time is found by dividing distance by speed.

$$\text{time } (t) = \frac{\text{distance}}{\text{speed}}$$

   When the jets pass each other:

$$\begin{aligned} \frac{1200 - x}{250} &= \frac{x}{350} \\ 350(1200 - x) &= 250x \\ 420\ 000 - 350x &= 250x \\ 600x &= 420\ 000 \\ x &= 700 \text{ km} \end{aligned}$$

   Now we know the distance travelled by the second jet when it passes the first jet, we can find the time:

$$\begin{aligned} t &= \frac{700 \text{ km}}{350 \text{ km·h}^{-1}} \\ &= 2 \text{ h} \end{aligned}$$

   It will take take the jets 2 hours to pass each other.

2. Two boats are moving towards each other from harbours that are 144 km apart. One boat is moving at 63 km·h^{-1} and the other boat at 81 km·h^{-1}. If both boats started their journey at the same time, how long will they take to pass each other?

   Solution:

   Notice that the sum of the distances for the two boats must be equal to the total distance when the boats meet: $d_1 + d_2 = d_{\text{total}} \longrightarrow d_1 + d_2 = 144$ km.

   This question is about distances, speeds, and times. The equation connecting these values is

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{- or -} \quad \text{distance} = \text{speed} \times \text{time}$$

   You want to know the amount of time needed for the boats to meet - let the time taken be t. Then you can write an expression for the distance each of the boats travels:

$$\begin{aligned}\text{For boat 1:} \quad d_1 &= s_1 t \\ &= 63t \\ \text{For boat 2:} \quad d_2 &= s_2 t \\ &= 81t\end{aligned}$$

   Now we can substitute the two expressions for the distances into the expression for the total distance:

$$\begin{aligned}d_1 + d_2 &= 144 \\ (63t) + (81t) &= 144 \\ 144t &= 144 \\ \therefore t &= \frac{144}{144} \\ &= 1\end{aligned}$$

   The boats will meet after 1 hour.

3. Zwelibanzi and Jessica are friends. Zwelibanzi takes Jessica's civil technology test paper and will not tell her what her mark is. He knows that Jessica dislikes word problems so he decides to tease her. Zwelibanzi says: "I have 12 marks more than you do and the sum of both our marks is equal to 148. What are our marks?"

   Solution:

   Let Zwelibanzi's mark be z and let Jessica's mark be j. Then

$$\begin{aligned}z &= j + 12 \\ z + j &= 148\end{aligned}$$

   Substitute the first equation into the second equation and solve:

$$\begin{aligned}z + j &= 148 \\ (j + 12) + j &= 148 \\ 2j &= 148 - 12 \\ \therefore j &= \frac{136}{2} \\ &= 68\end{aligned}$$

   Substituting this value back into the first equation gives:

$$\begin{aligned}z &= j + 12 \\ &= 68 + 12 \\ &= 80\end{aligned}$$

   Zwelibanzi achieved 80 marks and Jessica achieved 68 marks.

4. Kadesh bought 20 shirts at a total cost of R 980. If the large shirts cost R 50 and the small shirts cost R 40, how many of each size did he buy?

   Solution:

   Let x be the number of large shirts and $20-x$ the number of small shirts.

   Next we note the following:

   - He bought x large shirts for R 50

- He bought $20 - x$ small shirts for R 40
- He spent R 980 in total

We can represent the cost as:

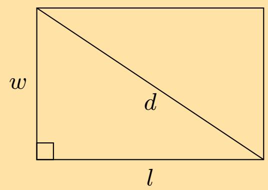
$$\begin{aligned}50x + 40(20 - x) &= 980\\ 50x + 800 - 40x &= 980\\ 10x &= 180\\ x &= 18\end{aligned}$$

Therefore Kadesh buys 18 large shirts and 2 small shirts.

5. The diagonal of a rectangle is 25 cm more than its width. The length of the rectangle is 17 cm more than its width. What are the dimensions of the rectangle?

Solution:

Let length $= l$, width $= w$ and diagonal $= d$. $\therefore d = w + 25$ and $l = w + 17$.



By the theorem of Pythagoras:

$$\begin{aligned}d^2 &= l^2 + w^2\\ \therefore w^2 &= d^2 - l^2\\ &= (w+25)^2 - (w+17)^2\\ &= w^2 + 50w + 625 - w^2 - 34w - 289\\ \therefore w^2 - 16w - 336 &= 0\\ (w+12)(w-28) &= 0\\ w = -12 \text{ or } w &= 28\end{aligned}$$

The width must be positive, therefore: width $w = 28$ cm length $l = (w + 17) = 45$ cm and diagonal $d = (w + 25) =$ 53 cm.

6. The sum of 27 and 12 is equal to 73 more than an unknown number. Find the unknown number.

Solution:

Let the unknown number $= x$.

$$\begin{aligned}27 + 12 &= x + 73\\ 39 &= x + 73\\ x &= -34\end{aligned}$$

The unknown number is -34.

7. A group of friends is buying lunch. Here are some facts about their lunch:
   - a milkshake costs R 7 more than a wrap
   - the group buys 8 milkshakes and 2 wraps
   - the total cost for the lunch is R 326

   Determine the individual prices for the lunch items.

   Solution:

   Let a milkshake be m and a wrap be w. From the given information we get the following equations:

$$\begin{aligned}m &= w + 7\\ 8m + 2w &= 326\end{aligned}$$

Substitute the first equation into the second equation and solve for w:

$$\begin{aligned} 8m + 2w &= 326 \\ 8(w+7) + 2w &= 326 \\ 8w + 56 + 2w &= 326 \\ 10w &= 326 - 56 \\ \therefore w &= \frac{270}{10} \\ &= 27 \end{aligned}$$

Substitute the value of w into the first equation and solve for m:

$$\begin{aligned} m &= w + 7 \\ &= 27 + 7 \\ &= 34 \end{aligned}$$

Therefore a milkshake costs R 34 and a wrap costs R 27.

8. The two smaller angles in a right-angled triangle are in the ratio of 1 : 2. What are the sizes of the two angles?
Solution:
Let $x =$ the smallest angle. Therefore the other angle $= 2x$.
We are given the third angle $= 90°$.

$$\begin{aligned} x + 2x + 90° &= 180° \text{ (sum of angles in a triangle)} \\ 3x &= 90° \\ x &= 30° \end{aligned}$$

The sizes of the angles are $30°$ and $60°$.

9. The length of a rectangle is twice the breadth. If the area is 128 cm^2, determine the length and the breadth.
Solution:
We are given length $l = 2b$ and $A = l \times b = 128$.
Substitute the first equation into the second equation and solve for b:

$$\begin{aligned} 2b \times b &= 128 \\ 2b^2 &= 128 \\ b^2 &= 64 \\ b &= \pm 8 \end{aligned}$$

But breadth must be positive, therefore $b = 8$.
Substitute this value into the first equation to solve for l:

$$\begin{aligned} l &= 2b \\ &= 2(8) \\ &= 16 \end{aligned}$$

Therefore $b = 8$ cm and $l = 2b = 16$ cm.

10. If 4 times a number is increased by 6, the result is 15 less than the square of the number. Find the number.
Solution:
Let the number $= x$. The equation that expresses the given information is:

$$\begin{aligned} 4x + 6 &= x^2 - 15 \\ x^2 - 4x - 21 &= 0 \\ (x-7)(x+3) &= 0 \\ x &= 7 \text{ or } x = -3 \end{aligned}$$

We are not told if the number is positive or negative. Therefore the number is 7 or -3.

11. The length of a rectangle is 2 cm more than the width of the rectangle. The perimeter of the rectangle is 20 cm. Find the length and the width of the rectangle.
Solution:
Let length $l = x$, width $w = x - 2$ and perimeter $= p$.

$$\begin{aligned} p &= 2l + 2w \\ &= 2x + 2(x - 2) \\ 20 &= 2x + 2x - 4 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

$l = 6$ cm and $w = l - 2 = 4$ cm.
length: 6 cm, width: 4 cm

12. Stephen has 1 litre of a mixture containing 69% salt. How much water must Stephen add to make the mixture 50% salt? Write your answer as a fraction of a litre.
Solution:
The new volume (x) of mixture must contain 50% salt, therefore:

$$\begin{aligned} 0{,}69 &= 0{,}5x \\ \therefore x &= \frac{0{,}69}{0{,}5} \\ x &= 2(0{,}69) \\ &= 1{,}38 \end{aligned}$$

The volume of the new mixture is 1,38 litre The amount of water (y) to be added is:

$$\begin{aligned} y &= x - 1{,}00 \\ &= 1{,}38 - 1{,}00 \\ &= 0{,}38 \end{aligned}$$

Therefore 0,38 litres of water must be added. To write this as a fraction of a litre: $0{,}38 = \frac{38}{100} = \frac{19}{50}$ litres
Therefore $\frac{19}{50}$ litres must be added.

13. The sum of two consecutive odd numbers is 20 and their difference is 2. Find the two numbers.
Solution:
Let the numbers be x and y.
Then the two equations describing the constraints are:

$$\begin{aligned} x + y &= 20 \\ x - y &= 2 \end{aligned}$$

Add the first equation to the second equation:

$$\begin{aligned} 2x &= 22 \\ x &= 11 \end{aligned}$$

Substitute into first equation:

$$\begin{aligned} 11 - y &= 2 \\ y &= 9 \end{aligned}$$

Therefore the two numbers are 9 and 11.

14. The denominator of a fraction is 1 more than the numerator. The sum of the fraction and its reciprocal is $\frac{5}{2}$. Find the fraction.
Solution:

Let the numerator be x. So the denominator is $x+1$.

$$\frac{x}{x+1}+\frac{x+1}{x}=\frac{5}{2}$$

Solve for x:

$$\begin{aligned}
\frac{x}{x+1}+\frac{x+1}{x}&=\frac{5}{2}\\
2x^2+2(x+1)^2&=5x(x+1)\\
2x^2+2(x^2+2x+1)&=5x^2+5x\\
2x^2+2x^2+4x+2&=5x^2+5x\\
x^2+x-2&=0\\
(x-1)(x+2)&=0\\
x&=1 \text{ of } x=-2
\end{aligned}$$

From this the fraction could be $\frac{1}{2}$ or $\frac{-2}{-1}$. For the second solution we can simplify the fraction to 2 and in this case the denominator is not 1 less than the numerator.
So the fraction is $\frac{1}{2}$.

15. Masindi is 21 years older than her daughter, Mulivhu. The sum of their ages is 37. How old is Mulivhu?
Solution:
Let Mulivhu be x years old. So Masindi is $x+21$ years old.

$$\begin{aligned}
x+x+21&=37\\
2x&=16\\
x&=8
\end{aligned}$$

Mulivhu is 8 years old.

16. Tshamano is now five times as old as his son Murunwa. Seven years from now, Tshamano will be three times as old as his son. Find their ages now.
Solution:
Let Murunwa be x years old. So Tshamano is $5x$ years old.
In 7 years time Murunwa's age will be $x+7$. Tshamano's age will be $5x+7$.

$$\begin{aligned}
5x+7&=3(x+7)\\
5x+7&=3x+21\\
2x&=14\\
x&=7
\end{aligned}$$

So Murunwa is 7 years old and Tshamano is 35 years old.
7 and 35 years old.

17. If adding one to three times a number is the same as the number, what is the number equal to?
Solution:
Let the number be x. Then:

$$\begin{aligned}
3x+1&=x\\
2x&=-1\\
x&=-\frac{1}{2}
\end{aligned}$$

18. If a third of the sum of a number and one is equivalent to a fraction whose denominator is the number and numerator is two, what is the number?
Solution:
Let the number be x. Then:

$$\frac{1}{3}(x+1)=\frac{2}{x}$$

Rearrange until we get a trinomial and solve for x:

$$\begin{aligned}
\frac{1}{3}(x+1) &= \frac{2}{x} \\
x+1 &= \frac{6}{x} \\
x^2+x &= 6 \\
x^2+x-6 &= 0 \\
(x-2)(x+3) &= 0 \\
\therefore x &= 2 \text{ or } x = -3
\end{aligned}$$

19. A shop owner buys 40 sacks of rice and mealie meal worth R 5250 in total. If the rice costs R 150 per sack and mealie meal costs R 100 per sack, how many sacks of mealie meal did he buy?
    Solution:

$$\begin{aligned}
x+y &= 40 \ (1) \\
150x+100y &= 5250 \ (2) \\
\\
&\text{look at } (1) \\
x &= 40-y \ (3) \\
&(3) \text{ into } (2) \\
150(40-y)+100y &= 5250 \\
-150y+100y &= 5250-6000 \\
-50y &= -750 \\
y &= 15 \\
\therefore 15 &\text{ sacks of melie meal were bought}
\end{aligned}$$

20. There are 100 bars of blue and green soap in a box. The blue bars weigh 50 g per bar and the green bars 40 g per bar. The total mass of the soap in the box is 4,66 kg. How many bars of green soap are in the box?
    Solution:

$$\begin{aligned}
x+y &= 100 \ (1) \\
50x+40y &= 4660 \ (2) \\
\\
&\text{look at } (1) \\
x &= 100-y \ (3) \\
&(3) \text{ into } (2) \\
50(100-y)+40y &= 4660 \\
-50y+40y &= 4660-5000 \\
-10y &= -340 \\
y &= 34 \\
\therefore 34 &\text{ sacks of melie meal were bought}
\end{aligned}$$

21. Lisa has 170 beads. She has blue, red and purple beads each weighing 13 g, 4 g and 8 g respectively. If there are twice as many red beads as there are blue beads and all the beads weigh 1,216 kg, how many beads of each type does Lisa have?
    Solution:

$$\begin{aligned} x+y+z &= 170 \qquad (1)\\ 13x+4y+8z &= 1216 \qquad (2)\\ y &= 2x \qquad (3) \end{aligned}$$

$$\begin{aligned} &(3) \text{ into } (1)\\ x+(2x)+z &= 170\\ 3x+z &= 170\\ z &= 170-3x \qquad (4)\\ &(3) \text{ into } \qquad (2)\\ 13x+4(2x)+8z &= 1216\\ 21x+8z &= 1216 \qquad (5)\\ &(4) \text{ into } (5)\\ 21x+8(170-3x) &= 1216\\ 21x+1360-24x &= 1216\\ -3x &= -144\\ x &= 48\\ y &= 2x = 96\\ z &= 170-3x = 26 \end{aligned}$$

$\therefore$ Lisa has 48 blue beads, 96 red beads and 36 purple beads,

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FDY 2. 2FDZ 3. 2FF2 4. 2FF3 5. 2FF4 6. 2FF5 7. 2FF6 8. 2FF7
9. 2FF8 10. 2FF9 11. 2FFB 12. 2FFC 13. 2FFD 14. 2FFF 15. 2FFG 16. 2FFH
17. 2FFJ 18. 2FFK 19. 2FFM 20. 2FFN 21. 2FFP

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4.6 Literal equations

Exercise 4 – 5:

1. Solve for x in the following formula: $2x+4y=2$.
   Solution:

$$\begin{aligned} 2x+4y &= 2\\ 2x &= 2-4y\\ \frac{1}{2}(2x) &= (2-4y)\frac{1}{2}\\ x &= 1-2y \end{aligned}$$

2. Make a the subject of the formula: $s=ut+\frac{1}{2}at^2$.
   Solution:

$$\begin{aligned} s &= ut+\frac{1}{2}a^2\\ s-ut &= \frac{1}{2}at^2\\ 2s-2ut &= at^2\\ \frac{2(s-ut)}{t^2} &= a \end{aligned}$$

Note restriction: $t \neq 0$

3. Solve for n: $pV = nRT$.

Solution:

$$\begin{aligned} pV &= nRT \\ \frac{pV}{RT} &= n \end{aligned}$$

Note restrictions: $R \neq 0$, $T \neq 0$

4. Make x the subject of the formula: $\frac{1}{b} + \frac{2b}{x} = 2$.

Solution:

$$\begin{aligned} \frac{1}{b} + \frac{2b}{x} &= 2 \\ \frac{x + b(2b)}{bx} &= 2 \\ x + 2b^2 &= 2bx \\ x - 2bx &= -2b^2 \\ x(1-2b) &= -2b^2 \\ x &= \frac{-2b^2}{1-2b} \end{aligned}$$

Note restriction: $1 \neq 2b$

5. Solve for $r : V = \pi r^2 h$.

Solution:

$$\begin{aligned} V &= \pi r^2 h \\ \frac{V}{\pi h} &= r^2 \\ \pm\sqrt{\frac{V}{\pi h}} &= r \end{aligned}$$

Note restriction: $h \neq 0$

6. Solve for h: $E = \frac{hc}{\lambda}$.

Solution:

$$\begin{aligned} E &= \frac{hc}{\lambda} \\ E\lambda &= hc \\ \frac{E\lambda}{c} &= h \end{aligned}$$

Note restriction: $c \neq 0$

7. Solve for h: $A = 2\pi rh + 2\pi r$.

Solution:

$$\begin{aligned} A &= 2\pi rh + 2\pi r \\ A - 2\pi r &= 2\pi rh \\ \frac{A - 2\pi r}{2\pi r} &= h \end{aligned}$$

Note restriction: $r \neq 0$

8. Make λ the subject of the formula: $t = \frac{D}{f\lambda}$.

Solution:

$$\begin{aligned} t &= \frac{D}{f\lambda} \\ t(\lambda) &= \frac{D}{f} \\ \lambda &= \frac{D}{tf} \end{aligned}$$

Note restrictions: $t \neq 0$, $f \neq 0$

9. Solve for m: $E = mgh + \frac{1}{2}mv^2$.
   Solution:

$$\begin{aligned} E &= mgh + \frac{1}{2}mv^2 \\ E &= m\left(gh + \frac{1}{2}v^2\right) \\ \frac{E}{gh + \frac{1}{2}v^2} &= m \end{aligned}$$

   Note restriction: $gh + \frac{1}{2}v^2 \neq 0$

10. Solve for x: $x^2 + x(a+b) + ab = 0$.
    Solution:

$$\begin{aligned} x^2 + x(a+b) + ab &= 0 \\ x^2 + xa + xb + ab &= 0 \\ (x+a)(x+b) &= 0 \\ x &= -a \text{ or } x = -b \end{aligned}$$

11. Solve for b: $c = \sqrt{a^2 + b^2}$.
    Solution:

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ c^2 &= a^2 + b^2 \\ c^2 - a^2 &= b^2 \\ b &= \pm\sqrt{c^2 - a^2} \end{aligned}$$

12. Make U the subject of the formula: $\frac{1}{V} = \frac{1}{U} + \frac{1}{W}$.
    Solution:

$$\begin{aligned} \frac{1}{V} &= \frac{1}{U} + \frac{1}{W} \\ \frac{UW}{UVW} &= \frac{VW}{UVW} + \frac{UV}{UVW} \\ UW &= VW + UV \\ UW - UV &= VW \\ U &= \frac{VW}{W-V} \end{aligned}$$

    Note restriction: $W \neq V$

13. Solve for r: $A = \pi R^2 - \pi r^2$.
    Solution:

$$A = \pi R^2 - \pi r^2$$
$$A - \pi R^2 = \pi r^2$$
$$\frac{A - \pi R^2}{\pi} = r^2$$
$$r = \pm\sqrt{\frac{A - \pi R^2}{\pi}}$$

14. $F = \frac{9}{5}C + 32°$ is the formula for converting temperature in degrees Celsius to degrees Fahrenheit. Derive a formula for converting degrees Fahrenheit to degrees Celcius.
Solution:

$$F = \frac{9}{5}C + 32°$$
$$F - 32° = \frac{9}{5}C$$
$$5(F - 32°) = 9C$$
$$\frac{5(F - 32°)}{9} = C$$

To convert degrees Fahrenheit to degrees Celsius we use: $C = \frac{5}{9}(F - 32°)$

15. $V = \frac{4}{3}\pi r^3$ is the formula for determining the volume of a soccer ball. Express the radius in terms of the volume.
Solution:

$$V = \frac{4}{3}\pi r^3$$
$$\frac{3}{4}V = \pi r^3$$
$$\frac{\frac{3}{4}V}{\pi} = r^3$$
$$\sqrt[3]{\frac{\frac{3}{4}V}{\pi}} = r$$

Therefore expressing the radius in terms of the volume gives: $r = \sqrt[3]{\frac{\frac{3}{4}V}{\pi}}$

16. Solve for x in: $x^2 - ax - 3x = 4 + a$
Solution:

$$x^2 - ax - 3x = 4 + a$$
$$x^2 - ax - 3x + a + 4 = 0$$
$$x^2 - x(a + 3) + (a + 4) = 0$$
$$(x + 1)(x - (a + 4)) = 0$$
$$\therefore x = a + 4 \text{ or } x = -1$$

17. Solve for x in: $ax^2 - 4a + bx^2 - 4b = 0$
Solution:

$$ax^2 - 4a + bx^2 - 4b = 0$$
$$a(x^2 - 4) + b(x^2 - 4) = 0$$
$$(a + b)(x^2 - 4) = 0$$
$$(a + b)(x - 2)(x + 2) = 0$$
$$\therefore x = 2 \text{ or } x = -2$$

18. Solve for x in $v^2 = u^2 + 2ax$ if $v = 2$, $u = 0{,}3$, $a = 0{,}5$

Solution:

$$
\begin{aligned}
v^2 &= u^2 + 2ax \\
2ax &= v^2 - u^2 \\
x &= \frac{v^2 - u^2}{2a} \\
x &= \frac{2^2 - 0{,}3^2}{2(0{,}5)} \\
x &= 3{,}91
\end{aligned}
$$

19. Solve for u in $f' = f\dfrac{v}{v-u}$ if $v = 13$, $f = 40$, $f' = 50$

Solution:

$$
\begin{aligned}
f' &= f\frac{v}{v-u} \\
f'(v-u) &= fv \\
v - u &= \frac{fv}{f'} \\
-u &= \frac{fv}{f'} - v \\
u &= v - \frac{fv}{f'} \\
u &= 13 - \frac{40(13)}{50} \\
u &= 2{,}6
\end{aligned}
$$

20. Solve for h in $I = \dfrac{bh^2}{12}$ if $b = 18$, $I = 384$

Solution:

$$
\begin{aligned}
I &= \frac{bh^2}{12} \\
h^2 &= \frac{12I}{b} \\
h &= \pm\sqrt{\frac{12I}{b}} \\
h &= \pm\sqrt{\frac{12(384)}{18}} \\
h &= \pm 16
\end{aligned}
$$

21. Solve for r_2 in $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$ if $R = \frac{3}{2}$, $r_1 = 2$

Solution:

$$
\begin{aligned}
\frac{1}{R} &= \frac{1}{r_1} + \frac{1}{r_2} \\
\frac{1}{r_2} &= \frac{1}{R} - \frac{1}{r_1} \\
\frac{1}{r_2} &= \frac{r_1 - R}{Rr_1} \\
Rr_1 &= r_2(r_1 - R) \\
\frac{Rr_1}{r_1 - R} &= r_2 \\
r_2 &= \frac{\frac{3}{2} \times 2}{2 - \frac{3}{2}} \\
&= \frac{3}{\frac{1}{2}} \\
&= 6
\end{aligned}
$$

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1. 2FFR	2. 2FFS	3. 2FFT	4. 2FFV	5. 2FFW	6. 2FFX
7. 2FFY	8. 2FFZ	9. 2FG2	10. 2FG3	11. 2FG4	12. 2FG5
13. 2FG6	14. 2FG7	15. 2FG8	16. 2FG9	17. 2FGB	18. 2FGC
19. 2FGD	20. 2FGF	21. 2FGG			

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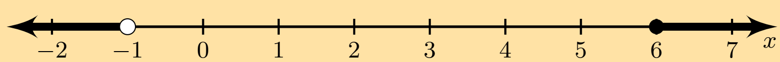
4.7 Solving linear inequalities

Interval notation

Exercise 4 – 6:

1. Look at the number line and write down the inequality it represents.

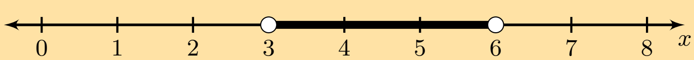
a)



Solution:

$x < -1$ and $x \geq 6; x \in \mathbb{R}$

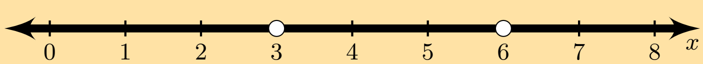
b)



Solution:

$3 < x < 6; x \in \mathbb{R}$

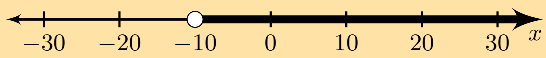
c)



Solution:

$x \neq 3; x \neq 6; x \in \mathbb{R}$

d)



Solution:

$x > -10; x \in \mathbb{R}$

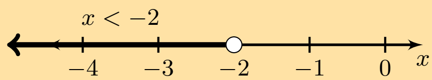
2. Solve for x and represent the answer on a number line and in interval notation.

   a) $3x + 4 > 5x + 8$

   Solution:

$$
\begin{aligned}
3x + 4 &> 5x + 8 \\
3x - 5x &> 8 - 4 \\
-2x &> 4 \\
2x &< -4 \\
x &< -2
\end{aligned}
$$

Represented on a number line:



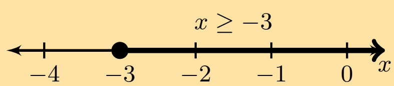
In interval notation: $(-\infty; -2)$

   b) $3(x - 1) - 2 \leq 6x + 4$

   Solution:

$$
\begin{aligned}
3(x - 1) - 2 &\leq 6x + 4 \\
3x - 5 &\leq 6x + 4 \\
3x - 6x &\leq 4 + 5 \\
-3x &\leq 9 \\
x &\geq -\frac{9}{3} \\
x &\geq -3
\end{aligned}
$$

Represented on a number line:



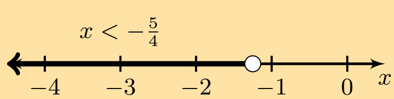
In interval notation: $[-3; \infty)$

   c) $\frac{x - 7}{3} > \frac{2x - 3}{2}$

   Solution:

$$
\begin{aligned}
\frac{x - 7}{3} &> \frac{2x - 3}{2} \\
2(x - 7) &> 3(2x - 3) \\
2x - 14 &> 6x - 9 \\
-4x &> 5 \\
x &< -\frac{5}{4}
\end{aligned}
$$

Represented on a number line:



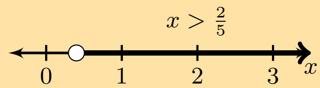
In interval notation: $(-\infty; -\frac{5}{4})$

d) $-4(x-1) < x+2$

Solution:

$$\begin{aligned} -4(x-1) &< x+2 \\ -4x+4 &< x+2 \\ -5x &< -2 \\ x &> \frac{2}{5} \end{aligned}$$

Represented on a number line:



In interval notation: $(\frac{2}{5}; \infty)$

e) $\frac{1}{2}x + \frac{1}{3}(x-1) \geq \frac{5}{6}x - \frac{1}{3}$

Solution:

$$\begin{aligned} \frac{1}{2}x + \frac{1}{3}(x-1) &\geq \frac{5}{6}x - \frac{1}{3} \\ \frac{1}{2}x + \frac{1}{3}x - \frac{1}{3} &\geq \frac{5}{6}x - \frac{1}{3} \\ \frac{1}{2}x + \frac{1}{3}x - \frac{5}{6}x &\geq \frac{1}{3} - \frac{1}{3} \\ \frac{3}{6}x + \frac{2}{6}x - \frac{5}{6}x &\geq 0 \\ 0x &\geq 0 \end{aligned}$$

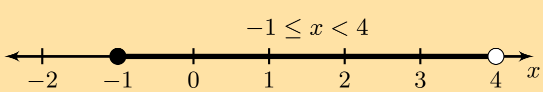
The inequality is true for all real values of x.

f) $-2 \leq x - 1 < 3$

Solution:

$$\begin{array}{ccccc} -2 & \leq & x-1 & < & 3 \\ -1 & \leq & x & < & 4 \end{array}$$

Represented on a number line:



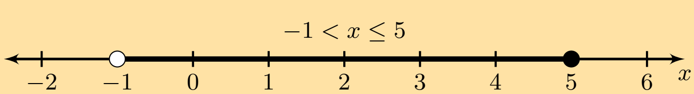
In interval notation: $[-1; 4)$

g) $-5 < 2x - 3 \leq 7$

Solution:

$$\begin{array}{ccccc} -5 & < & 2x-3 & \leq & 7 \\ -2 & < & 2x & \leq & 10 \\ -1 & < & x & \leq & 5 \end{array}$$

Represented on a number line:



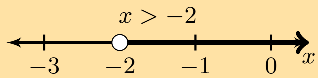
In interval notation: $(-1; 5]$

h) $7(3x+2)-5(2x-3)>7$

Solution:

$$\begin{aligned} 7(3x+2)-5(2x-3) &> 7 \\ 21x+14-10x+15 &> 7 \\ 11x &> -22 \\ x &> -2 \end{aligned}$$

Represented on a number line:



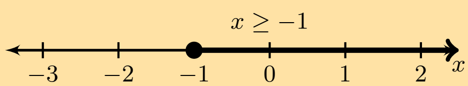
In interval notation: $(-2;\infty)$

i) $\frac{5x-1}{-6} \geq \frac{1-2x}{3}$

Solution:

$$\begin{aligned} \frac{5x-1}{-6} &\geq \frac{1-2x}{3} \\ 5x-1 &\geq -2(1-2x) \\ 5x-1 &\geq -2+4x \\ 5x-4x &\geq -1 \\ x &\geq -1 \end{aligned}$$

Represented on a number line:



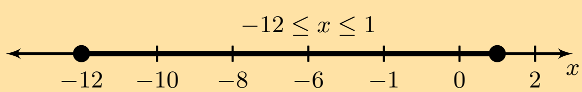
In interval notation: $[-1;\infty)$

j) $3 \leq 4-x \leq 16$

Solution:

$$\begin{array}{ccccc} 3 & \leq & 4-x & \leq & 16 \\ -1 & \leq & -x & \leq & 12 \\ 1 & \geq & x & \geq & -12 \end{array}$$

Represented on a number line:



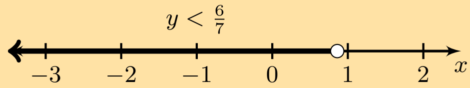
In interval notation: $[1;12]$

k) $\frac{-7y}{3} - 5 > -7$

Solution:

$$\begin{aligned} \frac{-7y}{3} - 5 &> -7 \\ -7y-15 &> -21 \\ -7y &> -6 \\ y &< \frac{6}{7} \end{aligned}$$

Represented on a number line:



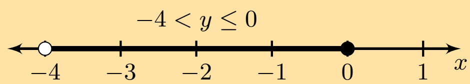
In interval notation: $(-\infty; \frac{6}{7})$

l) $1 \leq 1 - 2y < 9$

Solution:

$$\begin{array}{rcccl} 1 & \leq & 1-2y & < & 9 \\ 0 & \leq & -2y & < & 8 \\ 0 & \geq & y & > & -4 \\ -4 & < & y & \leq & 0 \end{array}$$

Represented on a number line:



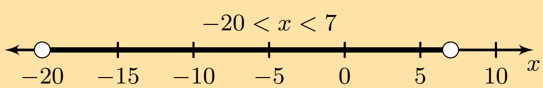
In interval notation: $(-4; 0]$

m) $-2 < \dfrac{x-1}{-3} < 7$

Solution:

$$\begin{array}{rcccl} -2 & < & \dfrac{x-1}{-3} & < & 7 \\ 6 & > & x-1 & > & -21 \\ 7 & > & x & > & -20 \\ -20 & < & x & < & 7 \end{array}$$

Represented on a number line:



In interval notation: $(-20; 7)$

3. Solve for x and show your answer in interval notation:

a) $2x - 1 < 3(x + 11)$

Solution:

$$\begin{aligned} 2x - 1 &< 3(x + 11) \\ 2x - 1 &< 3x + 33 \\ 2x - 3x &< 33 + 1 \\ -1x &< 34 \\ \therefore x &> -34 \end{aligned}$$

$$(-34; \infty)$$

b) $x - 1 < -4(x - 6)$

Solution:

$$\begin{aligned} x - 1 &< -4(x - 6) \\ x - 1 &< -4x + 24 \\ x + 4x &< 24 + 1 \\ 5x &< 25 \\ \therefore x &< 5 \end{aligned}$$

$$(-\infty; 5)$$

c) $\dfrac{x-1}{8} \leq \dfrac{2(x-2)}{3}$

Solution:

$$\begin{aligned}
\frac{x-1}{8} &\leq \frac{2(x-2)}{3} \\
3(x-1) &\leq 16(x-2) \\
3x-3 &\leq 16x-32 \\
3x-16x &\leq -32+3 \\
-13x &\leq -29 \\
\therefore x &\geq \frac{29}{13}
\end{aligned}$$

$x \in \left[\frac{29}{13}; \infty\right)$.

d) $\dfrac{x+2}{4} \leq \dfrac{-2(x-4)}{7}$

Solution:

$$\begin{aligned}
\frac{x+2}{4} &\leq \frac{-2(x-4)}{7} \\
7(x+2) &\leq -8(x-4) \\
7x+14 &\leq -8x+32 \\
7x+8x &\leq 32-14 \\
15x &\leq 18 \\
\therefore x &\leq \frac{6}{5}
\end{aligned}$$

$x \in \left(-\infty; \frac{6}{5}\right]$.

e) $\dfrac{1}{5}x - \dfrac{5}{4}(x+2) > \dfrac{1}{4}x + 3$

Solution:

$$\begin{aligned}
\frac{1}{5}x - \frac{5}{4}(x+2) &> \frac{1}{4}x + 3 \\
4x - 25(x+2) &> 5x + 60 \\
4x - 25x - 50 &> 5x + 60 \\
4x - 25x - 5x &> 60 + 50 \\
-26x &> 110 \\
\therefore x &< -\frac{55}{13}
\end{aligned}$$

The interval is:

$$\left(-\infty; -\frac{55}{13}\right)$$

f) $\dfrac{1}{5}x - \dfrac{2}{5}(x+3) \geq \dfrac{4}{2}x + 3$

Solution:

$$\begin{aligned}
\frac{1}{5}x - \frac{2}{5}(x+3) &\geq \frac{4}{2}x + 3 \\
2x - 4(x+3) &\geq 20x + 30 \\
2x - 4x - 12 &\geq 20x + 30 \\
2x - 4x - 20x &\geq 30 + 12 \\
-22x &\geq 42 \\
\therefore x &\leq -\frac{21}{11}
\end{aligned}$$

The interval is:

$$\left(-\infty; -\frac{21}{11}\right]$$

g) $4x + 3 < -3 \quad \text{or} \quad 4x + 3 > 5$

Solution:

Solve the inequality:

$$\begin{array}{rcl c rcl}
4x + 3 & < & -3 & \text{or} & 4x + 3 & > & 5 \\
4x & < & -3 - 3 & \text{or} & 4x & > & 5 - 3 \\
x & < & \frac{\frac{-3-3}{4}}{} & \text{or} & x & > & \frac{5-3}{4} \\
x & < & -\frac{3}{2} & \text{or} & x & > & \frac{1}{2}
\end{array}$$

$$\left(-\infty; -\frac{3}{2}\right) \cup \left(\frac{1}{2}; \infty\right)$$

h) $4 \geq -6x - 6 \geq -3$

Solution:

Solve the inequality:

$$\begin{array}{rcccl}
4 & \geq & -6x - 6 & \geq & -3 \\
4 + 6 & \geq & -6x & \geq & -3 + 6 \\
\frac{4+6}{-6} & \leq & x & \leq & \frac{-3+6}{-6} \\
-\frac{5}{3} & \leq & x & \leq & -\frac{1}{2}
\end{array}$$

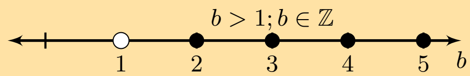
$$\left[-\frac{5}{3}; -\frac{1}{2}\right]$$

4. Solve for the unknown variable and show your answer on a number line.

a) $6b - 3 > b + 2, \ b \in \mathbb{Z}$

Solution:

$$\begin{aligned}
6b - 3 &> b + 2, \ b \in \mathbb{Z} \\
5b &> 5 \\
b &> 1
\end{aligned}$$

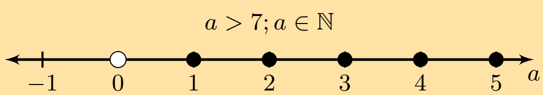


b) $3a - 1 < 4a + 6, \ a \in \mathbb{N}$

Solution:

$$\begin{aligned}
3a - 1 &< 4a + 6 \\
-a &< 7 \\
a &> -7
\end{aligned}$$

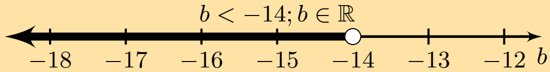
However we are told that $a \in \mathbb{N}$ and so $a > 0$.



c) $\dfrac{b-3}{2} + 1 < \dfrac{b}{4} - 4, \ b \in \mathbb{R}$

Solution:

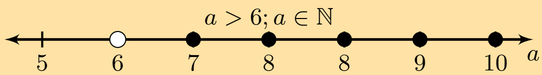
$$\begin{aligned}
\frac{b-3}{2} + 1 &< \frac{b}{4} - 4 \\
2b - 6 + 4 &< b - 16 \\
b &< -14
\end{aligned}$$



d) $\dfrac{4a+7}{3} - 5 > a - \dfrac{2}{3}$, $a \in \mathbb{N}$

Solution:

$$\begin{aligned} \frac{4a+7}{3} - 5 &> a - \frac{2}{3} \\ 4a + 7 - 15 &> 3a - 2 \\ a &> 6 \end{aligned}$$



For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FGJ	1b. 2FGK	1c. 2FGM	1d. 2FGN	2a. 2FGP	2b. 2FGQ
2c. 2FGR	2d. 2FGS	2e. 2FGT	2f. 2FGV	2g. 2FGW	2h. 2FGX
2i. 2FGY	2j. 2FGZ	2k. 2FH2	2l. 2FH3	2m. 2FH4	3a. 2FH5
3b. 2FH6	3c. 2FH7	3d. 2FH8	3e. 2FH9	3f. 2FHB	3g. 2FHC
3h. 2FHD	4a. 2FHF	4b. 2FHG	4c. 2FHH	4d. 2FHJ	

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4.8 Chapter summary

End of chapter Exercise 4 – 7:

1. Solve:

a) $5a - 7 = -2$

Solution:

$$\begin{aligned} 5a - 7 &= -2 \\ 5a &= -2 - (-7) \\ 5a &= 5 \\ a &= \frac{5}{5} \\ &= 1 \end{aligned}$$

b) $5m + 3 = -2$

Solution:

$$\begin{aligned} 5m + 3 &= -2 \\ 5m &= -2 - 3 \\ 5m &= -5 \\ m &= \frac{-5}{5} \\ &= -1 \end{aligned}$$

c) $1 = 4 - 3y$

Solution:

$$\begin{aligned} 1 &= 4 - 3y \\ 3y &= 4 - 1 \\ 3y &= 3 \\ y &= \frac{3}{3} \\ &= 1 \end{aligned}$$

d) $2(p-1) = 3(p+2)$

Solution:

$$\begin{aligned} 2(p-1) &= 3(p+2) \\ 2p - 2 &= 3p + 6 \\ p &= -8 \end{aligned}$$

e) $3 - 6k = 2k - 1$

Solution:

$$\begin{aligned} 3 - 6k &= 2k - 1 \\ 8k &= 4 \\ k &= \frac{1}{2} \end{aligned}$$

f) $2{,}1x + 3 = 4{,}1 - 3{,}3x$

Solution:

$$\begin{aligned} 2{,}1x + 3 &= 4{,}1 - 3{,}3x \\ 5{,}4x &= 1{,}1 \\ x &= \frac{11}{54} \end{aligned}$$

g) $m + 6(-m+1) + 8m = 0$

Solution:

$$\begin{aligned} m + 6(-m+1) + 8m &= 0 \\ m - 6m + 6 + 8m &= 0 \\ 3m &= -6 \\ m &= -2 \end{aligned}$$

h) $2k + 3 = 2 - 3(k+2)$

Solution:

$$\begin{aligned} 2k + 3 &= 2 - 3(k+3) \\ 2k + 3 &= 2 - 3k - 9 \\ 5k &= -10 \\ k &= -2 \end{aligned}$$

i) $3 + \frac{q}{5} = \frac{q}{2}$

Solution:

$$\begin{aligned} 3 + \frac{q}{5} &= \frac{q}{2} \\ 30 + 2q &= 5q \\ 3q &= 30 \\ q &= 10 \end{aligned}$$

j) $\frac{1}{2} = \frac{4z+1}{5z-6}$

Solution:

Note that $z \neq \frac{6}{5}$

$$\begin{aligned} \frac{1}{2} &= \frac{4z+1}{5z-6} \\ 8z+2 &= 5z-6 \\ 3z &= -8 \\ z &= -\frac{8}{3} \end{aligned}$$

k) $2 + \frac{a-4}{2} - \frac{a}{3} = 7$

Solution:

$$\begin{aligned} 2 + \frac{a-4}{2} - \frac{a}{3} &= 7 \\ \frac{3(a-4)-2a}{6} &= 5 \\ 3a - 12 - 2a &= 30 \\ a &= 42 \end{aligned}$$

l) $5 - \frac{2(m+4)}{m} = \frac{7}{m}$

Solution:

$$\begin{aligned} 5 - \frac{2(m+4)}{m} &= \frac{7}{m} \\ 5m - 2(m+4) &= 7 \\ 5m - 2m - 8 &= 7 \\ 3m &= 15 \\ m &= 5 \end{aligned}$$

m) $\frac{2}{t} - 2 - \frac{1}{2} = \frac{1}{2}\left(1 + \frac{2}{t}\right)$

Solution:

$$\begin{aligned} \frac{2}{t} - 2 - \frac{1}{2} &= \frac{1}{2}\left(1 + \frac{2}{t}\right) \\ \frac{2}{t} - 2 - \frac{1}{2} &= \frac{1}{2} + \frac{1}{t} \\ \frac{4}{t} - 4 - 1 &= 1 + \frac{2}{t} \\ 4 - 4t - t &= t + 2 \\ 6t &= 2 \\ t &= \frac{1}{3} \end{aligned}$$

2. Solve:

a) $b^2 + 6b - 27 = 0$

Solution:

$$\begin{aligned} b^2 + 6b - 27 &= 0 \\ (b-3)(b+9) &= 0 \\ b &= -9 \text{ or } b = 3 \end{aligned}$$

b) $-x^2 + 5x + 6 = 0$

Solution:

$$\begin{aligned} x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \\ x &= -1 \text{ or } x = 6 \end{aligned}$$

c) $-b^2 - 3b + 10 = 0$

Solution:

$$\begin{aligned} b^2 + 3b - 10 &= 0 \\ (b-2)(b+5) &= 0 \\ b &= -5 \text{ or } b = 2 \end{aligned}$$

d) $2b - 15 = (b+1)(b-6) - b^2$

Solution:

$$\begin{aligned} 2b - 15 &= (b+1)(b-6) - b^2 \\ 2b - 15 &= b^2 - 5b - 6 - b^2 \\ 7b &= 9 \\ b &= \frac{9}{7} \end{aligned}$$

e) $(5x+1)(x-3) = 0$

Solution:

$$\begin{aligned} (5x+1)(x-3) &= 0 \\ x &= -\frac{1}{5} \text{ or } x = 3 \end{aligned}$$

f) $5t - 1 = t^2 - (t+2)(t-2)$

Solution:

$$\begin{aligned} 5t - 1 &= t^2 - (t+2)(t-2) \\ 5t - 1 &= t^2 - t^2 + 4 \\ 5t - 1 &= 4 \\ 5t &= 5 \\ t &= 1 \end{aligned}$$

g) $\frac{a+2}{a-3} = \frac{a+8}{a+4}$

Solution:

Note restrictions: $a \neq 3; a \neq -4$.

$$\begin{aligned} \frac{a+2}{a-3} &= \frac{a+8}{a+4} \\ \frac{(a+2)(a+4)}{(a-3)(a+4)} &= \frac{(a+8)(a-3)}{(a-3)(a+4)} \\ (a+2)(a+4) &= (a+8)(a-3) \\ a^2 + 2a + 4a + (2)(4) &= a^2 - 3a + 8a + (-3)(8) \\ a^2 + 6a + 8 &= a^2 + 5a - 24 \\ 6a + 8 &= 5a - 24 \\ 6a - (5a) &= -24 - (8) \\ a &= \frac{-32}{1} \\ &= -32 \end{aligned}$$

h) $\dfrac{n+3}{n-2} = \dfrac{n-1}{n-7}$
Solution:
Note restrictions: $n \neq 2; n \neq 7$.

$$\begin{aligned}
\frac{n+3}{n-2} &= \frac{n-1}{n-7} \\
\frac{(n+3)(n-7)}{(n-2)(n-7)} &= \frac{(n-1)(n-2)}{(n-7)(n-2)} \\
(n+3)(n-7) &= (n-1)(n-2) \\
n^2 + 3n - 7n + (3)(-7) &= n^2 - 2n - 1n + (-2)(-1) \\
n^2 - 4n - 21 &= n^2 - 3n + 2 \\
-4n - 21 &= -3n + 2 \\
-4n - (-3n) &= 2 - (-21) \\
n &= \frac{23}{-1} \\
&= -23
\end{aligned}$$

i) $x^2 - 3x + 2 = 0$
Solution:

$$\begin{aligned}
x^2 - 3x + 2 &= 0 \\
(x-2)(x-1) &= \\
x &= 2 \text{ or } x = 1
\end{aligned}$$

j) $y^2 + y = 6$
Solution:

$$\begin{aligned}
y^2 + y &= 6 \\
y^2 + y - 6 &= 0 \\
(y+3)(y-2) &= 0 \\
y &= -3 \text{ or } y = 2
\end{aligned}$$

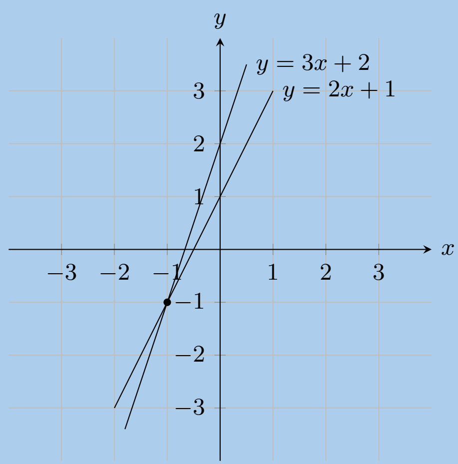
k) $0 = 2x^2 - 5x - 18$
Solution:

$$\begin{aligned}
0 &= 2x^2 - 5x - 18 \\
0 &= (2x+9)(x-2) \\
2x + 9 = 0 &\text{ or } x - 2 = 0 \\
x = -\frac{9}{2} &\text{ or } x = 2
\end{aligned}$$

l) $(d+4)(d-3) - d = (3d-2)^2 - 8d(d-1)$
Solution:

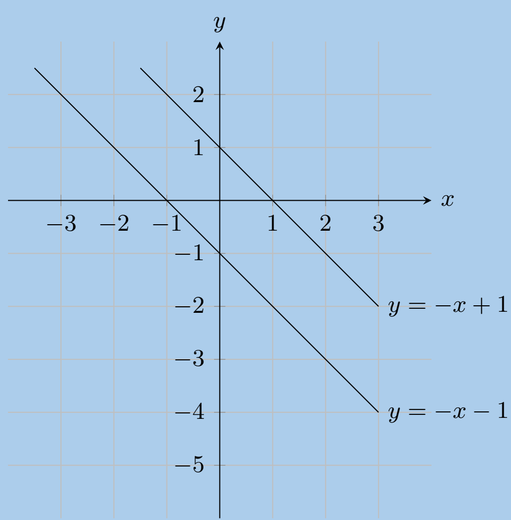
$$\begin{aligned}
(d+4)(d-3) - d &= (3d-2)^2 - 8d(d-1) \\
d^2 + d - 12 - d &= 9d^2 - 12d + 4 - 8d^2 + 8d \\
d^2 - 12 &= d^2 - 4d + 4 \\
4d &= 16 \\
d &= 4
\end{aligned}$$

3. Look at the graph below:



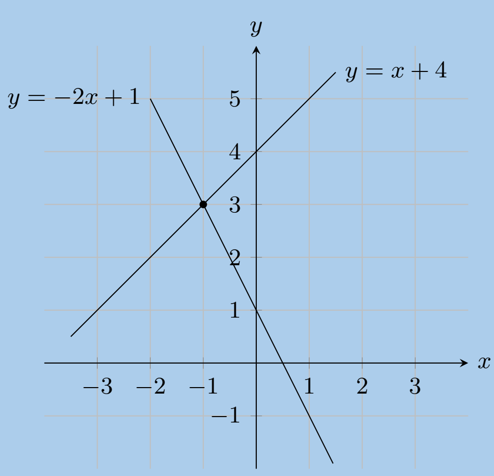
Solve the equations $y = 3x + 2$ and $y = 2x + 1$ simultaneously.
Solution:
From the graph we can see that the lines intersect at $x = -1$ and $y = -1$

4. Look at the graph below:



Solve the equations $y = -x + 1$ and $y = -x - 1$ simultaneously.
Solution:
The lines are parallel therefore there is no solution to x and y.

5. Look at the graph below:



Solve the equations $y = x + 4$ and $y = -2x + 1$ simultaneously.
Solution:
From the graph we can see that the lines intersect at $x = -1$ and $y = 3$

6. Solve the following simultaneous equations:
   a) $7x + 3y = 13$ and $2x - 3y = -4$
   Solution:
   Add the two equations to remove the y term and solve for x:

$$\begin{array}{rrcr} & 7x + 3y & = & 13 \\ + & (2x - 3y & = & -4) \\ \hline & 9x & = & 9 \end{array}$$

   Therefore $x = 1$.
   Substitute value of x into second equation:

$$\begin{aligned} 2x - 3y &= -4 \\ 2(1) - 3y &= -4 \\ 3y &= 6 \\ y &= 2 \end{aligned}$$

   Therefore $x = 1$ and $y = 2$.
   b) $10 = 2x + y$ and $y = x - 2$
   Solution:
   Substitute value of y into first equation:

$$\begin{aligned} 10 &= 2x + x - 2 \\ 10 &= 3x - 2 \\ 12 &= 3x \\ x &= 4 \end{aligned}$$

   Substitute value of x back into second equation:

$$\begin{aligned} y &= x - 2 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

   Therefore $x = 4$ and $y = 2$.
   c) $7x - 41 = 3y$ and $17 = 3x - y$
   Solution:
   Make y the subject of the first equation:

$$\begin{aligned} 17 &= 3x - y \\ y &= 3x - 17 \end{aligned}$$

   Substitute value of y into first equation:

$$\begin{aligned} 7x - 41 &= 3y \\ 7x - 41 &= 3(3x - 17) \\ 7x - 41 &= 9x - 51 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

   Substitute value of x back into second equation:

$$\begin{aligned} y &= 3x - 17 \\ y &= 3(5) - 17 \\ &= -2 \end{aligned}$$

Therefore $x = 5$ and $y = -2$.

d) $2x - 4y = 32$ and $7x + 2y = 32$

Solution:

Make x the subject of the first equation:

$$\begin{aligned} 2x - 4y &= 32 \\ 2x &= 32 + 4y \\ x &= \frac{32 + 4y}{2} \end{aligned}$$

Substitute value of x into second equation:

$$\begin{aligned} 7x + 2y &= 32 \\ 7\left(\frac{32 + 4y}{2}\right) + 2y &= 32 \\ 7(32 + 4y) + 2(2)y &= 32(2) \\ 224 + 28y + 4y &= 64 \\ 32y &= -160 \\ \therefore y &= -5 \end{aligned}$$

Substitute value of y back into first equation:

$$\begin{aligned} x &= \frac{32 + 4y}{2} \\ x &= \frac{32 + 4(-5)}{2} \\ x &= 6 \end{aligned}$$

Therefore $x = 6$ and $y = -5$.

e) $7x + 6y = -18$ and $4x + 12y = 24$

Solution:

Multiply the first equation by 2 so that the coefficient of y is the same as the second equation:

$$\begin{aligned} 7x + 6y &= -18 \\ 7(2)x + 6(2)y &= -18(2) \\ 14x + 12y &= -36 \end{aligned}$$

Subtract the second equation from the first equation:

$$\begin{array}{rrcr} & 14x + 12y & = & -36 \\ - & (4x + 12y & = & 24) \\ \hline & 10x & = & -60 \end{array}$$

Solve for x:

$$\begin{aligned} \therefore x &= \frac{-60}{10} \\ &= -6 \end{aligned}$$

Substitute the value of x into the first equation and solve for y:

$$\begin{aligned} 7x + 6y &= -18 \\ 7(-6) + 6y &= -18 \\ \therefore y &= \frac{-18 - 7(-6)}{6} \\ &= 4 \end{aligned}$$

Therefore $x = -6$ and $y = 4$.

f) $3x - 4y = -15$ and $12x + 5y = 66$

Solution:

Multiply the first equation by 4 so that the coefficient of x is the same as the second equation:

$$\begin{aligned} 3x - 4y &= -15 \\ 3(4)x - 4(4)y &= -15(4) \\ 12x - 16y &= -60 \end{aligned}$$

Subtract the second equation from the first equation:

$$\begin{array}{rrcr} & 12x - 16y & = & -60 \\ - & (12x + 5y & = & 66) \\ \hline & -21y & = & -126 \end{array}$$

Solve for y:

$$\begin{aligned} \therefore y &= \frac{-126}{-21} \\ &= 6 \end{aligned}$$

Substitute the value of y into the first equation and solve for x:

$$\begin{aligned} 3x - 4y &= -15 \\ 3x - 4(6) &= -15 \\ x &= \frac{-15 + 4(6)}{3} \\ &= 3 \end{aligned}$$

Therefore $x = 3$ and $y = 6$.

g) $x - 3y = -22$ and $5x + 2y = -25$

Solution:

Write the first equation in terms of x:

$$\begin{aligned} x - 3y &= -22 \\ x &= 3y - 22 \end{aligned}$$

Substitute the value of x into the second equation:

$$\begin{aligned} 5x + 2y &= -25 \\ 5(3y - 22) + 2y &= -25 \\ 15y - 110 + 2y &= -25 \\ 17y &= 85 \\ y &= 5 \end{aligned}$$

Substitute the value of y into the first equation and solve for x:

$$\begin{aligned} x - 3y &= -22 \\ x - 3(5) &= -22 \\ x &= -22 + 15 \\ &= -7 \end{aligned}$$

Therefore $x = -7$ and $y = 5$.

h) $3x + 2y = 46$ and $15x + 5y = 220$

Solution:

Make y the subject of the second equation:

$$\begin{aligned} 15x + 5y &= 220 \\ 3x + y &= 44 \\ y &= 44 - 3x \end{aligned}$$

Substitute the value of y into the first equation:

$$\begin{aligned} 3x + 2y &= 46 \\ 3x + 2(44 - 3x) &= 46 \\ 3x + 88 - 6x &= 46 \\ 42 &= 3x \\ x &= 14 \end{aligned}$$

Substitute the value of x into the second equation:

$$\begin{aligned} 3x + y &= 44 \\ 3(14) + y &= 44 \\ y &= 44 - 42 \\ &= 2 \end{aligned}$$

Therefore $x = 14$ and $y = 2$.

i) $6x + 3y = -63$ and $24x + 4y = -212$

Solution:

Multiply the first equation by 4 so that the coefficient of x is the same as the second equation:

$$\begin{aligned} 6x + 3y &= -63 \\ 6(4)x - 3(4)y &= -63(4) \\ 24x + 12y &= -252 \end{aligned}$$

Subtract the second equation from the first equation:

$$\begin{array}{rrcr} & 24x + 12y & = & -252 \\ - & (24x + 4y & = & -212) \\ \hline & 8y & = & -40 \end{array}$$

Solve for y:

$$\begin{aligned} \therefore y &= \frac{-40}{8} \\ &= -5 \end{aligned}$$

Substitute the value of y into the first equation and solve for x:

$$\begin{aligned} 6x + 3y &= -63 \\ 6x - 3(-5) &= -63 \\ x &= \frac{-63 + 15}{6} \\ &= -8 \end{aligned}$$

Therefore $x = -8$ and $y = -5$.

j) $5x - 6y = 11$ and $25x - 3y = 28$

Solution:

Multiply the first equation by 5 so that the coefficient of x is the same as the second equation:

$$\begin{aligned} 5x - 6y &= 11 \\ 5(5)x - 6(5)y &= 11(5) \\ 25x - 30y &= 55 \end{aligned}$$

Subtract the second equation from the first equation:

$$\begin{array}{rrcl} & 25x - 30y & = & 55 \\ - & (25x - 3y & = & 28) \\ \hline & -27y & = & 27 \end{array}$$

Solve for y:

$$\begin{aligned} \therefore y &= \frac{27}{-27} \\ &= -1 \end{aligned}$$

Substitute the value of y into the first equation and solve for x:

$$\begin{aligned} 5x - 6y &= 11 \\ 5x - 6(-1) &= 11 \\ x &= \frac{11-6}{5} \\ &= 1 \end{aligned}$$

Therefore $x = 1$ and $y = -1$.

k) $-9x + 3y = 4$ and $2x + 2y = 6$

Solution:

Make x the subject of the second equation:

$$\begin{aligned} 2x + 2y &= 6 \\ x &= 3 - y \end{aligned}$$

Substitute the value of x into the first equation:

$$\begin{aligned} -9x + 3y &= 4 \\ -9(3-y) + 3y &= 4 \\ -27 + 9y + 3y &= 4 \\ 12y &= 31 \\ y &= \frac{31}{12} \end{aligned}$$

Substitute the value of y into the second equation and solve for x:

$$\begin{aligned} x &= 3 - y \\ &= 3 - \frac{31}{12} \\ &= \frac{36-31}{12} \\ &= \frac{5}{12} \end{aligned}$$

Therefore $x = \frac{5}{12}$ and $y = \frac{31}{12}$.

l) $3x - 7y = -10$ and $10x + 2y = -6$

Solution:

Make y the subject of the second equation:

$$\begin{aligned} 10x + 2y &= -6 \\ 5x + y &= -3 \\ y &= -3 - 5x \end{aligned}$$

Substitute the value of y into the first equation:

$$\begin{aligned}
3x - 7y &= -10\\
3x - 7(-3 - 5x) &= -10\\
3x + 21 + 35x &= -10\\
38x &= -10 - 21\\
38x &= -31\\
x &= \frac{-31}{38}
\end{aligned}$$

Substitute the value of x into the second equation and solve for y:

$$\begin{aligned}
5x + y &= -3\\
5\left(\frac{-31}{38}\right) + y &= -3\\
y &= \frac{-114 + 155}{38}\\
&= \frac{41}{38}
\end{aligned}$$

Therefore $x = -\frac{31}{38}$ and $y = \frac{41}{38}$.

m) $2y = x + 8$ and $4y = 2x - 44$

Solution:

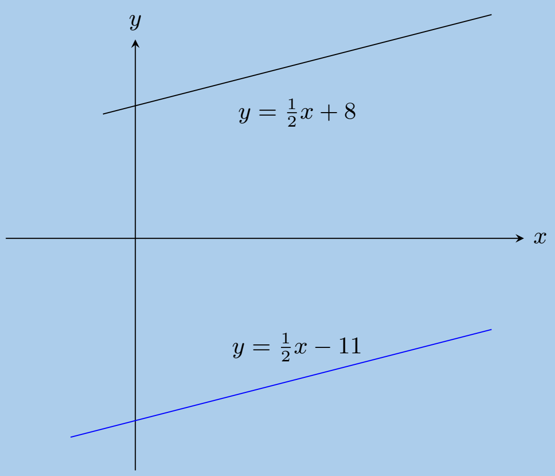
We note that the second equation has a common factor of 2:

$$\begin{aligned}
4y &= 2x - 44\\
2y &= x - 22
\end{aligned}$$

Now we can subtract the second equation from the first:

$$\begin{array}{rrcr}
 & 2y & = & x + 8\\
- & 2y & = & x - 22\\
\hline
 & 0 & = & -22
\end{array}$$

There is no solution for this system of equations. We can see this if we graph the two equations:



From the graph we see that the lines have the same gradient and do not intersect.
Therefore there is no solution.

n) $2a(a - 1) - 4 + a - b = 0$ and $2a^2 - a = b + 4$

Solution:

Look at the first equation

$$\begin{aligned}
2a(a - 1) - 4 + a - b &= 0\\
2a^2 - 2a - 4 + a - b &= 0\\
2a^2 - a &= b + 4
\end{aligned}$$

Note that this is the same as the second equation
a and b can be any real number except for 0.

o) $y = (x-2)^2$ and $x(x+3) - y = 3x + 4(x-1)$

Solution:

Look at the second equation:

$$\begin{aligned} x(x+3) - y &= 3x + 4(x-1) \\ x^2 + 3x - y &= 3x + 4x - 4 \\ x^2 - 4x + 4 &= y \\ y &= (x-2)^2 \end{aligned}$$

Note that this is the same as the first equation.
x and y can be any real number except for 0.

p) $\frac{x+1}{y} = 7$ and $\frac{x}{y+1} = 6$

Solution:

Note that $y \neq 0$ and $y \neq -1$

$$\begin{aligned} \frac{x+1}{y} &= 7 \qquad \text{equation 1} \\ \frac{x}{y+1} &= 6 \qquad \text{equation 2} \end{aligned}$$

Make x the subject of equation 1:

$$\begin{aligned} \frac{x+1}{y} &= 7 \\ x+1 &= 7y \\ x &= 7y - 1 \qquad \text{equation 3} \end{aligned}$$

Make x the subject of equation 2:

$$\begin{aligned} \frac{x}{y+1} &= 6 \\ x &= 6(y+1) \\ &= 6y + 6 \qquad \text{equation 4} \end{aligned}$$

Substitute equation 3 into equation 4:

$$\begin{aligned} 6y + 6 &= 7y - 1 \\ 6 + 1 &= 7y - 6y \\ y &= 7 \end{aligned}$$

Substitute the value of y into equation 3:

$$\begin{aligned} x &= 7(7) - 1 \\ &= 48 \end{aligned}$$

Therefore $x = 48$ and $y = 7$

q) $(x+3)^2 + (y-4)^2 = 0$

Solution:

Note that $(x+3)^2$ and $(y-4)^2$ are both greater than or equal to zero therefore for the equation to be true they must both equal zero.

$$(x+3)^2 = 0$$
$$x = -3$$

$$(y-4)^2 = 0$$
$$y = 4$$

$$\therefore x = -3 \text{ and } y = 4$$

7. Find the solutions to the following word problems:
   a) $\frac{7}{8}$ of a certain number is 5 more than $\frac{1}{3}$ of the number. Find the number.
   Solution:
   Let x be the number.

$$\begin{aligned} \frac{7}{8}x &= \frac{1}{3}x + 5 \\ 21x &= 8x + 120 \\ 13x &= 120 \\ x &= \frac{120}{13} \end{aligned}$$

   The number is $\frac{120}{13}$.
   b) Three rulers and two pens have a total cost of R 21,00. One ruler and one pen have a total cost of R 8,00. How much does a ruler cost and how much does a pen cost?
   Solution:
   Let the price of a ruler be r and the price of a pen be p.

$$\begin{aligned} 3r + 2p &= 21 \\ r + p &= 8 \end{aligned}$$

   From the second equation: $r = 8 - p$
   Substitute the value of r into the first equation:

$$\begin{aligned} 3(8-p) + 2p &= 21 \\ 24 - 3p + 2p &= 21 \\ p &= 3 \end{aligned}$$

   Substitute the value of p into the second equation:

$$\begin{aligned} r + 3 &= 8 \\ r &= 5 \end{aligned}$$

   Therefore each ruler costs R 5 and each pen costs R 3.
   c) A group of friends is buying lunch. Here are some facts about their lunch:
   - a hotdog costs R 6 more than a milkshake
   - the group buys 3 hotdogs and 2 milkshakes
   - the total cost for the lunch is R 143

   Determine the individual prices for the lunch items.
   Solution:
   Let the price of a hotdog be h and the price of a milkshake be m. From the given information we get:

$$\begin{aligned} h &= m + 6 \\ 3h + 2m &= 143 \end{aligned}$$

   Substitute the first equation into the second equation:

$$
\begin{aligned}
3h + 2m &= 143 \\
3(m+6) + 2m &= 143 \\
3m + 6(3) + 2m &= 143 \\
5m &= 143 - 18 \\
\therefore m &= \frac{125}{5} \\
&= 25
\end{aligned}
$$

Substitute the value of m into the first equation:

$$
\begin{aligned}
h &= m + 6 \\
&= 25 + 6 \\
&= 31
\end{aligned}
$$

The price of the hotdog is R 31 while a milkshake costs R 25.

d) Lefu and Monique are friends. Monique takes Lefu's business studies test paper and will not tell him what his mark is. She knows that Lefu dislikes word problems so she decides to tease him. Monique says: "I have 12 marks more than you do and the sum of both our marks is equal to 166. What are our marks?"

Solution:

Let Lefu's mark be l and let Monique's mark be m. Then

$$
\begin{aligned}
m &= l + 12 \\
l + m &= 166
\end{aligned}
$$

Substitute the first equation into the second equation and solve:

$$
\begin{aligned}
l + m &= 166 \\
l + (l + 12) &= 166 \\
2l &= 166 - 12 \\
\therefore l &= \frac{154}{2} \\
&= 77
\end{aligned}
$$

Substituting this value back into the first equation gives:

$$
\begin{aligned}
m &= l + 12 \\
&= 77 + 12 \\
&= 89
\end{aligned}
$$

The learners obtained the following marks: Lefu has 77 marks and Monique has 89 marks.

e) A man runs to the bus stop and back in 15 minutes. His speed on the way to the bus stop is 5 km·h^{-1} and his speed on the way back is 4 km·h^{-1}. Find the distance to the bus stop.

Solution:

Let D be the distance to the bus stop.

Speed $s_1 = 5$ km·h^{-1} and $s_2 = 4$ km·h^{-1}.

Distance is given by speed times time. The man runs the same distance to the bus stop as he does from the bus stop. Therefore:

$$
\begin{aligned}
D &= s \times t \\
D &= 5t_1 = 4t_2
\end{aligned}
$$

He takes a total of 15 minutes to run there and back so the total time is $t_1 + t_2 = 15$. However the speeds are given in kilometers per hour and so we must convert the time to hours. Therefore $t_1 + t_2 = 0{,}25$.

Next we note that $t_1 = \frac{D}{5}$ and $t_2 = \frac{D}{4}$.

Therefore:

$$\begin{aligned} \frac{D}{5} + \frac{D}{4} &= 0{,}25 \\ 4D + 5D &= 0{,}25(20) \\ 9D &= 5 \\ D &= \frac{5}{9} \end{aligned}$$

The bus stop is 0,56 km away.

f) Two trucks are travelling towards each other from factories that are 175 km apart. One truck is travelling at 82 km·h^{-1} and the other truck at 93 km·h^{-1}. If both trucks started their journey at the same time, how long will they take to pass each other?

Solution:

Notice that the sum of the distances for the two trucks must be equal to the total distance when the trucks meet: $D_1 + D_2 = d_{\text{total}} \longrightarrow D_1 + D_2 = 175$ km.

This question is about distances, speeds and times. The equation connecting these values is

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{- or -} \quad \text{distance} = \text{speed} \times \text{time}$$

You want to know the amount of time needed for the trucks to meet - let the time taken be t. Then you can write an expression for the distance each of the trucks travels:

$$\begin{aligned} \text{For truck 1:} \quad D_1 &= s_1 t \\ &= 82t \\ \text{For truck 2:} \quad D_2 &= s_2 t \\ &= 93t \end{aligned}$$

Now you have three different equations: you must solve them simultaneously; substitution is the easiest choice.

$$\begin{aligned} D_1 + D_2 &= 175 \\ (82t) + (93t) &= 175 \\ 175t &= 175 \\ \therefore t &= \frac{175}{175} \\ &= 1 \end{aligned}$$

The trucks will meet after 1 hours.

g) Zanele and Piet skate towards each other on a straight path. They set off 20 km apart. Zanele skates at 15 km·h^{-1} and Piet at 10 km·h^{-1}. How far will Piet have skated when they reach each other?

Solution:

Let x be the distance that Zanele skates and $20 - x$ the distance Piet skates.

Next we note the following information:

- Zanele skates x km at a speed of 15 km·h^{-1} in a time of $\frac{x}{15}$
- Piet skates $20 - x$ km at a speed of 10 km·h^{-1} in a time of $\frac{20-x}{10}$

$$\begin{aligned} \frac{x}{15} &= \frac{20 - x}{10} \\ 10x &= 15(20 - x) \\ 10x &= 300 - 15x \\ 25x &= 300 \\ x &= 12 \end{aligned}$$

Zanele will have skated 12 km and Piet will have skated 8 km when they reach other.

h) When the price of chocolates is increased by R 10, we can buy five fewer chocolates for R 300. What was the price of each chocolate before the price was increased?

Solution:

Let x be the original price of chocolates. The new price of x chocolates is R 300.

$$
\begin{aligned}
(x+10)\left(\frac{300}{x}-5\right) &= 300\\
300-5x+\frac{3000}{x}-50 &= 300\\
-5x+\frac{3000}{x}-50 &= 0\\
-5x^2+3000-50x &= 0\\
x^2+10x-600 &= 0\\
(x-20)(x+30) &= 0\\
x &= 20 \text{ or } x=-30
\end{aligned}
$$

Since price has to be positive the chocolates used to cost R 20.

i) A teacher bought R 11 300 worth of textbooks. The text books were for Science and Mathematics with each of them being sold at R 100 per book and R 125 per book respectively. If the teacher bought 97 books in total, how many Science books did she buy?

Solution:

$$
\begin{aligned}
x+y &= 97 \quad (1)\\
100x+125y &= 11\ 300 \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
&\text{look at } (1)\\
x &= 97-y \quad (3)\\
&(3) \text{ into } (2)\\
100(97-y)+125y &= 11\ 300\\
-100y+125y &= 11\ 300-9700\\
25y &= 1600\\
y &= 64\\
x &= 97-y\\
x &= 33
\end{aligned}
$$

She bought 33 science books.

j) Thom's mom bought R 91,50 worth of easter eggs. The easter eggs came in 3 different colours blue, green and yellow. The blue ones cost R 2 each, green ones R 1,50 each and yellow ones R 1 each. She bought three times as many yellow eggs as the green ones and 72 eggs in total. How many blue eggs did she buy?

Solution:

$$
\begin{aligned}
x+y+z &= 72\ (1)\\
2x+1{,}5y+z &= 91{,}5\ (2)\\
z &= 3y\ (3)\\
&(3) \text{ into } (1)\\
x+y+3y &= 72\\
x &= 72-4y\ (4)\\
&(3) \text{ into } (2)\\
2x+1{,}5y+3y &= 91{,}5\\
2x &= 91{,}5-4{,}5y\ (5)\\
&(4) \text{ into } (5)\\
2(72-4y) &= 91{,}5-4{,}5y\\
144-8y &= 91{,}5-4{,}5y\\
52{,}5 &= 3{,}5y\\
y &= 15\\
x &= 72-4(15)=12\\
\therefore\ &\text{Thom's mom bought 12 blue easter eggs}
\end{aligned}
$$

k) Two equivalent fractions both have their numerator as one. The denominator of one fraction is the sum of two and a number, while the other fraction is twice the number less 3. What is the number?

Solution:

$$\frac{1}{x+2} = \frac{1}{2x-3}$$
$$\text{Note } x \neq -2 \text{ and } x \neq \frac{3}{2}$$
$$x + 2 = 2x - 3$$
$$x = 5$$

8. Consider the following literal equations:

   a) Solve for x: $a - bx = c$

   Solution:

$$\begin{aligned} a - bx &= c \\ -bx &= c - a \\ -\frac{1}{b}(-bx) &= (c-a)\left(-\frac{1}{b}\right) \\ \therefore\ x &= \frac{a-c}{b}, b \neq 0 \end{aligned}$$

   b) Solve for I: $P = VI$.

   Solution:

$$\begin{aligned} P &= VI \\ \frac{P}{V} &= I \end{aligned}$$

   Note restriction: $V \neq 0$.

   c) Make m the subject of the formula: $E = mc^2$.

   Solution:

$$\begin{aligned} E &= mc^2 \\ \frac{E}{c^2} &= m \end{aligned}$$

   Note restriction: $c \neq 0$.

   d) Solve for t: $v = u + at$.

   Solution:

$$\begin{aligned} v &= u + at \\ v - u &= at \\ \frac{v-u}{a} &= t \end{aligned}$$

   Note restriction: $a \neq 0$.

   e) Make f the subject of the formula: $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$.

   Solution:

$$\begin{aligned} \frac{1}{u} + \frac{1}{v} &= \frac{1}{f} \\ \frac{f}{u} + \frac{f}{v} &= 1 \\ \frac{fv}{uv} + \frac{fu}{uv} &= 1 \\ fv + fu &= uv \\ f(v+u) &= uv \\ f &= \frac{uv}{v+u} \end{aligned}$$

Note restriction: $v \neq -u$.

f) Solve for y: $m = \dfrac{y-c}{x}$.

Solution:

$$\begin{aligned} m &= \frac{y-c}{x} \\ mx &= y - c \\ mx + c &= y \end{aligned}$$

g) Solve for x in: $ax - 4a + ab = 4b - bx - b^2 + 4c - cx - bc$

Solution:

$$\begin{aligned} ax - 4a + ab &= 4b - bx - b^2 + 4c - cx - bc \\ ax - 4a + ab + bx - 4b + b^2 + cx - 4c + bc &= 0 \\ a(x - 4 + b) + b(x - 4 + b) + c(x - 4 + b) &= 0 \\ (a + b + c)(x - 4 + b) &= 0 \\ \text{If } (a+b+c) \neq 0 \text{ then } x &= 4 - b \text{ If } a + b + c = 0, x \in \mathbb{R} \end{aligned}$$

h) Solve for r in $S = \dfrac{a}{1-r}$ if $a = 0{,}4$ and $S = 3$

Solution:

$$\begin{aligned} S &= \frac{a}{1-r} \\ 1 - r&\frac{a}{S} \\ r &= 1 - \frac{a}{S} \\ r &= 1 - \frac{0{,}4}{3} \\ r &= \frac{13}{15} \end{aligned}$$

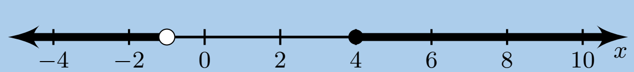
i) Solve for b in $I = \frac{1}{2}M(a^2 + b^2)$ if $a = 4$, $M = 8$, $I = 320$

Solution:

$$\begin{aligned} I &= \frac{1}{2}M(a^2 + b^2) \\ \frac{2I}{M} &= a^2 + b^2 \\ b^2 &= \frac{2I}{M} - a^2 \\ b &= \pm\sqrt{\frac{2I}{M} - a^2} \\ b &= \pm\sqrt{\frac{2(320)}{8} - 4^2} \\ b &= \pm\sqrt{80 - 16} \\ b &= \pm 8 \end{aligned}$$

9. Write down the inequality represented by the following:

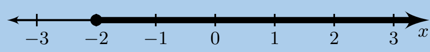
a)



Solution:

$x < -1$ and $x \geq 4; x \in \mathbb{R}$

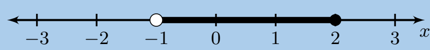
b)



Solution:

$x \geq -2; x \in \mathbb{R}$

c)



Solution:

$-1 < x \leq -2; x \in \mathbb{R}$

10. Solve for x and show your answer in interval notation

a) $-4x + 1 > -2(x - 15)$

Solution:

$$\begin{aligned} -4x + 1 &> -2(x - 15) \\ -4x + 1 &> -2x + 30 \\ -4x + 2x &> 30 - 1 \\ -2x &> 29 \\ \therefore x &< \frac{-29}{2} \end{aligned}$$

$$\left(-\infty; \frac{-29}{2}\right)$$

b) $\frac{x+2}{4} \leq \frac{-1(x+1)}{6}$

Solution:

$$\begin{aligned} \frac{x+2}{4} &\leq \frac{-1(x+1)}{6} \\ 6(x+2) &\leq -4(x+1) \\ 6x + 12 &\leq -4x - 4 \end{aligned}$$

Now solve. (Remember to flip the inequality symbol if you multiply or divide by a negative.)

$$\begin{aligned} 6x + 12 &\leq -4x - 4 \\ 6x + 4x &\leq -4 - 12 \\ 10x &\leq -16 \\ \therefore x &\leq \frac{-8}{5} \end{aligned}$$

$$\left(-\infty; \frac{-8}{5}\right]$$

c) $\frac{1}{4}x + \frac{2}{3}(x+1) \geq \frac{2}{5}x + 2$

Solution:

$$\begin{aligned} \frac{1}{4}x + \frac{2}{3}(x+1) &\geq \frac{2}{5}x + 2 \\ 15x + 40(x+1) &\geq 24x + 120 \\ 15x + 40x + 40 &\geq 24x + 120 \\ 15x + 40x - 24x &\geq 120 - 40 \\ 31x &\geq 80 \\ \therefore x &\geq \frac{80}{31} \end{aligned}$$

$$\left[\frac{80}{31}; \infty\right)$$

d) $3x - 3 > 14$ or $3x - 3 < -2$

Solution:

Solve the inequality:

$$\begin{array}{rclcrcl} 3x - 3 & > & 14 & \text{or} & 3x - 3 & < & -2 \\ 3x & > & 14 + 3 & \text{or} & 3x & < & -2 + 3 \\ x & > & \frac{14+3}{3} & \text{or} & x & < & \frac{-2+3}{3} \\ x & > & \frac{17}{3} & \text{or} & x & < & \frac{1}{3} \end{array}$$

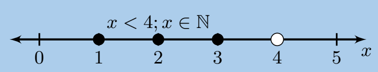
$$\left(-\infty; \frac{1}{3}\right) \cup \left(\frac{17}{3}; \infty\right)$$

11. Solve and represent your answer on a number line

a) $2x - 3 < \dfrac{3x - 2}{2},\ x \in \mathbb{N}$

Solution:

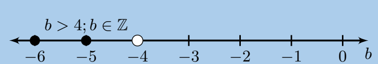
$$\begin{aligned} 2x - 3 &< \frac{3x - 2}{2} \\ 4x - 6 &< 3x - 2 \\ x &< 4 \end{aligned}$$



b) $3(1 - b) - 4 + b > 7 + b,\ b \in \mathbb{Z}$

Solution:

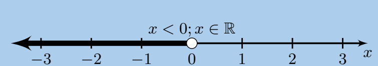
$$\begin{aligned} 3(1 - b) - 4 + b &> 7 + b \\ 3 - 3b - 4 + b &> 7 + b \\ -2b &> 8 \\ b &< -4 \end{aligned}$$



c) $1 - 5x > 4(x + 1) - 3,\ x \in \mathbb{R}$

Solution:

$$\begin{aligned} 1 - 5x &> 4(x + 1) - 3 \\ 1 - 5x &> 4x + 4 - 3 \\ -9x &> 0 \\ x &< 0 \end{aligned}$$



12. Solve for the unknown variable

a) $2 + 2\frac{1}{3}(x + 4) = \frac{1}{5}(3 - x) + \frac{1}{6}$

Solution:

$$\begin{aligned} 2 + 2\frac{1}{3}(x + 4) &= \frac{1}{5}(3 - x) + \frac{1}{6} \\ 2 + 2\frac{1}{3}x + 9\frac{1}{3} &= \frac{3}{5} - \frac{1}{5}x + \frac{1}{6} \\ 2\frac{1}{3}x + \frac{1}{5}x &= \frac{3}{5} + \frac{1}{6} - 2 - 9\frac{1}{3} \\ \frac{38}{15}x &= -\frac{317}{30} \\ x &= -\frac{317}{76} \end{aligned}$$

b) $36 - (x-4)^2 = 0$

Solution:

$$\begin{aligned} 36 - (x-4)^2 &= 0 \\ (6 + x - 4)(6 - (x-4)) &= 0 \\ (2+x)(10-x) &= 0 \\ \therefore x &= -2 \text{ or } x = 10 \end{aligned}$$

c) $64 - (a+3)^2 = 0$

Solution:

$$\begin{aligned} 64 - (a+3)^2 &= 0 \\ (8 - a - 3)(8 + a + 3) &= 0 \\ (5-a)(11+a) &= 0 \\ \therefore a &= 5 \text{ or } a = -11 \end{aligned}$$

d) $\frac{1}{2}x - \frac{2}{x} = 0$

Solution:

Note $x \neq 0$

$$\begin{aligned} \frac{1}{2}x - \frac{2}{x} &= 0 \\ x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ \therefore x &= 2 \text{ or } x = -2 \end{aligned}$$

e) $a - 3 = 2\left(\frac{6}{a} + 1\right)$

Solution:

Note $a \neq 0$

$$\begin{aligned} a - 3 &= 2\left(\frac{6}{a} - 1\right) \\ a - 3 &= \frac{12}{a} - 2 \\ a^2 - 3a &= 12 - 2a \\ a^2 - a - 12 &= 0 \\ (a-4)(a+3) &= 0 \\ \therefore a &= 4 \text{ or } a = -3 \end{aligned}$$

f) $a - \frac{6}{a} = -1$

Solution:

Note $a \neq 0$

$$\begin{aligned} a - \frac{6}{a} &= -1 \\ a^2 - 6 &= -a \\ a^2 + a - 6 &= 0 \\ (a-2)(a+3) &= 0 \\ \therefore a &= 2 \text{ or } a = -3 \end{aligned}$$

g) $(a+6)^2 - 5(a+6) - 24 = 0$

Solution:

$$
\begin{aligned}
(a+6)^2 - 5(a+6) - 24 &= 0 \\
((a+6)-8)((a+6)+3) &= 0 \\
(a-2)(a+9) &= 0 \\
\therefore a &= 2 \text{ or } a = -9
\end{aligned}
$$

h) $a^4 - 4a^2 + 3 = 0$
Solution:

$$
\begin{aligned}
a^4 - 4a^2 + 3 &= 0 \\
(a^2-3)(a^2-1) &= 0 \\
(a-\sqrt{3})(a+\sqrt{3})(a-1)(a+1) &= 0 \\
\therefore b &= \pm\sqrt{3} \text{ or } b = \pm 1
\end{aligned}
$$

i) $9y^4 - 13y^2 + 4 = 0$
Solution:

$$
\begin{aligned}
9y^4 - 13y^2 + 4 &= 0 \\
(9y^2-4)(y^2-1) &= 0 \\
(3y-2)(3y+2)(y-1)(y+1) &= 0 \\
\therefore y &= \pm\frac{2}{3} \text{ or } y = \pm 1
\end{aligned}
$$

j) $\dfrac{(b+1)^2 - 16}{b+5} = 1$
Solution:
Note $b \neq -5$

$$
\begin{aligned}
\frac{(b+1)^2 - 16}{b+5} &= 1 \\
b^2 + 2b + 1 - 16 &= b + 5 \\
b^2 + b - 20 &= 0 \\
(b-4)(b+5) &= 0 \\
\therefore b &= 4
\end{aligned}
$$

k) $\dfrac{a^2 + 8a + 7}{a+7} = 2$
Solution:
Note $a \neq -7$

$$
\begin{aligned}
\frac{a^2 + 8a + 7}{a+7} &= 2 \\
a^2 + 8a + 7 &= 2a + 14 \\
a^2 + 6a - 7 &= 0 \\
(a-1)(a+7) &= 0 \\
\therefore a &= 1
\end{aligned}
$$

l) $5x + 2 \leq 4(2x-1)$
Solution:

$$
\begin{aligned}
5x + 2 &\leq 4(2x-1) \\
5x + 2 &\leq 8x - 4 \\
-3x &\leq -6 \\
x &\geq 2
\end{aligned}
$$

m) $\frac{4x-2}{6} > 2x+1$

Solution:

$$\begin{aligned} \frac{4x-2}{6} &> 2x+1 \\ 4x-2 &> 12x+6 \\ -8x &> 8 \\ x &< -1 \end{aligned}$$

n) $\frac{x}{3} - 14 > 14 - \frac{x}{7}$

Solution:

$$\begin{aligned} \frac{x}{3} - 14 &> 14 - \frac{x}{7} \\ 7x - 294 &> 294 - 3x \\ 10x &> 588 \\ x &> \frac{588}{10} \end{aligned}$$

o) $\frac{1-a}{2} - \frac{2-a}{3} \geq 1$

Solution:

$$\begin{aligned} \frac{1-a}{2} - \frac{2-a}{3} &\geq 1 \\ 3-3a-4+2a &\geq 6 \\ -a &\geq 7 \\ a &\leq -7 \end{aligned}$$

p) $-5 \leq 2k+1 < 5$

Solution:

$$\begin{array}{ccccc} -5 & \leq & 2k+1 & < & 5 \\ -6 & \leq & 2k & < & 4 \\ -3 & \leq & k & < & 2 \end{array}$$

q) $x - 1 = \frac{42}{x}$

Solution:

Note that $x \neq 0$.

$$\begin{aligned} x - 1 &= \frac{42}{x} \\ x^2 - x &= 42 \\ x^2 - x - 42 &= 0 \\ (x-7)(x+6) &= 0 \\ x &= 7 \text{ or } x = -6 \end{aligned}$$

r) $(x+1)^2 = (x+1)(2x+3)$

Solution:

$$\begin{aligned} (x+1)^2 &= (x+1)(2x+3) \\ x^2+2x+1 &= 2x^2+3x+2x+3 \\ 0 &= 2x^2-x^2+5x-2x+3-1 \\ x^2+3x+2 &= 0 \\ (x+1)(x+2) &= 0 \end{aligned}$$

$$\therefore x = -1 \text{ or } x = -2$$

s) $3ax + 2a - ax = 5ax - 6a$
Solution:

$$
\begin{aligned}
3ax + 2a - ax &= 5ax - 6a \\
0 &= 5ax - 3ax - 6a - 2a + ax \\
3ax - 8a &= 0 \\
a(3x - 8) &= 0
\end{aligned}
$$

$$\therefore x = \frac{3}{8} \text{ if } a \neq 0, x \in \mathbb{R}$$

Note that you cannot simply divide through by a, because it is not stated that $a \neq 0$, so the value of a may be 0 and you cannot divide by zero.

t) $\frac{ax}{b} - \frac{bx}{a} = \frac{a}{b} + 1$
Solution:
Note that in this solution a and b are denominators, this means that they are no equal to zero.

$$
\begin{aligned}
\frac{ax}{b} - \frac{bx}{a} &= \frac{a}{b} + 1 \\
(ab)\frac{ax}{b} - (ab)\frac{bx}{a} &= (ab)\frac{a}{b} + (ab)1 \\
a^2x - b^2x &= a^2 + ab \\
x(a^2 - b^2) &= a(a + b) \\
x(a + b)(a - b) &= a(a + b) \\
x &= \frac{a(a + b)}{(a + b)(a - b)} \\
\therefore x &= \frac{a}{(a - b)} \text{ for } a, b \neq 0 \text{ and } a \neq b
\end{aligned}
$$

u) $3x^2 - xy - 2y^2 = 0$
Solution:

$$
\begin{aligned}
3x^2 - xy - 2y^2 &= 0 \\
(3x + 2y)(x - y) &= 0 \\
\therefore x &= -\frac{2}{3}y \text{ or } x = y
\end{aligned}
$$

v) $x(2x + 1) = 1$
Solution:

$$
\begin{aligned}
x(2x + 1) &= 1 \\
2x^2 + x - 1 &= 0 \\
(2x - 1)(x + 1) &= 0 \\
\therefore x &= \frac{1}{2}y \text{ or } x = -1
\end{aligned}
$$

w) $\frac{2x - 5}{(x + 2)(x - 4)} = \frac{1}{2(x - 4)}$
Solution:
Note that $x \neq 4$ and $x \neq -2$, because the numerator cannot be zero

$$\frac{2x-5}{(x+2)(x-4)} = \frac{1}{2(x-4)}$$
$$2(2x-5) = x+2$$
$$4x-10 = x+2$$
$$3x = 12$$
$$x = 4$$
$$\therefore \text{ no solution since } x \neq 4$$

x) $x^2 + 1 = 0$

Solution:

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \sqrt{-1}$$
$$\therefore \text{ no solution since } \sqrt{-1} \text{ is not a real number}$$

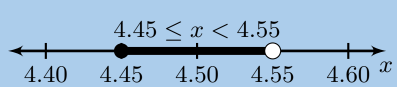
y) $\frac{x+4}{3} - 2 > \frac{x-3}{2} - \frac{x+1}{4}$

Solution:

$$\frac{x+4}{3} - 2 > \frac{x-3}{2} - \frac{x+1}{4}$$
$$12(\frac{x+4}{3}) - (12)(2) > 12(\frac{x-3}{2}) - 12(\frac{x+1}{4})$$
$$4x+16-24 > 6x-18-3x-3$$
$$4x+3x-6x+16-24+18+3 > 0$$
$$x+13 > 0$$
$$x > -13$$
$$\therefore x > -13$$

13. After solving an equation, Luke gave his answer as 4,5 rounded to one decimal digit. Show on a number line the interval in which his solution lay.

Solution:



For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FHM	1b. 2FHN	1c. 2FHP	1d. 2FHQ	1e. 2FHR	1f. 2FHS	1g. 2FHT	1h. 2FHV
1i. 2FHW	1j. 2FHX	1k. 2FHY	1l. 2FHZ	1m. 2FJ2	2a. 2FJ3	2b. 2FJ4	2c. 2FJ5
2d. 2FJ6	2e. 2FJ7	2f. 2FJ8	2g. 2FJ9	2h. 2FJB	2i. 2FJC	2j. 2FJD	2k. 2FJF
2l. 2FJG	3. 2FJH	4. 2FJJ	5. 2FJK	6a. 2FJM	6b. 2FJN	6c. 2FJP	6d. 2FJQ
6e. 2FJR	6f. 2FJS	6g. 2FJT	6h. 2FJV	6i. 2FJW	6j. 2FJX	6k. 2FJY	6l. 2FJZ
6m. 2FK2	6n. 2FK3	6o. 2FK4	6p. 2FK5	6q. 2FK6	7a. 2FK7	7b. 2FK8	7c. 2FK9
7d. 2FKB	7e. 2FKC	7f. 2FKD	7g. 2FKF	7h. 2FKG	7i. 2FKH	7j. 2FKJ	7k. 2FKK
8a. 2FKM	8b. 2FKN	8c. 2FKP	8d. 2FKQ	8e. 2FKR	8f. 2FKS	8g. 2FKT	8h. 2FKV
8i. 2FKW	9a. 2FKX	9b. 2FKY	9c. 2FKZ	10a. 2FM2	10b. 2FM3	10c. 2FM4	10d. 2FM5
11a. 2FM6	11b. 2FM7	11c. 2FM8	12a. 2FM9	12b. 2FMB	12c. 2FMC	12d. 2FMD	12e. 2FMF
12f. 2FMG	12g. 2FMH	12h. 2FMJ	12i. 2FMK	12j. 2FMM	12k. 2FMN	12l. 2FMP	12m. 2FMQ
12n. 2FMR	12o. 2FMS	12p. 2FMT	12q. 2FMV	12r. 2FMW	12s. 2FMX	12t. 2FMY	12u. 2FMZ
12v. 2FN2	12w. 2FN3	12x. 2FN4	12y. 2FN5	13. 2FN6			

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CHAPTER 5

Trigonometry

5.1 Introduction

- Content covered in this chapter includes defining the trigonometric ratios and extending these definitions to any angle. Also covered is the definitions of the reciprocals of the trigonometric ratios. Both the trigonometric ratios and their reciprocals are solved for several special angles. In addition simple trigonometric equations are covered.
- Solving problems in two-dimensions using trigonometry is only covered later in the year and the content for this can be found in chapter 11.
- Similarity of triangles is fundamental to the trigonometric ratios
- Trigonometric ratios are independent of the lengths of the sides and instead depend only on the angles
- Doubling a ratio has a different effect from doubling an angle.
- Emphasise the value and importance of making sketches, where appropriate.
- Remind learners that angles in the Cartesian plane are always measured from the positive x-axis.
- When working with angles on the Cartesian plane remind learners to check that their answers are within the correct quadrant.
- Calculator skills are very important in this chapter. Methods for CASIO calculators are shown but practical demonstration may be required. For a SHARP calculator the keys are generally the same although the SHIFT key is now the 2ndF key.
- We will refer to sine, cosine, tangent, secant, cosecant and cotangent as trigonometric ratios rather than as trigonometric functions. Both these terms are correct though but for the nature of the content in this chapter the term ratio better captures the content and is likely to be more accessible to learners at this stage.

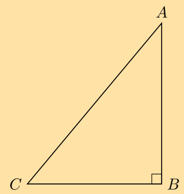
Fabumaths has some useful links and content for trigonometry.

5.2 Similarity of triangles

5.3 Defining the trigonometric ratios

Exercise 5 – 1:

1. Complete each of the following:



a) $\sin \hat{A} =$

Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\hat{A}$ is directly opposite the angle $\hat{A}$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\hat{A}$.

$$\sin \hat{A} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CB}{AC}$$

b) $\cos \hat{A} =$

Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\hat{A}$ is directly opposite the angle $\hat{A}$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\hat{A}$.

$$\cos \hat{A} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$

c) $\tan \hat{A} =$

Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\hat{A}$ is directly opposite the angle $\hat{A}$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\hat{A}$.

$$\tan \hat{A} = \frac{\text{opposite}}{\text{adjacent}} = \frac{CB}{AB}$$

d) $\sin \hat{C} =$

Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\hat{C}$ is directly opposite the angle $\hat{C}$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\hat{C}$.

$$\sin \hat{C} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{AC}$$

e) $\cos \hat{C} =$

Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\hat{C}$ is directly opposite the angle $\hat{C}$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\hat{C}$.

$$\cos \hat{C} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{CB}{AC}$$

f) $\tan \hat{C} =$

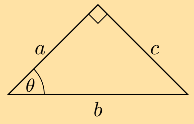
Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\hat{C}$ is directly opposite the angle $\hat{C}$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\hat{C}$.

$$\tan \hat{C} = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{CB}$$

2. In each of the following triangles, state whether a, b and c are the hypotenuse, opposite or adjacent sides of the triangle with respect to θ.

a)



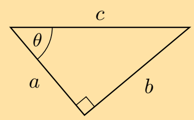
Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle θ is directly opposite the angle θ. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle θ.

- a is the adjacent side

- b is the hypotenuse
- c is the opposite side

b)

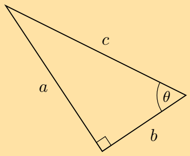


Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle θ is directly opposite the angle θ. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle θ.

- a is the adjacent side
- b is the opposite side
- c is the hypotenuse

c)

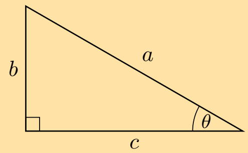


Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle θ is directly opposite the angle θ. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle θ.

- a is the opposite side
- b is the adjacent side
- c is the hypotenuse

d)

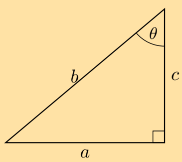


Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle θ is directly opposite the angle θ. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle θ.

- a is the hypotenuse
- b is the opposite side
- c is the adjacent side

e)

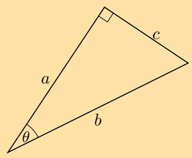


Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle θ is directly opposite the angle θ. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle θ.

- a is the opposite side
- b is the hypotenuse
- c is the adjacent side

f)

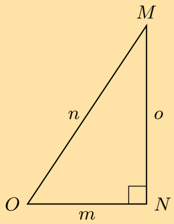


Solution:

First find the right angle, the hypotenuse is always directly opposite the right angle. The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle θ is directly opposite the angle θ. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle θ.

- a is the adjacent side
- b is the hypotenuse
- c is the opposite side

3. Consider the following diagram:



Without using a calculator, answer each of the following questions.

a) Write down $\cos \hat{O}$ in terms of m, n and o.

Solution:

- m is the adjacent side
- n is the hypotenuse
- o is the opposite side

$$\cos \hat{O} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{m}{o}$$

b) Write down $\tan \hat{M}$ in terms of m, n and o.

Solution:

- m is the opposite side
- n is the hypotenuse
- o is the adjacent side

$$\tan \hat{M} = \frac{\text{opposite}}{\text{adjacent}} = \frac{m}{o}$$

c) Write down $\sin \hat{O}$ in terms of m, n and o.

Solution:

- m is the adjacent side
- n is the hypotenuse
- o is the opposite side

$$\sin \hat{O} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{o}{n}$$

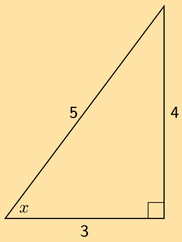
d) Write down $\cos \hat{M}$ in terms of m, n and o.

Solution:

- m is the opposite side
- n is the hypotenuse
- o is the adjacent side

$\cos \hat{M} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{o}{n}$

4. Find x in the diagram in three different ways. You do not need to calculate the value of x, just write down the appropriate ratio for x.



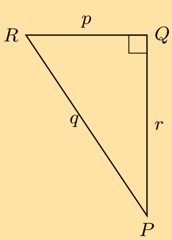
Solution:

- Side of length 4 is the opposite side
- Side of length 5 is the hypotenuse
- Side of length 3 is the adjacent side

Notice that the hypotenuse is the longest side as we would expect.

$$\sin x = \frac{4}{5}$$
$$\cos x = \frac{3}{5}$$
$$\tan x = \frac{4}{3}$$

5. Which of these statements is true about $\triangle PQR$?



a) $\sin \hat{R} = \frac{p}{q}$ b) $\tan \hat{Q} = \frac{r}{p}$ c) $\cos \hat{P} = \frac{r}{q}$ d) $\sin \hat{P} = \frac{p}{r}$

Solution:

We first find the opposite and adjacent sides with respect to $\hat{P}$ and $\hat{R}$:

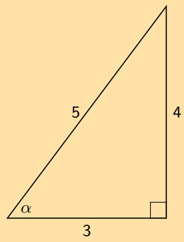
- p is the opposite side to $\hat{P}$ and the adjacent side to $\hat{R}$
- q is the hypotenuse
- r is the adjacent side to $\hat{P}$ and the opposite side to $\hat{R}$

We also note that:

- $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$

Looking at each of the given ratios we can see that only $\cos \hat{P} = \frac{r}{q}$ is correct.

6. Sarah wants to find the value of α in the triangle below. Which statement is a correct line of working?



a) $\sin\alpha = \frac{4}{5}$
b) $\cos\left(\frac{3}{5}\right) = \alpha$
c) $\tan\alpha = \frac{5}{4}$
d) $\cos 0{,}8 = \alpha$

Solution:

Sarah first needs to identify the hypotenuse, opposite and adjacent sides in the triangle. She then needs to write down a trigonometric ratio that will allow her to find α.

$\sin\alpha = \frac{4}{5}$ is one such ratio that will help her find α. From the given list of options this is the only correct line of reasoning.

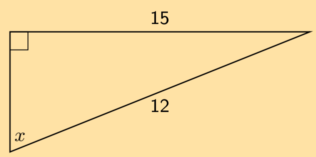
$\cos\left(\frac{3}{5}\right) = \alpha$ has the angle and the lengths of the sides switched around.

$\tan\alpha = \frac{3}{4}$ uses the wrong sides with respect to α for $\tan$.

$\cos 0{,}8 = \alpha$ uses the wrong sides with respect to α for $\cos$. Note that you can reduce the fraction to a decimal number but you need to first write the correct fraction.

7. Explain what is wrong with each of the following diagrams.

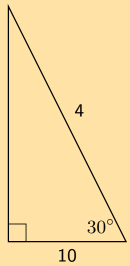
a)



Solution:

The hypotenuse is too small. The hypotenuse is the longest side of the right-angled triangle and in this case one side of the triangle is given as being larger than the hypotenuse.

b)



Solution:

The hypotenuse is too small. The hypotenuse is the longest side of the right-angled triangle and in this case one side of the triangle is given as being larger than the hypotenuse.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FN9	2a. 2FNB	2b. 2FNH	2c. 2FNC	2d. 2FND	2e. 2FNF
2f. 2FNG	3a. 2FNJ	3b. 2FNK	3c. 2FNM	3d. 2FNN	4. 2FNP
5. 2FNQ	6. 2FNR	7a. 2FNT	7b. 2FNS		

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5.4 Reciprocal ratios

5.5 Calculator skills

Exercise 5 – 2:

1. Use your calculator to determine the value of the following (correct to 2 decimal places):

   a) $\tan 65°$

   Solution:

$$\begin{aligned}\tan 65° &= 2{,}1445069...\\ &\approx 2{,}14\end{aligned}$$

   b) $\sin 38°$

   Solution:

$$\begin{aligned}\sin 38° &= 0{,}615661...\\ &\approx 0{,}62\end{aligned}$$

   c) $\cos 74°$

   Solution:

$$\begin{aligned}\cos 74° &= 0{,}275637...\\ &\approx 0{,}28\end{aligned}$$

   d) $\sin 12°$

   Solution:

$$\begin{aligned}\sin 12° &= 0{,}20791...\\ &\approx 0{,}21\end{aligned}$$

   e) $\cos 26°$

   Solution:

$$\begin{aligned}\cos 26° &= 0{,}898794...\\ &\approx 0{,}90\end{aligned}$$

   f) $\tan 49°$

   Solution:

$$\begin{aligned}\tan 49° &= 1{,}150368...\\ &\approx 1{,}15\end{aligned}$$

   g) $\sin 305°$

   Solution:

$$\begin{aligned}\sin 305° &= -0{,}81915...\\ &\approx -0{,}82\end{aligned}$$

   h) $\tan 124°$

   Solution:

$$\begin{aligned}\tan 124° &= -1{,}482560...\\ &\approx -1{,}48\end{aligned}$$

i) $\sec 65°$
Solution:

$$\begin{aligned}\sec 65° &= \frac{1}{\cos 65°}\\ &= 2{,}36620...\\ &\approx 2{,}37\end{aligned}$$

j) $\sec 10°$
Solution:

$$\begin{aligned}\sec 10° &= \frac{1}{\cos 10°}\\ &= 1{,}01542...\\ &\approx 1{,}02\end{aligned}$$

k) $\sec 48°$
Solution:

$$\begin{aligned}\sec 48° &= \frac{1}{\cos 48°}\\ &= 1{,}49447...\\ &\approx 1{,}49\end{aligned}$$

l) $\cot 32°$
Solution:

$$\begin{aligned}\cot 32° &= \frac{1}{\tan 32°}\\ &= 1{,}6003334...\\ &\approx 1{,}60\end{aligned}$$

m) $\text{cosec } 140°$
Solution:

$$\begin{aligned}\text{cosec } 140° &= \frac{1}{\sin 140°}\\ &= 1{,}555724...\\ &\approx 1{,}56\end{aligned}$$

n) $\text{cosec } 237°$
Solution:

$$\begin{aligned}\text{cosec } 237° &= \frac{1}{\sin 237°}\\ &= -1{,}192363...\\ &\approx -1{,}19\end{aligned}$$

o) $\sec 231°$
Solution:

$$\begin{aligned}\sec 231° &= \frac{1}{\cos 231°}\\ &= -1{,}589016...\\ &\approx -1{,}59\end{aligned}$$

p) cosec $226°$
Solution:

$$\begin{aligned}\text{cosec } 226° &= \frac{1}{\sin 226°}\\ &= -1{,}390164...\\ &\approx -1{,}39\end{aligned}$$

q) $\frac{1}{4}\cos 20°$
Solution:

$$\begin{aligned}\frac{1}{4}\cos 20° &= \frac{1}{4}(0{,}939692...)\\ &= 0{,}234923...\\ &\approx 0{,}23\end{aligned}$$

r) $3\tan 40°$
Solution:

$$\begin{aligned}3\tan 40° &= 3(0{,}83909963...)\\ &= 2{,}517298894...\\ &\approx 2{,}52\end{aligned}$$

s) $\frac{2}{3}\sin 90°$
Solution:

$$\begin{aligned}\frac{2}{3}\sin 90° &= \frac{2}{3}(1)\\ &= 0{,}66666...\\ &\approx 0{,}67\end{aligned}$$

t) $\frac{5}{\cos 4{,}3°}$
Solution:

$$\begin{aligned}\frac{5}{\cos 4{,}3°} &= \frac{5}{0{,}9971...}\\ &\approx 5{,}01\end{aligned}$$

u) $\sqrt{\sin 55°}$
Solution:

$$\begin{aligned}\sqrt{\sin 55°} &= \sqrt{0{,}81915...}\\ &\approx 0{,}91\end{aligned}$$

v) $\frac{\sin 90°}{\cos 90°}$
Solution:

$$\begin{aligned}\frac{\sin 90°}{\cos 90°} &= \frac{1}{0}\\ &\text{undefined}\end{aligned}$$

w) $\tan 35° + \cot 35°$
Solution:

$$\begin{aligned}\tan 35° + \cot 35° &= 0{,}7002... + \frac{1}{\tan 35°}\\ &= 0{,}7002... + 1{,}4281...\\ &\approx 2{,}13\end{aligned}$$

x) $\dfrac{2 + \cos 310°}{2 + \sin 87°}$

Solution:

$$\begin{aligned}\frac{2 + \cos 310°}{2 + \sin 87°} &= \frac{2{,}64278...}{2{,}99862...}\\ &\approx 0{,}88\end{aligned}$$

y) $\sqrt{4 \sec 99°}$

Solution:

$$\begin{aligned}\sqrt{4 \sec 99°} &= \sqrt{\frac{4}{\cos 99°}}\\ &= \sqrt{-25{,}5698...}\\ &\text{non-real}\end{aligned}$$

z) $\sqrt{\dfrac{\cot 103° + \sin 1090°}{\sec 10° + 5}}$

Solution:

$$\begin{aligned}\sqrt{\frac{\cot 85° + \sin 1090°}{\sec 10° + 5}} &= \sqrt{\frac{\frac{1}{\tan 85°} + \sin 1090°}{\frac{1}{\cos 10°} + 5}}\\ &= \sqrt{\frac{0{,}2611...}{6{,}015...}}\\ &= \sqrt{0{,}043411...}\\ &\approx 0{,}21\end{aligned}$$

2. If $x = 39°$ and $y = 21°$, use a calculator to determine whether the following statements are true or false:

a) $\cos x + 2 \cos x = 3 \cos x$

Solution:

LHS:

$$\begin{aligned}\cos x + 2\cos x &= \cos 39° + 2\cos 39°\\ &= 0{,}7771... + 1{,}55429...\\ &= 2{,}3314...\\ &\approx 2{,}33\end{aligned}$$

RHS:

$$\begin{aligned}3\cos x &= 3\cos 39°\\ &= 2{,}3314...\\ &\approx 2{,}33\end{aligned}$$

Therefore the statement is true.

b) $\cos 2y = \cos y + \cos y$

Solution:

LHS:

$$\cos 2y = \cos 2(21°)$$
$$= 0{,}7431...$$
$$\approx 0{,}74$$

RHS:

$$\cos y + \cos y = \cos 21° + \cos 21°$$
$$= 0{,}93358... + 0{,}93358...$$
$$= 1{,}86716...$$
$$\approx 1{,}86$$

Therefore the statement is false.

c) $\tan x = \frac{\sin x}{\cos x}$

Solution:

LHS:

$$\tan x = \tan 39°$$
$$= 0{,}809784...$$
$$\approx 0{,}81$$

RHS:

$$\frac{\sin x}{\cos x} = \frac{\sin 39°}{\cos 39°}$$
$$= \frac{0{,}62932...}{0{,}777145...}$$
$$= 0{,}80978...$$
$$\approx 0{,}81$$

Therefore the statement is true.

d) $\cos(x + y) = \cos x + \cos y$

Solution:

LHS:

$$\cos(x + y) = \cos 39° + 21°$$
$$\approx 0{,}5$$

RHS:

$$\cos x + \cos y = \cos 39° + \cos 21°$$
$$= 0{,}777145... + 0{,}933358...$$
$$= 1{,}71072...$$
$$\approx 1{,}71$$

Therefore the statement is false.

3. Solve for x in $5^{\tan x} = 125$.

Solution:

To solve this problem we need to recall from exponents that if $a^x = a^y$ then $x = y$. Then we note that $125 = 5^3$. Now we can solve the problem:

$$5^{\tan x} = 5^3$$
$$\therefore \tan x = 3$$
$$x = 71{,}56505...$$
$$\approx 71{,}57$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FNW	1b. 2FNX	1c. 2FNY	1d. 2FNZ	1e. 2FP2	1f. 2FP3
1g. 2FP4	1h. 2FP5	1i. 2FP6	1j. 2FP7	1k. 2FP8	1l. 2FP9
1m. 2FPB	1n. 2FPC	1o. 2FPD	1p. 2FPF	1q. 2FPG	1r. 2FPH
1s. 2FPJ	1t. 2FPK	1u. 2FPM	1v. 2FPN	1w. 2FPP	1x. 2FPQ
1y. 2FPR	1z. 2FPS	2. 2FPT	3. 2FPV		

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5.6 Special angles

Exercise 5 – 3:

1. Select the closest answer for each expression from the list provided:

a) $\cos 45^\circ$

$$\tfrac{1}{2} \quad 1 \quad \sqrt{2} \quad \tfrac{1}{\sqrt{3}} \quad \tfrac{1}{\sqrt{2}}$$

Solution:

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

b) $\sin 45^\circ$

$$\sqrt{2} \quad \tfrac{1}{2} \quad \tfrac{1}{\sqrt{2}} \quad \tfrac{\sqrt{3}}{1} \quad 1$$

Solution:

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

c) $\tan 30^\circ$

$$\tfrac{1}{2} \quad \tfrac{\sqrt{3}}{1} \quad \tfrac{1}{\sqrt{2}} \quad \tfrac{\sqrt{3}}{2} \quad \tfrac{1}{\sqrt{3}}$$

Solution:

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

d) $\tan 60^\circ$

$$\tfrac{\sqrt{3}}{2} \quad \tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{3}} \quad \tfrac{\sqrt{3}}{1} \quad \tfrac{1}{1}$$

Solution:

$$\tan 60^\circ = \frac{\sqrt{3}}{1}$$

e) $\cos 45^\circ$

$$\tfrac{\sqrt{3}}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{\sqrt{3}} \quad \tfrac{1}{\sqrt{2}} \quad \sqrt{2}$$

Solution:

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

f) $\tan 30^\circ$

$\frac{1}{\sqrt{2}}$ $\frac{1}{2}$ $\frac{\sqrt{3}}{2}$ $\frac{1}{1}$ $\frac{1}{\sqrt{3}}$

Solution:

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

g) $\tan 30^\circ$

$\frac{1}{\sqrt{3}}$ $\frac{1}{2}$ $\frac{1}{\sqrt{2}}$ $\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{1}$

Solution:

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

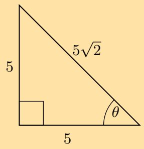
h) $\cos 60^\circ$

$\frac{1}{\sqrt{3}}$ $\frac{1}{2}$ $\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{1}$ $\frac{1}{\sqrt{2}}$

Solution:

$$\cos 60^\circ = \frac{1}{2}$$

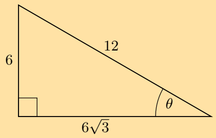
2. Solve for $\cos\theta$ in the following triangle, in surd form:



Solution:

$$\begin{aligned}\cos\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\ &= \frac{5}{5\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

3. Solve for $\tan\theta$ in the following triangle, in surd form:



Solution:

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}}\\ &= \frac{6}{6\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

4. Calculate the value of the following without using a calculator:

   a) $\sin 45° \times \cos 45°$

      Solution:

      For both ratios the angle given is 45°. This is one of the special angles. We note that $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$ using special angles.

$$\begin{aligned}\sin 45° \times \cos 45° &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{2}\end{aligned}$$

   b) $\cos 60° + \tan 45°$

      Solution:

      We are given angles of 45° and 60°. These are both special angles. We note that $\cos 60° = \frac{1}{2}$ and $\tan 45° = 1$ using special angles.

$$\begin{aligned}\cos 60° + \tan 45° &= \frac{1}{2} + 1\\ &= \frac{3}{2}\end{aligned}$$

   c) $\sin 60° - \cos 60°$

      Solution:

      For both ratios the angle given is 60°. This is one of the special angles. We note that $\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$ using special angles.

$$\begin{aligned}\sin 60° - \cos 60° &= \frac{\sqrt{3}}{2} - \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2}\end{aligned}$$

5. Evaluate the following without using a calculator. Select the closest answer from the list provided.

   a) $\tan 45° \div \sin 60°$

$$\frac{2}{\sqrt{3}} \quad \frac{\sqrt{3}}{1} \quad \frac{\sqrt{2}}{\sqrt{3}} \quad \frac{1}{1} \quad \frac{1}{2}$$

      Solution:

      We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$\begin{aligned}\tan 45° \div \sin 60° &= \frac{1}{1} \div \frac{\sqrt{3}}{2}\\ &= 1 \times \frac{2}{\sqrt{3}}\\ &= \frac{2}{\sqrt{3}}\end{aligned}$$

   b) $\tan 30° - \sin 60°$

$$0 \quad \frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2\sqrt{3}} \quad \frac{\sqrt{3}}{2}$$

      Solution:

      We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$
\begin{aligned}
\tan 30^\circ - \sin 60^\circ &= \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2} \\
&= \frac{2 - \left(\sqrt{3}\right)\left(\sqrt{3}\right)}{(2)\left(\sqrt{3}\right)} \\
&= \frac{2-3}{2\sqrt{3}} \\
&= \frac{-1}{2\sqrt{3}}
\end{aligned}
$$

c) $\sin 30^\circ - \tan 45^\circ - \sin 30^\circ$

$-\frac{\sqrt{3}}{2}$ $\quad -1$ $\quad -\frac{1}{\sqrt{3}}$ $\quad -\frac{\sqrt{3}}{1}$ $\quad -\frac{7}{2\sqrt{3}}$

Solution:
We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$
\begin{aligned}
\sin 30^\circ - \tan 45^\circ - \sin 30^\circ &= \frac{1}{2} - \frac{1}{1} - \frac{1}{2} \\
&= \frac{1-2-1}{2} \\
&= -1
\end{aligned}
$$

d) $\tan 30^\circ \div \tan 30^\circ \div \sin 45^\circ$

$\frac{\sqrt{3}}{1}$ $\quad \frac{2\sqrt{3}}{\sqrt{2}}$ $\quad \frac{2}{\sqrt{3}}$ $\quad \frac{\sqrt{2}}{1}$ $\quad \frac{2\sqrt{2}}{\sqrt{3}}$

Solution:
We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$
\begin{aligned}
\tan 30^\circ \div \tan 30^\circ \div \sin 45^\circ &= \frac{1}{\sqrt{3}} \div \frac{1}{\sqrt{3}} \div \frac{1}{\sqrt{2}} \\
&= \left(\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{1}\right) \div \frac{1}{\sqrt{2}} \\
&= 1 \times \frac{\sqrt{2}}{1} \\
&= \frac{\sqrt{2}}{1}
\end{aligned}
$$

e) $\sin 45^\circ \div \sin 30^\circ \div \cos 45^\circ$

$\frac{\sqrt{2}}{\sqrt{3}}$ $\quad \frac{1}{\sqrt{2}}$ $\quad \frac{4}{\sqrt{3}}$ $\quad 2$ $\quad \frac{2\sqrt{2}}{\sqrt{3}}$

Solution:
We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$
\begin{aligned}
\sin 45^\circ \div \sin 30^\circ \div \cos 45^\circ &= \frac{1}{\sqrt{2}} \div \frac{1}{2} \div \frac{1}{\sqrt{2}} \\
&= \left(\frac{1}{\sqrt{2}} \times \frac{2}{1}\right) \div \frac{1}{\sqrt{2}} \\
&= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{1} \\
&= 2
\end{aligned}
$$

f) $\tan 60° - \tan 60° - \sin 60°$

$$-\frac{1}{\sqrt{3}} \quad -\frac{1}{2} \quad -\frac{1}{\sqrt{2}} \quad -\frac{1}{1} \quad \frac{-\sqrt{3}}{2}$$

Solution:
We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$\begin{aligned}
\tan 60° - \tan 60° - \sin 60° &= \frac{\sqrt{3}}{1} - \frac{\sqrt{3}}{1} - \frac{\sqrt{3}}{2} \\
&= \frac{(2)\left(\sqrt{3}\right) - (2)\left(\sqrt{3}\right) - \sqrt{3}}{2} \\
&= \frac{-\sqrt{3}}{2}
\end{aligned}$$

g) $\cos 45° - \sin 60° - \sin 45°$

$$-\frac{1}{2} \quad -\frac{1}{\sqrt{2}} \quad -\frac{7}{2\sqrt{3}} \quad \frac{-\sqrt{3}}{2} \quad -\frac{1}{\sqrt{3}}$$

Solution:
We need to use special angles to help us solve this problem. First write down each ratio using special angles and then simplify the answer.

$$\begin{aligned}
\cos 45° - \sin 60° - \sin 45° &= \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \\
&= \frac{2 - \left(\sqrt{3}\right)\left(\sqrt{2}\right) - 2}{(2)\left(\sqrt{2}\right)} \\
&= \frac{-\sqrt{3}\sqrt{2}}{2\sqrt{2}} \\
&= \frac{-\sqrt{3}}{2}
\end{aligned}$$

6. Use special angles to show that:

a) $\dfrac{\sin 60°}{\cos 60°} = \tan 60°$

Solution:
We are told to use special angles, so we first write each ratio using special angles and then simplify each side of the equation.
LHS:

$$\begin{aligned}
\frac{\sin 60°}{\cos 60°} &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
&= \frac{\sqrt{3}}{2} \times \frac{2}{1} \\
&= \sqrt{3}
\end{aligned}$$

RHS:

$$\tan 60° = \sqrt{3}$$

Therefore the equation is true.

b) $\sin^2 45° + \cos^2 45° = 1$

Solution:
We are told to use special angles, so we first write each ratio using special angles and then simplify each side of the equation.
LHS:

$$\sin^2 45° + \cos^2 45° = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

RHS $= 1$
Therefore the equation is true.

c) $\cos 30° = \sqrt{1 - \sin^2 30°}$
Solution:
We are told to use special angles, so we first write each ratio using special angles and then simplify each side of the equation.
LHS:

$$\cos 30° = \frac{\sqrt{3}}{2}$$

RHS:

$$\sqrt{1 - \sin^2 30°} = \sqrt{1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$
$$= \sqrt{1 - \frac{1}{4}}$$
$$= \sqrt{\frac{3}{4}}$$
$$= \frac{\sqrt{3}}{2}$$

Therefore the equation is true.

7. Use the definitions of the trigonometric ratios to answer the following questions:

a) Explain why $\sin\alpha \leq 1$ for all values of α.
Solution:
The sine ratio is defined as $\frac{\text{opposite}}{\text{hypotenuse}}$. In any right-angled triangle, the hypotenuse is the side of longest length. Therefore the maximum length of the opposite side is equal to the length of the hypotenuse. The maximum value of the sine ratio is then $\frac{\text{hypotenuse}}{\text{hypotenuse}} = 1$.

b) Explain why $\cos\alpha$ has a maximum value of 1.
Solution:
The cosine ratio is defined as $\frac{\text{adjacent}}{\text{hypotenuse}}$. In any right-angled triangle, the hypotenuse is the side of longest length. Therefore the maximum length of the adjacent side is equal to the length of the hypotenuse. The maximum value of the cosine ratio is then $\frac{\text{hypotenuse}}{\text{hypotenuse}} = 1$.

c) Is there a maximum value for $\tan\alpha$?
Solution:
The tangent ratio is defined as $\frac{\text{opposite}}{\text{adjacent}}$. Since the opposite and adjacent sides can have any value (so long as the length of the side is less than or equal to the length of the hypotenuse), there is no maximum value for the tangent ratio.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FPW	1b. 2FPX	1c. 2FPY	1d. 2FPZ	1e. 2FQ2	1f. 2FQ3
1g. 2FQ4	1h. 2FQ5	2. 2FQ6	3. 2FQ7	4a. 2FQ8	4b. 2FQ9
4c. 2FQB	5a. 2FQC	5b. 2FQD	5c. 2FQF	5d. 2FQG	5e. 2FQH
5f. 2FQJ	5g. 2FQK	6a. 2FQM	6b. 2FQN	6c. 2FQP	7a. 2FQQ
7b. 2FQR	7c. 2FQS				

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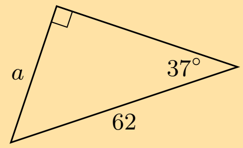
Finding lengths

Exercise 5 – 4:

1. In each triangle find the length of the side marked with a letter. Give your answers correct to 2 decimal places.

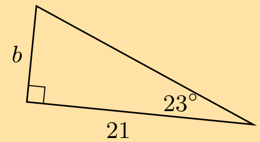
a)



Solution:

$$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \sin 37° &= \frac{a}{62} \\ 62(0{,}6018\ldots) &= a \\ a &= 36{,}10890\ldots \\ &\approx 36{,}11 \end{aligned}$$

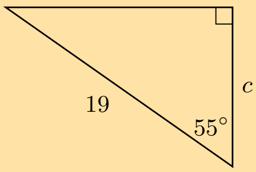
b)



Solution:

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan 23° &= \frac{b}{21} \\ 21(0{,}42447\ldots) &= b \\ b &= 8{,}91397\ldots \\ &\approx 8{,}91 \end{aligned}$$

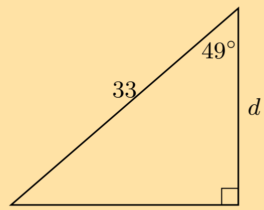
c)



Solution:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 55° = \frac{c}{19}$$
$$19(0{,}5735...) = c$$
$$c = 10{,}89795...$$
$$\approx 10{,}90$$

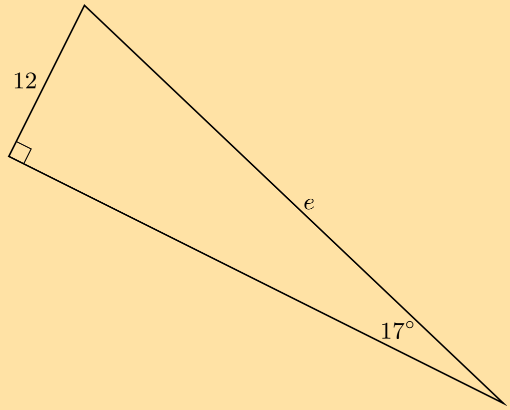
d)



Solution:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 49° = \frac{d}{33}$$
$$33(0{,}65605...) = d$$
$$d = 21{,}64994...$$
$$\approx 21{,}65$$

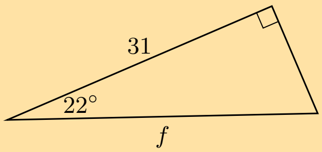
e)



Solution:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin 17° = \frac{e}{12}$$
$$12(0{,}29237...) = e$$
$$e = 3{,}50846...$$
$$\approx 3{,}51$$

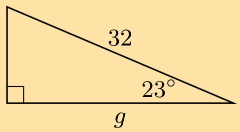
f)



Solution:

$$\begin{aligned}
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\cos 22° &= \frac{31}{f} \\
f(0{,}92718...) &= 31 \\
f &= 33{,}434577... \\
&\approx 33{,}43
\end{aligned}$$

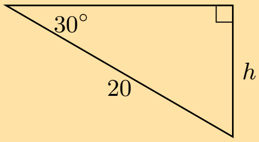
g)



Solution:

$$\begin{aligned}
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\cos 23° &= \frac{g}{32} \\
32(0{,}92050...) &= g \\
g &= 29{,}4561... \\
&\approx 29{,}46
\end{aligned}$$

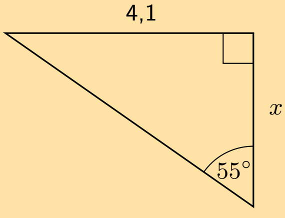
h)



Solution:

$$\begin{aligned}
\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\sin 30° &= \frac{h}{20} \\
20(0{,}5) &= h \\
h &\approx 10
\end{aligned}$$

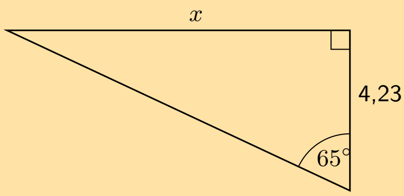
i)



Solution:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan 55^\circ = \frac{4{,}1}{x}$$
$$x = 2{,}87$$

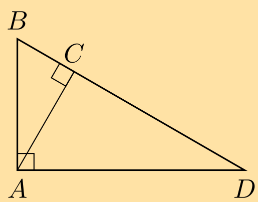
j)



Solution:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan 65^\circ = \frac{x}{4{,}23}$$
$$x = 9{,}06$$

2. Write down two ratios for each of the following in terms of the sides: AB; BC; BD; AD; DC and AC.



a) $\sin \hat{B}$
Solution:
We note that triangles ABC and ABD both contain angle B so we can use these triangles to write down the ratios:

$$\sin \hat{B} = \frac{AC}{AB} = \frac{AD}{BD}$$

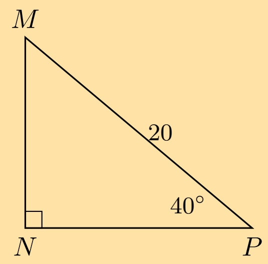
b) $\cos \hat{D}$
Solution:
We note that triangles ACD and ABD both contain angle D so we can use these triangles to write down the ratios:

$$\cos \hat{D} = \frac{AD}{BD} = \frac{CD}{AD}$$

c) $\tan \hat{B}$
Solution:
We note that triangles ABC and ABD both contain angle B so we can use these triangles to write down the ratios:

$$\tan \hat{B} = \frac{AC}{BC} = \frac{AD}{AB}$$

3. In $\triangle MNP$, $\hat{N} = 90°$, $MP = 20$ and $\hat{P} = 40°$. Calculate NP and MN (correct to 2 decimal places).
Solution:
Sketch the triangle:



To find MN we use the sine ratio:

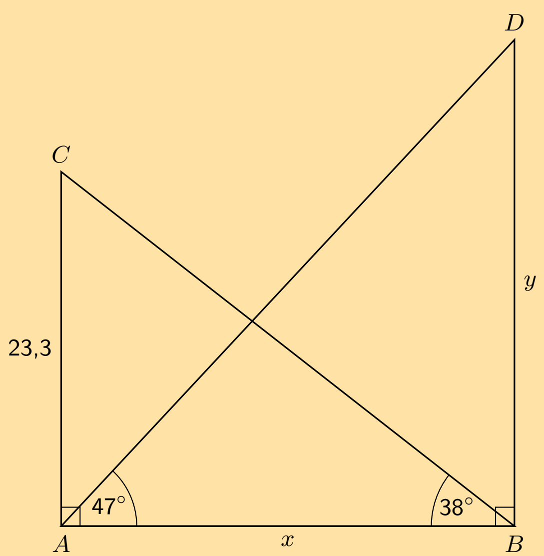
$$\begin{aligned}
\sin \hat{P} &= \frac{MN}{MP} \\
\sin 40° &= \frac{MN}{20} \\
20(0{,}642787...) &= MN \\
MN &= 12{,}8557... \\
&\approx 12{,}86
\end{aligned}$$

To find NP we can use the cosine ratio:

$$\begin{aligned}
\cos \hat{P} &= \frac{NP}{MP} \\
\cos 40° &= \frac{NP}{20} \\
20(0{,}76604...) &= NP \\
NP &= 15{,}32088... \\
&\approx 15{,}32
\end{aligned}$$

Therefore $MN = 12{,}86$ and $NP = 15{,}32$

4. Calculate x and y in the following diagram.



Solution:
To find x we use $\triangle ABC$ and the tangent ratio. To find y we use $\triangle ABD$ and the tangent ratio.

$$\tan 38° = \frac{23,3}{x}$$
$$x = \frac{23,3}{\tan 38°}$$
$$= 29,82264...$$
$$\approx 29,82$$

$$\tan 47° = \frac{y}{29,82264...}$$
$$y = 29,82264... \tan 47°$$
$$= 31,98086...$$
$$\approx 31,98$$

Therefore $x = 29,82$ and $y = 31,98$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FQV 1b. 2FQW 1c. 2FQX 1d. 2FQY 1e. 2FQZ 1f. 2FR2
1g. 2FR3 1h. 2FR4 1i. 2FR5 1j. 2FR6 2. 2FR7 3. 2FR8
4. 2FR9

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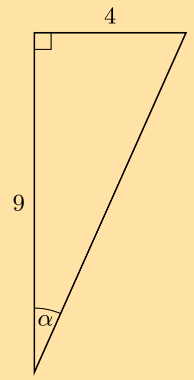
Finding an angle

Exercise 5 – 5:

Determine α in the following right-angled triangles:

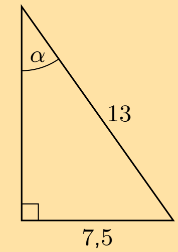
1.



Solution:

$$\tan \alpha = \frac{4}{9}$$
$$= 0,4444...$$
$$\alpha = 23,9624...$$
$$\approx 23,96°$$

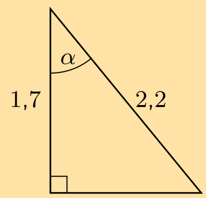
2.



Solution:

$$\begin{aligned}\sin\alpha &= \frac{7{,}5}{13}\\ &= 0{,}5769...\\ \alpha &= 35{,}2344...\\ &\approx 35{,}23°\end{aligned}$$

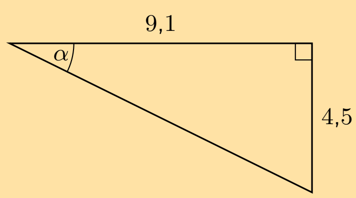
3.



Solution:

$$\begin{aligned}\sin\alpha &= \frac{1{,}7}{2{,}2}\\ \alpha &= 39{,}4005...\\ &\approx 39{,}40°\end{aligned}$$

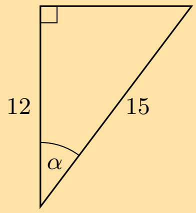
4.



Solution:

$$\begin{aligned}\tan\alpha &= \frac{4{,}5}{9{,}1}\\ &= 0{,}49450...\\ \alpha &= 26{,}3126...\\ &\approx 26{,}31°\end{aligned}$$

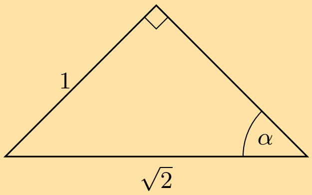
5.



Solution:

$$\begin{aligned} \cos\alpha &= \frac{12}{15} \\ &= 0{,}8 \\ \alpha &= 36{,}869897... \\ &\approx 36{,}87^\circ \end{aligned}$$

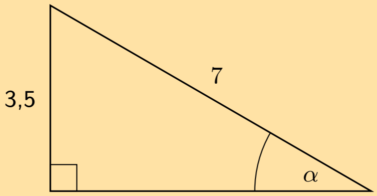
6.



Solution:

$$\begin{aligned} \sin\alpha &= \frac{1}{\sqrt{2}} \\ &= 0{,}7071... \\ \alpha &= 45^\circ \end{aligned}$$

7.



Solution:

$$\begin{aligned} \sin\alpha &= \frac{3{,}5}{7} \\ &= 0{,}5 \\ \alpha &= 30^\circ \end{aligned}$$

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1. 2FRB 2. 2FRC 3. 2FRD 4. 2FRF 5. 2FRG 6. 2FRH
7. 2FRJ

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If learners get a math error on their calculator encourage them to think about what might have happened. It is also important to ensure that they know they must write down no solution rather than math error when this happens.

Exercise 5 – 6:

1. Determine the angle (correct to 1 decimal place):

   a) $\tan\theta = 1{,}7$

   Solution:

$$\begin{aligned}\tan\theta &= 1{,}7\\ \theta &= 59{,}5344...\\ &\approx 59{,}5^\circ\end{aligned}$$

   b) $\sin\theta = 0{,}8$

   Solution:

$$\begin{aligned}\sin\theta &= 0{,}8\\ \theta &= 53{,}1301...\\ &\approx 53{,}1^\circ\end{aligned}$$

   c) $\cos\alpha = 0{,}32$

   Solution:

$$\begin{aligned}\cos\alpha &= 0{,}32\\ \alpha &= 71{,}3370...\\ &\approx 71{,}3^\circ\end{aligned}$$

   d) $\tan\beta = 4{,}2$

   Solution:

$$\begin{aligned}\tan\beta &= 4{,}2\\ \beta &= 76{,}60750...\\ &\approx 76{,}6^\circ\end{aligned}$$

   e) $\tan\theta = 5\frac{3}{4}$

   Solution:

$$\begin{aligned}\tan\theta &= 5\frac{3}{4}\\ &= 5{,}75\\ \theta &= 80{,}13419...\\ &\approx 80{,}1^\circ\end{aligned}$$

   f) $\sin\theta = \frac{2}{3}$

   Solution:

$$\begin{aligned}\sin\theta &= \frac{2}{3}\\ &= 0{,}666...\\ \theta &= 41{,}8103...\\ &\approx 41{,}8^\circ\end{aligned}$$

   g) $\cos\beta = 1{,}2$

   Solution:

$$\begin{aligned}\cos\beta &= 1{,}2\\ &\text{no solution}\end{aligned}$$

h) $4\cos\theta = 3$
Solution:

$$\begin{aligned} 4\cos\theta &= 3 \\ \cos\theta &= \frac{3}{4} \\ &= 0{,}75 \\ \theta &= 41{,}40962\ldots \\ &\approx 41{,}4° \end{aligned}$$

i) $\cos 4\theta = 0{,}3$
Solution:

$$\begin{aligned} \cos 4\theta &= 0{,}3 \\ 4\theta &= 72{,}54239\ldots \\ \theta &= 18{,}135599\ldots \\ &\approx 18{,}1° \end{aligned}$$

j) $\sin\beta + 2 = 2{,}65$
Solution:

$$\begin{aligned} \sin\beta + 2 &= 2{,}65 \\ \sin\beta &= 0{,}65 \\ \beta &= 40{,}54160\ldots \\ &\approx 40{,}5° \end{aligned}$$

k) $2\sin\theta + 5 = 0{,}8$
Solution:

$$\begin{aligned} 2\sin\theta + 5 &= 0{,}8 \\ 2\sin\theta &= -4{,}2 \\ \sin\theta &= -2{,}1 \\ &\text{no solution} \end{aligned}$$

l) $3\tan\beta = 1$
Solution:

$$\begin{aligned} 3\tan\beta &= 1 \\ \tan\beta &= \frac{1}{3} \\ &= 0{,}3333\ldots \\ \beta &= 18{,}434948\ldots \\ &\approx 18{,}4° \end{aligned}$$

m) $\sin 3\alpha = 1{,}2$
Solution:

$$\begin{aligned} \sin 3\alpha &= 1{,}2 \\ &\text{no solution} \end{aligned}$$

n) $\tan\frac{\theta}{3} = \sin 48°$
Solution:

$$\begin{aligned}
\tan\frac{\theta}{3} &= \sin 48^\circ \\
&= 0{,}7431... \\
\frac{\theta}{3} &= 36{,}61769... \\
\theta &= 109{,}8530... \\
&\approx 109{,}9^\circ
\end{aligned}$$

o) $\frac{1}{2}\cos 2\beta = 0{,}3$

Solution:

$$\begin{aligned}
\frac{1}{2}\cos 2\beta &= 0{,}3 \\
\cos 2\beta &= 0{,}6 \\
2\beta &= 53{,}1301... \\
\beta &= 26{,}56505... \\
&\approx 26{,}6^\circ
\end{aligned}$$

p) $2\sin 3\theta + 1 = 2{,}6$

Solution:

$$\begin{aligned}
2\sin 3\theta + 1 &= 2{,}6 \\
2\sin 3\theta &= 1{,}6 \\
\sin 3\theta &= 0{,}8 \\
3\theta &= 53{,}1301... \\
\theta &= 17{,}71003... \\
&\approx 17{,}7^\circ
\end{aligned}$$

2. If $x = 16^\circ$ and $y = 36^\circ$, use your calculator to evaluate each of the following, correct to 3 decimal places.

a) $\sin(x-y)$

Solution:

$$\begin{aligned}
\sin(x-y) &= \sin(16-36) \\
&= \sin(-20) \\
&= -0{,}3420201... \\
&\approx -0{,}342
\end{aligned}$$

b) $3\sin x$

Solution:

$$\begin{aligned}
3\sin x &= 3\sin(16) \\
&= 0{,}826912... \\
&\approx 0{,}827
\end{aligned}$$

c) $\tan x - \tan y$

Solution:

$$\begin{aligned}
\tan x - \tan y &= \tan(16) - \tan(36) \\
&= -0{,}439797... \\
&\approx -0{,}440
\end{aligned}$$

d) $\cos x + \cos y$

Solution:

$$\begin{aligned}\cos x+\cos y&=\cos(16)+\cos(36)\\&=1{,}77027...\\&\approx 1{,}770\end{aligned}$$

e) $\frac{1}{3}\tan y$
Solution:

$$\begin{aligned}\frac{1}{3}\tan y&=\frac{1}{3}\tan(36)\\&=0{,}24218...\\&\approx 0{,}242\end{aligned}$$

f) cosec $(x-y)$
Solution:

$$\begin{aligned}\text{cosec }(x-y)&=\text{cosec }(16-36)\\&=\text{cosec }(-20)\\&=\frac{1}{\sin(-20)}\\&=-2{,}92380...\\&\approx -2{,}924\end{aligned}$$

g) $2\cos x+\cos 3y$
Solution:

$$\begin{aligned}2\cos x+\cos 3y&=2\cos(16)+\cos(3(36))\\&=2\cos 16+\cos 108\\&=1{,}61350...\\&\approx 1{,}614\end{aligned}$$

h) $\tan(2x-5y)$
Solution:

$$\begin{aligned}\tan(2x-5y)&=\tan(2(16)-5(36))\\&=\tan(-148)\\&=0{,}624869...\\&\approx 0{,}625\end{aligned}$$

3. In each of the following find the value of x correct to two decimal places.

a) $\sin x=0{,}814$
Solution:

$$\begin{aligned}\sin x&=0{,}814\\x&=54{,}48860...\\&\approx 54{,}49°\end{aligned}$$

b) $\sin x=\tan 45°$
Solution:

$$\begin{aligned}\sin x&=\tan 45°\\&=1\\x&=90°\end{aligned}$$

c) $\tan 2x = 3{,}123$
Solution:

$$
\begin{aligned}
\tan 2x &= 3{,}123 \\
2x &= 72{,}244677... \\
x &= 36{,}12233... \\
&\approx 36{,}12°
\end{aligned}
$$

d) $\tan x = 3\sin 41°$
Solution:

$$
\begin{aligned}
\tan x &= 3\sin 41° \\
&= 1{,}96817... \\
x &= 63{,}06558... \\
&\approx 63{,}07°
\end{aligned}
$$

e) $\sin(2x + 45) = 0{,}123$
Solution:

$$
\begin{aligned}
\sin(2x + 45°) &= 0{,}123 \\
2x + 45 &= 7{,}06527... \\
2x &= -37{,}9347... \\
x &= -18{,}9673... \\
&\approx -18{,}97°
\end{aligned}
$$

f) $\sin(x - 10°) = \cos 57°$
Solution:

$$
\begin{aligned}
\sin(x - 10°) &= \cos 57° \\
&= 0{,}54463... \\
x - 10 &= 33 \\
x &= 43°
\end{aligned}
$$

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1a. 2FRK	1b. 2FRM	1c. 2FRN	1d. 2FRP	1e. 2FRQ	1f. 2FRR
1g. 2FRS	1h. 2FRT	1i. 2FRV	1j. 2FRW	1k. 2FRX	1l. 2FRY
1m. 2FRZ	1n. 2FS2	1o. 2FS3	1p. 2FS4	2a. 2FS5	2b. 2FS6
2c. 2FS7	2d. 2FS8	2e. 2FS9	2f. 2FSB	2g. 2FSC	2h. 2FSD
3a. 2FSF	3b. 2FSG	3c. 2FSH	3d. 2FSJ	3e. 2FSK	3f. 2FSM

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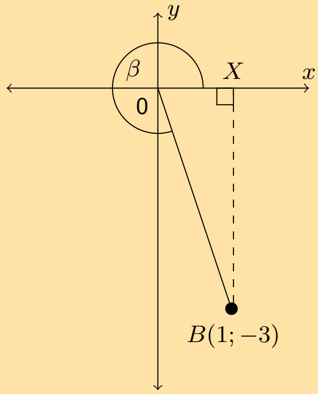
5.8 Defining ratios in the Cartesian plane

Exercise 5 – 7:

1. B is a point in the Cartesian plane. Determine without using a calculator:



a) OB
Solution:
OB is the hypotenuse of $\triangle BOX$. We can calculate the length of OB using the theorem of Pythagoras:

$$\begin{aligned} OB^2 &= OX^2 + XB^2 \\ &= (1)^2 + (3)^2 \\ &= 10 \\ OB &= \sqrt{10} \end{aligned}$$

b) $\cos\beta$
Solution:
From the diagram and the first question we know that $x = 1$, $y = -3$ and $r = \sqrt{10}$.

$$\begin{aligned} \cos\beta &= \frac{x}{r} \\ &= \frac{1}{\sqrt{10}} \end{aligned}$$

c) cosec β
Solution:
From the diagram and the first question we know that $x = 1$, $y = -3$ and $r = \sqrt{10}$.

$$\begin{aligned} \text{cosec}\ \beta &= \frac{r}{y} \\ &= \frac{\sqrt{10}}{-3} \end{aligned}$$

d) $\tan\beta$
Solution:
From the diagram and the first question we know that $x = 1$, $y = -3$ and $r = \sqrt{10}$.

$$\begin{aligned} \tan\beta &= \frac{y}{x} \\ &= \frac{-3}{1} \\ &= -3 \end{aligned}$$

2. If $\sin\theta = 0{,}4$ and θ is an obtuse angle, determine:

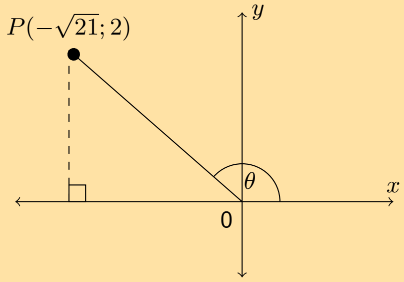
a) $\cos\theta$
Solution:
We first need to determine x, y and r.

$$\begin{aligned}\sin\theta &= 0{,}4\\ &= \frac{4}{10}\\ &= \frac{2}{5}\\ &= \frac{y}{r}\end{aligned}$$

Therefore $y = 2$ and $r = 5$.

$$\begin{aligned}x^2 &= r^2 - y^2\\ &= (5)^2 - (2)^2\\ &= 21\\ x &= \pm\sqrt{21}\end{aligned}$$

We are told that the angle is obtuse. An obtuse angle is greater than 90° but less than 180°. Therefore the angle is in the second quadrant and x is negative. Therefore $x = -\sqrt{21}$.
Next draw a sketch:



Now we can determine $\cos\theta$:

$$\begin{aligned}\cos\theta &= \frac{x}{r}\\ &= \frac{-\sqrt{21}}{5}\end{aligned}$$

b) $\sqrt{21}\tan\theta$

Solution:

From the first question we have a sketch of the angle and x, y and r.

$$\begin{aligned}\sqrt{21}\tan\theta &= \sqrt{21}\left(\frac{y}{x}\right)\\ &= \sqrt{21}\left(\frac{2}{-\sqrt{21}}\right)\\ &= -2\end{aligned}$$

3. Given $\tan\theta = \frac{t}{2}$, where $0° \leq \theta \leq 90°$. Determine the following in terms of t:

a) $\sec\theta$

Solution:

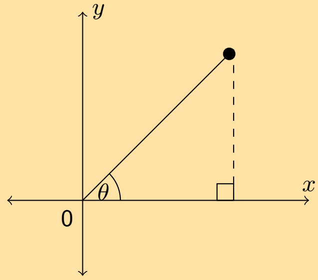
We first need to determine x, y and r. We are given $\tan\theta = \frac{t}{2}$ and so we can use this to find x and y.

$$\begin{aligned}\tan\theta &= \frac{t}{2}\\ \frac{y}{x} &= \frac{t}{2}\end{aligned}$$

Therefore $y = t$ and $x = 2$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (2)^2 + (t)^2 \\ &= 4 + t^2 \\ r &= \sqrt{4 + t^2} \end{aligned}$$

We are told that $0° \leq \theta \leq 90°$. Therefore the angle is in the first quadrant. Even though we do not know the value of t we can draw a rough sketch:



Now we can determine $\sec\theta$:

$$\begin{aligned} \sec\theta &= \frac{r}{x} \\ &= \frac{\sqrt{t^2 + 4}}{2} \end{aligned}$$

b) $\cot\theta$

Solution:

From the first question we have a sketch of the angle and x, y and r.

$$\begin{aligned} \cot\theta &= \frac{x}{y} \\ &= \frac{2}{t} \end{aligned}$$

c) $\cos^2\theta$

Solution:

From the first question we have a sketch of the angle and x, y and r.

$$\begin{aligned} \cos^2\theta &= \left(\frac{x}{r}\right)^2 \\ &= \left(\frac{2}{\sqrt{t^2 + 4}}\right)^2 \\ &= \frac{4}{t^2 + 4} \end{aligned}$$

d) $\tan^2\theta - \sec^2\theta$

Solution:

From the first question we have a sketch of the angle and x, y and r.

$$\begin{aligned} \tan^2\theta - \sec^2\theta &= \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2 \\ &= \left(\frac{t}{2}\right)^2 - \left(\frac{\sqrt{t^2 + 4}}{2}\right)^2 \\ &= \frac{t^2}{4} - \frac{t^2 + 4}{4} \\ &= \frac{t^2 - t^2 - 4}{4} \\ &= -1 \end{aligned}$$

4. Given: $10\cos\beta + 8 = 0$ and $180° < \beta < 360°$. Determine the value of:

a) $\cos\beta$

Solution:

We are given an equation with $\cos\beta$ in it. We can therefore rearrange this equation to find $\cos\beta$:

$$\begin{aligned} 10\cos\beta + 8 &= 0 \\ \cos\beta &= \frac{-8}{10} \\ &= \frac{-4}{5} \end{aligned}$$

b) $\dfrac{3}{\tan\beta} + 2\sin^2\beta$

Solution:

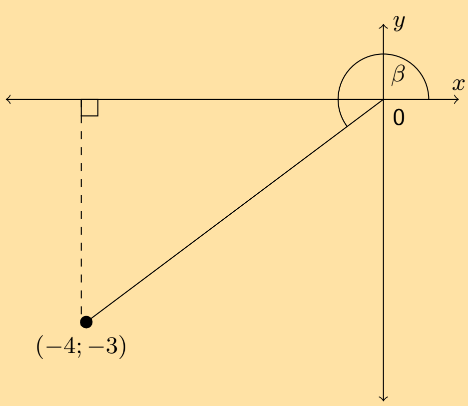
We first need to determine x, y and r. In the first question we found that $\cos\beta = \frac{-4}{5}$ and so we can use this to find x and r.

$$\begin{aligned} \cos\beta &= \frac{-4}{5} \\ \frac{x}{r} &= \frac{-4}{5} \end{aligned}$$

Therefore $x = -4$ and $r = 5$.

$$\begin{aligned} y^2 &= r^2 - x^2 \\ &= (5)^2 - (-4)^2 \\ &= 25 - 16 \\ y &= \pm 3 \end{aligned}$$

We are told that $180° < \beta < 360°$. Therefore the angle is in the third quadrant and $y = -3$. We can draw a rough sketch of the angle:



We can now find $\frac{3}{\tan\beta} + 2\sin^2\beta$:

$$
\begin{aligned}
\frac{3}{\tan\beta} + 2\sin^2\beta &= \frac{3}{\frac{y}{x}} + 2\left(\frac{y}{r}\right)^2 \\
&= \frac{3x}{y} + \frac{2y^2}{r^2} \\
&= \frac{3(-4)}{-3} + \frac{2(-3)^2}{(5)^2} \\
&= \frac{-12}{3} + \frac{18}{25} \\
&= -4 + \frac{18}{25} \\
&= \frac{-100+18}{25} \\
&= \frac{-82}{25}
\end{aligned}
$$

5. If $\sin\theta = -\frac{15}{17}$ and $\cos\theta < 0$ find the following, without the use of a calculator:

   a) $\cos\theta$

   Solution:

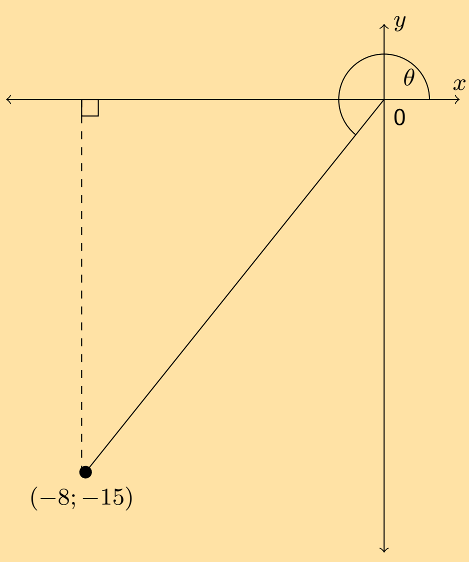
   We first need to determine x, y and r. We are given $\sin\theta = -\frac{15}{17}$ and so we can use this to find y and r.

$$
\begin{aligned}
\sin\theta &= -\frac{15}{17} \\
\frac{y}{r} &= -\frac{15}{17}
\end{aligned}
$$

   Therefore $y = -15$ and $r = 17$ (remember that r cannot be negative).

$$
\begin{aligned}
x^2 &= r^2 - y^2 \\
&= (17)^2 - (-15)^2 \\
&= 289 - 225 \\
x &= \pm 8
\end{aligned}
$$

   We are told that $\cos\theta < 0$. Therefore the angle is in either the second or the third quadrant. From the value of y we see that the angle must lie in the third quadrant and $x = -8$.



   Now we can determine $\cos\theta$:

$$
\begin{aligned}
\cos\theta &= \frac{x}{r} \\
&= \frac{-8}{17}
\end{aligned}
$$

b) $\tan\theta$

Solution:

From the first part we have $x = -8$, $y = -15$ and $r = 17$ so we can find $\tan\theta$.

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ &= \frac{-15}{-8}\\ &= \frac{15}{8}\end{aligned}$$

c) $\cos^2\theta + \sin^2\theta$

Solution:

From the first part we have $x = -8$, $y = -15$ and $r = 17$ so we can find $\cos^2\theta + \sin^2\theta$.

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\\ &= \frac{x^2}{r^2} + \frac{y^2}{r^2}\\ &= \frac{x^2 + y^2}{r^2}\\ &= \frac{(-8)^2 + (-15)^2}{(17)^2}\\ &= \frac{64 + 225}{289}\\ &= 1\end{aligned}$$

6. Find the value of $\sin A + \cos A$ without using a calculator, given that $13\sin A - 12 = 0$, where $\cos A < 0$.

Solution:

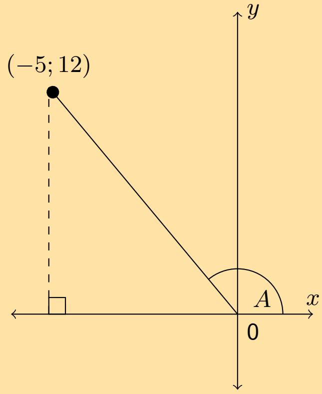
We first need to determine x, y and r. We are given $13\sin A - 12 = 0$ and so we can use this to find y and r.

$$\begin{aligned}13\sin A - 12 &= 0\\ \sin A &= \frac{12}{13}\end{aligned}$$

Therefore $y = 12$ and $r = 13$.

$$\begin{aligned}x^2 &= r^2 - y^2\\ &= (13)^2 - (12)^2\\ &= 169 - 144\\ x &= \pm 5\end{aligned}$$

We are told that $\cos A < 0$. Therefore the angle is in either the second or the third quadrant. From the value of y we see that the angle must lie in the second quadrant.



Now we can determine $\sin A + \cos A$:

$$\begin{aligned}\sin A + \cos A &= \frac{y}{r} + \frac{x}{r} \\ &= \frac{y+x}{r} \\ &= \frac{12-5}{13} \\ &= \frac{7}{13}\end{aligned}$$

7. If $17\cos\theta = -8$ and $\tan\theta > 0$ determine the following with the aid of a diagram (not a calculator):

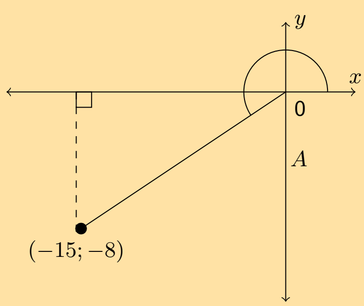
   a) $\dfrac{\cos\theta}{\sin\theta}$
   Solution:
   We first need to determine x, y and r. We are given $17\cos\theta = -8$ and so we can use this to find x and r.

$$\begin{aligned}17\cos\theta &= -8 = 0 \\ \cos\theta &= \frac{-8}{17}\end{aligned}$$

   Therefore $x = -8$ and $r = 17$.

$$\begin{aligned}y^2 &= r^2 - x^2 \\ &= (17)^2 - (8)^2 \\ y &= \pm 15\end{aligned}$$

   We are told that $\tan\theta > 0$. Therefore the angle is in either the first or the third quadrant. From the value of x we see that the angle must lie in the third quadrant.



   Now we can determine $\dfrac{\cos\theta}{\sin\theta}$:

$$\begin{aligned}\frac{\cos\theta}{\sin\theta} &= \cos\theta \times \frac{1}{\sin\theta} \\ &= \frac{y}{r} \times \frac{1}{\frac{x}{r}} \\ &= \frac{y}{r} \times \frac{r}{x} \\ &= \frac{y}{x} \\ &= \frac{-8}{-15} \\ &= \frac{8}{15}\end{aligned}$$

   b) $17\sin\theta - 16\tan\theta$
   Solution:
   From the first part we have that $x = -15$, $y = -8$ and $r = 17$.

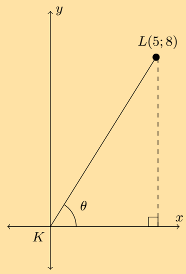
$$\begin{aligned}17\sin\theta - 16\tan\theta &= 17\frac{y}{r} - 16\frac{y}{x}\\ &= 17\left(\frac{-15}{17}\right) - 16\left(\frac{-15}{-8}\right)\\ &= -15 - 2(15)\\ &= -45\end{aligned}$$

8. L is a point with co-ordinates $(5;8)$ on a Cartesian plane. LK forms an angle, θ, with the positive x-axis. Set up a diagram and use it to answer the following questions.

   a) Find the distance LK.

   Solution:

   We are given $L(5;8)$. Therefore the angle lies in the first quadrant. We can sketch this and use our sketch to find x, y and r.

   Therefore $x = 5$ and $y = 8$. We can calculate r using the theorem of Pythagoras. From the diagram we note that $LK = r$.

$$\begin{aligned}LK^2 &= 5^2 + 8^2\\ &= 89\\ LK &= \sqrt{89}\end{aligned}$$

   b) $\sin\theta$

   Solution:

   From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\sin\theta &= \frac{y}{r}\\ &= \frac{8}{\sqrt{89}}\end{aligned}$$

   c) $\cos\theta$

   Solution:

   From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\cos\theta &= \frac{x}{r}\\ &= \frac{5}{\sqrt{89}}\end{aligned}$$

   d) $\tan\theta$

   Solution:

   From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ &= \frac{8}{5}\end{aligned}$$

e) $\text{cosec}\,\theta$

Solution:

From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\text{cosec}\,\theta &= \frac{r}{y}\\ &= \frac{\sqrt{89}}{8}\end{aligned}$$

f) $\sec\theta$

Solution:

From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\sec\theta &= \frac{r}{x}\\ &= \frac{\sqrt{89}}{5}\end{aligned}$$

g) $\cot\theta$

Solution:

From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\cot\theta &= \frac{x}{y}\\ &= \frac{5}{8}\end{aligned}$$

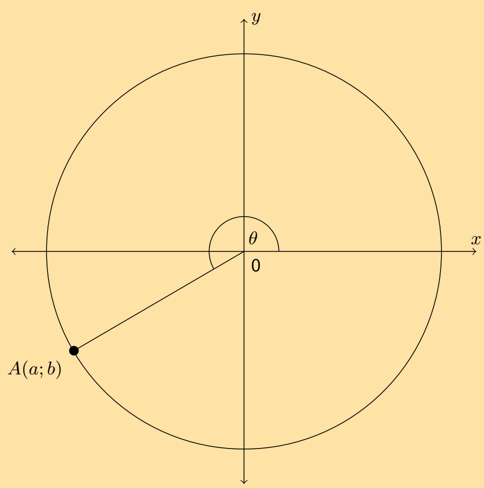
h) $\sin^2\theta + \cos^2\theta$

Solution:

From the previous question we have that $x = 5$, $y = 8$ and $r = \sqrt{89}$.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2\\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2}\\ &= \frac{y^2 + x^2}{r^2}\\ &= \frac{64 + 25}{89}\\ &= 1\end{aligned}$$

9. Given the following diagram and that $\cos\theta = -\frac{24}{25}$.



a) State two sets of possible values of a and b.

Solution:

We first need to use the given information to find a possible set of values for a and b.

Using $\cos\theta = -\frac{24}{25}$ and the fact that $\cos\theta = \frac{x}{r}$ we can determine that $x = -24$ and $r = 25$. Now we can find y:

$$\begin{aligned} y^2 &= r^2 - x^2 \\ &= (25)^2 - (24)^2 \\ &= 625 - 576 \\ &= 49 \\ y &= \pm 7 \end{aligned}$$

From the diagram we see that y must be negative.

This gives us one possible set of values for a and b: $a = 24$ and $b = -7$.

Now we note that we can simply double the size of the circle and the trigonometric ratios will stay the same. We could even multiply the radius of the circle by any integer and the trigonometric ratios will still remain the same.

Therefore the possible sets of values for $A(a,b)$ are multiples of $(-24;-7)$. Two possible sets are $(-24;-7)$ and $(-48;-14)$.

b) If $OA = 100$, state the values of a and b.

Solution:

First note that in the original diagram $OA = 25$. Now we are multiplying OA by 4. This also means that the x and y values must be multiplied by 4.

Therefore $a = 4(-24) = -96$ and $b = 4(-7) = -28$.

c) Hence determine without the use of a calculator the value of $\sin\theta$.

Solution:

The question states: "hence". This means we must use the scaled values for a and b not the original values. We know that $x = -96$, $y = -28$ and $r = 100$.

$$\begin{aligned} \sin\theta &= \frac{y}{r} \\ &= \frac{-28}{100} \\ &= \frac{-7}{25} \end{aligned}$$

Notice how the answer reduced to the original values of y and r as we would expect from the first question.

10. If $\tan\alpha = \frac{5}{-12}$ and $0° \leq \alpha \leq 180°$, determine **without the use of a calculator** the value of $\frac{12}{\cos\alpha}$

Solution:

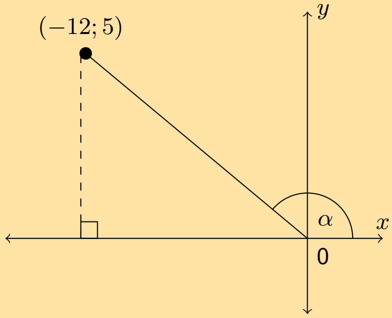
We first need to determine x, y and r. We are given $\tan\alpha = \frac{5}{-12}$ and so we can use this to find x and y.

$$\begin{aligned} \tan\alpha &= \frac{5}{-12} \\ \frac{y}{x} &= \frac{5}{-12} \end{aligned}$$

Therefore $y = 5$ and $x = -12$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-12)^2 + (5)^2 \\ &= 144 + 25 \\ r &= 13 \end{aligned}$$

We are told that $0° \leq \alpha \leq 180°$. Therefore the angle is in either the first or the second quadrant. From the values of x and y we see that the angle must lie in the second quadrant.



Now we can determine $\frac{12}{\cos\alpha}$:

$$\begin{aligned}\frac{12}{\cos\alpha} &= \frac{12}{\frac{x}{r}} \\ &= \frac{12r}{x} \\ &= \frac{12(13)}{-12} \\ &= -13\end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FSP 2. 2FSQ 3. 2FSR 4. 2FSS 5. 2FST 6. 2FSV
7. 2FSW 8. 2FSX 9. 2FSY 10. 2FSZ

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5.9 Chapter summary

End of chapter Exercise 5 – 8:

1. State whether each of the following trigonometric ratios has been written correctly.

   a) $\sin\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$

   Solution:

   We recall the definition of the sine ratio: $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$. Therefore this trigonometric ratio has not been written correctly.

   b) $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

   Solution:

   We recall the definition of the tangent ratio: $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$. Therefore this trigonometric ratio has been written correctly.

   c) $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$

   Solution:

   We recall the definition of the secant ratio: $\sec\theta = \frac{\text{hypotenuse}}{\text{opposite}}$. Therefore this trigonometric ratio has not been written correctly.

2. Use your calculator to evaluate the following expressions to two decimal places:

a) $\tan 80°$

Solution:

$$\begin{aligned}\tan 80° &= 5{,}6712... \\ &\approx 5{,}67\end{aligned}$$

b) $\cos 73°$

Solution:

$$\begin{aligned}\cos 73° &= 0{,}29237... \\ &\approx 0{,}29\end{aligned}$$

c) $\sin 17°$

Solution:

$$\begin{aligned}\sin 17° &= 0{,}2923... \\ &\approx 0{,}29\end{aligned}$$

d) $\tan 313°$

Solution:

$$\begin{aligned}\tan 313° &= -1{,}07236... \\ &\approx -1{,}07\end{aligned}$$

e) $\cos 138°$

Solution:

$$\begin{aligned}\cos 138° &= -0{,}743144... \\ &\approx -0{,}74\end{aligned}$$

f) $\sec 56°$

Solution:

$$\begin{aligned}\sec 56° &= \frac{1}{\cos 56°} \\ &= \frac{1}{0{,}5591...} \\ &= 1{,}78829... \\ &\approx 1{,}79\end{aligned}$$

g) $\cot 18°$

Solution:

$$\begin{aligned}\cot 18° &= \frac{1}{\tan 18°} \\ &= \frac{1}{0{,}32491...} \\ &= 3{,}07768... \\ &\approx 3{,}08\end{aligned}$$

h) $\operatorname{cosec} 37°$

Solution:

$$\begin{aligned}\operatorname{cosec} 37° &= \frac{1}{\sin 37°} \\ &= \frac{1}{0{,}6018...} \\ &= 1{,}66164... \\ &\approx 1{,}66\end{aligned}$$

i) $\sec 257°$
Solution:

$$\begin{aligned}\sec 257° &= \frac{1}{\cos 257°}\\ &= \frac{1}{-0{,}224951...}\\ &= -4{,}445411...\\ &\approx -4{,}45\end{aligned}$$

j) $\sec 304°$
Solution:

$$\begin{aligned}\sec 304° &= \frac{1}{\cos 304°}\\ &= \frac{1}{0{,}559193...}\\ &= 1{,}788292...\\ &\approx 1{,}79\end{aligned}$$

k) $3\sin 51°$
Solution:

$$\begin{aligned}3\sin 51° &= 2{,}3314...\\ &\approx 2{,}33\end{aligned}$$

l) $4\cot 54° + 5\tan 44°$
Solution:

$$\begin{aligned}4\cot 54° + 5\tan 44° &= \frac{4}{\tan 54°} + 5\tan 44°\\ &= 7{,}7346...\\ &\approx 7{,}73\end{aligned}$$

m) $\frac{\cos 205°}{4}$
Solution:

$$\begin{aligned}\frac{\cos 205°}{4} &= -0{,}22657...\\ &\approx -0{,}23\end{aligned}$$

n) $\sqrt{\sin 99°}$
Solution:

$$\begin{aligned}\sqrt{\sin 99°} &= \sqrt{0{,}98768...}\\ &= 0{,}9938...\\ &\approx 0{,}99\end{aligned}$$

o) $\sqrt{\cos 687° + \sin 120°}$
Solution:

$$\begin{aligned}\sqrt{\cos 687° + \sin 120°} &= \sqrt{1{,}7046...}\\ &= 1{,}3056...\\ &\approx 1{,}31\end{aligned}$$

p) $\dfrac{\tan 70^\circ}{\text{cosec } 1^\circ}$

Solution:

$$\begin{aligned}\frac{\tan 70^\circ}{\text{cosec } 1^\circ} &= \tan 70^\circ \times \frac{1}{\frac{1}{\sin 1^\circ}} \\ &= \tan 70^\circ \times \sin 1^\circ \\ &= 0{,}04795... \\ &\approx 0{,}05\end{aligned}$$

q) $\sec 84^\circ + 4 \sin 0{,}4^\circ \times 50 \cos 50^\circ$

Solution:

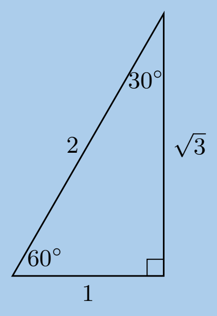
$$\begin{aligned}\sec 84^\circ + 4 \sin 0{,}4^\circ \times 50 \cos 50^\circ &= \frac{1}{\cos 84^\circ} + 4 \sin 0{,}4^\circ \times 50 \cos 50^\circ \\ &= 9{,}56677... + 0{,}89749... \\ &= 10{,}46426... \\ &\approx 10{,}46\end{aligned}$$

r) $\dfrac{\cos 40^\circ}{\sin 35^\circ} + \tan 38^\circ$

Solution:

$$\begin{aligned}\frac{\cos 40^\circ}{\sin 35^\circ} + \tan 38^\circ &= 1{,}3355... + 0{,}7812... \\ &= 2{,}1168... \\ &\approx 2{,}12\end{aligned}$$

3. Use the triangle below to complete the following:



a) $\sin 60^\circ =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

b) $\cos 60^\circ =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\cos 60^\circ = \frac{1}{2}$$

c) $\tan 60^\circ =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

d) $\sin 30° =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\sin 30° = \frac{1}{2}$$

e) $\cos 30° =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\cos 30° = \frac{\sqrt{3}}{2}$$

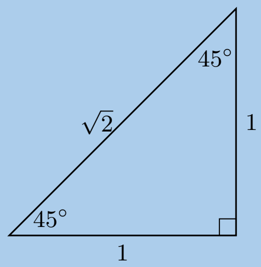
f) $\tan 30° =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\tan 30° = \frac{1}{\sqrt{3}}$$

4. Use the triangle below to complete the following:



a) $\sin 45° =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\sin 45° = \frac{1}{\sqrt{2}}$$

b) $\cos 45° =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\cos 45° = \frac{1}{\sqrt{2}}$$

c) $\tan 45° =$

Solution:

Remember to first identify the hypotenuse, opposite and adjacent sides for the given angle. Then write down the correct fraction for each ratio. You can confirm your answer by using your calculator to find the value of the ratio for that angle.

$$\tan 45° = \frac{1}{1} = 1$$

5. Evaluate the following without using a calculator. Select the closest answer from the list provided.

   a) $\sin 60° - \tan 60°$

$$0 \quad -\frac{1}{2} \quad \frac{2}{\sqrt{3}} \quad -\frac{\sqrt{3}}{2} \quad -\frac{2}{\sqrt{3}}$$

Solution:

$$\begin{aligned}\sin 60° - \tan 60° &= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{1} \\ &= \frac{\sqrt{3} - 2\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

   b) $\tan 30° - \cos 30°$

$$0 \quad -\frac{1}{2\sqrt{3}} \quad \frac{\sqrt{3}}{2} \quad -\frac{2}{\sqrt{3}} \quad -\frac{\sqrt{3}}{2}$$

Solution:

$$\begin{aligned}\tan 30° - \cos 30° &= \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2} \\ &= \frac{2 - \left(\sqrt{3}\right)\left(\sqrt{3}\right)}{2\sqrt{3}} \\ &= -\frac{1}{2\sqrt{3}}\end{aligned}$$

   c) $\tan 60° - \sin 60° - \tan 60°$

$$-\frac{\sqrt{3}}{2} \quad -\frac{\sqrt{3}}{1} \quad -\frac{1}{2} \quad -\frac{1}{1} \quad -\frac{1}{\sqrt{2}}$$

Solution:

$$\begin{aligned}\tan 60° - \sin 60° - \tan 60° &= \frac{\sqrt{3}}{1} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{1} \\ &= \frac{2\sqrt{3} - \sqrt{3} - 2\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

   d) $\sin 30° \times \sin 30° \times \sin 30°$

$$\frac{1}{2} \quad \frac{1}{2\sqrt{3}} \quad \frac{1}{8} \quad \frac{1}{4} \quad \frac{\sqrt{3}}{4\sqrt{2}}$$

Solution:

$$\begin{aligned}\sin 30° \times \sin 30° \times \sin 30° &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8}\end{aligned}$$

e) $\sin 45° \times \tan 45° \times \tan 60°$

$$\frac{3}{2\sqrt{2}} \quad \frac{\sqrt{3}}{8} \quad \frac{3}{4} \quad \frac{\sqrt{3}}{\sqrt{2}} \quad \frac{1}{4}$$

Solution:

$$\begin{aligned}\sin 45° \times \tan 45° \times \tan 60° &= \frac{1}{\sqrt{2}} \times \frac{1}{1} \times \frac{\sqrt{3}}{1} \\ &= \frac{\sqrt{3}}{\sqrt{2}}\end{aligned}$$

f) $\cos 60° \times \cos 45° \times \tan 60°$

$$\frac{\sqrt{3}}{2\sqrt{2}} \quad \frac{\sqrt{3}}{4} \quad \frac{3}{4\sqrt{2}} \quad \frac{1}{2} \quad \frac{1}{4\sqrt{3}}$$

Solution:

$$\begin{aligned}\cos 60° \times \cos 45° \times \tan 60° &= \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{1} \\ &= \frac{\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

g) $\tan 45° \times \sin 60° \times \tan 45°$

$$\frac{\sqrt{3}}{2} \quad \frac{3}{8} \quad \frac{1}{3} \quad \frac{\sqrt{3}}{2\sqrt{2}} \quad \frac{1}{4\sqrt{3}}$$

Solution:

$$\begin{aligned}\tan 45° \times \sin 60° \times \tan 45° &= \frac{1}{1} \times \frac{\sqrt{3}}{2} \times \frac{1}{1} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

h) $\cos 30° \times \cos 60° \times \sin 60°$

$$\frac{3}{8} \quad \frac{3}{2\sqrt{2}} \quad \frac{\sqrt{3}}{4\sqrt{2}} \quad \frac{1}{2\sqrt{3}} \quad \frac{1}{4\sqrt{3}}$$

Solution:

$$\begin{aligned}\cos 30° \times \cos 60° \times \sin 60° &= \frac{\sqrt{3}}{2} \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \\ &= \frac{3}{8}\end{aligned}$$

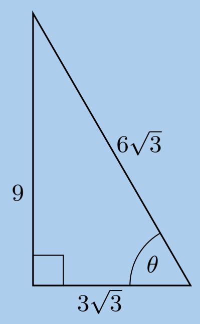
6. Without using a calculator, determine the value of:

$$\sin 60° \cos 30° - \cos 60° \sin 30° + \tan 45°$$

Solution:
These are all special angles.

$$\begin{aligned}\sin 60° \cos 30° - \cos 60° \sin 30° + \tan 45° &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + 1 \\ &= \frac{3}{4} - \frac{1}{4} + 1 \\ &= \frac{2}{4} + 1 \\ &= \frac{3}{2}\end{aligned}$$

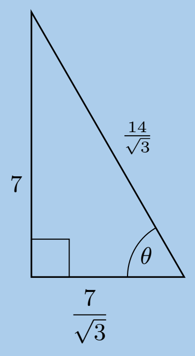
7. Solve for $\sin\theta$ in the following triangle, in surd form:



Solution:

$$\begin{aligned}\sin\theta &= \frac{9}{6\sqrt{3}}\\ &= \frac{3}{2\sqrt{3}}\end{aligned}$$

8. Solve for $\tan\theta$ in the following triangle, in surd form:



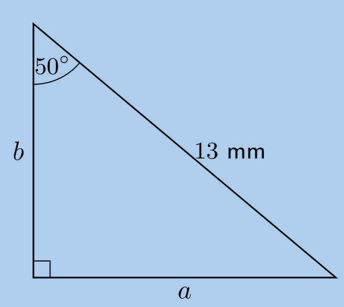
Solution:

$$\begin{aligned}\tan\theta &= \frac{7}{\frac{7}{\sqrt{3}}}\\ &= 7 \times \frac{\sqrt{3}}{7}\\ &= \sqrt{3}\end{aligned}$$

9. A right-angled triangle has hypotenuse 13 mm. Find the length of the other two sides if one of the angles of the triangle is 50°.

Solution:

First draw a diagram:



Next we get:

$$\begin{aligned}
\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\sin 50° &= \frac{a}{13} \\
a &= 13\sin 50° \\
&= 9{,}9585... \\
&\approx 9{,}96 \text{ mm}
\end{aligned}$$

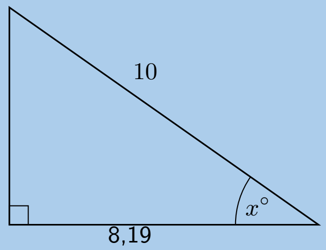
Now we can use the theorem of Pythagoras to find the other side:

$$\begin{aligned}
b^2 &= c^2 - a^2 \\
&= (13)^2 - (9{,}9585...)^2 \\
&= 69{,}8267... \\
b &= 8{,}3562... \\
&= 8{,}36 \text{ mm}
\end{aligned}$$

Therefore the other two sides are 9,96 mm and 8,35 mm.

10. Solve for x to the nearest integer.

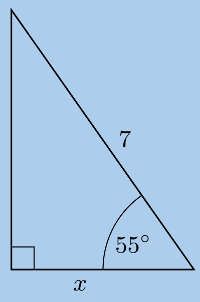
a)



Solution:

$$\begin{aligned}
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\cos x &= \frac{8{,}19}{10} \\
&= 0{,}819 \\
x &= 35{,}0151... \\
&\approx 35°
\end{aligned}$$

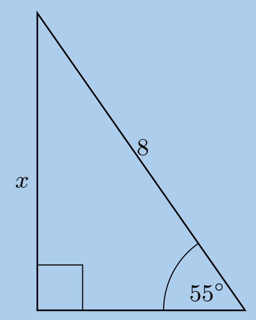
b)



Solution:

$$\begin{aligned}
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\cos 55^\circ &= \frac{x}{7} \\
7\cos 55^\circ &= x \\
x &= 4{,}01503... \\
&\approx 4
\end{aligned}$$

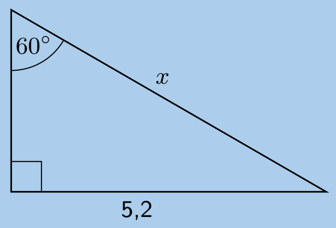
c)



Solution:

$$\begin{aligned}
\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\sin 55^\circ &= \frac{x}{8} \\
8\sin 55^\circ &= x \\
x &= 6{,}55321... \\
&\approx 7
\end{aligned}$$

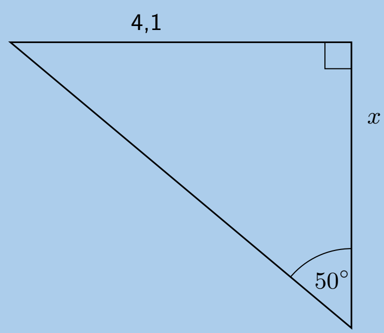
d)



Solution:

$$\begin{aligned}
\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\sin 60^\circ &= \frac{5{,}2}{x} \\
x &= \frac{5{,}2}{\sin 60^\circ} \\
&= 6{,}00444... \\
&\approx 6
\end{aligned}$$

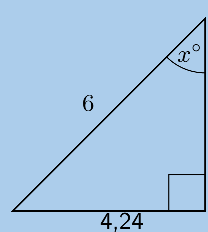
e)



Solution:

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}}\\ \tan 50^\circ &= \frac{4{,}1}{x}\\ x &= \frac{4{,}1}{\tan 50^\circ}\\ &= 3{,}4403...\\ &\approx 3\end{aligned}$$

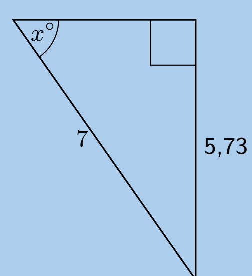
f)



Solution:

$$\begin{aligned}\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\ \sin x &= \frac{4{,}24}{6}\\ &= 0{,}7067...\\ x &= 44{,}96434...\\ &\approx 45^\circ\end{aligned}$$

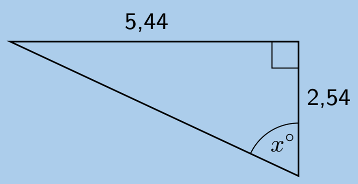
g)



Solution:

$$\begin{aligned}
\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\
\sin x &= \frac{5{,}73}{7}\\
&= 0{,}81857\ldots\\
x &= 54{,}9420\ldots\\
&\approx 55^\circ
\end{aligned}$$

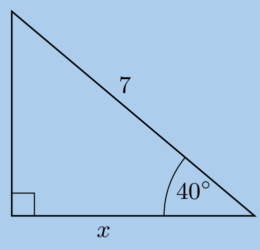
h)



Solution:

$$\begin{aligned}
\tan\theta &= \frac{\text{opposite}}{\text{adjacent}}\\
\tan x &= \frac{5{,}44}{2{,}54}\\
&= 2{,}14173\ldots\\
x &= 64{,}9715\ldots\\
&\approx 65^\circ
\end{aligned}$$

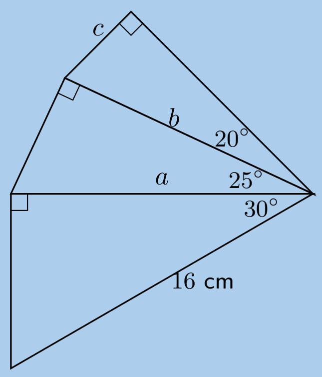
i)

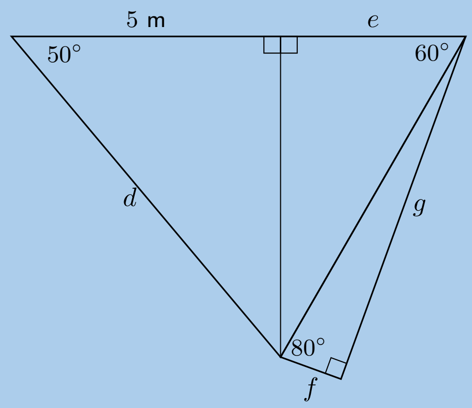


Solution:

$$\begin{aligned}
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}\\
\cos 40^\circ &= \frac{x}{7}\\
7\cos 40^\circ &= x\\
x &= 5{,}36231\ldots\\
&\approx 5
\end{aligned}$$

11. Calculate the unknown lengths in the diagrams below:





Solution:

For all of these we use the appropriate trigonometric ratio or the theorem of Pythagoras to solve.

To find a and b we use $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$:

$$\begin{aligned} \cos 30° &= \frac{a}{16} \\ a &= 16\cos 30° \\ &\approx 13{,}86 \text{ cm} \end{aligned}$$

$$\begin{aligned} \cos 25° &= \frac{b}{13{,}86} \\ b &= 13{,}86\cos 25° \\ &\approx 12{,}56 \text{ cm} \end{aligned}$$

To find c we use $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$:

$$\begin{aligned} \sin 20° &= \frac{c}{12{,}56} \\ c &= 12{,}56\sin 20° \\ &\approx 4{,}30 \text{ cm} \end{aligned}$$

To find d we use $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$$\begin{aligned} \cos 50° &= \frac{5}{d} \\ d\cos 50 &= 5 \\ d &= \frac{5}{\cos 50°} \\ &\approx 7{,}78 \text{ cm} \end{aligned}$$

Next we use the theorem of Pythagoras to find the third side, so we can use trig functions to find e:

$$\begin{aligned} (5)^2 + (7{,}78)^2 &= 85{,}5284 \\ \sqrt{85{,}5284} &\approx 9{,}25 \end{aligned}$$

We use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find e:

$$\begin{aligned} \tan 60° &= \frac{9{,}25}{e} \\ e\tan 60° &= 9{,}25 \\ e &= \frac{9{,}25}{\tan 60°} \\ &\approx 5{,}34 \text{ cm} \end{aligned}$$

Next we use the theorem of Pythagoras to find the third side, so we can use trig functions to find f and g:

$$\begin{aligned}(5{,}34)^2 + (7{,}78)^2 &= 89{,}0044...\\ \sqrt{89{,}0044...} &\approx 9{,}44\end{aligned}$$

We use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find g:

$$\begin{aligned}\tan 80° &= \frac{9{,}44}{g}\\ g\tan 80° &= 9{,}44\\ g &= \frac{9{,}44}{\tan 80°}\\ &\approx 1{,}66 \text{ cm}\end{aligned}$$

And finally we find f using the theorem of Pythagoras:

$$\begin{aligned}f^2 &= (9{,}44)^2 - (1{,}65)^2\\ f &= \sqrt{86{,}39}\\ &\approx 9{,}29 \text{ cm}\end{aligned}$$

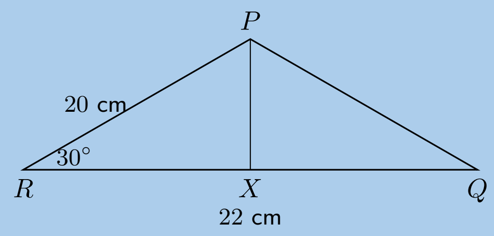
The final answers are: $a = 13{,}86$, $b = 12{,}56$, $c = 4{,}30$, $d = 7{,}78$, $e = 5{,}34$, $f = 9{,}29$ and $g = 1{,}66$.

12. In $\triangle PQR$, $PR = 20$ cm, $QR = 22$ cm and $P\hat{R}Q = 30°$. The perpendicular line from P to QR intersects QR at X. Calculate:

a) the length XR

Solution:

First draw a sketch:



Since we are told that $PX \perp QR$ we can use $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ to find XR.

$$\begin{aligned}\cos 30° &= \frac{XR}{20}\\ XR &= 20\cos 30°\\ &= 17{,}3205...\\ &\approx 17{,}32 \text{ cm}\end{aligned}$$

b) the length PX

Solution:

We can use $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ to find PX.

$$\begin{aligned}\sin 30° &= \frac{PX}{20}\\ PX &= 20\sin 30°\\ &= 9{,}999...\\ &\approx 10 \text{ cm}\end{aligned}$$

c) the angle $Q\hat{P}X$

Solution:

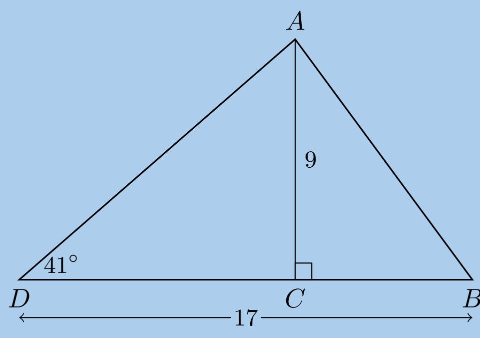
We know the length of QR and we have found the length of XR, so we can work out the length of QX:

$$
\begin{aligned}
QX &= QR - XR \\
&= (22) - (17{,}32) \\
&= 4{,}68
\end{aligned}
$$

Since we know two sides and an angle we can use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find the angle:

$$
\begin{aligned}
\tan(Q\hat{P}X) &= \frac{4{,}68}{10} \\
&= 0{,}468 \\
Q\hat{P}X &= 25{,}0795... \\
&\approx 25{,}08°
\end{aligned}
$$

13. In the following triangle find the size of $A\hat{B}C$.



Solution:
We use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find DC:

$$
\begin{aligned}
\tan 41° &= \frac{9}{DC} \\
DC &= 9\tan 41° \\
&= 7{,}8235...
\end{aligned}
$$

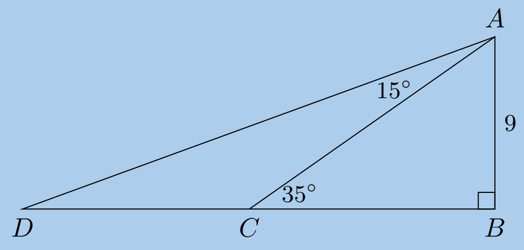
Next we find BC:

$$
\begin{aligned}
BC &= BD - DC \\
&= 17 - 7{,}8235... \\
&= 9{,}1764...
\end{aligned}
$$

And then we use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find the angle:

$$
\begin{aligned}
\tan A\hat{B}C &= \frac{9}{9{,}1764...} \\
&= 0{,}98077... \\
A\hat{B}C &= 44{,}439... \\
&\approx 44{,}44°
\end{aligned}
$$

14. In the following triangle find the length of side CD:



Solution:

We use the angles in a triangle to find $C\hat{A}B$:

$$C\hat{A}B = 180° - 90° - 35° = 55°$$

Then we find $D\hat{A}B$:

$$D\hat{A}B = 15° + 55° = 70°$$

Now we can use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find BC:

$$\begin{aligned} \tan 35° &= \frac{9}{BC} \\ BC &= \frac{9}{\tan 35°} \\ BC &= 12{,}85 \end{aligned}$$

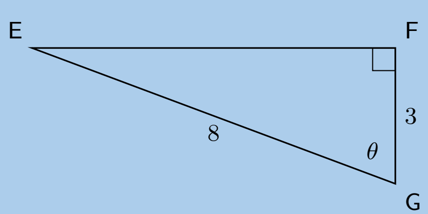
Then we find BD also using $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$:

$$\begin{aligned} \tan 70° &= \frac{BD}{9} \\ BD &= 9\tan 70° \\ BD &= 24{,}73 \end{aligned}$$

Finally we can find CD:

$$\begin{aligned} CD &= BD - BC \\ &= 24{,}73 - 12{,}85 \\ &= 11{,}88 \end{aligned}$$

15. Determine



a) The length of EF

Solution:

$$
\begin{aligned}
GE^2 &= EF^2 + FG^2 \\
EF^2 &= GE^2 - FG^2 \\
EF &= \sqrt{GE^2 - FG^2} \\
&= \sqrt{8^2 - 3^2} \\
&= \sqrt{64 - 9} \\
&= \sqrt{55}
\end{aligned}
$$

b) $\tan(90^\circ - \theta)$

Solution:

We note that $\hat{G} = \theta$ and $\hat{F} = 90^\circ$, therefore $\hat{E} = 90^\circ - \theta$. So we need to find $\tan \hat{E}$:

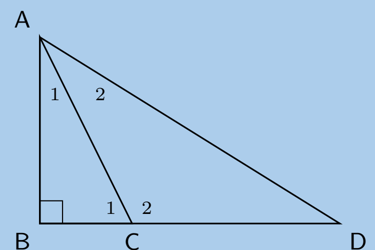
$$
\tan(90^\circ - \theta) = \frac{3}{\sqrt{55}}
$$

c) The value of θ

Solution:

$$
\begin{aligned}
\cos\theta &= \frac{3}{8} \\
\theta &= \cos^{-1}\frac{3}{8} \\
\theta &= 67{,}976^\circ
\end{aligned}
$$

16. Given that $\hat{D} = x,\ \hat{C}_1 = 2x,\ BC = 12{,}2$ cm, $AB = 24{,}6$ cm. Calculate CD.



Solution:

We first calculate $\hat{C}_1$ by using the given information about AB and BC.

$$
\begin{aligned}
\tan\hat{C}_1 &= \frac{AB}{BC} \\
&= \frac{24{,}6}{12{,}2} \\
\hat{C}_1 &= 63{,}62257...
\end{aligned}
$$

Next we find $\hat{D}$:

$$
\begin{aligned}
\hat{D} &= \frac{\hat{C}_1}{2} \\
&= \frac{63{,}62257...}{2} \\
&= 31{,}8107...
\end{aligned}
$$

Now we can calculate BD:

$$\tan \hat{D} = \frac{AB}{BD}$$
$$BD = \frac{AB}{\tan \hat{D}}$$
$$= \frac{24{,}6}{\tan 31{,}8107...}$$
$$= 39{,}65906...$$

Finally we can calculate CD:

$$\begin{aligned} CD &= BD - BC \\ &= 39{,}65906... - 12{,}2 \\ &= 27{,}45906... \\ &\approx 27{,}46 \text{ cm} \end{aligned}$$

17. Solve for θ if θ is a positive, acute angle:

a) $2 \sin \theta = 1{,}34$

Solution:

$$\begin{aligned} 2 \sin \theta &= 1{,}34 \\ \sin \theta &= 0{,}67 \\ \theta &= 42{,}06706 \ldots \\ &= 42{,}07° \end{aligned}$$

b) $1 - \tan \theta = -1$

Solution:

$$\begin{aligned} 1 - \tan \theta &= -1 \\ - \tan \theta &= -2 \\ \tan \theta &= 2 \\ \theta &= 63{,}43494 \ldots \\ &= 63{,}43° \end{aligned}$$

c) $\cos 2\theta = \sin 40°$

Solution:

$$\begin{aligned} \cos 2\theta &= \sin 40° \\ &= 0{,}64278 \ldots \\ 2\theta &= 50 \\ \theta &= 25° \end{aligned}$$

d) $\sec \theta = 1{,}8$

Solution:

$$\begin{aligned} \sec \theta &= 1{,}8 \\ \frac{1}{\cos \theta} &= 1{,}8 \\ 1 &= 1{,}8 \cos \theta \\ \frac{1}{1{,}8} &= \cos \theta \\ \theta &= 56{,}25101... \\ &\approx 56{,}25° \end{aligned}$$

e) $\cot 4\theta = \sin 40°$
Solution:

$$\begin{aligned}
\cot 4\theta &= \sin 40° \\
\cot 4\theta &= 0{,}642787... \\
\frac{1}{\tan 4\theta} &= 0{,}642787... \\
\frac{1}{0{,}642787...} &= \tan 4\theta \\
4\theta &= 57{,}2675... \\
\theta &= 14{,}3168... \\
&\approx 14{,}32°
\end{aligned}$$

f) $\sin 3\theta + 5 = 4$
Solution:

$$\begin{aligned}
\sin 3\theta + 5 &= 4 \\
\sin 3\theta &= -1 \\
3\theta &= 90 \\
\theta &= 30°
\end{aligned}$$

g) $\cos(4 + \theta) = 0{,}45$
Solution:

$$\begin{aligned}
\cos(4 + \theta) &= 0{,}45 \\
4 + \theta &= 63{,}25631... \\
\theta &= 59{,}25631... \\
&\approx 59{,}26
\end{aligned}$$

h) $\frac{\sin \theta}{\cos \theta} = 1$
Solution:
First we note that:

$$\begin{aligned}
\frac{\sin \theta}{\cos \theta} &= \sin \theta \times \frac{1}{\cos \theta} \\
&= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} \\
&= \frac{\text{opposite}}{\text{adjacent}} \\
&= \tan \theta
\end{aligned}$$

Now we can solve for θ:

$$\begin{aligned}
\frac{\sin \theta}{\cos \theta} &= 1 \\
\tan \theta &= 1 \\
\theta &= 45°
\end{aligned}$$

18. If $a = 29°$, $b = 38°$ and $c = 47°$, use your calculator to evaluate each of the following, correct to 2 decimal places.

a) $\tan(a + c)$
Solution:

$$\begin{aligned}
\tan(a + c) &= \tan(29 + 47) \\
&= \tan 76 \\
&= 4{,}0107... \\
&\approx 4{,}01
\end{aligned}$$

b) cosec $(c - b)$

Solution:

$$\begin{aligned}\text{cosec } (c - b) &= \sin(47 - 38)\\ &= \text{cosec } 9\\ &= \frac{1}{\sin 9}\\ &= 6{,}3924...\\ &\approx 6{,}39\end{aligned}$$

c) $\sin(a \times b \times c)$

Solution:

$$\begin{aligned}\sin(a \times b \times c) &= \sin((29)(38)(47))\\ &= \sin(114)\\ &= 0{,}9135...\\ &\approx 0{,}91\end{aligned}$$

d) $\tan a + \sin b + \cos c$

Solution:

$$\begin{aligned}\tan a + \sin b + \cos c &= \tan 29 + \sin 38 + \cos 47\\ &= 1{,}8519...\\ &\approx 1{,}85\end{aligned}$$

19. If $3 \tan \alpha = -5$ and $0° < \alpha < 270°$, use a sketch to determine:

a) $\cos \alpha$

Solution:

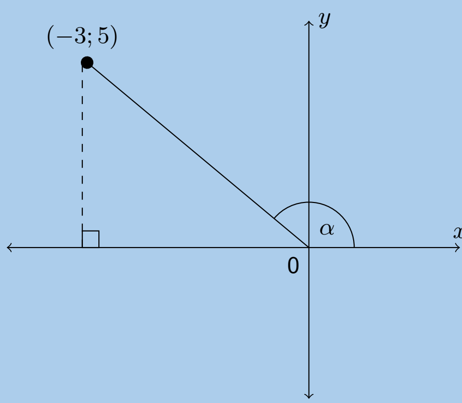
Find x, y and r

$$\begin{aligned}3 \tan \alpha &= -5\\ \tan \alpha &= \frac{-5}{3}\end{aligned}$$

Therefore $x = -3$ and $y = 5$.

$$\begin{aligned}r^2 &= x^2 + y^2\\ &= (-3)^2 + (5)^2\\ &= 34\\ r &= \sqrt{34}\end{aligned}$$

Draw a sketch:



Now we can find $\cos\alpha$:

$$\begin{aligned}\cos\alpha &= \frac{x}{r}\\ &= \frac{-3}{\sqrt{34}}\end{aligned}$$

b) $\tan^2\alpha - \sec^2\alpha$

Solution:

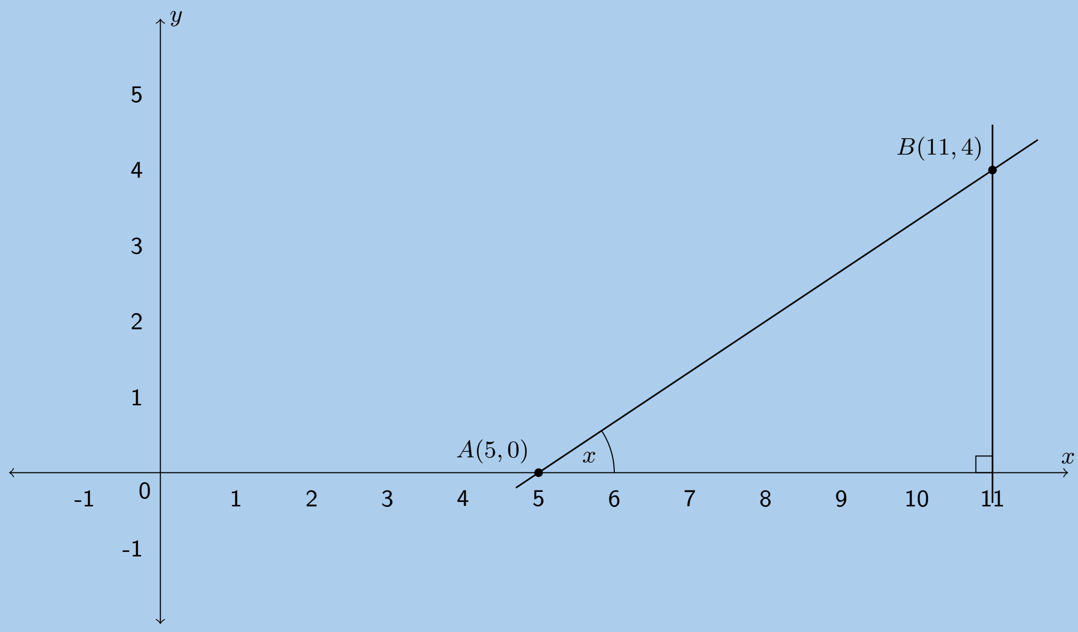
We have x, y and r from the first question.

$$\begin{aligned}\tan^2\alpha - \sec^2\alpha &= \left(\frac{y}{x}\right)^2 - \left(\frac{r}{x}\right)^2\\ &= \left(\frac{5}{-3}\right)^2 - \left(\frac{\sqrt{34}}{-3}\right)^2\\ &= \frac{25}{9} - \frac{34}{9}\\ &= \frac{-9}{9}\\ &= -1\end{aligned}$$

20. Given $A(5;0)$ and $B(11;4)$, find the angle between the line through A and B and the x-axis.

Solution:

First draw a diagram:



Next we note that the distance from B to the x-axis is 4 (B is 4 units up from the x-axis) and that the distance from A to C is $11 - 5 = 6$ units.

We use the tangent ratio to find the angle:

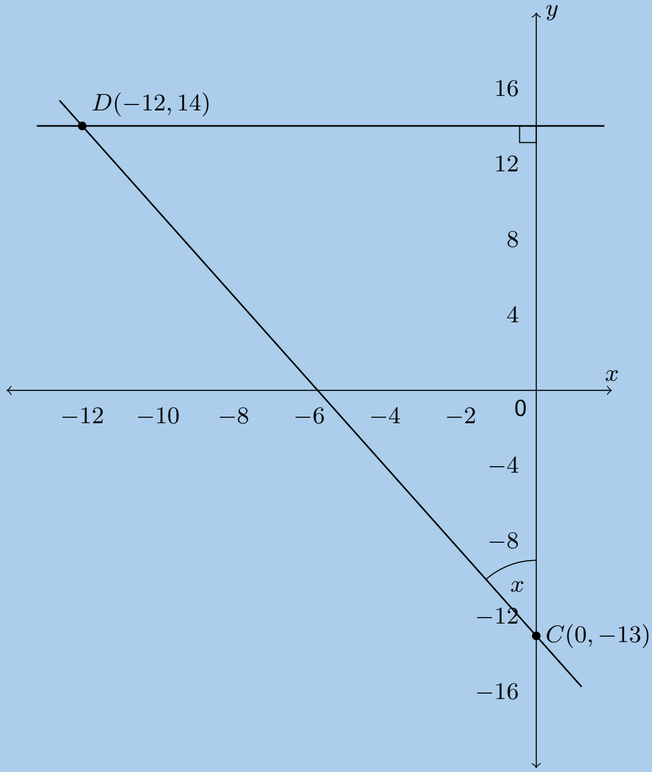
$$\begin{aligned}\tan x &= \frac{4}{6}\\ \tan x &= 0{,}66666\ldots\\ x &= 33{,}69°\end{aligned}$$

Therefore the angle between line AB and the x-axis is 33,69°.

21. Given $C(0;-13)$ and $D(-12;14)$, find the angle between the line through C and D and the y-axis.

Solution:

First draw a diagram:



Next we note that the distance from D to the x-axis is 12 (although D is $(-12; 14)$ the distance is positive). The distance from C to the point where the perpendicular line from D intercepts the y-axis is $14 - (-13) = 27$ units. We use the tangent ratio to find the angle:

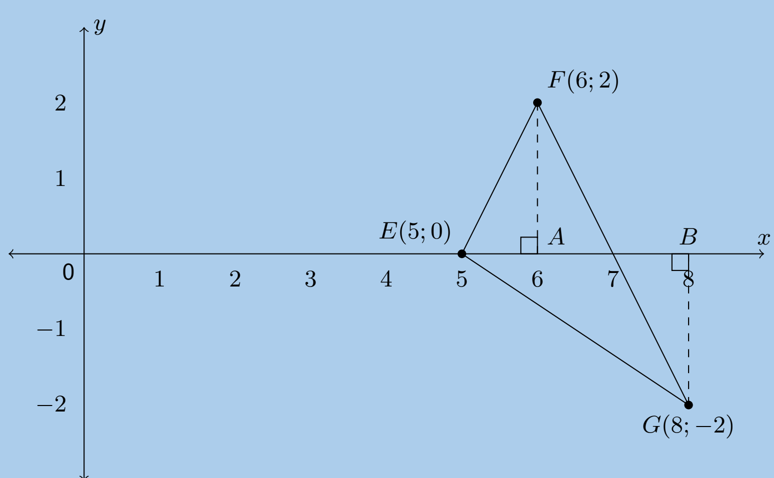
$$\begin{aligned}
\tan x &= \frac{12}{27} \\
\tan x &= 0{,}4444\ldots \\
x &= 23{,}96°
\end{aligned}$$

Therefore the angle between line CD and the x-axis is 23,96°.

22. Given the points $E(5; 0)$, $F(6; 2)$ and $G(8; -2)$. Find the angle $F\hat{E}G$.

Solution:

First draw a sketch:



To find $F\hat{E}G$ we look at $\triangle FEA$ and $\triangle GEB$ in turn. These two triangles will each give one part of the angle that we want.

In triangle FEA we can use the tangent ratio. FA is 2 units and EA is 1 unit.

$$\begin{aligned}\tan F\hat{E}X &= \frac{2}{1} \\ F\hat{E}X &= 63{,}43°\end{aligned}$$

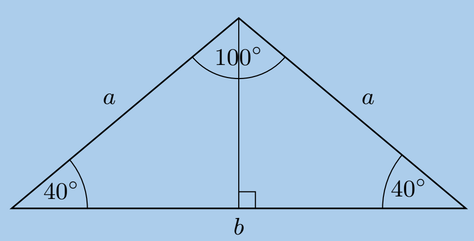
In triangle GEB we also use the tangent ratio. GB is 2 units and EB is 3 units.

$$\begin{aligned}\tan G\hat{E}X &= \frac{2}{3} \\ G\hat{E}X &= 33{,}69°\end{aligned}$$

Now we add these two angles together to get the angle we want to find:

$$\begin{aligned}G\hat{E}X + F\hat{E}X &= F\hat{E}G \\ F\hat{E}G &= 33{,}69° + 63{,}43° \\ &= 97{,}12°\end{aligned}$$

23. A triangle with angles 40°, 40° and 100° has a perimeter of 20 cm. Find the length of each side of the triangle.
Solution:
First draw a sketch:



We construct a perpendicular bisector and now we have a right-angled triangle to work with. We can use either of these two triangles.
We know $2a + b = 20$. Rearranging gives: $b = 2(10 - a)$. We can use the cos ratio to find a:

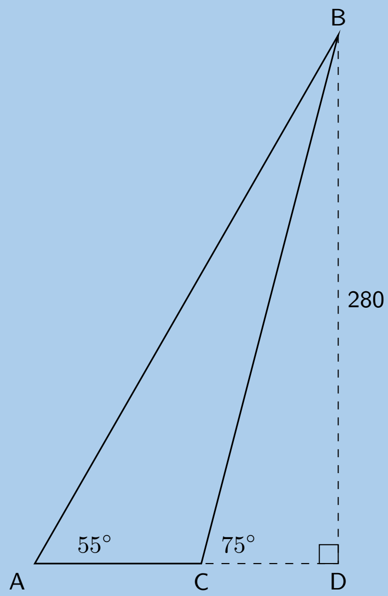
$$\begin{aligned}\cos 40° &= \frac{\frac{b}{2}}{a} \\ 0{,}77 &= \frac{\frac{2(10-a)}{2}}{a} \\ &= \frac{10-a}{a} \\ 0{,}77a &= 10 - a \\ a &= 5{,}65 \text{ cm}\end{aligned}$$

From the perimeter we get:

$$b = 2(10 - 5{,}65) = 8{,}7 \text{ cm}$$

Therefore the lengths of the sides are 8,7 cm, 5,65 cm and 5,65 cm.

24. Determine the area of $\triangle ABC$.



Solution:

Let the right angled vertex be D

$$\begin{aligned}\tan 55 &= \frac{280}{AD}\\ AD &= \frac{280}{\tan 55}\\ AD &= 196{,}058\end{aligned}$$

$$\begin{aligned}\tan 75 &= \frac{280}{CD}\\ CD &= \frac{280}{\tan 75}\\ CD &= 75{,}026\end{aligned}$$

$$\begin{aligned}AC &= AD - CD\\ AC &= 121{,}032\end{aligned}$$

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2}\text{base} \times \text{height}\\ \text{Area of } \triangle ABC &= \frac{1}{2} \times 121{,}032 \times 280\\ \therefore \text{Area of } \triangle ABC &= 16944 \text{ units}^2\end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FT3	1b. 2FT4	1c. 2FT5	2a. 2FT6	2b. 2FT7	2c. 2FT8
2d. 2FT9	2e. 2FTB	2f. 2FTC	2g. 2FTD	2h. 2FTF	2i. 2FTG
2j. 2FTH	2k. 2FTJ	2l. 2FTK	2m. 2FTM	2n. 2FTN	2o. 2FTP
2p. 2FTQ	2q. 2FTR	2r. 2FTS	3. 2FTT	4. 2FTV	5a. 2FTW
5b. 2FTX	5c. 2FTY	5d. 2FTZ	5e. 2FV2	5f. 2FV3	5g. 2FV4
5h. 2FV5	6. 2FV6	7. 2FV7	8. 2FV8	9. 2FV9	10a. 2FVB
10b. 2FVC	10c. 2FVD	10d. 2FVF	10e. 2FVG	10f. 2FVH	10g. 2FVJ
10h. 2FVK	10i. 2FVM	11. 2FVN	12. 2FVP	13. 2FVQ	14. 2FVR
15. 2FVS	16. 2FVT	17a. 2FVV	17b. 2FVW	17c. 2FVX	17d. 2FVY
17e. 2FVZ	17f. 2FW2	17g. 2FW3	17h. 2FW4	18. 2FW5	19. 2FW6
20. 2FW7	21. 2FW8	22. 2FW9	23. 2FWB	24. 2FWC	

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CHAPTER 6

Functions

6.1 Introduction

- This chapter covers the concept of a function and representing functions using tables, graphs, words and formulae. Straight line graphs were covered in grade 9 and are revised here. Parabolas, hyperbolas and exponential graphs are introduced here. Graphs for sine, cosine and tangent functions are also introduced here.
- A more formal definition of a function is only covered in grade 12. At this level learners should know the terms independent (input) and dependent (output) variables as well as how these vary.
- Summaries should be compiled for each type of graph and include the effects of a (vertical stretch and/or reflection in x) and q (vertical shift).
- Remember that graphs in some practical applications might be discrete or continuous.
- Encourage learners to state restrictions, particularly for quadratic functions.
- Learners must understand that $y = \sqrt{x}$ has no real solutions for $x < 0$.
- Sketching graphs is based on knowing the effects of a and q and using these to determine the shape of the graph.

A tool such as mathsisfun function-grapher can be used to plot graphs for classroom use. If you use this tool for plotting trigonometric graphs the values on the x-axis will not be in degrees.

Exercise 6 – 1:

1. Write the following in set notation:

   a) $(-\infty; 7]$
   Solution:
   $\{x : x \in \mathbb{R}, x \leq 7\}$

   b) $[-13; 4)$
   Solution:
   $\{x : x \in \mathbb{R}, -13 \leq x < 4\}$

   c) $(35; \infty)$
   Solution:
   $\{x : x \in \mathbb{R}, x > 35\}$

   d) $[\frac{3}{4}; 21)$
   Solution:
   $\left\{x : x \in \mathbb{R}, \frac{3}{4} \leq x < 21\right\}$

   e) $[-\frac{1}{2}; \frac{1}{2}]$
   Solution:
   $\left\{x : x \in \mathbb{R}, -\frac{1}{2} \leq x \leq \frac{1}{2}\right\}$

   f) $(-\sqrt{3}; \infty)$
   Solution:
   $\left\{x : x \in \mathbb{R}, x > -\sqrt{3}\right\}$

2. Write the following in interval notation:

   a) $\{p : p \in \mathbb{R}, p \leq 6\}$
   Solution:
   $(-\infty; 6]$

   b) $\{k : k \in \mathbb{R}, -5 < k < 5\}$
   Solution:
   $(-5; 5)$

   c) $\left\{x : x \in \mathbb{R}, x > \frac{1}{5}\right\}$
   Solution:
   $(\frac{1}{5}; \infty)$

   d) $\{z : z \in \mathbb{R}, 21 \leq x < 41\}$
   Solution:
   $[21; 41)$

3. Complete the following tables and identify the function.

a)

x	1	2	3	4	5	6
y	5	10		20		

Solution:

x	1	2	3	4	5	6
y	5	10	15	20	25	30

$$y = 5x$$

b)

x	1		3	4		6
y	5	5			5	5

Solution:

x	1	2	3	4	5	6
y	5	5	5	5	5	5

$$y = 5$$

c)

x	2			8	10	12
y	1	2	3			6

Solution:

x	2	4	6	8	10	12
y	1	2	3	4	5	6

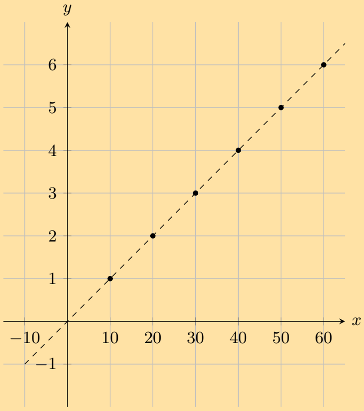
$$y = \frac{1}{2}x$$

4. Plot the following points on a graph.

a)

x	1	2	3	4	5	6
y	0,1	0,2	0,3	0,4	0,5	0,6

Solution:

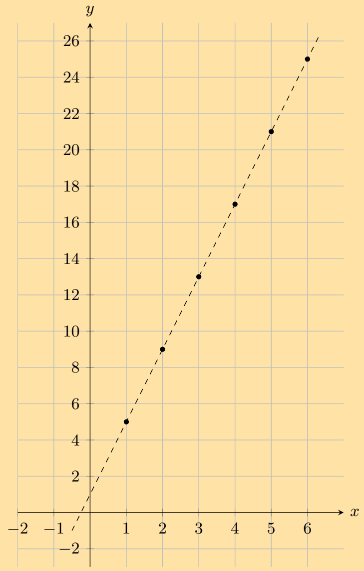


Note that this graph is scaled. Each value for x and y has been multiplied by 10. This process does not change the function, but it stretches the graph, thereby making it easier to read.

b)

x	1	2	3	4	5	6
y	5	9	13	17	21	25

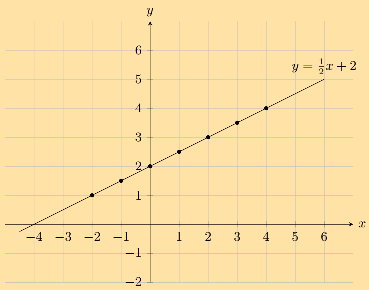
Solution:



5. Create a table of values from the function given and then plot the function. Your table must have at least 5 ordered pairs.

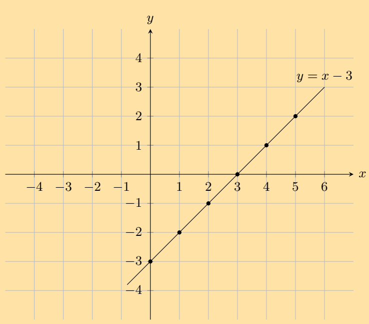
   a) $y = \frac{1}{2}x + 2$
   Solution:

x	-2	-1	0	1	2	3	4
y	1	$1{,}5$	2	$2{,}5$	3	$3{,}5$	4



   b) $y = x - 3$
   Solution:

x	0	1	2	3	4	5
y	-3	-2	-1	0	1	2



6. If the functions $f(x) = x^2 + 1$; $g(x) = x - 4$; $h(x) = 7 - x^2$; $k(x) = 3$ are given, find the value of the following:

   a) $f(-1)$
   Solution:

$$
\begin{aligned}
f(x) &= x^2 + 1 \\
f(-1) &= (-1)^2 + 1 \\
&= 2
\end{aligned}
$$

b) $g(-7)$
Solution:

$$
\begin{aligned}
g(x) &= x - 4 \\
g(-7) &= (-7) - 4 \\
&= -11
\end{aligned}
$$

c) $h(3)$
Solution:

$$
\begin{aligned}
h(x) &= 7 - x^2 \\
h(3) &= 7 - (3)^2 \\
&= -2
\end{aligned}
$$

d) $k(100)$
Solution:

$$
\begin{aligned}
k(x) &= 3 \\
k(100) &= 3
\end{aligned}
$$

Regardless of the value of x, the output is always 3.

e) $f(-2) + h(2)$
Solution:

$$
\begin{aligned}
f(x) + h(x) &= x^2 + 1 + 7 - x^2 \\
f(-2) + h(2) &= (-2)^2 + 1 + 7 - (2)^2 \\
&= 8
\end{aligned}
$$

f) $k(-5) + h(3)$
Solution:

$$
\begin{aligned}
k(x) + h(x) &= 3 + 7 - x^2 \\
k(-5) + h(3) &= 3 + 7 - (3)^2 \\
&= 1
\end{aligned}
$$

g) $f(g(1))$
Solution:

$$
\begin{aligned}
g(x) &= x - 4 \\
g(1) &= (1) - 4 \\
&= -3 \\
\therefore f(g(1)) &= f(-3) \\
f(x) &= x^2 + 1 \\
&= (-3)^2 + 1 \\
&= 10
\end{aligned}
$$

h) $k(f(6))$
Solution:

$$
\begin{aligned}
f(x) &= x^2 + 1 \\
f(6) &= (6)^2 + 1 \\
&= 37 \\
\therefore k(f(6)) &= k(37) \\
k(x) &= 3 \\
k(f(6)) &= 3
\end{aligned}
$$

Regardless of the value of x, the output is always 3.

7. The cost of petrol and diesel per litre are given by the functions P and D, where:

$$
\begin{aligned}
P &= 13{,}61V \\
D &= 12{,}46V
\end{aligned}
$$

Use this information to answer the following:

a) Evaluate $P(8)$
Solution:

$$
\begin{aligned}
P(8) &= 13{,}61(8) \\
&= \text{R } 108{,}88
\end{aligned}
$$

b) Evaluate $D(16)$
Solution:

$$
\begin{aligned}
D(16) &= 12{,}46(16) \\
&= \text{R } 199{,}36
\end{aligned}
$$

c) How many litres of petrol can you buy with R 300?
Solution:

$$
\begin{aligned}
P(V) &= 300 \\
13{,}61V &= 300 \\
V &= 22{,}043 \text{ L}
\end{aligned}
$$

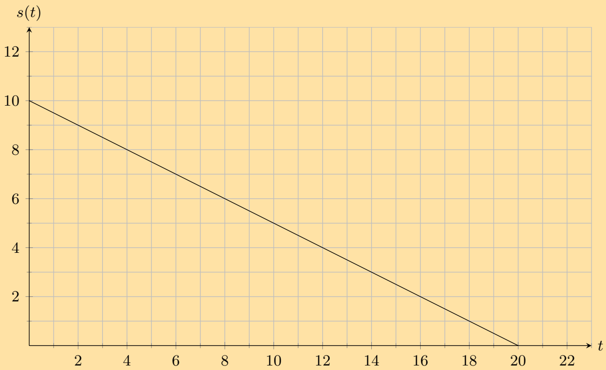
d) How many litres of petrol can you buy with R 275?
Solution:

$$
\begin{aligned}
D(V) &= 275 \\
12{,}46V &= 275 \\
V &= 22{,}071 \text{ L}
\end{aligned}
$$

e) How much more expensive is petrol than diesel? Show you answer as a function.
Solution:

$$
\begin{aligned}
P(V) - D(V) &= 13{,}61V - 12{,}46V \\
&= 1{,}15V
\end{aligned}
$$

8. A ball is rolling down a 10 m slope. The graph below shows the relationship between the distance and the time.



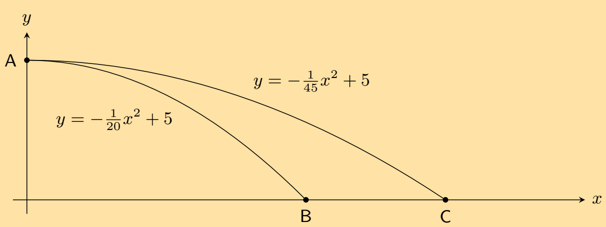
Use this information to answer the following:

a) After 6 s how much further does the ball have to roll?
Solution:
7 m

b) What is the range of the function?
Solution:
0 m $\leq s(t) \leq$ 10 m

c) What is the domain of the function, and what does it represent?
Solution:
The domain is 0 s $\leq t \leq$ 20 s. It represents the total time taken to reach the bottom of the slope.

9. James and Themba both throw a stone from the top of a building into a river. The path travelled by the stones can be described by quadratic equations. $y = -\frac{1}{20}x^2 + 5$ describes the path of the stone thrown by James and $y = -\frac{1}{45}x^2 + 5$ describes the path of Themba's stone.



a) What is the height of the building that they stood on?
Solution:
Both functions have a maximum value of 5 m. This can be found by letting $x = 0$ in each of the two functions and is represented by point A on the graph above.

b) How far did James throw his stone before it hit the river surface?
Solution:

$$
\begin{aligned}
y &= -\frac{1}{20}x^2 + 5 \\
0 &= -\frac{1}{20}x^2 + 5 \\
\frac{1}{20}x^2 - 5 &= 0 \\
x^2 - 100 &= 0 \\
(x - 10)(x + 10) &= 0 \\
\therefore x &= 10 \text{ m}
\end{aligned}
$$

James threw his stone 10 m before it hit the river surface.

c) How much farther did Themba throw his stone before it hit the river surface?
Solution:

$$y = -\frac{1}{45}x^2 + 5$$
$$0 = -\frac{1}{45}x^2 + 5$$
$$\frac{1}{45}x^2 - 5 = 0$$
$$x^2 - 225 = 0$$
$$(x - 15)(x + 15) = 0$$
$$\therefore x = 15 \text{ m}$$

Themba threw his stone 15 m before it hit the river surface.
Therefore Themba threw his stone 5 m farther than James did.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2FWD	1b. 2FWF	1c. 2FWG	1d. 2FWH	1e. 2FWJ	1f. 2FWK	2a. 2FWM	2b. 2FWN
2c. 2FWP	2d. 2FWQ	3a. 2FWR	3b. 2FWS	3c. 2FWT	4a. 2FWV	4b. 2FWW	5a. 2FWX
5b. 2FWY	6. 2FWZ	7. 2FX2	8. 2FX3	9. 2FX4			

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6.2 Linear functions

Exercise 6 – 2:

1. Determine the x-intercept and the y-intercept of the following equations.

   a) $y = x - 1$

   Solution:

$$y = x - 1$$
$$y = (0) - 1$$
$$y = -1$$
$$\therefore c = -1$$

$$y = x - 1$$
$$(0) = x - 1$$
$$1 = x$$
$$1 = x$$

   x-intercept $= 1$ and y-intercept $= -1$

   b) $y = x + 2$

   Solution:

$$y = x + 2$$
$$y = (0) + 2$$
$$y = 2$$
$$\therefore c = 2$$

$$y = x + 2$$
$$(0) = x + 2$$
$$-2 = x$$
$$-2 = x$$

x-intercept $= -2$ and y-intercept $= 2$

c) $y = x - 3$

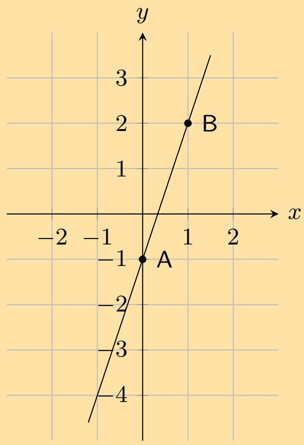
Solution:

$$
\begin{aligned}
y &= x - 3 \\
y &= (0) - 3 \\
y &= -3 \\
\therefore c &= -3
\end{aligned}
$$

$$
\begin{aligned}
y &= x - 3 \\
(0) &= x - 3 \\
3 &= x \\
3 &= x
\end{aligned}
$$

x-intercept $= 3$ and y-intercept $= -3$

2. In the graph below there is a function with the equation $y = mx + c$. Determine the values of m (the gradient of the line) and c (the y-intercept of the line).



Solution:

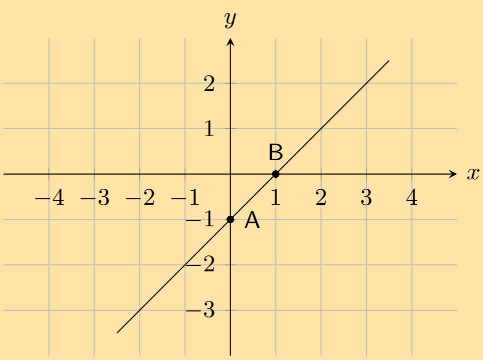
To determine m, we use the coordinates of any other point on the line apart from the one used for the y-intercept. In this solution, we have chosen the coordinates of point B which are $(1; 2)$.

From the y-intercept $c = -1$.

$$
\begin{aligned}
y &= mx + c \\
2 &= m(1) - 1 \\
2 &= m - 1 \\
3 &= m
\end{aligned}
$$

$m = 3$ and $c = -1$.

3. The graph below shows a function with the equation $y = mx + c$. Determine the values of m (the gradient of the line) and c (the y-intercept of the line).



Solution:

To determine m, we use the coordinates of any other point on the line apart from the one used for the y-intercept. In this solution, we have chosen the coordinates of point B which are $(1; 2)$.

From the y-intercept $c = -1$.

$$\begin{aligned} y &= mx + c \\ 0 &= m(1) - 1 \\ 0 &= m - 1 \\ 1 &= m \end{aligned}$$

$m = 1$, and $c = -1$.

4. List the x and y-intercepts for the following straight line graphs. Indicate whether the graph is increasing or decreasing:

   a) $y = x + 1$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the points $(0; 1)$ and $(-1; 0)$. The graph is increasing $(m > 0)$.

   b) $y = x - 1$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the points $(0; -1)$ and $(1; 0)$. The graph is increasing $(m > 0)$.

   c) $h(x) = 2x + 1$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the points $(0; -1)$ and $\left(\frac{1}{2}; 0\right)$. The graph is increasing $(m > 0)$.

   d) $y + 3x = 1$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the points $(0; 1)$ and $\left(\frac{1}{3}; 0\right)$. The graph is decreasing $(m < 0)$.

   e) $3y - 2x = 6$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the points $(0; 2)$ and $(-3; 0)$. The graph is increasing $(m > 0)$.

   f) $k(x) = -3$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the point $(0; 3)$. The graph is horizontal.

   g) $x = 3y$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the same point for both intercepts: $(0; 0)$. The graph is increasing $(m > 0)$.

   h) $\frac{x}{2} - \frac{y}{3} = 1$

      Solution:

      To find the x-intercept we set $y = 0$ and to find the y-intercept we set $x = 0$. This gives the points $(0; -3)$ and $(2; 0)$. The graph is increasing $(m > 0)$.

5. State whether the following are true or not.

   a) The gradient of $2y = 3x - 1$ is 3.

      Solution:

      False

      $$\begin{aligned} 2y &= 3x - 1 \\ y &= \frac{3}{2}x - 12 \end{aligned}$$

      Therefore the gradient is $\frac{3}{2}$.

   b) The y-intercept of $y = x + 4$ is 4.

Solution:
True

c) The gradient of $2 - y = 2x - 1$ is -2.
Solution:
True

d) The gradient of $y = \frac{1}{2}x - 1$ is -1.
Solution:
False

$$m = \frac{1}{2}$$

e) The y-intercept of $2y = 3x - 6$ is 6.
Solution:
False

$$\begin{aligned} 2y &= 3x - 6 \\ y &= \frac{3}{2}x - 3 \end{aligned}$$

6. Write the following in standard form ($y = mx + c$):

a) $2y + 3x = 1$
Solution:

$$\begin{aligned} 2y + 3x &= 1 \\ 2y &= 1 - 3x \\ y &= -\frac{3}{2}x + \frac{1}{2} \end{aligned}$$

b) $3x - y = 5$
Solution:

$$\begin{aligned} 3x - y &= 5 \\ -y &= 5 - 3x \\ y &= -3x + 5 \end{aligned}$$

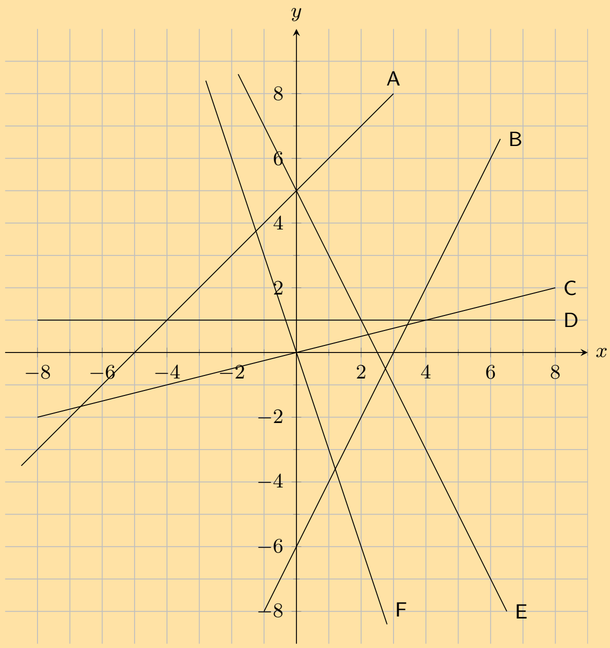
c) $3y - 4 = x$
Solution:

$$\begin{aligned} 3y - 4 &= x \\ 3y &= x + 4 \\ y &= \frac{1}{3}x + \frac{4}{3} \end{aligned}$$

d) $y + 2x - 3 = 1$
Solution:

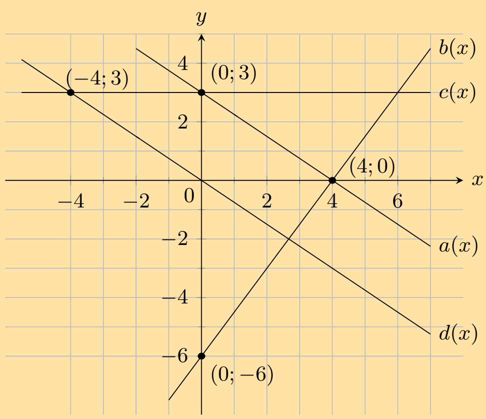
$$\begin{aligned} y + 2x - 3 &= 1 \\ y &= -2x + 4 \end{aligned}$$

7. Look at the graphs below. Each graph is labelled with a letter. In the questions that follow, match any given equation with the label of a corresponding graph.



a) $y = 5 - 2x$
   Solution:
   E
b) $x + 5$
   Solution:
   A
c) $y = 2x - 6$
   Solution:
   B
d) $y = -3x$
   Solution:
   F
e) $y = 1$
   Solution:
   D
f) $y = \frac{1}{2}x$
   Solution:
   C

8. For the functions in the diagram below, give the equation of each line:



a) $a(x)$
   Solution:

The y-intercept is $(0;3)$, therefore $c = 3$.

$$\begin{aligned} y &= mx + 3 \\ 0 &= 4m + 3 \\ \therefore m &= \frac{-3}{4} \end{aligned}$$

Therefore $a(x) = -\frac{3}{4}x + 3$

b) $b(x)$

Solution:

The y-intercept is $(0;-6)$, therefore $c = -6$.

$$\begin{aligned} y &= mx - 6 \\ 0 &= 4m - 6 \\ \therefore m &= \frac{3}{2} \end{aligned}$$

Therefore $b(x) = \frac{3}{2}x - 6$

c) $c(x)$

Solution:

The y-intercept is $(0;3)$, therefore $c = 3$.

$$\begin{aligned} y &= mx + 3 \\ 3 &= -4m + 3 \\ 0 &= -4m \\ \therefore m &= 0 \end{aligned}$$

Therefore $c(x) = 3$

d) $d(x)$

Solution:

The y-intercept is $(0;0)$, therefore $c = 0$.

$$\begin{aligned} y &= mx \\ 3 &= -4m \\ \therefore m &= \frac{-3}{4} \end{aligned}$$

Therefore $d(x) = -\frac{3}{4}x$

9. Sketch the following functions on the same set of axes, using the dual intercept method. Clearly indicate the coordinates of the intercepts with the axes and the point of intersection of the two graphs: $x+2y-5=0$ and $3x-y-1=0$.

Solution:

For $x + 2y - 5 = 0$:

We first write the equation in standard form: $y = -\frac{1}{2}x + \frac{5}{2}$. From this we see that the y-intercept is $\frac{5}{2}$. The x-intercept is 5.

For $3x - y - 1 = 0$:

We first write the equation in standard form: $y = 3x - 1$. From this we see that the y-intercept is -1. The x-intercept is $\frac{1}{3}$.

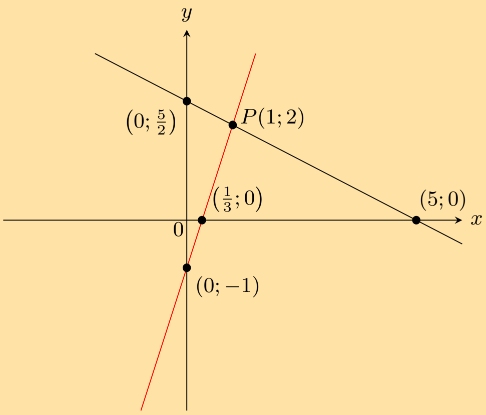
To find the point of intersection we need to solve the two equations simultaneously. We can use the standard form of the first equation and substitute this value of y into the second equation:

$$\begin{aligned} 3x + \frac{1}{2}x - \frac{5}{2} - 1 &= 0 \\ \frac{7}{2}x &= \frac{7}{2} \\ x &= 1 \end{aligned}$$

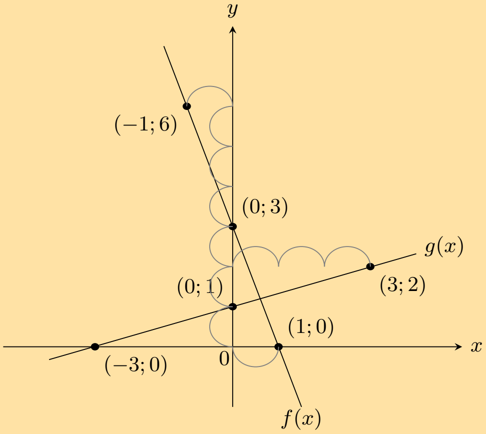
Substitute the value of x back into the first equation:

$$\begin{aligned} x + 2y - 5 &= 0 \\ 1 + 2y - 5 &= 0 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

Therefore the graphs intersect at $(1;2)$.
Now we can sketch the graphs:



10. On the same set of axes, draw the graphs of $f(x) = 3 - 3x$ and $g(x) = \frac{1}{3}x + 1$ using the gradient-intercept method.
Solution:
For $f(x) = 3 - 3x$ the y-intercept is 3. The gradient is -3.
To get the second point we start at $(0;3)$ and move 3 units up and 1 unit to the left. This gives the second point $(-1;6)$. Or we can move 3 units down and 1 unit right to get $(1;0)$.
For $g(x) = \frac{1}{3}x + 1$ the y-intercept is 1. The gradient is $\frac{1}{3}$.
To get the second point we start at $(0;1)$ and move 1 unit up and 3 units to the right. This gives the second point $(3;2)$. Or we can move 1 unit down and 3 units left to get $(-3;0)$.
Now we can sketch the graphs.



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1a. 2FX5	1b. 2FX6	1c. 2FX7	2. 2FX8	3. 2FX9	4a. 2FXB
4b. 2FXC	4c. 2FXD	4d. 2FXF	4e. 2FXG	4f. 2FXH	4g. 2FXJ
4h. 2FXK	5a. 2FXM	5b. 2FXN	5c. 2FXP	5d. 2FXQ	5e. 2FXR
6a. 2FXS	6b. 2FXT	6c. 2FXV	6d. 2FXW	7. 2FXX	8a. 2FXY
8b. 2FXZ	8c. 2FY2	8d. 2FY3	9. 2FY4	10. 2FY5	

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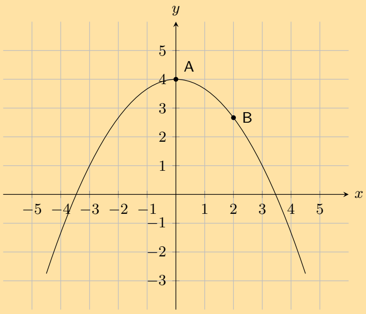
Exercise 6 – 3:

1. The graph below shows a quadratic function with the following form: $y = ax^2 + q$.
   Two points on the parabola are shown: **Point A**, the turning point of the parabola, at $(0; 4)$, and **Point B** is at $\left(2; \frac{8}{3}\right)$.
   Calculate the values of a and q.

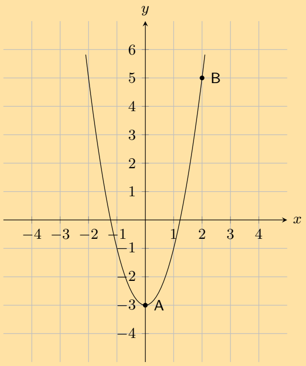


Solution:
The value of q is 4.

$$\begin{aligned} y &= ax^2 + 4 \\ \left(\frac{8}{3}\right) &= a(2)^2 + 4 \quad \longleftarrow \quad \text{substitute in the coordinates of a point!} \\ \frac{8}{3} &= 4a + 4 \\ \frac{8}{3} - 4 &= 4a \\ -\frac{4}{3} &= 4a \\ -\frac{1}{3} &= a \end{aligned}$$

$$a = -\frac{1}{3}; \ q = 4$$

2. The graph below shows a quadratic function with the following form: $y = ax^2 + q$.
   Two points on the parabola are shown: **Point A**, the turning point of the parabola, at $(0; -3)$, and **Point B** is at $(2; 5)$.
   Calculate the values of a and q.



Solution:
The value of q is -3.

$$\begin{aligned} y &= ax^2 - 3 \\ (5) &= a(2)^2 - 3 \longleftarrow \text{substitute in the coordinates of a point!} \\ 5 &= 4a - 3 \\ 5 + 3 &= 4a \\ 8 &= 4a \\ 2 &= a \end{aligned}$$

$$a = 2;\ q = -3$$

3. Given the following equation:
   $y = 5x^2 - 2$

   a) Calculate the y-coordinate of the y-intercept.
   Solution:

$$\begin{aligned} y &= ax^2 + q \\ &= 5x^2 - 2 \\ &= 5(0)^2 - 2 \\ &= 0 - 2 \end{aligned}$$

   The y-coordinate of the y-intercept is -2.

   b) Now calculate the x-intercepts. Your answer must be correct to 2 decimal places.
   Solution:

$$\begin{aligned} y &= 5x^2 - 2 \\ (0) &= 5x^2 - 2 \\ -5x^2 &= -2 \\ x^2 &= \frac{-2}{-5} \\ x &= \pm\sqrt{\frac{2}{5}} \end{aligned}$$

Therefore: $x = +\sqrt{\frac{2}{5}}$ and $x = -\sqrt{\frac{2}{5}}$
$x = -0{,}63$ and $x = 0{,}63$

   The x-intercepts are $(-0{,}63; 0)$ and $(0{,}63; 0)$.

4. Given the following equation:
   $y = -2x^2 + 1$

   a) Calculate the y-coordinate of the y-intercept.
   Solution:

$$\begin{aligned} y &= ax^2 + q \\ &= -2x^2 + 1 \\ &= -2(0)^2 + 1 \\ &= 0 + 1 \end{aligned}$$

   The y-coordinate of the y-intercept is 1.

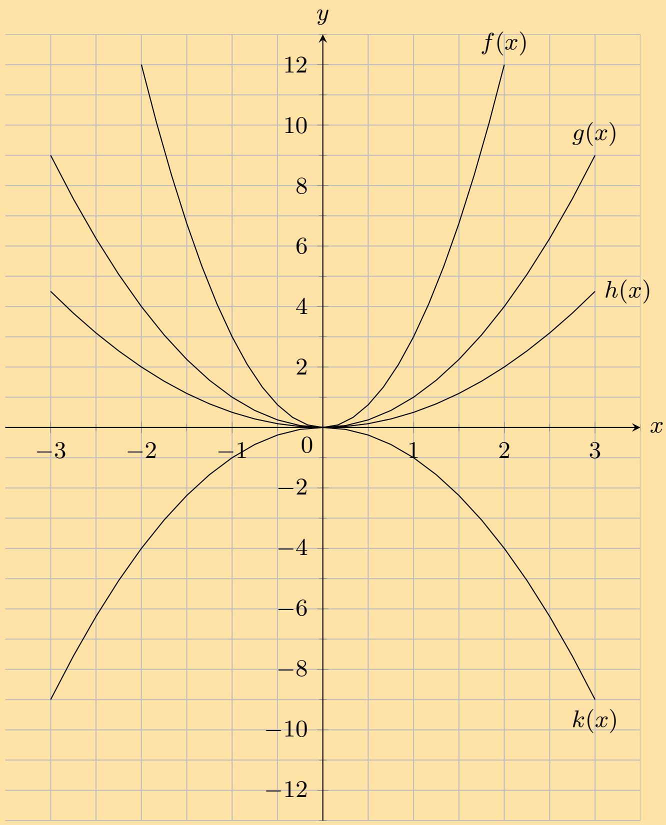
   b) Now calculate the x-intercepts. Your answer must be correct to 2 decimal places.
   Solution:

$$y = -2x^2 + 1$$
$$(0) = -2x^2 + 1$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\sqrt{\frac{1}{2}}$$
Therefore: $x = +\sqrt{\frac{1}{2}}$ and $x = -\sqrt{\frac{1}{2}}$

$x = -0{,}71$ and x

The x-intercepts are $(-0{,}71; 0)$ and $(0{,}71; 0)$.

5. Given the following graph, identify a function that matches each of the following equations:



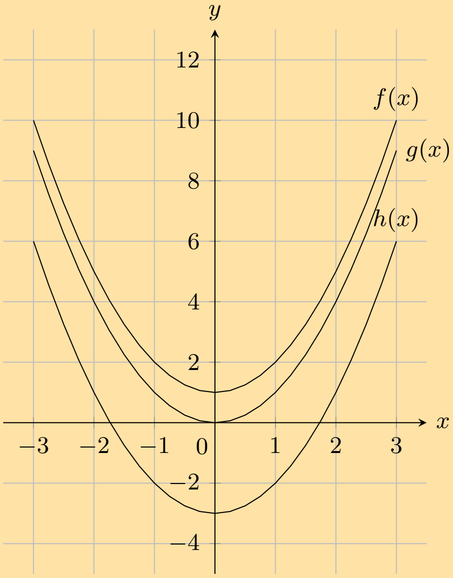
a) $y = 0{,}5x^2$
Solution:
$h(x)$

b) $y = x^2$
Solution:
$g(x)$

c) $y = 3x^2$
Solution:
$f(x)$

d) $y = -x^2$
Solution:
$k(x)$

6. Given the following graph, identify a function that matches each of the following equations:

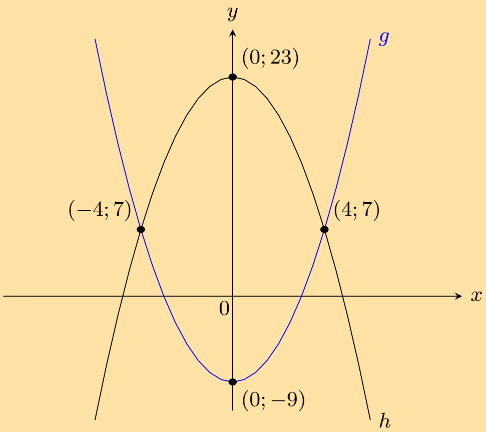


a) $y = x^2 - 3$
Solution:
$h(x)$

b) $y = x^2 + 1$
Solution:
$f(x)$

c) $y = x^2$
Solution:
$g(x)$

7. Two parabolas are drawn: $g : y = ax^2 + p$ and $h : y = bx^2 + q$.



a) Find the values of a and p.
Solution:
p is the y-intercept of the function $g(x)$, therefore $p = -9$
To find a we use one of the points on the graph (e.g. $(4; 7)$):

$$
\begin{aligned}
y &= ax^2 - 9 \\
7 &= a(4^2) - 9 \\
16a &= 16 \\
\therefore a &= 1
\end{aligned}
$$

$a = 1$; $p = -9$

b) Find the values of b and q.

Solution:

q is the y-intercept of the function $h(x)$, therefore $q = 23$

To find b, we use one of the points on the graph (e.g. $(4; 7)$):

$$\begin{aligned} y &= bx^2 = 23 \\ 7 &= b(4^2) + 23 \\ 16b &= -16 \\ \therefore b &= -1 \end{aligned}$$

$b = -1$; $q = 23$

c) Find the values of x for which $g(x) \geq h(x)$.

Solution:

These are the points where g lies above h.

From the graph we see that g lies above h when: $x \leq -4$ or $x \geq 4$

d) For what values of x is g increasing?

Solution:

g increases from the turning point $(0; -9)$, i.e. for $x \geq 0$.

8. Show that if $a < 0$ the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \leq q\}$.

Solution:

Because the square of any number is always positive we get: $x^2 \geq 0$.

If we multiply by a where $(a < 0)$ then the sign of the inequality is reversed: $ax^2 \leq 0$

Adding q to both sides gives $ax^2 + q \leq q$

And so $f(x) \leq q$

This gives the range as $(-\infty; q]$.

9. Draw the graph of the function $y = -x^2 + 4$ showing all intercepts with the axes.

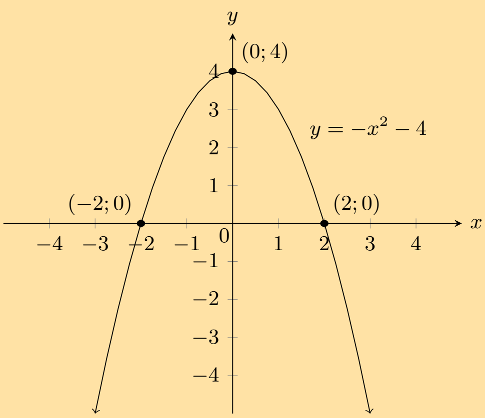
Solution:

The y-intercept is $(0; 4)$. The x-intercepts are given by setting $y = 0$:

$$\begin{aligned} 0 &= -x^2 + 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Therefore the x-intercepts are: $(2; 0)$ and $(-2; 0)$.

Now we can sketch the graph:



For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FY6 2. 2FY7 3. 2FY8 4. 2FY9 5. 2FYB 6. 2FYC
7. 2FYD 8. 2FYF 9. 2FYG

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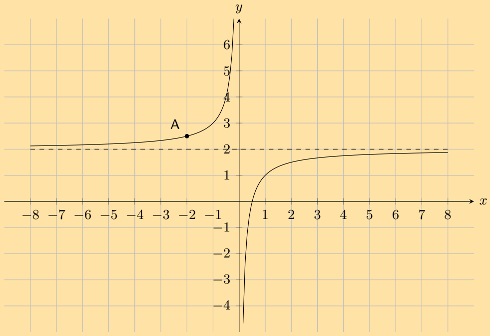
Exercise 6 – 4:

1. The following graph shows a hyperbolic equation of the form $y = \frac{a}{x} + q$. **Point A** is shown at $\left(-2; \frac{5}{2}\right)$. Calculate the values of a and q.



Solution:

$$q = 2$$

$$\begin{aligned} y &= \frac{a}{x} + 2 \\ \left(\frac{5}{2}\right) &= \frac{a}{(-2)} + 2 \\ -2\left(\frac{5}{2}\right) &= \left[\frac{a}{-2} + 2\right](-2) \\ -5 &= a - 4 \\ -1 &= a \end{aligned}$$

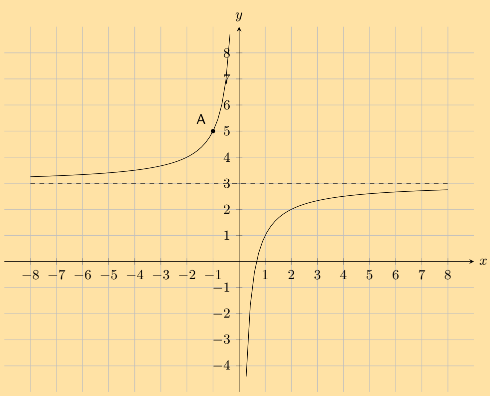
Therefore

$$a = -1$$

and $q = 2$

The equation is $y = -\frac{1}{x} + 2$.

2. The following graph shows a hyperbolic equation of the form $y = \frac{a}{x} + q$. **Point A** is shown at $(-1; 5)$. Calculate the values of a and q.



Solution:

$$q = 3$$

$$\begin{aligned} y &= \frac{a}{x} + 3 \\ (5) &= \frac{a}{(-1)} + 3 \\ -1(5) &= \left[\frac{a}{-1} + 3\right](-1) \\ -5 &= a - 3 \\ -2 &= a \end{aligned}$$

Therefore

$$a = -2$$

and $q = 3$

The equation is $y = -\frac{2}{x} + 3$.

3. Given the following equation:

$$y = \frac{3}{x} + 2$$

   a) Determine the location of the y-intercept.
   Solution:

$$\begin{aligned} y &= \frac{3}{x} + 2 \\ &= \frac{3}{(0)} + 2 \end{aligned}$$

undefined

   There is no y-intercept.

   b) Determine the location of the x-intercept. Give your answer as a fraction.
   Solution:

$$\begin{aligned} y &= \frac{3}{x} + 2 \\ (0) &= \frac{3}{x} + 2 \\ (x)(0) &= \left[\frac{3}{x} + 2\right](x) \\ 0 &= 3 + 2x \\ -3 &= 2x \\ x &= -\frac{3}{2} \end{aligned}$$

4. Given the following equation:

$$y = -\frac{2}{x} - 2$$

   a) Determine the location of the y-intercept.
   Solution:

$$\begin{aligned} y &= -\frac{2}{x} - 2 \\ y &= -\frac{2}{(0)} - 2 \end{aligned}$$

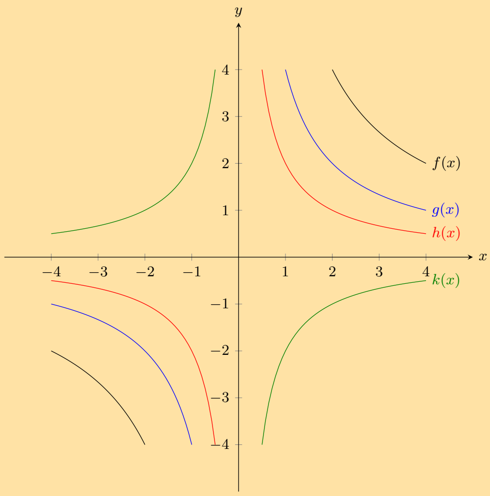
undefined

   There is no y-intercept.

b) Determine the location of the x-intercept.
Solution:

$$\begin{aligned} y &= -\frac{2}{x} - 2 \\ (0) &= -\frac{2}{x} - 2 \\ (x)(0) &= \left[-\frac{2}{x} - 2\right](x) \\ 0 &= -2 - 2x \\ 2 &= -2x \\ x &= -1 \end{aligned}$$

5. Given the following graph, identify a function that matches each of the following equations:



a) $y = \frac{2}{x}$
Solution:
$h(x)$

b) $y = \frac{4}{x}$
Solution:
$g(x)$

c) $y = -\frac{2}{x}$
Solution:
$k(x)$

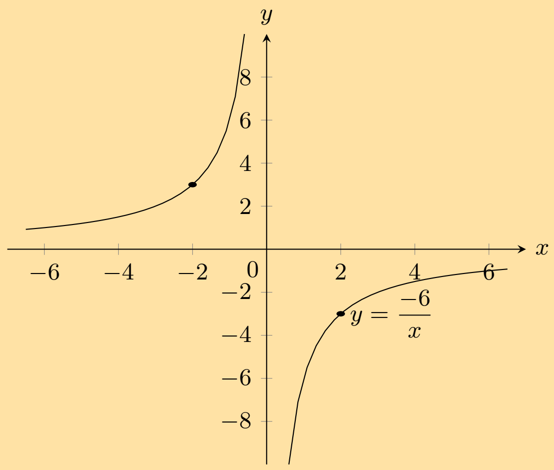
d) $y = \frac{8}{x}$
Solution:
$f(x)$

6. Given the function: $xy = -6$.

a) Draw the graph.
Solution:
a is negative and so the function lies in the second and fourth quadrants.
There is no y-intercept or x-intercept.
Instead we can plot the graph from a table of values.

x	-2	-1	1	2
y	3	6	-6	-3

Now we can plot the graph:



b) Does the point $(-2;3)$ lie on the graph? Give a reason for your answer.
Solution:
If we substitute the point $(-2;3)$ into each side of the equation we get:

$$RHS = -6$$
$$LHS = xy = (-2)(3) = -6$$

This satisfies the equation therefore the point does lie on the graph.

c) If the x-value of a point on the graph is 0,25 what is the corresponding y-value?
Solution:
Substitute in the value of x:

$$\begin{aligned} y &= \frac{-6}{0{,}25} \\ &= \frac{-6}{\frac{1}{4}} \\ &= -6 \times \frac{4}{1} \\ &= -24 \end{aligned}$$

d) What happens to the y-values as the x-values become very large?
Solution:
The y-values decrease as the x-values become very large. The larger the denominator (x), the smaller the result of the fraction (y).

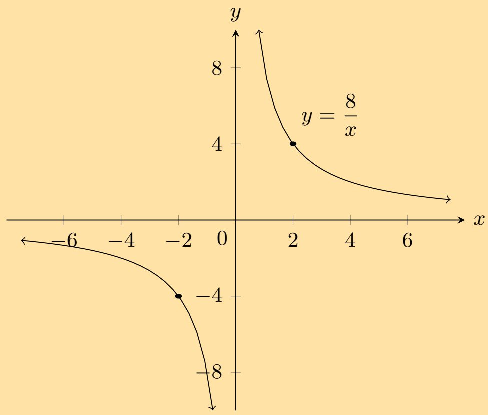
e) Give the equation of the asymptotes.
Solution:
The graph is not vertically or horizontally shifted, therefore the asymptotes are $y = 0$ and $x = 0$.

f) With the line $y = -x$ as a line of symmetry, what is the point symmetrical to $(-2;3)$?
Solution:
Across the line of symmetry $y = -x$, the point symmetrical to $(-2;3)$ is $(-3;2)$.

7. Given the function: $h(x) = \frac{8}{x}$.

a) Draw the graph.
Solution:
a is positive and so the function lies in the first and third quadrants.
There is no y-intercept and no x-intercept.
Instead we can plot the graph from a table of values.

x	-2	-1	1	2
y	-4	-8	8	4



b) How would the graph of $g(x) = \frac{8}{x} + 3$ compare with that of $h(x) = \frac{8}{x}$? Explain your answer fully.

Solution:

The graph $g(x) = \frac{8}{x} + 3$ is the graph of $h(x) = \frac{8}{x}$, vertically shifted upwards by 3 units. They would be the same shape but the asymptote of $g(x)$ would be $y = 3$, instead of $y = 0$ (for $h(x)$) and the axis of symmetry would be $y = -x + 3$ instead of $y = -x$.

c) Draw the graph of $y = \frac{8}{x} + 3$ on the same set of axes, showing asymptotes, axes of symmetry and the coordinates of one point on the graph.

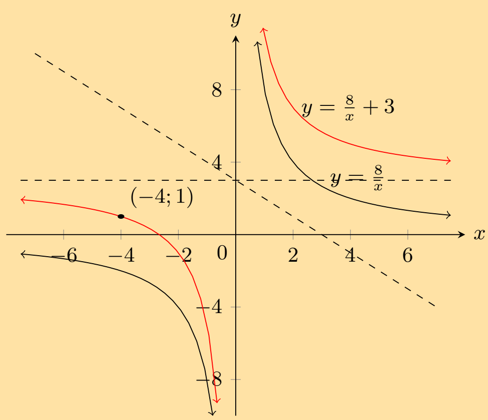
Solution:

a is positive and so the function lies in the first and third quadrants.

For $y = \frac{8}{x} + 3$ there is no y-intercept. The x-intercept is at $-\frac{8}{3}$.

We can plot the graph from a table of values.

x	-4	-2	2	4
y	1	-1	7	5



8. Sketch the functions given and describe the transformation performed on the first function to obtain the second function. Show all asymptotes.

a) $y = \frac{1}{x}$ and $\frac{3}{x}$

Solution:

a is positive for both graphs and so both graphs lie in the first and third quadrants.

For both graphs there is no y-intercept or x-intercept.

Instead we can plot the graph from a table of values.

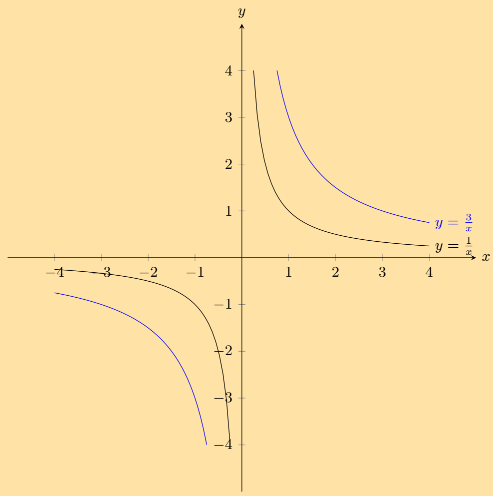
$y = \frac{1}{x}$:

x	-2	-1	1	2
y	$-\frac{1}{2}$	-1	1	$\frac{1}{2}$

$y = \frac{3}{x}$:

x	-2	-1	1	2
y	$-\frac{3}{2}$	-3	3	$\frac{3}{2}$

The asymptotes are $y = 0$ and $x = 0$.
Now we can plot the graphs:



Magnification by 3

b) $y = \dfrac{6}{x}$ and $\dfrac{6}{x} - 1$

Solution:

a is positive for both graphs and so both graphs lie in the first and third quadrants.

For both graphs there is no y-intercept. For $y = \frac{6}{x}$ there is no x-intercept. For $y = \frac{6}{x} - 1$ the x-intercept is $(6; 0)$.

We can plot the graphs from a table of values.

$y = \frac{6}{x}$:

x	-2	-1	1	2
y	-3	-6	6	3

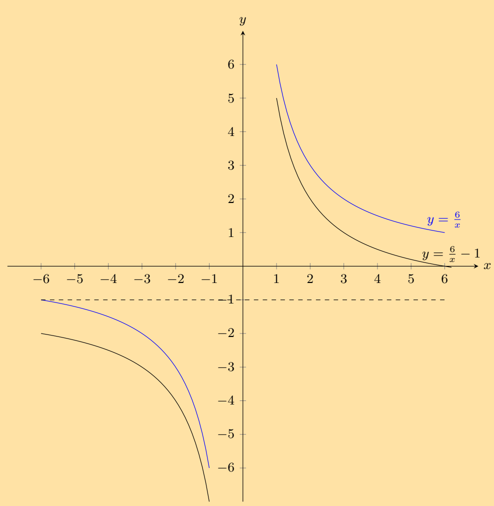
$y = \frac{6}{x} - 1$:

x	-2	-1	1	2
y	-4	-7	5	2

The asymptotes for $y = \frac{6}{x}$ are $y = 0$ and $x = 0$.

The asymptotes for $y = \frac{6}{x} - 1$ are $y = -1$ and $x = 0$.

Now we can plot the graphs:



Translation along the y-axis by -1

c) $y = \frac{5}{x}$ and $-\frac{5}{x}$

Solution:

$y = \frac{5}{x}$:

a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept and no x-intercept.

We can plot the graph from a table of values.

x	-2	-1	1	2
y	$-\frac{5}{2}$	-5	5	$\frac{5}{2}$

The asymptotes are $y = 0$ and $x = 0$.

$y = -\frac{5}{x}$:

a is negative and so the graph lies in the second and fourth quadrants.

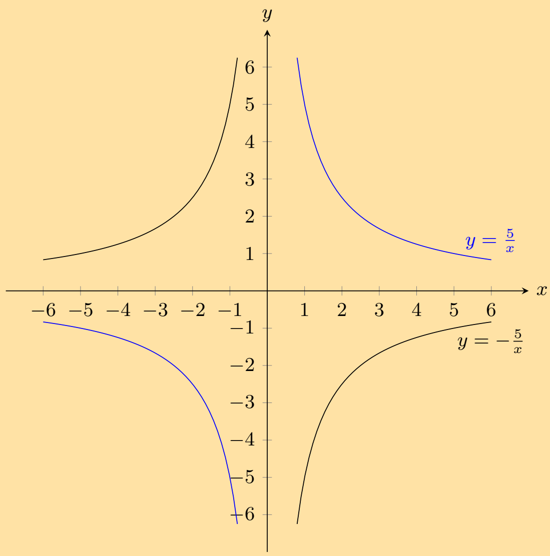
There is no y-intercept and no x-intercept.

We can plot the graph from a table of values.

x	-2	-1	1	2
y	$\frac{5}{2}$	5	-5	$-\frac{5}{2}$

The asymptotes are $y = 0$ and $x = 0$.

Now we can plot the graphs:



Reflection on the x-axis or reflection on the y-axis.

d) $y = \frac{1}{x}$ and $\frac{1}{2x}$

Solution:

a is positive for both graphs and so both graphs lie in the first and third quadrants.

For both graphs there is no y-intercept and no x-intercept.

We can plot the graphs from a table of values.

$y = \frac{1}{x}$:

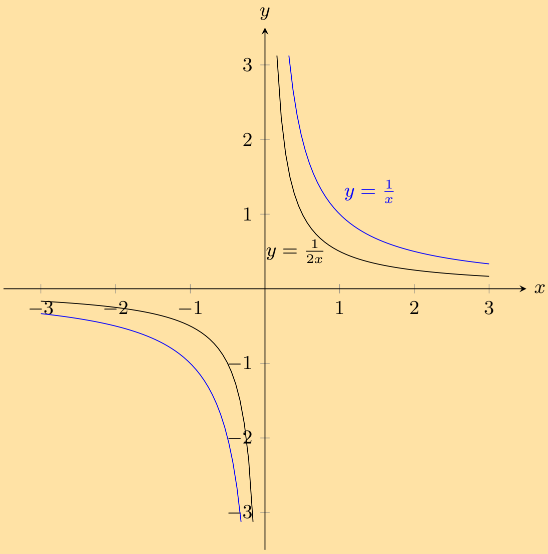
x	-2	-1	1	2
y	$-\frac{1}{2}$	-1	1	$\frac{1}{2}$

$y = \frac{1}{2x}$:

x	-2	-1	1	2
y	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$

The asymptotes for both graphs are $y = 0$ and $x = 0$.

Now we can plot the graphs:



Reduction by 2

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1. 2FYH 2. 2FYJ 3. 2FYK 4. 2FYM 5. 2FYN 6. 2FYP
7. 2FYQ 8a. 2FYR 8b. 2FYS 8c. 2FYT 8d. 2FYV

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6.5 Exponential functions

CAPS states to only investigate the effects of a and q on an exponential graph. However it is also important for learners to see that b has a different effect on the graph depending on if $b > 1$ or $0 < b < 1$.

For this reason the effect of b is included in the investigation so that learners can see what happens when $b > 1$ and when $0 < b < 1$.

Also note that the above worked example further reinforces the effects of b on the exponential graph.

Exercise 6 – 5:

1. Given the following equation:
   $y = -\frac{2}{3}.(3)^x + 1$

   a) Calculate the y-intercept. Your answer must be correct to 2 decimal places.
   Solution:

$$\begin{aligned} y &= \left(-\frac{2}{3}\right).(3)^x + 1 \\ &= \left(-\frac{2}{3}\right).(3)^{(0)} + 1 \\ &= \left(-\frac{2}{3}\right).(1) + 1 \\ &= (-0{,}66666...) + 1 \\ &= 0{,}33 \end{aligned}$$

The y-intercept is $(0; 0{,}33)$.

b) Now calculate the x-intercept. Estimate your answer to one decimal place if necessary.

Solution:

We calculate the x-intercept by letting $y = 0$. Then we solve for x:

$$
\begin{aligned}
0 &= \left(-\frac{2}{3}\right).(3)^x + 1 \\
-1 &= \left(-\frac{2}{3}\right)(3)^x \\
\left(-\frac{3}{2}\right)(-1) &= \left(-\frac{3}{2}\right)\left(-\frac{2}{3}\right).(3)^x \\
\frac{3}{2} &= 3^x
\end{aligned}
$$

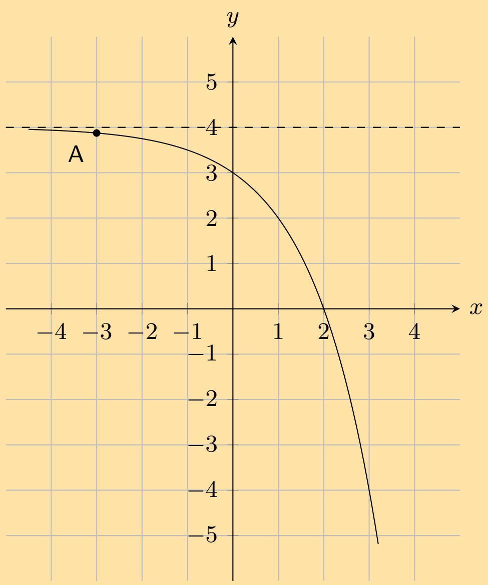
To find the answer we try different values of x:

$$
\begin{aligned}
\text{Try: } 3^{-1} &= 0{,}333... \\
\text{Try: } 3^{0} &= 1 \\
\text{Try: } 3^{1} &= 3
\end{aligned}
$$

We can see that the exponent must be between 0 and 1. Next we try values starting with 0,1 and see what the value of the exponent is. Doing this we find that $x = 0{,}4$.

The x-intercept is $(0{,}4; 0)$.

2. The graph here shows an exponential function with the equation $y = a \ . \ 2^x + q$. One point is given on the curve: **Point A** is at $(-3; 3{,}875)$. Determine the values of a and q, correct to the nearest integer.



Solution:

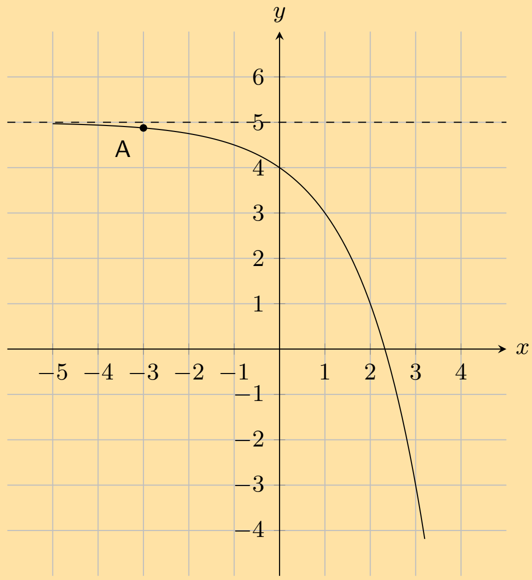
The asymptote lies at $y = 4$. Therefore q is 4.

At this point we know that the equation for the graph must be $y = a \ . \ 2^x + 4$.

$$
\begin{aligned}
y &= a(2)^x + 4 \\
(3{,}875) &= a(2)^{(-3)} + 4 \\
3{,}875 - 4 &= a(2)^{(-3)} \\
-0{,}125 &= a(0{,}125) \\
-1 &= a
\end{aligned}
$$

Therefore $a = -1$ and $q = 4$

3. Below you see a graph of an exponential function with the equation $y = a \ . \ 2^x + q$. One point is given on the curve: **Point A** is at $(-3; 4{,}875)$. Calculate the values of a and q, correct to the nearest integer.



Solution:

The asymptote lies at $y = 5$. Therefore q is 5.

At this point we know that the equation for the graph must be $y = a \ . \ 2^x + 5$.

$$\begin{aligned} y &= a(2)^x + 5 \\ (4{,}875) &= a(2)^{(-3)} + 5 \\ 4{,}875 - 5 &= a(2)^{(-3)} \\ -0{,}125 &= a(0{,}125) \\ -1 &= a \end{aligned}$$

Therefore $a = -1$ and $q = 5$.

4. Given the following equation:

$y = \frac{1}{4} \ . \ (4)^x - 1$

a) Calculate the y-intercept. Your answer must be correct to 2 decimal places.

Solution:

$$\begin{aligned} y &= \left(\frac{1}{4}\right) \ . \ (4)^x - 1 \\ &= \left(\frac{1}{4}\right) \ . \ (4)^{(0)} - 1 \\ &= \left(\frac{1}{4}\right) \ . \ (1) - 1 \\ &= (0{,}25) - 1 \\ &= -0{,}75 \end{aligned}$$

Therefore the y-intercept is $(0; -0{,}75)$.

b) Now calculate the x-intercept.

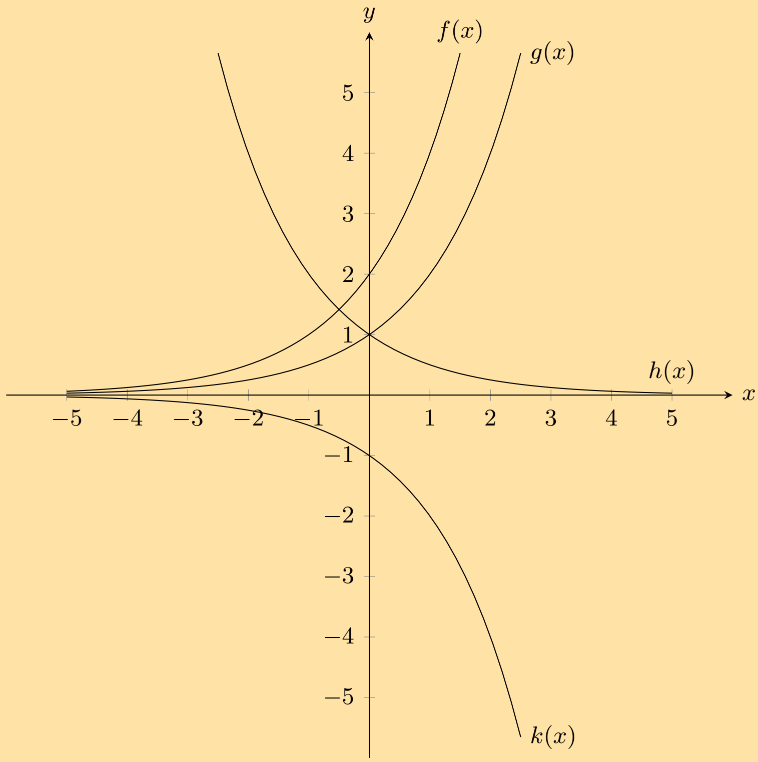
Solution:

We calculate the x-intercept by letting $y = 0$. Then start to solve for x.

$$0 = \left(\frac{1}{4}\right) . (4)^x - 1$$
$$1 = \left(\frac{1}{4}\right) . (4)^x$$
$$(4)(1) = (4)\left(\frac{1}{4}\right) . (4)^x$$
$$4^1 = 4^x$$
$$x = 1$$

Therefore the x-intercept is $(1; 0)$.

5. Given the following graph, identify a function that matches each of the following equations:



a) $y = 2^x$
Solution:
$g(x)$

b) $y = -2^x$
Solution:
$k(x)$

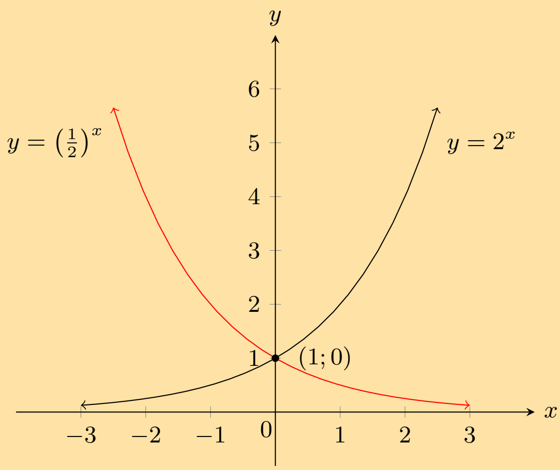
c) $y = 2 . 2^x$
Solution:
$f(x)$

d) $y = \left(\frac{1}{2}\right)^x$
Solution:
$h(x)$

6. Given the functions $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$.

a) Draw the graphs on the same set of axes.
Solution:
For $y = 2^x$:
a is positive and greater than 1 and so the graph curves upwards. The y-intercept is $(0; 1)$. There is no x-intercept. The asymptote is the line $x = 0$.

For $y = \left(\frac{1}{2}\right)^x$:
a is positive and less than 1 and so the graph curves downwards. The y-intercept is $(0;1)$. There is no x-intercept. The asymptote is the line $x = 0$.
The graph is:



b) Is the x-axis an asymptote or an axis of symmetry to both graphs? Explain your answer.
Solution:
The x-axis is an asymptote to both graphs because both approach the x-axis but never touch it.

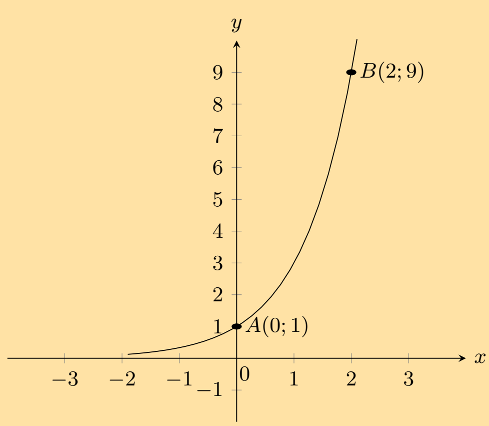
c) Which graph can be described by the equation $y = 2^{-x}$? Explain your answer.
Solution:
$y = \left(\frac{1}{2}\right)^x$ can be described by the equation $y = 2^{-x}$ because $y = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x}$.

d) Solve the equation $2^x = \left(\frac{1}{2}\right)^x$ graphically and check your answer is correct by using substitution.
Solution:
The graphs intersect at the point $(0;1)$. If we substitute these values into each side of the equation we get:

$$\text{LHS: } 2^x = 2^0 = 1 \text{ and}$$
$$\text{RHS: } (\frac{1}{2})^x = (\frac{1}{2})^0 = 1$$

LHS = RHS, therefore the answer is correct.

7. The curve of the exponential function f in the accompanying diagram cuts the y-axis at the point $A(0;1)$ and passes through the point $B(2;9)$.



a) Determine the equation of the function f.
Solution:
The general form of the equation is $f(x) = a\ .\ b^x + q$.
We are given $A(0;1)$ and $B(2;9)$.
The asymptote is at $y = 0$ and so $q = 0$.

Substitute in the values of point A:

$$1 = a \,.\, b^0$$
$$1 = a$$

Substitute in the values of point B:

$$9 = b^2$$
$$3^2 = b^2$$
$$\therefore b = 3$$

Therefore the equation is $f(x) = 3^x$.

b) Determine the equation of the function $h(x)$, the reflection of $f(x)$ in the x-axis.
Solution:
$h(x) = -3^x$

c) Determine the range of $h(x)$.
Solution:
Range: $(-\infty; 0)$

d) Determine the equation of the function $g(x)$, the reflection of $f(x)$ in the y-axis.
Solution:
$g(x) = 3^{-x}$

e) Determine the equation of the function $j(x)$ if $j(x)$ is a vertical stretch of $f(x)$ by $+2$ units.
Solution:
$j(x) = 2 \,.\, 3^x$

f) Determine the equation of the function $k(x)$ if $k(x)$ is a vertical shift of $f(x)$ by -3 units.
Solution:
$k(x) = 3^x - 3$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2FYX 2. 2FYY 3. 2FYZ 4. 2FZ2 5. 2FZ3 6. 2FZ4 7. 2FZ5

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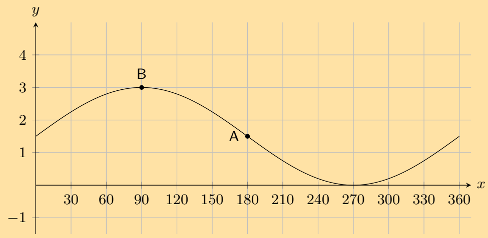
6.6 Trigonometric functions

Exercise 6 – 6:

1. Shown the following graph of the following form: $y = a \sin\theta + q$ where **Point A** is at $(180°; 1{,}5)$, and **Point B** is at $(90°; 3)$, find the values of a and q.



Solution:

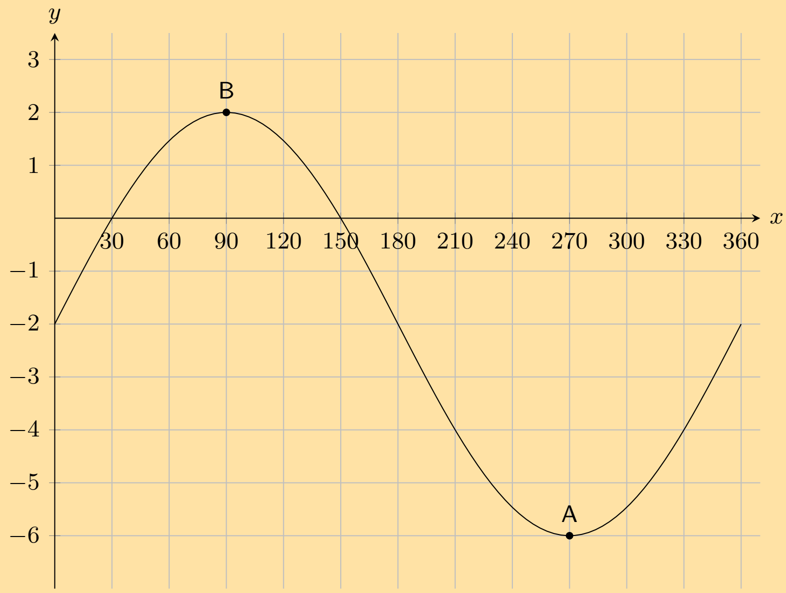
To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point A is at $(180°; 1{,}5)$. For an unshifted sine graph point A would be at $(180°; 0)$. For this graph we see that this point has been shifted up by 1,5 or $\frac{3}{2}$ spaces. Therefore $q = \frac{3}{2}$.

To find a we note that the y-value at the middle (point A) is 1,5, while the y-value at the top (point B) is 3. We can find the amplitude by working out the distance from the top of the graph to the middle of the graph: $3 - 1{,}5 = 1{,}5$. Therefore $a = \frac{3}{2}$.

The complete equation for the graph shown in this question is $y = \frac{3}{2}\sin\theta + \frac{3}{2}$.

Therefore $a = \frac{3}{2}$ and $q = \frac{3}{2}$

2. Shown the following graph of the following form: $y = a\sin\theta + q$ where **Point A** is at $(270°; -6)$, and **Point B** is at $(90°; 2)$, determine the values of a and q.



Solution:

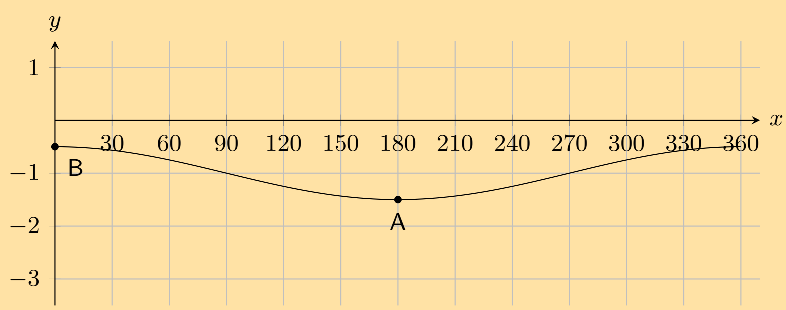
To find a we note that the y-value at the bottom (point A) is -6, while the y-value at the top (point B) is 2. We can find the amplitude by working out the distance from the top of the graph to the bottom of the graph and then dividing this by 2 since this distance is twice the amplitude: $\frac{2-(-6)}{2} = 4$. Therefore $a = 4$.

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point B is at $(90°; 2)$. For an unshifted sine graph with the same a value (i.e. $4\sin\theta$) point B would be at $(90°; 4)$. For this graph we see that this point has been shifted down by 2 spaces. Therefore $q = 2$.

The complete equation for the graph shown in this question is $y = 4\sin\theta - 2$.

Therefore $a = 4$ and $q = -2$

3. The graph below shows a trigonometric equation of the following form: $y = a\cos\theta + q$. Two points are shown on the graph: **Point A** at $(180°; -1{,}5)$, and **Point B**: $(0°; -0{,}5)$. Calculate the values of a (the amplitude of the graph) and q (the vertical shift of the graph).



Solution:

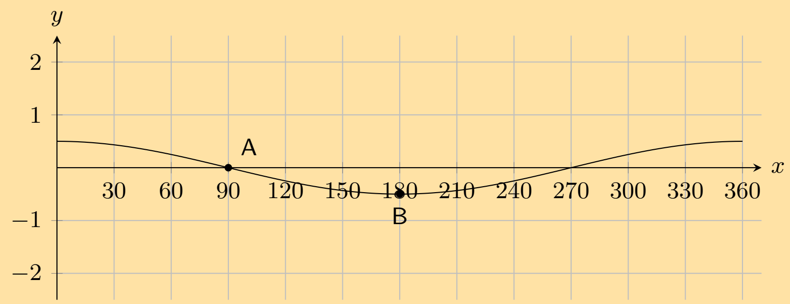
To find a we note that the y-value at the bottom (point A) is $-1{,}5$, while the y-value at the top (point B) is $-0{,}5$. We can find the amplitude by working out the distance from the top of the graph to the bottom of the graph and then dividing this by 2 since this distance is twice the amplitude: $\frac{-0{,}5-(-1{,}5)}{2} = \frac{1}{2}$. Therefore $a = \frac{1}{2}$.

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point B is at $(0°; -0{,}5)$. For an unshifted cosine graph with the same a value (i.e. $\frac{1}{2}\cos\theta$) point B would be at $(0°; 0{,}5)$. For this graph we see that this point has been shifted down by 1 space. Therefore $q = 1$.

The complete equation for the graph shown in this question is $y = \frac{1}{2}\cos\theta - 1$.

Therefore $a = \frac{1}{2}$, and $q = -1$.

4. The graph below shows a trigonometric equation of the following form: $y = a\cos\theta + q$. Two points are shown on the graph: **Point A** at $(90°; 0{,}0)$, and **Point B**: $(180°; -0{,}5)$. Calculate the values of a (the amplitude of the graph) and q (the vertical shift of the graph).



Solution:

To find a we note that the y-value at the bottom (point B) is $-0{,}5$, while the y-value at the middle (point A) is 0. We can find the amplitude by working out the distance from the top of the graph to the middle of the graph: $0 - (-0{,}5) = \frac{1}{2}$. Therefore $a = \frac{1}{2}$.

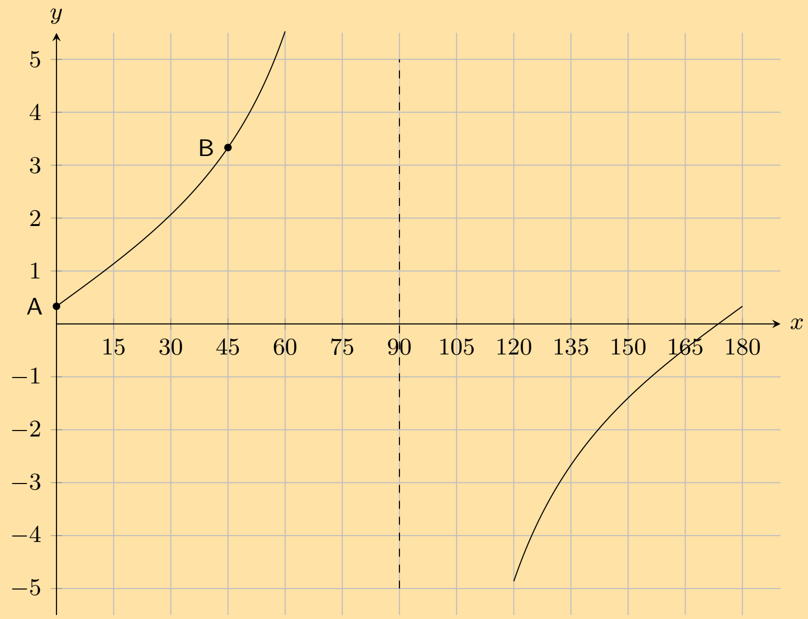
To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point A is at $(90°; 0)$. For an unshifted cosine graph with the same a value (i.e. $\frac{1}{2}\cos\theta$) point B would be at $(0°; 0)$. For this graph we see that this point has not been shifted. Therefore $q = 0$.

The complete equation for the graph shown in this question is $y = \frac{1}{2}\cos\theta$.

Therefore $a = \frac{1}{2}$, and $q = 0$.

5. On the graph below you see a tangent curve of the following form: $y = a\tan\theta + q$. Two points are labelled on the curve: **Point A** is at $\left(0°; \frac{1}{3}\right)$, and **Point B** is at $\left(45°; \frac{10}{3}\right)$.

Calculate, or otherwise determine, the values of a and q.



Solution:

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point A is at $\left(0°; \frac{1}{3}\right)$. For an unshifted tangent graph point A would be at $(0°; 0)$. For this graph we see that this point been shifted upwards by a $\frac{1}{3}$. Therefore $q = \frac{1}{3}$.

To find a we can substitute point B into the equation for the tangent graph:

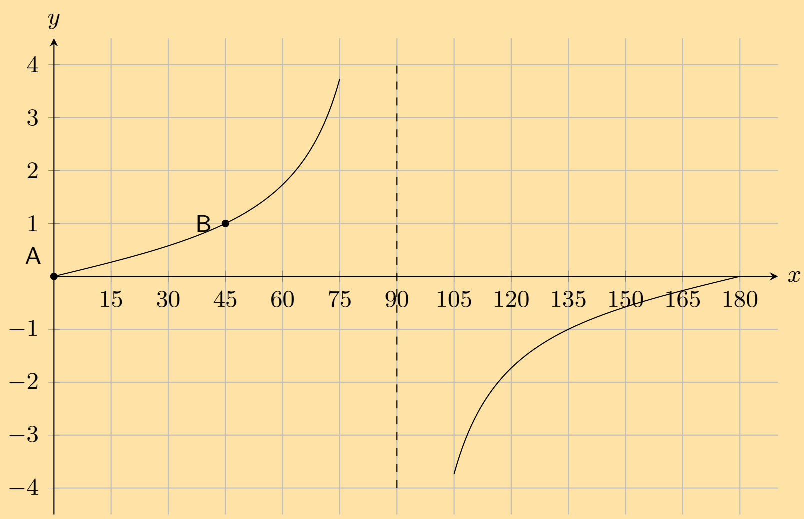
$$
\begin{aligned}
y &= a\tan\theta + \frac{1}{3} \\
\left(\frac{10}{3}\right) &= a\tan 45^\circ + \frac{1}{3} \\
\frac{10}{3} &= a(1) + \frac{1}{3} \\
\frac{10}{3} - \frac{1}{3} &= a \\
3 &= a
\end{aligned}
$$

The complete equation is: $y = 3\tan\theta + \frac{1}{3}$.

Therefore $a = 3$ and $q = \frac{1}{3}$.

6. The graph below shows a tangent curve with an equation of the form $y = a\tan\theta + q$. Two points are labelled on the curve: **Point A** is at $(0^\circ; 0)$, and **Point B** is at $(45^\circ; 1)$.

   Find a and q.



Solution:

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point A is at $(0^\circ; 0)$. For an unshifted tangent graph point A would be at $(0^\circ; 0)$. For this graph we see that the graph has not been shifted. Therefore $q = 0$.

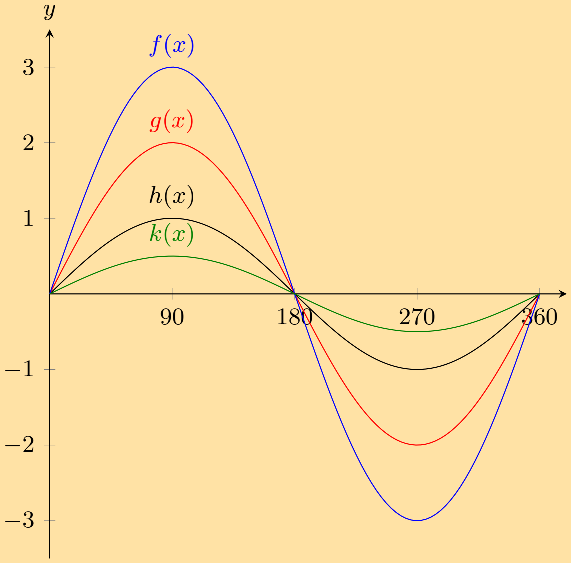
To find a we can substitute point B into the equation for the tangent graph:

$$
\begin{aligned}
y &= a\tan\theta \\
1 &= a\tan 45^\circ \\
1 &= a(1) \\
1 &= a
\end{aligned}
$$

The complete equation is: $y = \tan\theta$.

Therefore $a = 1$ and $q = 0$.

7. Given the following graph, identify a function that matches each of the following equations:



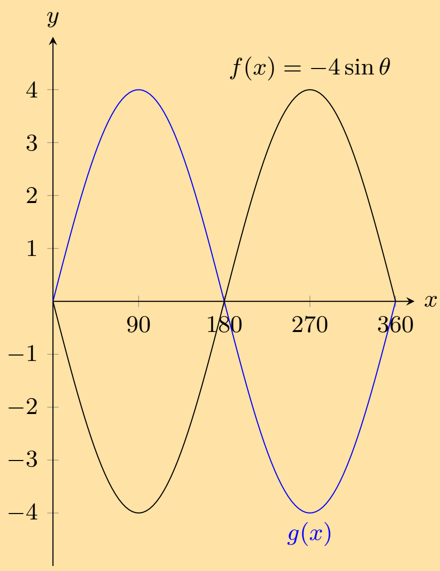
a) $y = \sin\theta$
Solution:
$h(x)$

b) $y = \frac{1}{2}\sin\theta$
Solution:
$k(x)$

c) $y = 3\sin\theta$
Solution:
$f(x)$

d) $y = 2\sin\theta$
Solution:
$g(x)$

8. The graph below shows functions $f(x)$ and $g(x)$

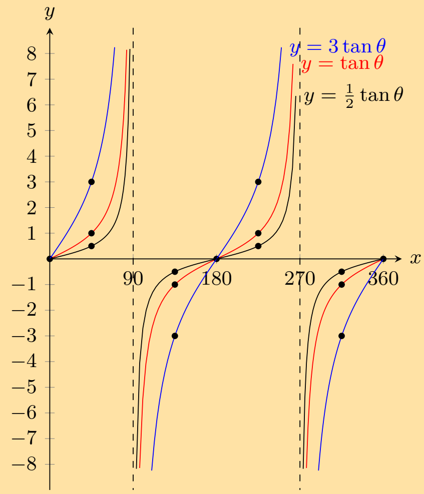


What is the equation for $g(x)$?
Solution:
$g(x) = 4\sin\theta$

9. With the assistance of the table below sketch the three functions on the same set of axes.

θ	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\tan\theta$	0	1	undefined	-1	0	1	undefined	-1	0
$3\tan\theta$	0	3	undefined	-3	0	3	undefined	-3	0
$\frac{1}{2}\tan\theta$	0	$\frac{1}{2}$	undefined	$-\frac{1}{2}$	0	$\frac{1}{2}$	undefined	$-\frac{1}{2}$	0

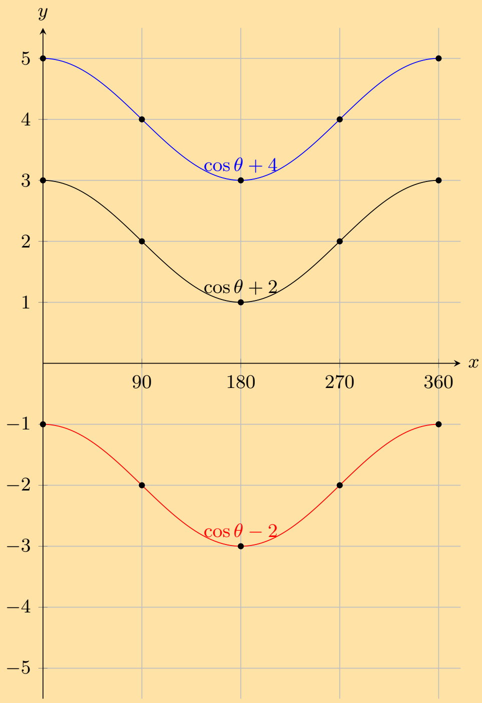
Solution:
We are given a table with values and so we plot each of these points and join them with a smooth curve.



10. With the assistance of the table below sketch the three functions on the same set of axes.

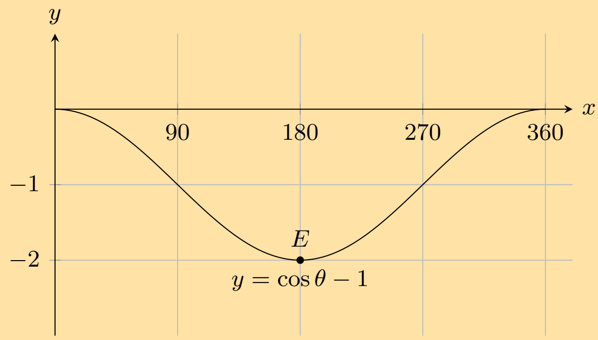
θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\cos\theta - 2$	-1	-2	-3	-2	-1
$\cos\theta + 4$	5	4	2	4	5
$\cos\theta + 2$	3	2	1	2	3

Solution:
We are given a table with values and so we plot each of these points and join them with a smooth curve.



11. State the coordinates at E and the range of the function.

a)

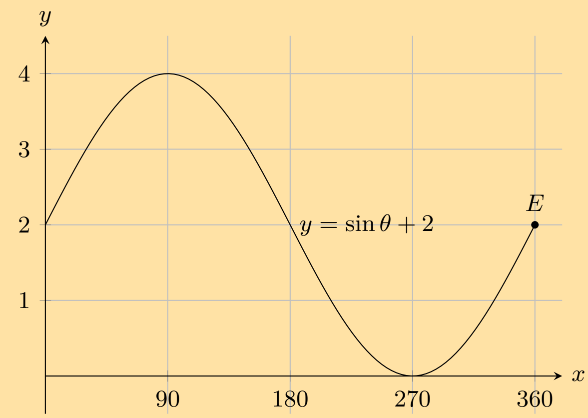


Solution:

To find the coordinates of E we read the value off the graph. To find the range we note that this is a cosine graph and so the maximum value occurs at $0°$ (and at $360°$). The minimum value occurs at $180°$. So we read off the value of y at $0°$ and at $180°$.

Therefore $E(180°; -2)$ and $-2 \leq y \leq 0$.

b)

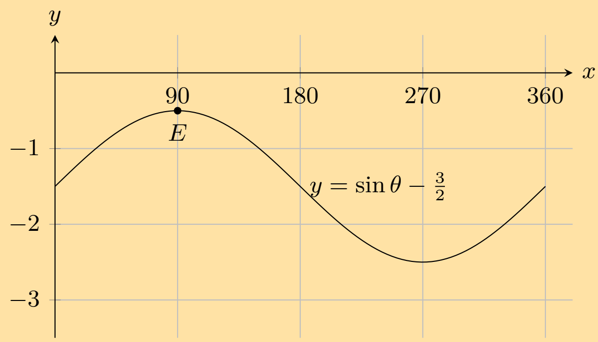


Solution:

To find the coordinates of E we read the value off the graph. To find the range we note that this is a sine graph and so the maximum value occurs at $90°$. The minimum value occurs at $270°$. So we read off the value of y at $90°$ and at $270°$.

Therefore $E(360°; 2)$ and $0 \leq y \leq 4$

c)

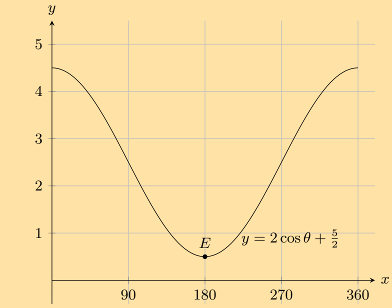


Solution:

To find the coordinates of E we read the value off the graph. To find the range we note that this is a sine graph and so the maximum value occurs at $90°$. The minimum value occurs at $270°$. So we read off the value of y at $90°$ and at $270°$.

Therefore $E(90°; -0{,}5)$ and $-2{,}5 \leq y \leq -0{,}5$

d)

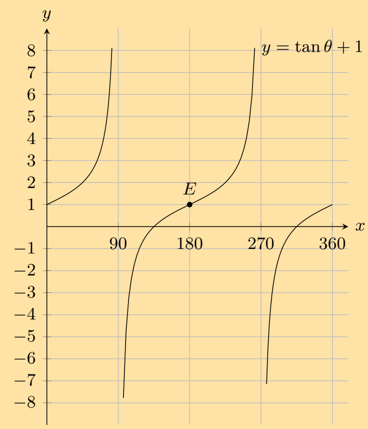


Solution:

To find the coordinates of E we read the value off the graph. To find the range we note that this is a cosine graph and so the maximum value occurs at $0°$ (and at $360°$). The minimum value occurs at $180°$. So we read off the value of y at $0°$ and at $180°$.

Therefore $E(180°; 0{,}5)$ and $0{,}5 \leq y \leq 4{,}5$

12. State the coordinates at E and the domain and range of the function in the interval shown.



Solution:

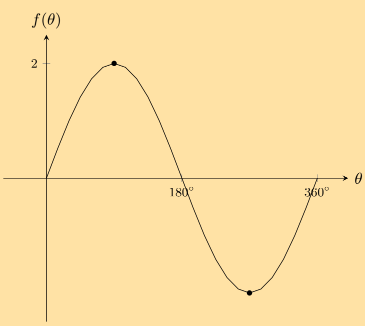
$E(180°; 1)$, range $y \in \mathbb{R}$ and domain $0 \leq \theta \leq 360, x \neq 90, x \neq 270$

13. Using your knowledge of the effects of a and q, sketch each of the following graphs, without using a table of values, for $\theta \in [0°; 360°]$

a) $y = 2\sin\theta$

Solution:

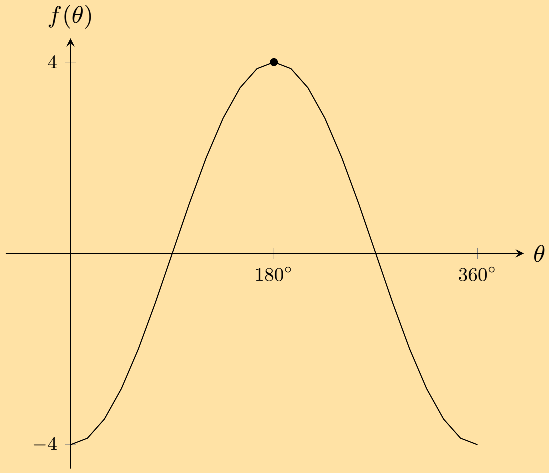
In this case $q = 0$ and so the basic sine graph is not shifted up or downwards. We also note that $a = 2$ and so the graph is stretched by 2 units. The maximum value will be 2 and the minimum value will be -2.



b) $y = -4\cos\theta$

Solution:

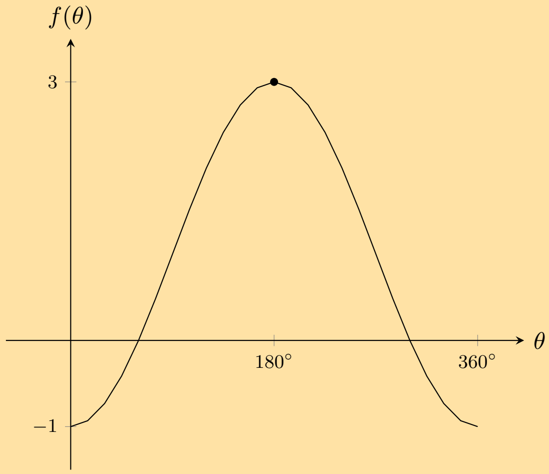
In this case $q = 0$ and so the basic cosine graph is not shifted up or downwards. We also note that $a = -4$ and so the graph is stretched by -4 units. The maximum value will be 4 and the minimum value will be -4.



c) $y = -2\cos\theta + 1$

Solution:

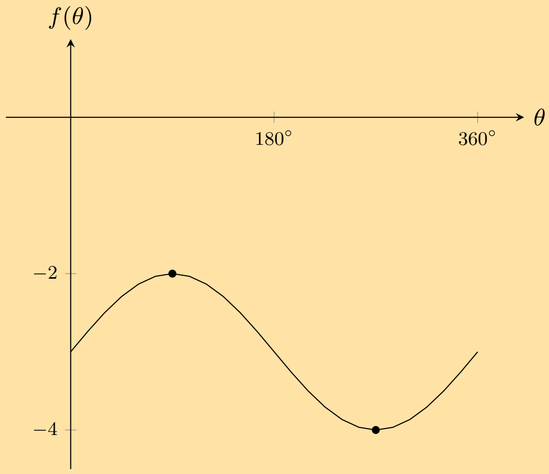
In this case $q = 1$ and so the basic cosine graph is shifted upwards by 1 unit. We also note that $a = -2$ and so the graph is stretched by -2 units. The maximum value will be 3 and the minimum value will be -1.



d) $y = \sin\theta - 3$

Solution:

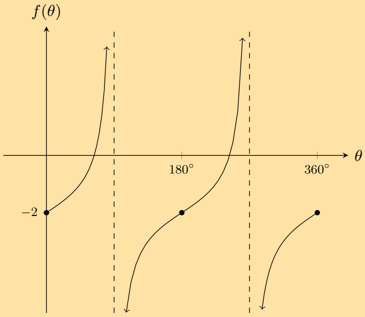
In this case $q = -3$ and so the basic sine graph is shifted downwards by 3 units. We also note that $a = 1$ and so the graph is not stretched. The maximum value will be -2 and the minimum value will be -4.



e) $y = \tan\theta - 2$

Solution:

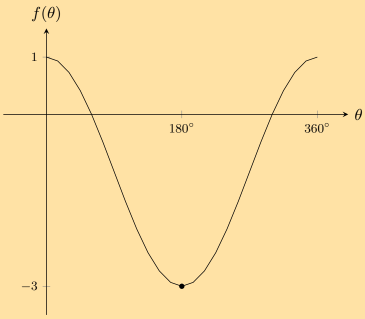
In this case $q = -2$ and so the basic tangent graph is shifted downwards by 2 units. We also note that $a = 1$ and so the graph is not stretched. When $\theta = 0°$, $y = -2$. Similarly when $\theta = 180°$, $y = -2$ and when $\theta = 360°$, $y = -2$.



f) $y = 2\cos\theta - 1$

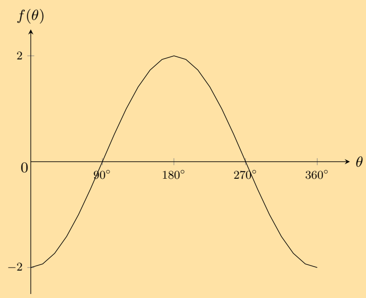
Solution:

In this case $q = -1$ and so the basic cosine graph is shifted downwards by 1 units. We also note that $a = 2$ and so the graph is stretched by 2 units. The maximum value will be 1 and the minimum value will be -3.



14. Give the equations for each of the following graphs:

a)

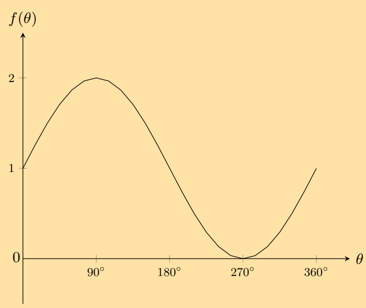


Solution:

The general form of a cosine graph is $y = a\cos\theta + q$. We note that in this case the graph is not shifted. We also note the graph is stretched by -2 units.

Therefore $y = -2\cos\theta$.

b)

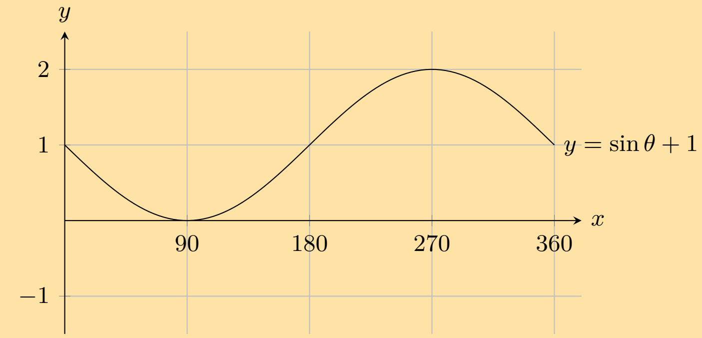


Solution:

The general form of a sine graph is $y = a\sin\theta + q$. We note that in this case the graph is shifted upwards by 1 unit. We also note the graph is not stretched.

Therefore $y = \sin\theta + 1$.

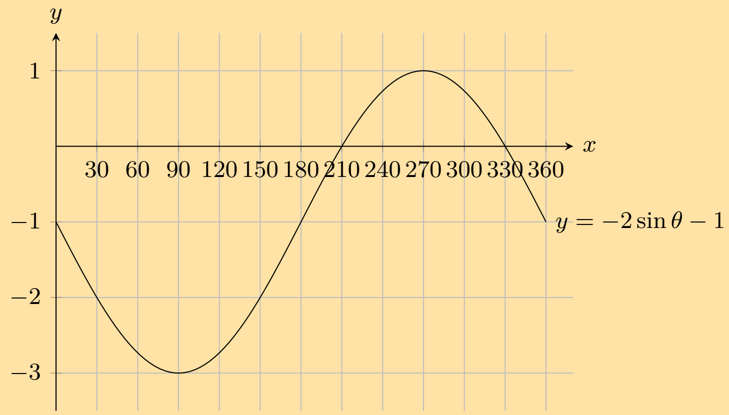
15. For which values of θ is the function increasing, in the interval shown?



Solution:

$90° < \theta < 270°$

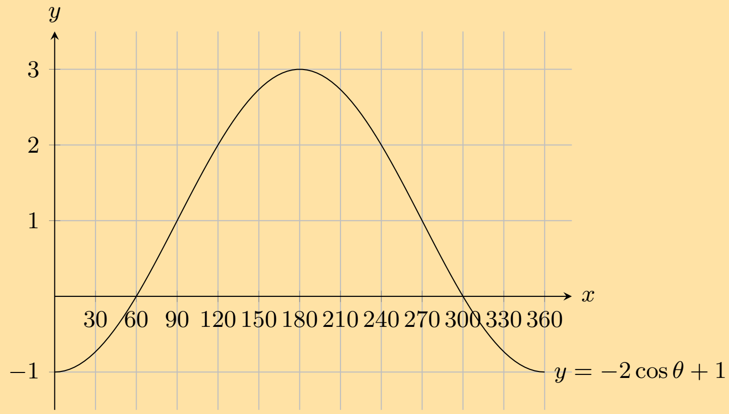
16. For which values of θ is the function negative, in the interval shown?



Solution:

$0° < \theta < 210°$ and $330° < \theta < 360°$

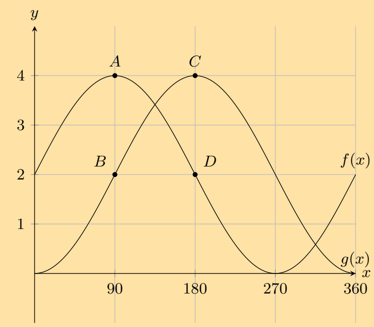
17. For which values of θ is the function positive, in the interval shown?



Solution:

$60° < \theta < 300°$

18. Given the following graph.



a) State the coordinates at A, B, C and D.
Solution:
We can read the values off the graph:
$A = (90°; 4)$, $B = (90°; -2)$, $C = (180°; 4)$ and $D = (180°; -2)$

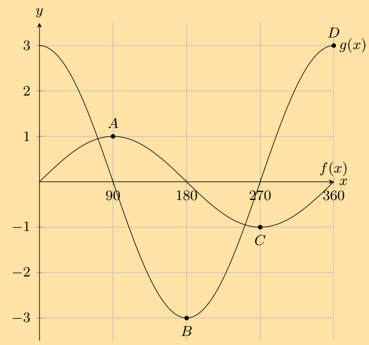
b) How many times in this interval does $f(x)$ intersect $g(x)$.
Solution:
2

c) What is the amplitude of $f(x)$.
Solution:
2

d) Evaluate: $f(360°) - g(360°)$.
Solution:
Read off the value of $f(360°)$ and $g(360°)$ from the graph. Then subtract $g(360°)$ from $f(360°)$.

$$\begin{aligned} f(360°) - g(360°) &= 2 - 0 \\ &= 2 \end{aligned}$$

19. Given the following graph.



a) State the coordinates at A, B, C and D.
Solution:
We read the values off the graph:
$A = (90°; 1)$, $B = (180°; -3)$, $C = (270°; -1)$ and $D = (360°; 3)$

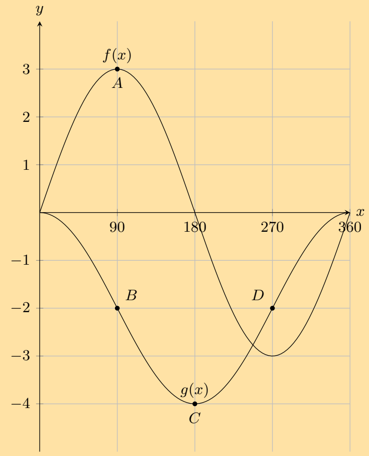
b) How many times in this interval does $f(x)$ intersect $g(x)$.
Solution:
2

c) What is the amplitude of $g(x)$.
Solution:
3

d) Evaluate: $f(90°) - g(90°)$.
Solution:
Read off the value of $f(90°)$ and $g(90°)$ from the graph. Then subtract $g(90°)$ from $f(90°)$.

$$\begin{aligned} f(90°) - g(90°) &= 1 - 0 \\ &= 1 \end{aligned}$$

20. Given the following graph:



a) State the coordinates at A, B, C and D.
Solution:
We read the values off the graph:
$A = (90°;3)$, $B = (90°;2)$, $C = (180°;-4)$ and $D = (270°;2)$

b) How many times in this interval does $f(x)$ intersect $g(x)$.
Solution:
3

c) What is the amplitude of $g(x)$.
Solution:
2

d) Evaluate: $f(270°) - g(270°)$.
Solution:
Read off the value of $f(270°)$ and $g(270°)$ from the graph. Then subtract $g(270°)$ from $f(270°)$.

$$\begin{aligned} f(270°) - g(270°) &= -3 - (-2) \\ &= -1 \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FZ6	2. 2FZ7	3. 2FZ8	4. 2FZ9	5. 2FZB	6. 2FZC
7. 2FZD	8. 2FZF	9. 2FZG	10. 2FZH	11a. 2FZJ	11b. 2FZK
11c. 2FZM	11d. 2FZN	12. 2FZP	13a. 2FZQ	13b. 2FZR	13c. 2FZS
13d. 2FZT	13e. 2FZV	13f. 2FZW	14a. 2FZX	14b. 2FZY	15. 2FZZ
16. 2G22	17. 2G23	18. 2G24	19. 2G25	20. 2G26	

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6.7 Interpretation of graphs

Exercise 6 – 7:

1. Plot the following functions on the same set of axes and clearly label all the points at which the functions intersect.

a) $y = x^2 + 1$ and $y = 3^x$

Solution:

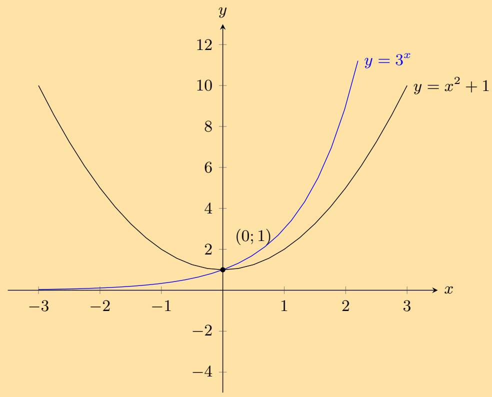
The y-intercept for each graph is:

$0^2 + 1 = 1$

$3^0 = 1$

This is also the only point of intersection.

For both graphs there is no x-intercept.



b) $y = x$ and $y = \frac{2}{x}$

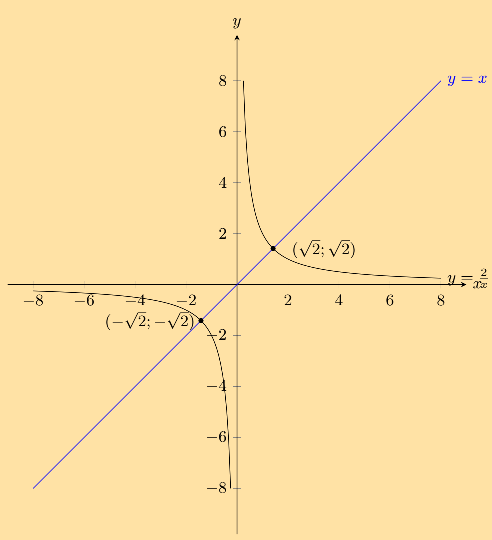
Solution:

$y = x$ is a basic straight line graph. For $y = \frac{2}{x}$ there is no y-intercept and no x-intercept. We note that this is a hyperbolic graph that has been stretched by 2 units.

The points of intersection are:

$$\begin{aligned} x &= \frac{2}{x} \\ x^2 &= 2 \\ x^2 - 2 &= 0 \\ (x - \sqrt{2})(x + \sqrt{2}) &= 0 \\ x &= \sqrt{2} \text{ or } x = -\sqrt{2} \\ y &= \sqrt{2} \text{ or } y = -\sqrt{2} \end{aligned}$$

The graphs intersect at $\left(\sqrt{2}; \sqrt{2}\right)$ and $\left(-\sqrt{2}; -\sqrt{2}\right)$.



c) $y = x^2 + 3$ and $y = 6$

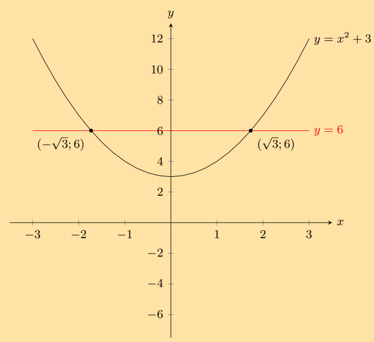
Solution:

$y = 6$ is a horizontal line through $(0;6)$. For $y = x^2 + 3$ the y-intercept is $(0;3)$ and there are no x-intercepts. From the value of q we see that this is a basic parabola that has been shifted upwards by 3 units.

The points of intersection are:

$$\begin{aligned} x^2 + 3 &= 6 \\ x^2 - 3 &= 0 \\ (x - \sqrt{3})(x + \sqrt{3}) &= 0 \\ x &= \sqrt{3} \text{ or } x = -\sqrt{3} \\ y &= 6 \end{aligned}$$

The graphs intersect at $\left(\sqrt{3};6\right)$ and $\left(-\sqrt{3};-6\right)$.



d) $y = -x^2$ and $y = \dfrac{8}{x}$

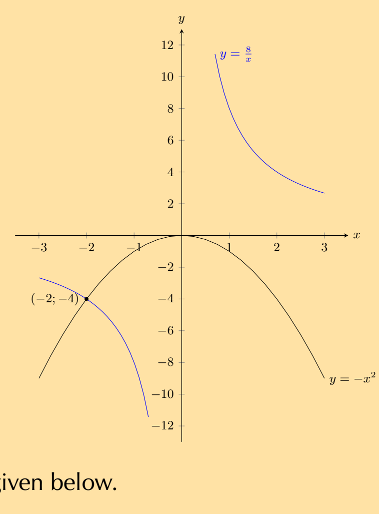
Solution:

$y = -x^2$ is a parabola that has been reflected about the x-axis. For $y = \frac{8}{x}$ there is no y-intercept and there is no x-intercepts. From the value of a we see that this is a basic hyperbola that has been stretched by 8 units.

The points of intersection are:

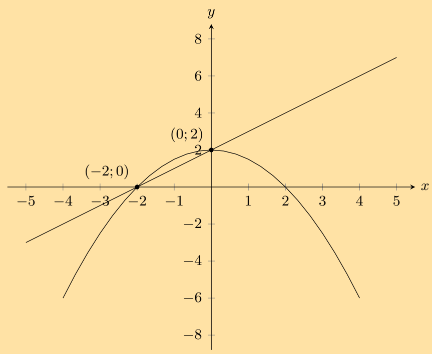
$$\begin{aligned} -x^2 &= \frac{8}{x} \\ x^3 &= -8 \\ x &= -2 \\ y &= \frac{8}{-2} \\ y &= -4 \end{aligned}$$

The graphs intersect at $(-2;-4)$.



2. Determine the equations for the graphs given below.

a)



Solution:

For the straight line graph we have the x and y-intercepts. The y-intercept gives $c = 2$. Now we can calculate the gradient of the straight line graph:

$$\begin{aligned} y &= mx + 2 \\ m &= \frac{2 - 0}{0 - (-2)} \\ &= 1 \end{aligned}$$

Therefore the equation of the straight line graph is $y = x + 2$.

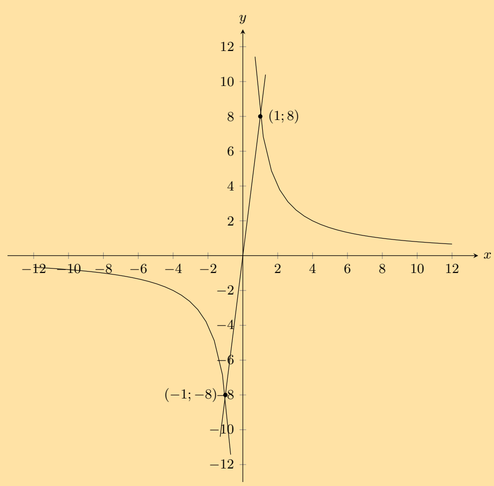
For the parabola we also have the x and y-intercepts. The y-intercept gives $q = 2$. Now we can calculate a:

$$\begin{aligned} y &= ax^2 + 2 \\ 0 &= a(-2)^2 + 2 \\ -2 &= 4a \\ a &= -\frac{1}{2} \end{aligned}$$

Therefore the equation of the parabola is $y = -\frac{1}{2}x^2 + 2$.

The equations for the two graphs are $y = x + 2$ and $y = -\frac{1}{2}x^2 + 2$

b)



Solution:

For the straight line graph we notice that it passes through $(0; 0)$ and so $c = 0$.

We have two points on the straight line graph and so we can calculate the gradient, m:

$$
\begin{aligned}
y &= mx + 0 \\
m &= \frac{8-(-8)}{1-(-1)} \\
m &= 8
\end{aligned}
$$

The equation of the straight line graph is $y = 8x$.
For the hyperbola we note that the graph is not shifted either upwards or downwards. Therefore $q = 0$. Now we can calculate a:

$$
\begin{aligned}
y &= \frac{a}{x} \\
8 &= \frac{a}{1} \\
a &= 8
\end{aligned}
$$

Therefore the equation of the hyperbola is $y = \frac{8}{x}$.
The equations for the two graphs are $y = 8x$ and $y = \frac{8}{x}$.

3. Choose the correct answer:

   a) The range of $y = 2\sin\theta + 1$ is:

      i. $1 \leq \theta \leq 2$
      ii. $-2 \leq \theta \leq 2$
      iii. $-1 \leq \theta \leq 3$
      iv. $-2 \leq \theta \leq 3$

   Solution:
   (iii)

   b) The range of $y = 2\cos\theta - 4$ is:

      i. $-6 \leq \theta \leq 2$
      ii. $-4 \leq \theta \leq -2$
      iii. $-6 \leq \theta \leq 1$
      iv. $-6 \leq \theta \leq -2$

   Solution:
   (iv)

   c) The y-intercept of $2^x + 1$ is:

      i. 3
      ii. 1
      iii. 2
      iv. 0

   Solution:
   (iii)

   d) Which of the following passes through $(1; 7)$?

      i. $y = \frac{7}{x}$
      ii. $y = 2x + 3$
      iii. $y = \frac{4}{x}$
      iv. $y = x^2 + 1$

   Solution:
   (i)

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2G27 1b. 2G28 1c. 2G29 1d. 2G2B 2a. 2G2C 2b. 2G2D
3a. 2G2F 3b. 2G2G 3c. 2G2H 3d. 2G2J

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End of chapter Exercise 6 – 8:

1. Complete the following tables and identify the function.

   a)

x		2	3	4		6
y	3	6		12	15	

Solution:

x	1	2	3	4	5	6
y	3	6	9	12	15	18

$$y = 3x$$

   b)

x	1			4	5	6
y	-3	-2	-1		1	2

Solution:

x	1	2	3	4	5	6
y	-3	-2	-1	0	1	2

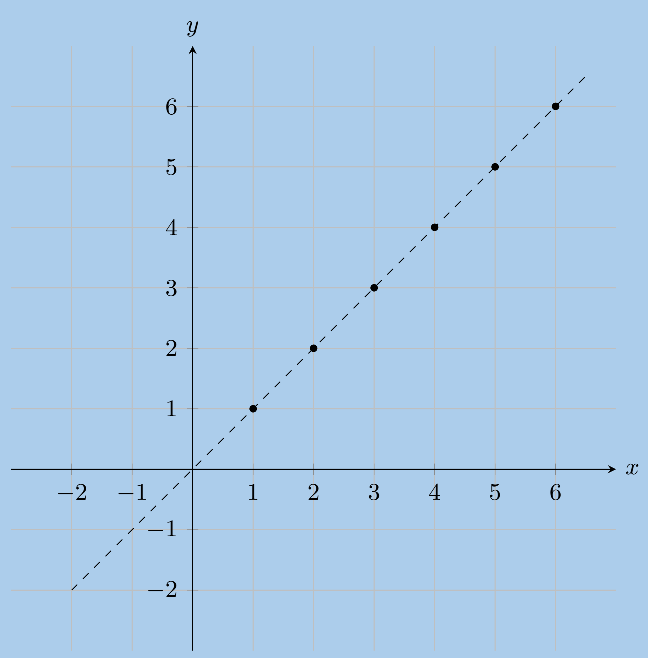
$$y = x - 4$$

2. Plot the following points on a graph.

   a)

x	1	2	3	4	5	6
y	1	2	3	4	5	6

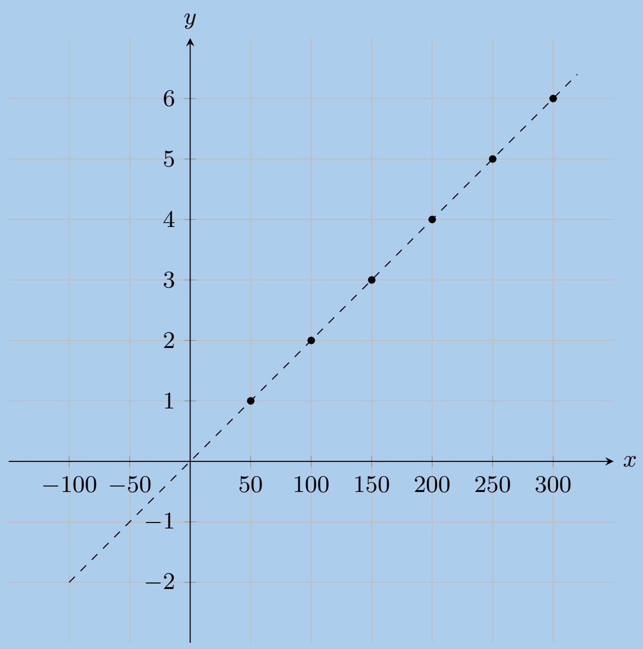
Solution:



   b)

x	50	100	150	200	250	300
y	1	2	3	4	5	6

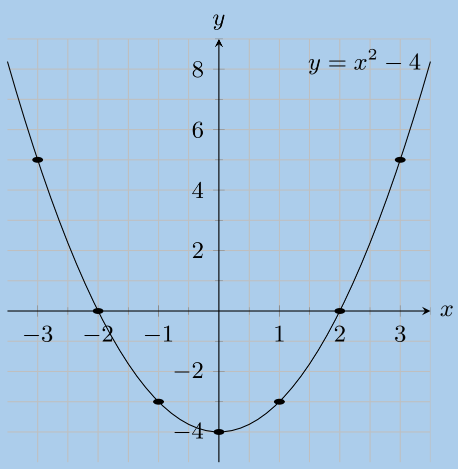
Solution:



3. Create a table of values from the function given and then plot the function. Your table must have at least 5 ordered pairs.

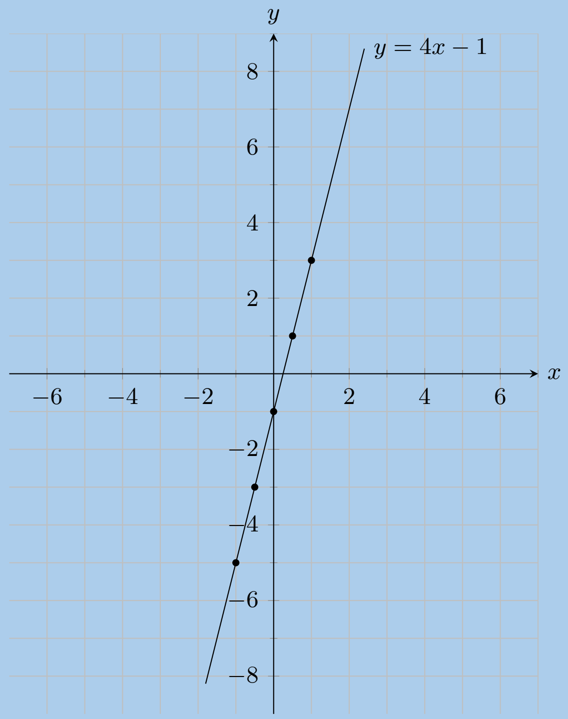
   a) $x^2 - 4$
   Solution:

x	-3	-2	-1	0	1	2	3
y	5	0	-3	-4	-3	0	5



   b) $y = 4x - 1$
   Solution:

x	-1	$-0{,}5$	0	$0{,}5$	1
y	-5	-3	-1	1	3



4. Determine the y-intercept and the x-intercepts of the function.

   a) $y = -3x - 5$
      Solution:

$$\begin{aligned} y &= -3x - 5 \\ y &= -3(0) - 5 \\ y &= -5 \\ \therefore c &= -5 \end{aligned}$$

$$\begin{aligned} y &= -3x - 5 \\ (0) &= -3x - 5 \\ 5 &= -3x \\ -\frac{5}{3} &= x \end{aligned}$$

      x-intercept $= -\frac{5}{3}$ and y-intercept $= -5$

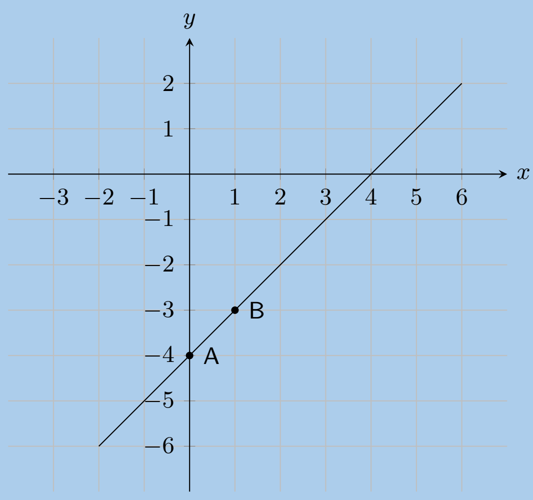
   b) $y = 2x + 4$
      Solution:

$$\begin{aligned} y &= 2x + 4 \\ y &= 2(0) + 4 \\ y &= 4 \\ \therefore c &= 4 \end{aligned}$$

$$\begin{aligned} y &= 2x + 4 \\ (0) &= 2x + 4 \\ -4 &= 2x \\ -2 &= x \end{aligned}$$

      x-intercept $= -2$ and y-intercept $= 4$

5. The graph below shows an equation, which has the form $y = mx + c$. Calculate or otherwise find the values of m (the gradient of the line) and c (the y-intercept of the line).

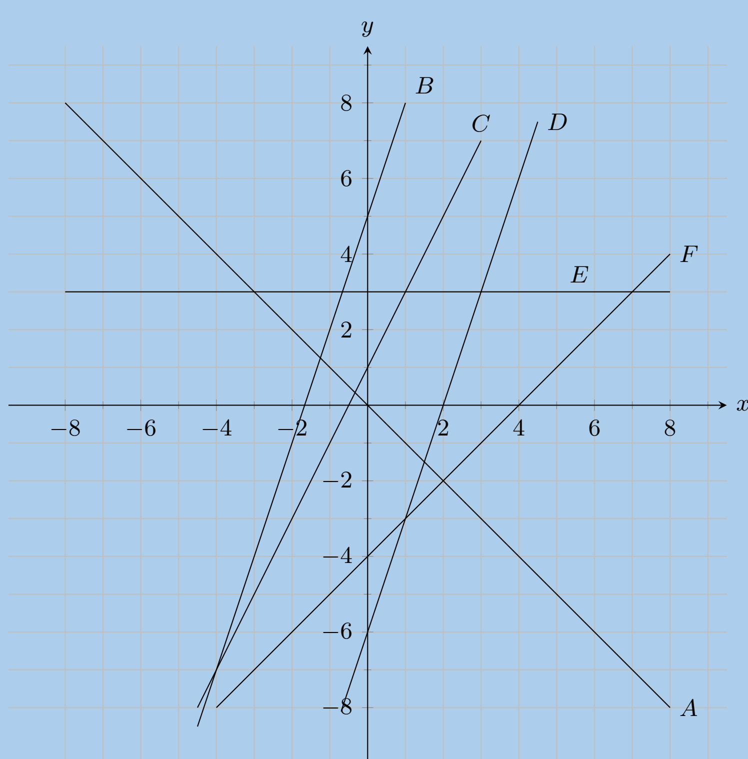


Solution:
Point A is the y-intercept. Point A has co-ordinates $(0; -4)$ and so $c = -4$.

$$\begin{aligned} y &= mx + c \\ (-3) &= m(1) - 4 \\ -3 &= m - 4 \\ 1 &= m \end{aligned}$$

Therefore $m = 1$, and $y = x - 4$.

6. Look at the graphs below. Each graph is labelled with a letter. In the questions that follow, match any given equation with the label of a corresponding graph.



a) $y = 3$
Solution:
E

b) $y = 3x + 5$
Solution:
B

c) $y = -x$
Solution:
A

d) $y = 2x + 1$
Solution:
C

e) $y = x - 4$
Solution:
F

f) $y = 3x - 6$
Solution:
D

7. State whether the following is true or not

a) The y-intercept of $y + 5 = x$ is -5.
Solution:
True

b) The gradient of $-y = x + 2$ is 1.
Solution:
False

$$\begin{aligned} -y &= x + 2 \\ y &= -x - 2 \end{aligned}$$

c) The gradient of $-4y = 3$ is 1.
Solution:
False

$$\begin{aligned} -4y &= 3 \\ y &= -\frac{3}{4} \end{aligned}$$

8. Write the following in standard form:

a) $2y - 5x = 6$
Solution:

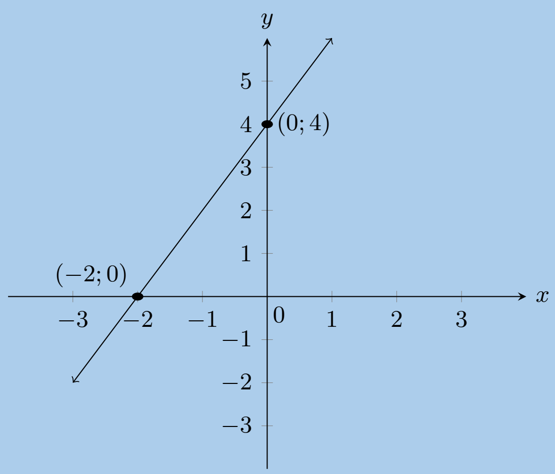
$$\begin{aligned} 2y - 5x &= 6 \\ 2y &= 5x + 6 \\ y &= \frac{5}{2}x + 3 \end{aligned}$$

b) $6y - 3x = 5x + 1$
Solution:

$$\begin{aligned} 6y - 3x &= 5x + 1 \\ 6y &= 8x + 1 \\ y &= \frac{4}{3}x + \frac{1}{6} \end{aligned}$$

9. Sketch the graphs of the following:

a) $y = 2x + 4$
Solution:
The y-intercept is $(0; 4)$ and the x-intercept is $(-2; 0)$.



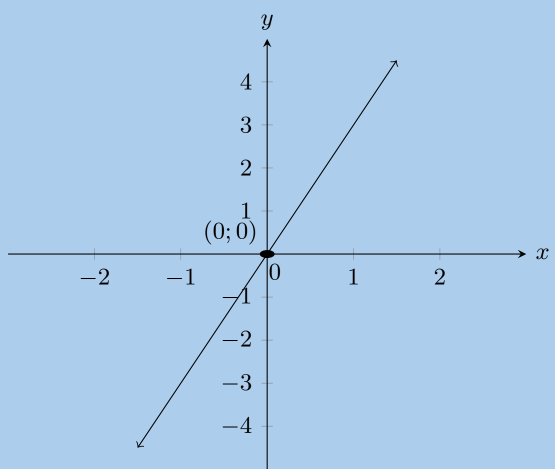
b) $y - 3x = 0$

Solution:

Write the equation in standard form: $y = 3x$.

The y-intercept is $(0; 0)$ and the x-intercept is $(0; 0)$.

We note the following pairs of values: $(1; 3)$, $(2; 6)$, $(-1; -3)$ and $(-2; -6)$. Now we can draw the graph.

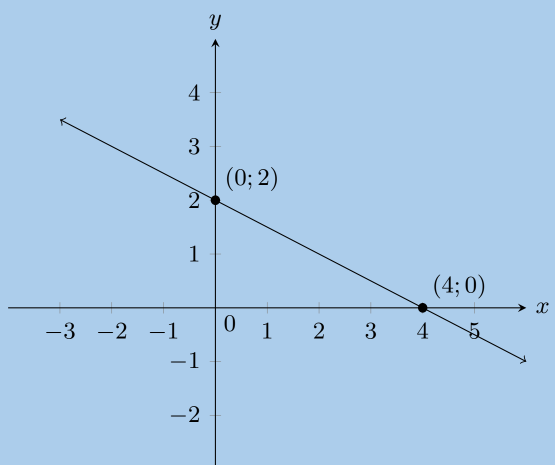


c) $2y = 4 - x$

Solution:

Write the equation in standard form: $y = -\frac{1}{2}x + 2$.

The y-intercept is $(0; 2)$ and the x-intercept is $(4; 0)$.



10. The function for how much water a tap dispenses is given by: $V = 60t$, where x and V are in seconds and mL respectively.
Use this information to answer the following:

a) Evaluate $V(2)$.
Solution:

$$\begin{aligned} V(2) &= 60(2) \\ &= 120 \text{ mL} \end{aligned}$$

b) Evaluate $V(10)$.
Solution:

$$\begin{aligned} V(10) &= 60(10) \\ &= 600 \text{ mL} \end{aligned}$$

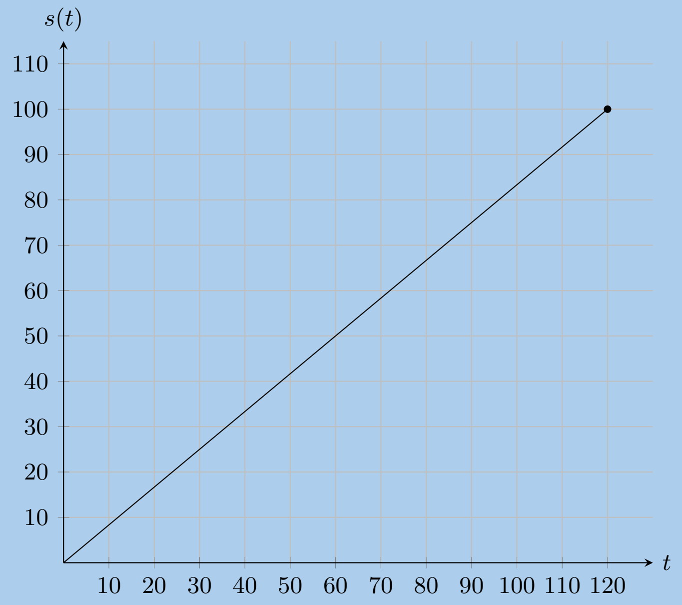
c) How long will it take to fill a 2 L bottle of water?
Solution:

$$\begin{aligned} V(t) &= 2000t \\ t &= \frac{2000}{60} \\ t &= 33{,}33s \end{aligned}$$

d) How much water can the tap dispense in 4 s?
Solution:

$$\begin{aligned} V(4) &= 60(4) \\ &= 240 \text{ mL} \end{aligned}$$

11. The graph below shows the distance travelled by a car over time, where $s(t)$ is distance in km and t is time in minutes.



Use this information to answer the following:

a) What distance did the car travel in an hour?
Solution:
50km

b) What is the domain of the function?

Solution:

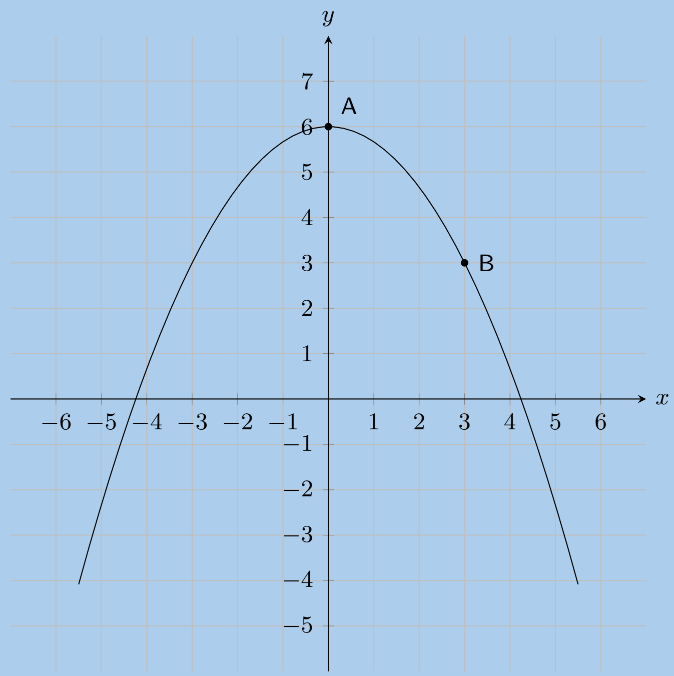
The domain is $0 \leq t \leq 120$ min.

c) What is the range of the function? What does it represent?

Solution:

The range is $0 \leq s \leq 100$ km, it represents the total distance travelled.

12. On the graph here you see a function of the form: $y = ax^2 + q$.
Two points on the parabola are shown: **Point A**, the turning point of the parabola, at $(0;6)$, and **Point B** is at $(3;3)$.
Calculate the values of a and q.



Solution:

The value of q is 6.

$$
\begin{aligned}
y &= ax^2 + 6 \\
(3) &= a(3)^2 + 6 \quad \longleftarrow \quad \text{substitute in the coordinates of a point!} \\
3 &= 9a + 6 \\
3 - 6 &= 9a \\
-3 &= 9a \\
-\frac{1}{3} &= a
\end{aligned}
$$

$$a = -\frac{1}{3};\ q = 6$$

13. Given the following equation:
$y = -5x^2 + 3$

a) Calculate the y-coordinate of the y-intercept.

Solution:

$$
\begin{aligned}
y &= ax^2 + q \\
&= -5x^2 + 3 \\
&= -5(0)^2 + 3 \\
&= 0 + 3
\end{aligned}
$$

The y-coordinate of the y-intercept is 3.

b) Now calculate the x-intercepts. Your answer must be correct to 2 decimal places.

Solution:

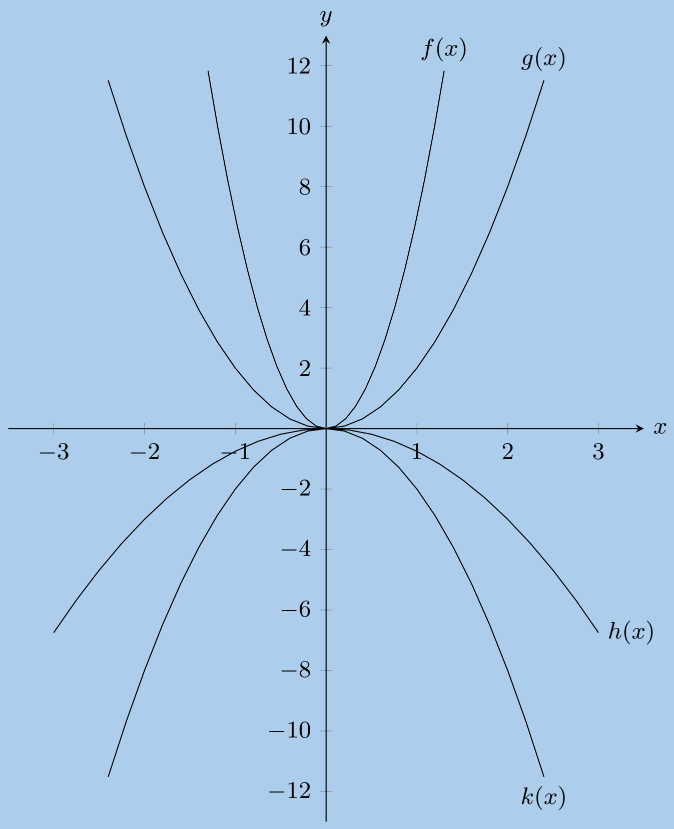
$$\begin{aligned} y &= -5x^2 + 3 \\ (0) &= -5x^2 + 3 \\ 5x^2 &= 3 \\ x^2 &= \frac{3}{5} \\ x &= \pm\sqrt{\frac{3}{5}} \end{aligned}$$

$$\text{Therefore: } x = +\sqrt{\frac{3}{5}} \text{ and } x = -\sqrt{\frac{3}{5}}$$

$$x = -0{,}77 \text{ and } x = 0{,}77$$

The x-intercepts are $(-0{,}77; 0)$ and $(0{,}77; 0)$.

14. Given the following graph, identify a function that matches each of the given equations:



a) $y = -2x^2$

Solution:

$k(x)$

b) $y = 2x^2$

Solution:

$g(x)$

c) $y = -0{,}75x^2$

Solution:

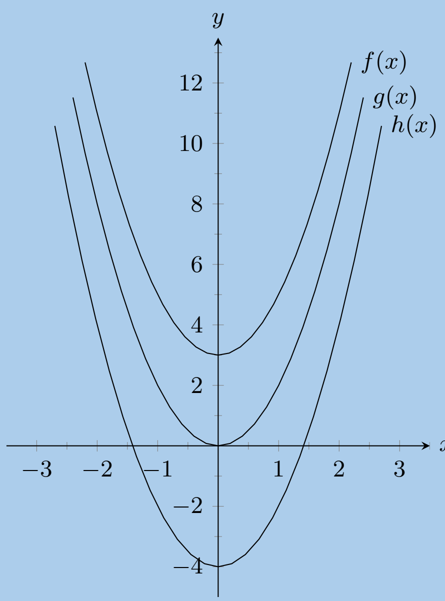
$h(x)$

d) $y = 7x^2$

Solution:

$f(x)$

15. Given the following graph, identify a function that matches each of the given equations:



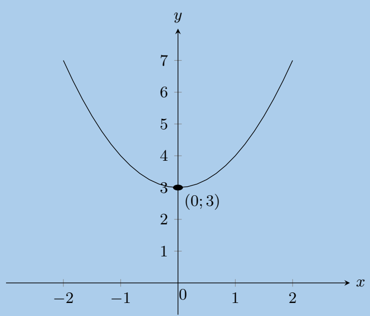
a) $y = 2x^2$
Solution:
$g(x)$

b) $y = 2x^2 + 3$
Solution:
$f(x)$

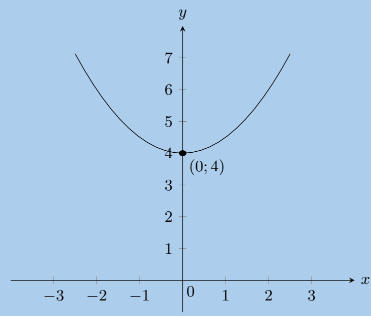
c) $y = 2x^2 - 4$
Solution:
$h(x)$

16. Sketch the following functions:

a) $y = x^2 + 3$
Solution:
The y-intercept is $(0; 3)$. There are no x-intercepts.
a is positive and so the graph is a smile with a minimum turning point at $(0; 3)$.



b) $y = \frac{1}{2}x^2 + 4$
Solution:
The y-intercept is $(0; 4)$. There are no x-intercepts.
a is positive and so the graph is a smile with a minimum turning point at $(0; 4)$.

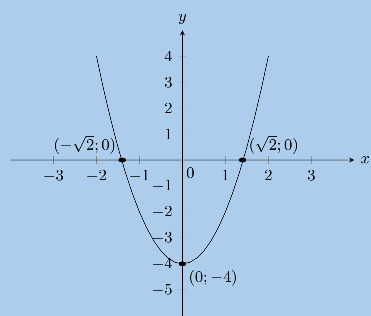


c) $y = 2x^2 - 4$

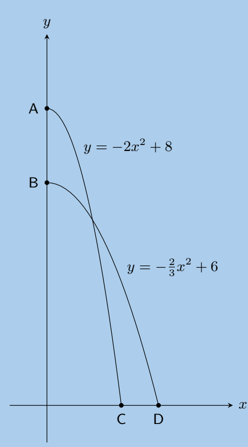
Solution:

The y-intercept is $(0;-4)$. The x-intercepts are at $(\sqrt{2};0)$ and $(-\sqrt{2};0)$.

a is positive and so the graph is a smile with a minimum turning point at $(0;-4)$.



17. Sebastian and Lucas dive into a pool from different heights. Their midair paths can be described by the following quadratic equations: $y = -2x^2 + 8$ for Sebastian and $y = -\frac{2}{3}x^2 + 6$ for Lucas.



a) From what height did Sebastian dive?

Solution:

Maximum value of $y = -2x^2 + 8$ is 8 m

b) From what height did Lucas dive?

Solution:

Maximum value of $y = -\frac{2}{3}x^2 + 6$ is 6 m

c) How far from the pool wall did Lucas land?

Solution:

$$
\begin{aligned}
y &= -\frac{2}{3}x^2 + 6\\
0 &= -\frac{2}{3}x^2 + 6\\
\frac{3}{2}x^2 - 6 &= 0\\
x^2 - 9 &= 0\\
(x-3)(x+3) &= 0\\
\therefore x &= 3 \text{ m}
\end{aligned}
$$

Lucas landed 3 m from the pool wall.

d) How much closer to the pool wall did Sebastian land compared to Lucas?

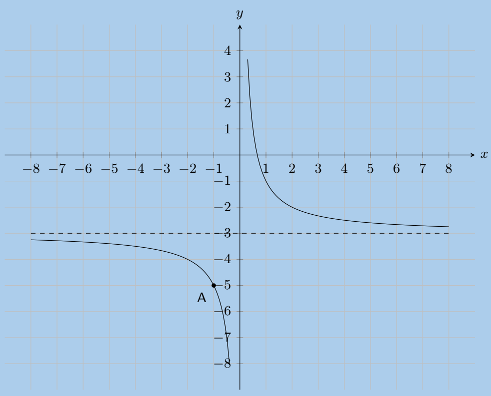
Solution:

$$
\begin{aligned}
y &= -2x^2 + 8\\
0 &= -2x^2 + 8\\
2x^2 - 8 &= 0\\
x^2 - 4 &= 0\\
(x-2)(x+2) &= 0\\
\therefore x &= 2 \text{ m}
\end{aligned}
$$

Sebastian landed 2 m from the pool wall.

Therefore Sebastian landed 1 m closer to the wall than Lucas.

18. The following graph shows a hyperbolic equation of the form $y = \frac{a}{x} + q$. **Point A** is shown at $(-1; -5)$. Calculate the values of a and q.



Solution:

$$q = -3$$

$$
\begin{aligned}
y &= \frac{a}{x} - 3\\
(-5) &= \frac{a}{(-1)} - 3
\end{aligned}
$$

$$
\begin{aligned}
-1\,(-5) &= \left[\frac{a}{-1} - 3\right](-1)\\
5 &= a + 3\\
2 &= a
\end{aligned}
$$

Therefore $a = 2$ and $q = -3$.

The equation is $y = \frac{2}{x} - 3$.

19. Given the following equation:

$$y = -\frac{3}{x} + 4$$

a) Determine the location of the y-intercept.
Solution:

$$\begin{aligned} y &= -\frac{3}{x} + 4 \\ y &= -\frac{3}{(0)} + 4 \\ &\text{no solution} \end{aligned}$$

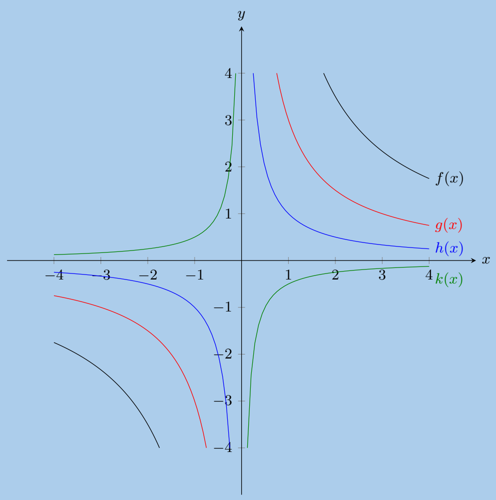
There is no y-intercept.

b) Determine the location of the x-intercept.
Solution:

$$\begin{aligned} y &= -\frac{3}{x} + 4 \\ (0) &= -\frac{3}{x} + 4 \\ (x)(0) &= \left[-\frac{3}{x} + 4\right](x) \\ 0 &= -3 + 4x \\ 3 &= 4x \\ x &= \frac{3}{4} \end{aligned}$$

The x-intercept is at $\left(\frac{3}{4}; 0\right)$.

20. Given the following graph, identify a function that matches each of the given equations:



a) $y = -\frac{1}{2x}$
Solution:
$k(x)$

b) $y = \frac{7}{x}$
Solution:
$f(x)$

c) $y = \frac{3}{x}$
Solution:
$g(x)$

d) $y = \frac{1}{x}$

Solution:

$h(x)$

21. Sketch the following functions and identify the asymptotes:

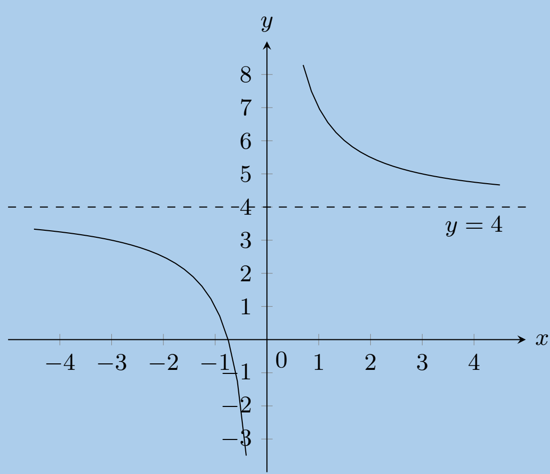
a) $y = -\frac{3}{x} + 4$

Solution:

The asymptote is $y = 4$.

a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept. The x-intercept is at $\left(\frac{3}{4}; 0\right)$.



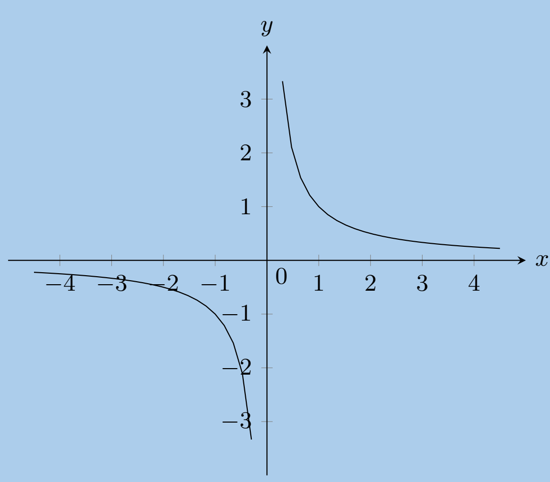
b) $y = \frac{1}{x}$

Solution:

The asymptote is $y = 0$.

a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept and no x-intercept.



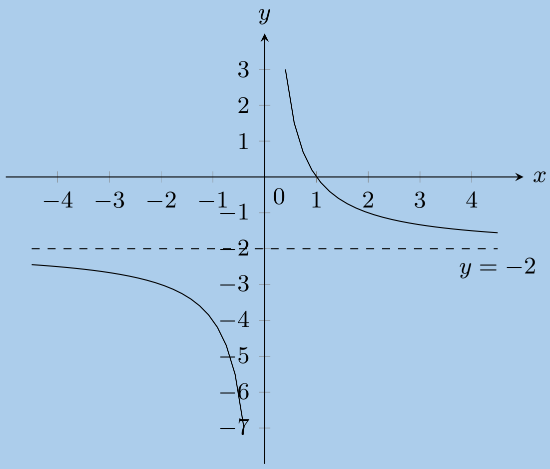
c) $y = \frac{2}{x} - 2$

Solution:

The asymptote is $y = -2$.

a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept. The x-intercept is at $(1; 0)$.



22. Sketch the functions given and describe the transformation used to obtain the second function. Show all asymptotes.

a) $y = \frac{2}{x}$ and $\frac{2}{x} + 2$

Solution:

$y = \frac{2}{x}$:

The asymptote is $y = 0$.

a is positive and so the graph lies in the first and third quadrants.

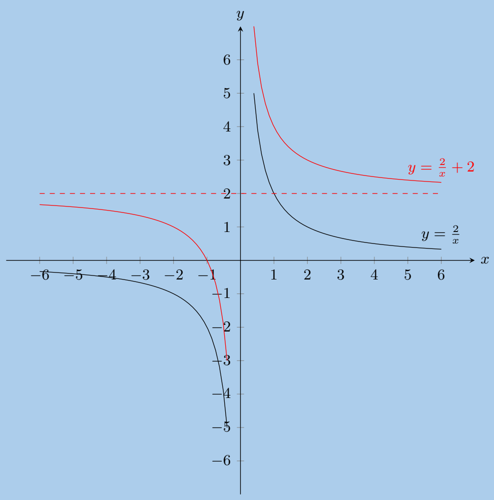
There is no y-intercept and no x-intercept.

$y = \frac{2}{x} + 2$:

The asymptote is $y = 2$.

a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept. The x-intercept is at $(-1; 0)$.



Translation by 2 in the positive y-direction.

b) $y = \frac{2}{x}$ and $\frac{1}{2x}$

Solution:

$y = \frac{2}{x}$:

The asymptote is $y = 0$.

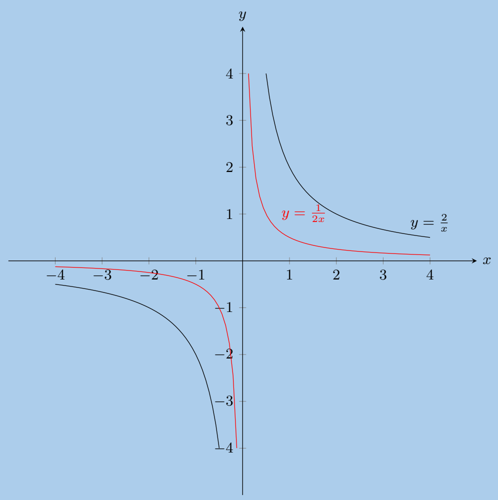
a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept and no x-intercept.

$y = \frac{1}{2x}$:

The asymptote is $y = 0$.

a is positive and so the graph lies in the first and third quadrants.
There is no y-intercept and no x-intercept.



Reduction by 4

c) $y = \frac{3}{x}$ and $y = \frac{3x+3}{x}$

Solution:

First simplify the second equation:

$y = \frac{3x+3}{x} = \frac{3}{x} + 1$

$y = \frac{3}{x}$:

The asymptote is $y = 0$.

a is positive and so the graph lies in the first and third quadrants.

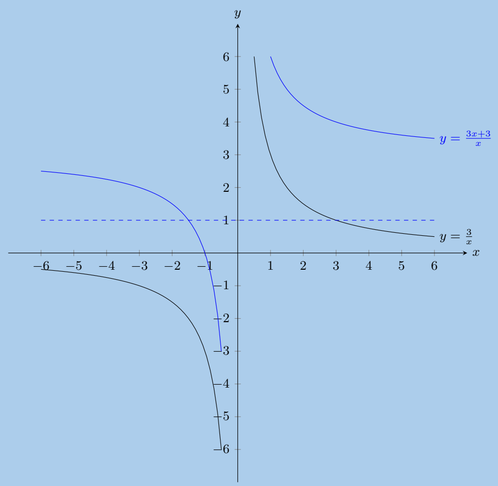
There is no y-intercept and no x-intercept.

$y = \frac{3}{x} + 1$:

The asymptote is $y = 1$.

a is positive and so the graph lies in the first and third quadrants.

There is no y-intercept. The x-intercept is at $(-1; 0)$.



Translation by 3 units in the positive y-direction.

d) $y = \frac{3}{x}$ and $y = -\frac{3}{x}$

Solution:

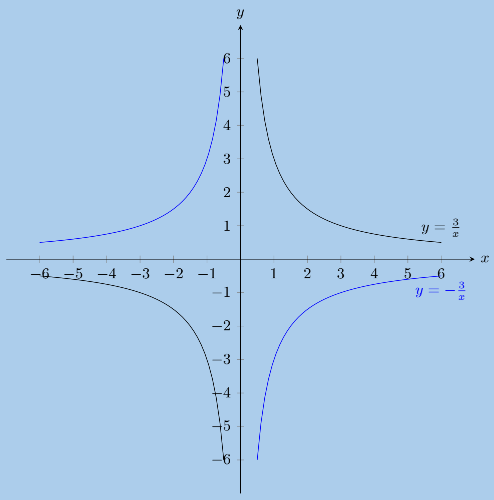
$y = \frac{3}{x}$:

The asymptote is $y = 0$.

a is positive and so the graph lies in the first and third quadrants.
There is no y-intercept and no x-intercept.

$y = -\frac{3}{x}$:

The asymptote is $y = 0$.
a is negative and so the graph lies in the second and fourth quadrants.
There is no y-intercept and no x-intercept.



Reflection on x-axis

23. Given the following equation:
$y = -\frac{1}{2} \cdot (4)^x + 3$

a) Calculate the y-intercept. Your answer must be correct to 2 decimal places.
Solution:

$$\begin{aligned} y &= \left(-\frac{1}{2}\right).(4)^x + 3 \\ &= \left(-\frac{1}{2}\right).(4)^{(0)} + 3 \\ &= \left(-\frac{1}{2}\right).(1) + 3 \\ &= (-0{,}5) + 3 \\ &= 2{,}50 \end{aligned}$$

The y-intercept is $(0; 2{,}50)$.

b) Now calculate the x-intercept. Estimate your answer to one decimal place if necessary.
Solution:
We calculate the x-intercept by letting $y = 0$. Then start to solve for x.

$$\begin{aligned} 0 &= \left(-\frac{1}{2}\right) \cdot (4)^x + 3 \\ -3 &= \left(-\frac{1}{2}\right) \cdot (4)^x \\ (-2)(-3) &= (-2)\left(-\frac{1}{2}\right) \cdot (4)^x \\ 6 &= 4^x \end{aligned}$$

Try: $4^0 = 1$
Try: $4^1 = 4$
Try: $4^2 = 16$

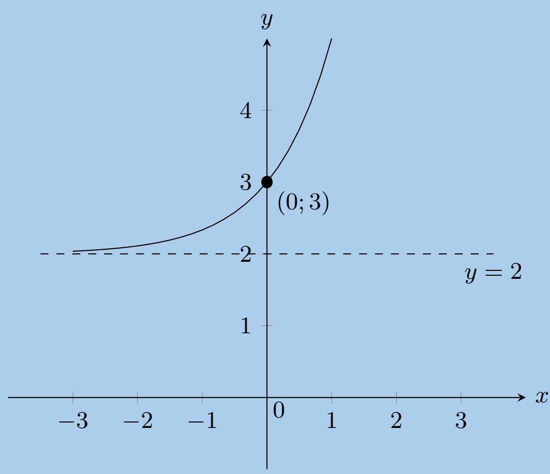
We can see that the exponent must be between 1 and 2. By trial and error we get 1,3. Therefore the x-intercept is $(1{,}3;0)$.

24. Sketch the following functions and identify the asymptotes:

a) $y = 3^x + 2$

Solution:

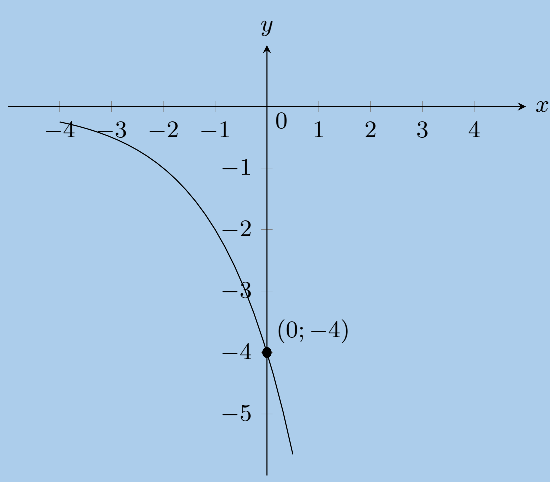
The y-intercept is $(0;2)$. There is no x-intercept. The asymptote is at $y = 2$.
$a > 1$ therefore the graph curves upwards.



b) $y = -4 \times 2^x$

Solution:

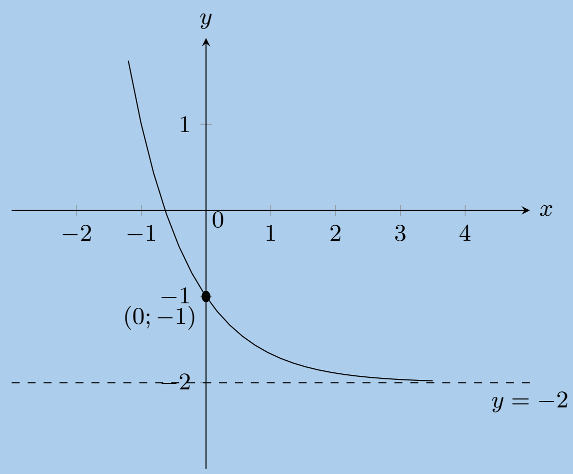
The y-intercept is $(0;-4)$. There is no x-intercept. The asymptote is at $y = 0$.
$a < 1$ therefore the graph curves downwards.



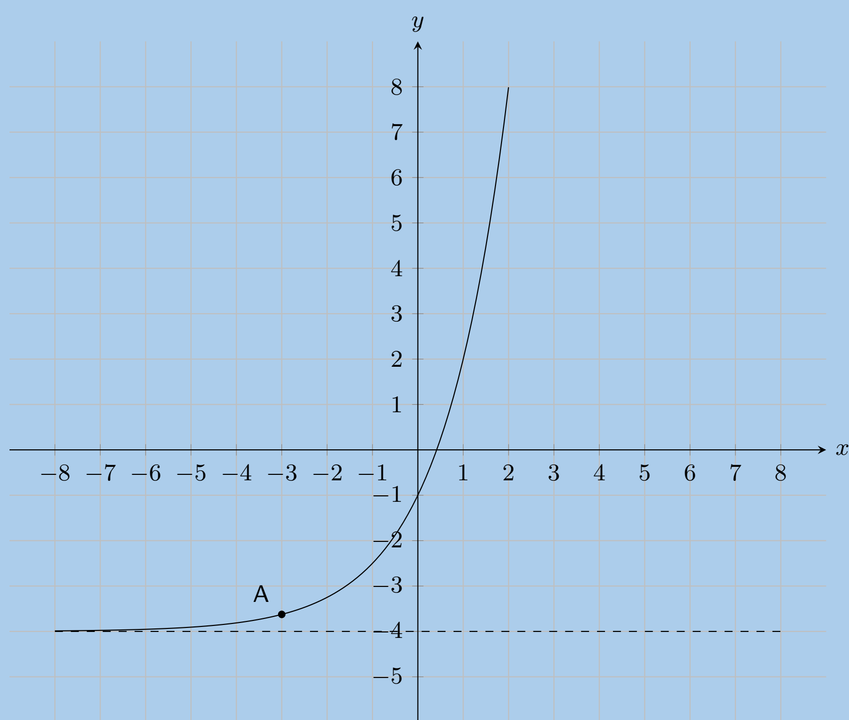
c) $y = \left(\frac{1}{3}\right)^x - 2$

Solution:

The y-intercept is $(0;-2)$. The x-intercept is $(0{,}6;0)$. The asymptote is at $y = -2$.
$0 < a < 1$ therefore the graph curves downwards.



25. The form of the curve graphed below is $y = a\ .\ 2^x + q$. One point is given on the curve: **Point A** is at $(-3; -3{,}625)$. Find the values of a and q, correct to the nearest integer.



Solution:

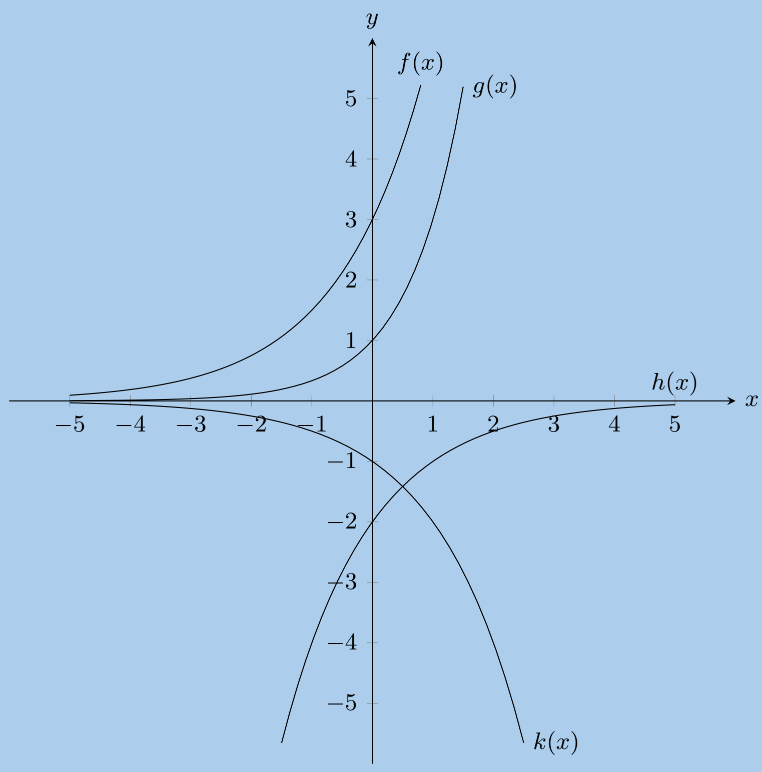
The asymptote lies at $y = -4$. Therefore q is -4.

At this point we know that the equation for the graph must be $y = a.2^x - 4$.

$$\begin{aligned} y &= a(2)^x - 4 \\ (-3{,}625) &= a(2)^{(-3)} - 4 \\ -3{,}625 + 4 &= a(2)^{(-3)} \\ 0{,}375 &= a(0{,}125) \\ 3 &= a \end{aligned}$$

$a = 3$ and $q = -4$

26. Given the following graph, identify a function that matches each of the given equations



a) $y = -2\left(\frac{1}{2}\right)^{x}$
   Solution:
   $h(x)$
b) $y = 3{,}2^{x}$
   Solution:
   $f(x)$
c) $y = -2^{x}$
   Solution:
   $k(x)$
d) $y = 3^{x}$
   Solution:
   $g(x)$

27. Use the functions $f(x) = 3 - x, g(x) = 2x^2 - 4; h(x) = 3^x - 4; k(x) = \frac{3}{2x} - 1$, to find the value of the following:

a) $f(7)$
   Solution:

$$\begin{aligned} f(1) &= 3 - (7) \\ &= -4 \end{aligned}$$

b) $g(1)$
   Solution:

$$\begin{aligned} g(1) &= 2(1)^2 - 4 \\ &= -2 \end{aligned}$$

c) $h(-4)$
   Solution:

$$h(-4) = 3^{-4} - 4$$
$$= -\frac{323}{81}$$

d) $k(5)$

Solution:

$$k(5) = \frac{3}{2(5)} - 1$$
$$= -\frac{7}{10}$$

e) $f(-1) + h(-3)$

Solution:

$$f(-1) + h(-3) = 3 - (-1) + 3^{-3} - 4$$
$$= \frac{1}{27}$$

f) $h(g(-2))$

Solution:

$$\begin{aligned} g(-2) &= 2(-2)^2 - 4 \\ &= 4 \\ \therefore h(g(-2)) &= h(4) \\ &= 3^4 - 4 \\ &= 77 \end{aligned}$$

g) $k(f(6))$

Solution:

$$\begin{aligned} f(6) &= 3 - (6) \\ &= -3 \\ \therefore k(f(6)) &= k(-3) \\ &= \frac{3}{2(-3)} - 1 \\ &= -\frac{3}{2} \end{aligned}$$

28. Determine whether the following statements are true or false. If the statement is false, give reasons why.

a) The given or chosen y-value is known as the independent variable.

Solution:

False, the given or chosen y-value is the dependent variable because it's value depends on the independent variable x.

b) A graph is said to be continuous if there are breaks in the graph.

Solution:

False, a graph is said to be continuous if there are no breaks in it.

c) Functions of the form $y = ax + q$ are straight lines.

Solution:

True

d) Functions of the form $y = \frac{a}{x} + q$ are exponential functions.

Solution:

False, functions of the form $y = \frac{a}{x} + q$ are hyperbolic functions.

e) An asymptote is a straight line which a graph will intersect at least once.

Solution:

False, an asymptote is a straight line that a graph will never intersect.

f) Given a function of the form $y = ax + q$, to find the y-intercept let $x = 0$ and solve for y.

Solution:

True

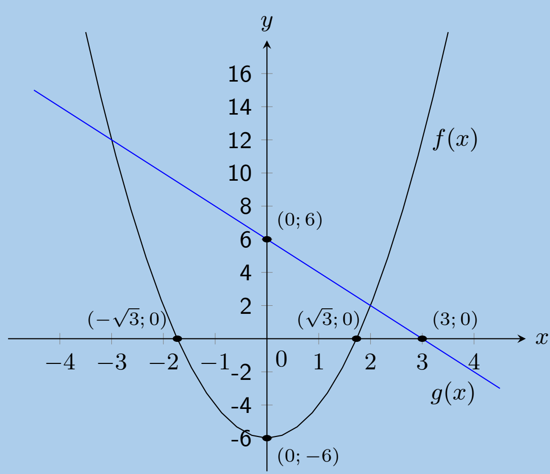
29. Given the functions $f(x) = 2x^2 - 6$ and $g(x) = -2x + 6$.

a) Draw f and g on the same set of axes.

Solution:

For $g(x)$ the y-intercept is $(0; 6)$ and the x-intercept is $(3; 0)$.

For $f(x)$ the y-intercept is $(0; -6)$ and the x-intercepts are $\left(\sqrt{3}; 0\right)$ and $\left(-\sqrt{3}; 0\right)$.



b) Calculate the points of intersection of f and g.

Solution:

The x-values of the points of intersection can be found by setting $f(x) = g(x)$:

$$\begin{aligned} 2x^2 - 6 &= -2x + 6 \\ 2x^2 + 2x - 12 &= 0 \\ x^2 + x - 6 &= 0 \\ (x-2)(x+3) &= 0 \\ \therefore x &= 2 \text{ and } x = -3 \end{aligned}$$

The y-values can be obtained by substituting the x-values into either equation:

$$\begin{aligned} g(x) &= -2(-3) + 6 = 12 \\ g(x) &= -2(2) + 6 = 2 \end{aligned}$$

Therefore the points of intersection are $(-3; 12)$ and $(2; 2)$.

c) Use your graphs and the points of intersection to solve for x when:

i. $f(x) > 0$

ii. $g(x) < 0$

iii. $f(x) \leq g(x)$

Solution:

i.

$$\begin{aligned} \text{Let } f(x) &= 0 \\ 2x^2 - 6 &= 0 \\ 2x^2 &= 6 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Therefore, for $f(x) > 0$, $x \in (-\infty; \sqrt{3}) \cup (\sqrt{3}; \infty)$.

ii.

$$\begin{aligned} \text{Let } g(x) &= 0 \\ -2x + 6 &= 0 \\ -2x &= -6 \\ x &= 3 \end{aligned}$$

Therefore, for $g(x) < 0$, $x \in (3; \infty)$.

iii. This is found by looking at where the graph of $f(x)$ lies underneath the graph of $g(x)$.
For $f(x) \leq g(x)$, $x \in [-3; 2]$.

d) Give the equation of the reflection of f in the x-axis.
Solution:
$y = -2x^2 + 6$

30. After a ball is dropped, the rebound height of each bounce decreases. The equation $y = 5(0{,}8)^x$ shows the relationship between the number of bounces x and the height of the bounce y for a certain ball. What is the approximate height of the fifth bounce of this ball to the nearest tenth of a unit?
Solution:
For the fifth bounce $x = 5$. Now we can solve for y:

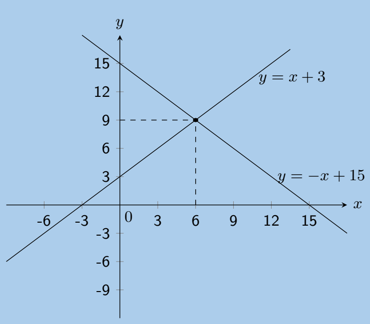
$$\begin{aligned} y &= 5(0{,}8)^x \\ &= 5\left(\frac{4}{5}\right)^5 \\ &= 5\left(\frac{1024}{3125}\right) \\ &= 5(0{,}38) \\ &= 1{,}6 \text{ units} \end{aligned}$$

Therefore the approximate height of the fifth bounce is 1,6 units

31. Mark had 15 coins in R 5 and R 2 pieces. He had 3 more R 2 coins than R 5 coins. He wrote a system of equations to represent this situation, letting x represent the number of R 5 coins and y represent the number of R 2 coins. Then he solved the system by graphing.

a) Write down the system of equations.
Solution:
Let $x =$ R 5 coins and $y =$ R 2 coins. Then the system of equations is:
$x + y = 15$; $y = x + 3$

b) Draw their graphs on the same set of axes.
Solution:
For $x + y = 15$ the y-intercept is $(0; 15)$ and the x-intercept is $(15; 0)$.
For $y = x + 3$ the y-intercept is $(0; 3)$ and the x-intercept is $(-3; 0)$.



c) Use your sketch to determine how many R 5 and R 2 pieces Mark had.
Solution:
From the sketch we see that the graphs intersect at $(6; 9)$. Checking algebraically we get:
Substitute the value of $y = -x + 15$ into the second equation:

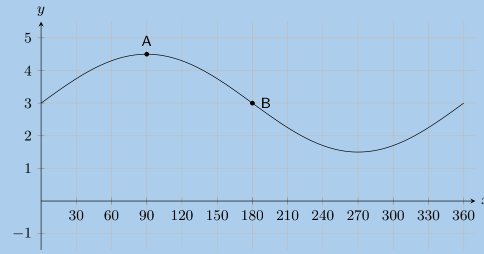
$$
\begin{aligned}
-x + 15 &= x + 3 \\
-2x &= -12 \\
\therefore x &= 6
\end{aligned}
$$

Substitute the value of x back into the first equation:

$$
\begin{aligned}
y &= -(6) + 15 \\
&= 9
\end{aligned}
$$

Mark has 6 R 5 coins and 9 R 2 coins.

32. Shown the following graph of the following form: $y = a \sin \theta + q$ where **Point A** is at $(90°; 4{,}5)$, and **Point B** is at $(180°; 3)$, determine the values of a and q.



Solution:

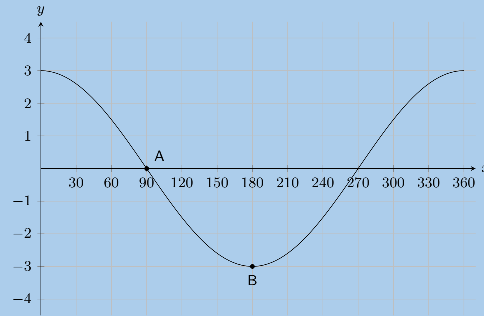
To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point B is at $(180°; 3)$. For an unshifted sine graph point B would be at $(180°; 0)$. For this graph we see that this point has been shifted up by 3 spaces. Therefore $q = 3$.

To find a we note that the y-value at the middle (point B) is 3, while the y-value at the top (point A) is 4,5. We can find the amplitude by working out the distance from the top of the graph to the middle of the graph: $4{,}5 - 3 = 1{,}5$. Therefore $a = \frac{3}{2}$.

The complete equation for the graph shown in this question is $y = \frac{3}{2} \sin \theta + 3$.

Therefore $a = \frac{3}{2}$ and $q = 3$.

33. The graph below shows a trigonometric equation of the following form: $y = a \cos \theta + q$. Two points are shown on the graph: **Point A** at $(90°; 0)$, and **Point B**: $(180°; -3)$. Calculate the values of a (the amplitude of the graph) and q (the vertical shift of the graph).



Solution:

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point A is at $(90°; 0)$. For an unshifted cosine graph point A would be at $(90°; 0)$. For this graph we see that this point has not been shifted. Therefore $q = 0$.

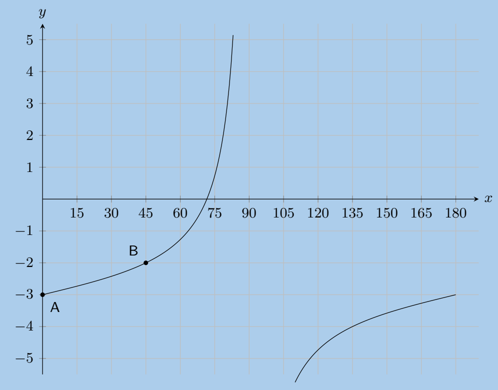
To find a we note that the y-value at the middle (point A) is 0, while the y-value at the bottom (point B) is -3. We can find the amplitude by working out the distance from the middle of the graph to the bottom of the graph: $0 - (-3) = 3$. Therefore $a = 3$.

The complete equation for the graph shown in this question is $y = 3 \cos \theta$.

Therefore $a = 3$ and $q = 0$.

34. On the graph below you see a tangent curve of the following form: $y = a \tan \theta + q$. Two points are labelled on the curve: **Point A** is at $(0°; -3)$, and **Point B** is at $(45°; -2)$.

Calculate, or otherwise determine, the values of a and q.



Solution:

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance point A is at $(0°; -3)$. For an unshifted tangent graph point A would be at $(0°; 0)$. For this graph we see that this point been shifted downwards by 3. Therefore $q = -3$.

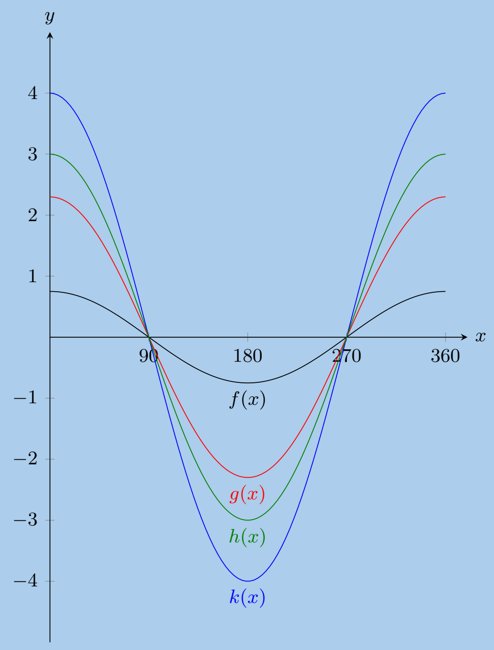
To find a we can substitute point B into the equation for the tangent graph:

$$\begin{aligned} y &= a \tan\theta - 3 \\ (-2) &= a \tan 45° - 3 \\ -2 &= a(-1) - 3 \\ -2 + 3 &= -a \\ -1 &= -a \\ 1 &= a \end{aligned}$$

The complete equation is: $y = \tan\theta - 3$.

Therefore $a = 1$ and $q = -3$.

35. Given the following graph, identify a function that matches each of the given equations:



a) $y = 2{,}3 \cos\theta$

Solution:

$g(x)$

b) $y = 0{,}75 \cos\theta$

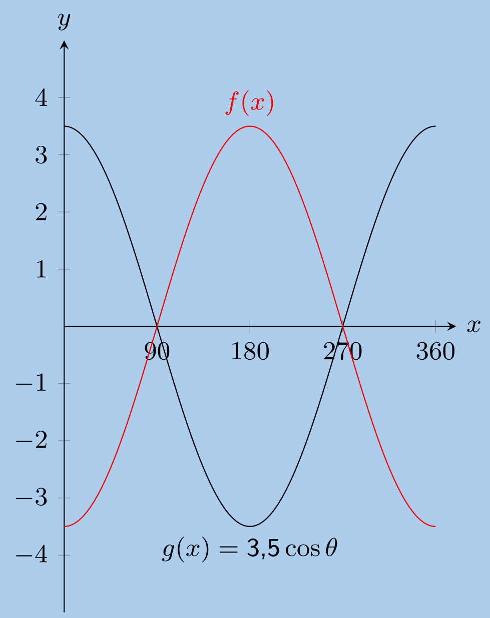
Solution:

$f(x)$

c) $y = 4\cos\theta$
Solution:
$k(x)$

d) $y = 3\cos\theta$
Solution:
$h(x)$

36. The graph below shows functions $f(x)$ and $g(x)$.

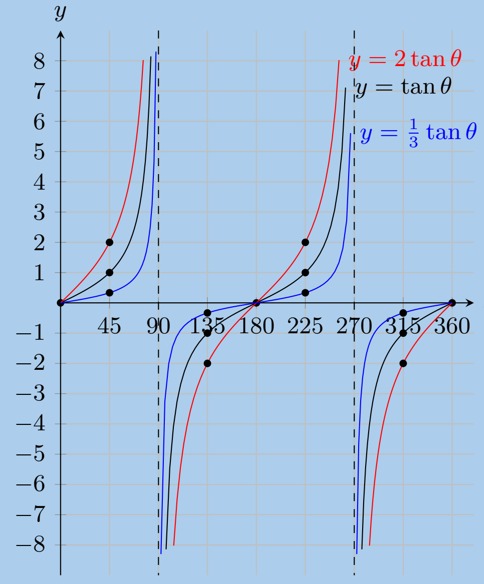


What is the equation for $f(x)$?
Solution:
$f(x) = -3{,}5\cos\theta$

37. With the assistance of the table below sketch the three functions on the same set of axes.

θ	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\tan\theta$	0	1	undefined	-1	0	1	undefined	-1	0
$2\tan\theta$	0	2	undefined	-2	0	2	undefined	-2	0
$\frac{1}{3}\tan\theta$	0	$\frac{1}{3}$	undefined	$-\frac{1}{3}$	0	$\frac{1}{3}$	undefined	$-\frac{1}{3}$	0

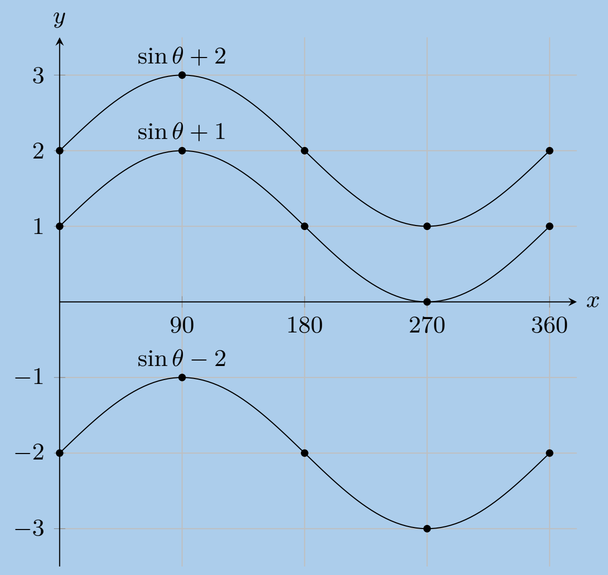
Solution:



38. With the assistance of the table below sketch the three functions on the same set of axes.

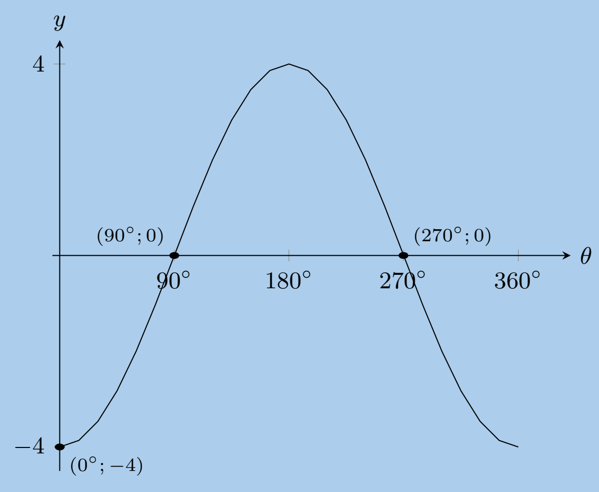
θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta + 1$	1	2	1	0	1
$\sin\theta + 2$	3	2	2	1	2
$\sin\theta - 2$	-2	-1	-2	-3	-2

Solution:

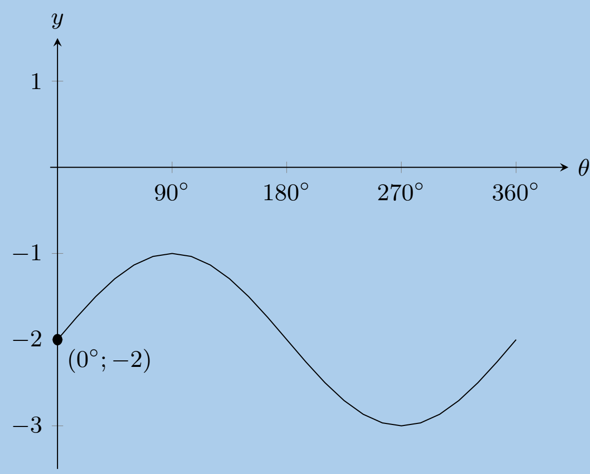


39. Sketch graphs of the following trigonometric functions for $\theta \in [0°; 360°]$. Show intercepts and asymptotes.

a) $y = -4\cos\theta$
Solution:
The y-intercept is $(0°; -4)$. The x-intercepts are $(90°; 0)$ and $(270°; 0)$. There are no asymptotes.
The graph is not shifted up or down since $q = 0$. The graph is stretched by 4 and reflected in the x-axis.



b) $y = \sin\theta - 2$
Solution:
The y-intercept is $(0°; -2)$. There are no x-intercepts. There are no asymptotes.
The graph is shifted down by -2 since $q = -2$. The graph is not stretched since $a = 1$.

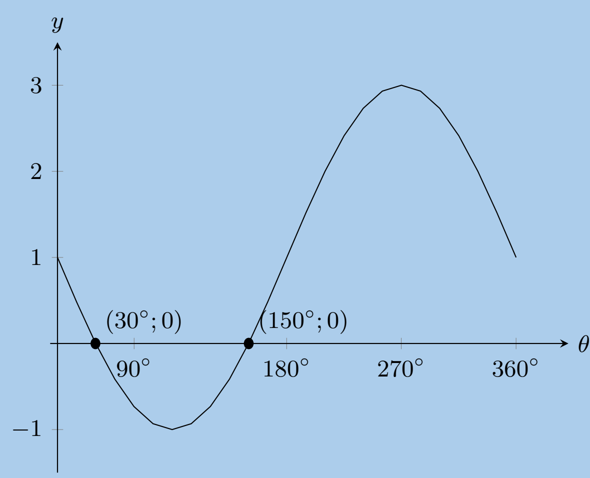


c) $y = -2\sin\theta + 1$

Solution:

The y-intercept is $(0°; 1)$. The x-intercepts are $(30°; 0)$ and $(150°; 0)$. There are no asymptotes.

The graph is shifted up by 1 since $q = 1$. The graph is stretched by -2 and reflected in the x-axis since $a = -2$.

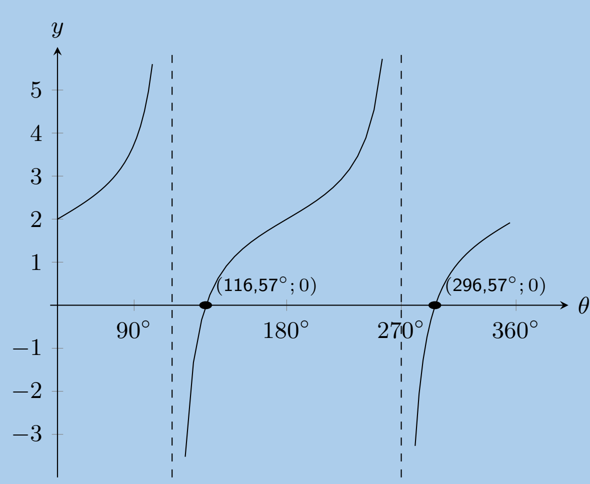


d) $y = \tan\theta + 2$

Solution:

The y-intercept is $(0°; 2)$. The x-intercepts are $(116{,}57°; 0)$ and $(296{,}57°; 0)$. The asymptotes are $x = 90°$ and $x = 270°$.

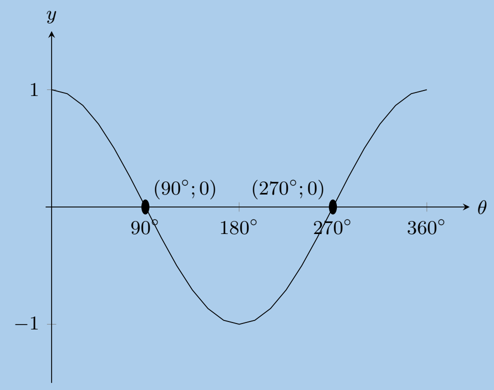
The graph is shifted up by 2 since $q = 2$. The graph is not stretched since $a = 1$.



e) $y = \frac{\cos\theta}{2}$

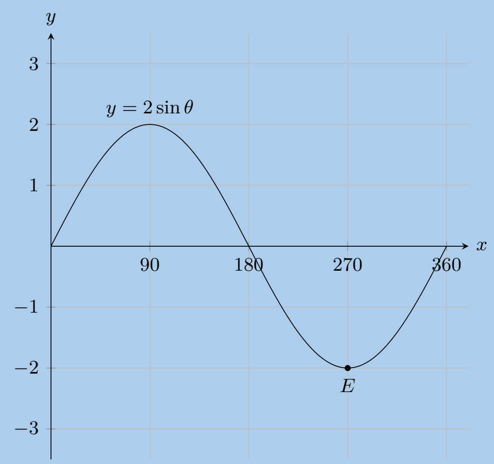
Solution:

The y-intercept is $(0°; 0{,}5)$. The x-intercepts are $(90°; 0)$ and $(270°; 0)$. There are no asymptotes.
The graph is not shifted upwards or downwards since $q = 0$. The graph is stretched by 0,5 since $a = \frac{1}{2}$.



40. State the coordinates at E and the range of the function.

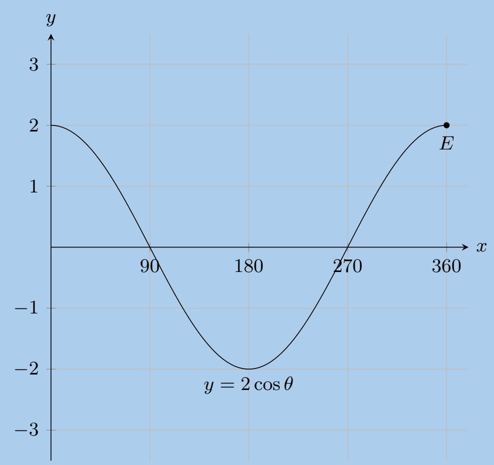
a)



Solution:

$E = (270°; -2)$ and $-2 \leq y \leq 2$

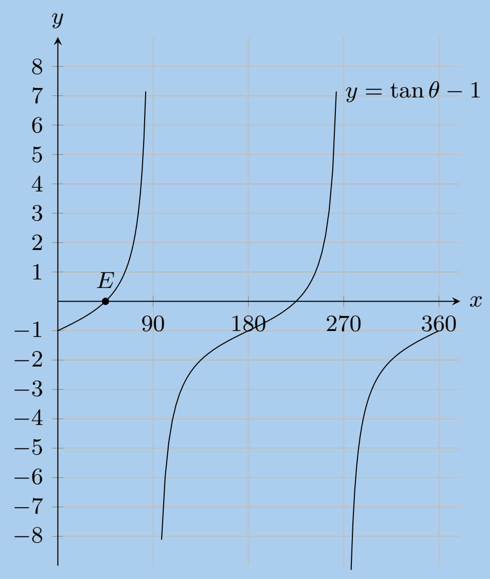
b)



Solution:

$E = (360°; 2)$ and $-2 \leq y \leq 2$

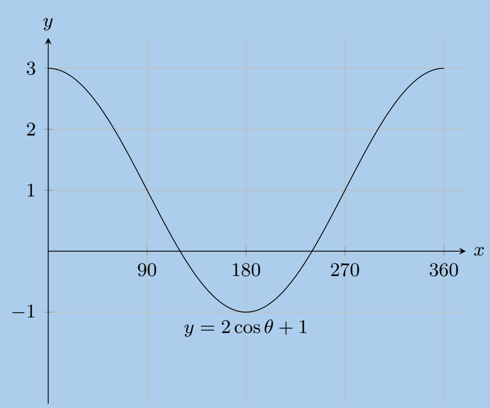
41. State the coordinates at E and the domain and range of the function in the interval shown.



Solution:

$E = (45°; 0)$, range $y \in \mathbb{R}$ and domain $0 \leq x \leq 360, x \neq 90, \theta \neq 270$

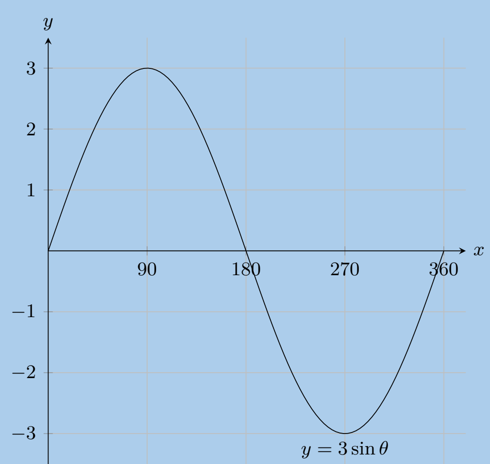
42. For which values of θ is the function decreasing, in the interval shown?



Solution:

$0° < \theta < 180°$

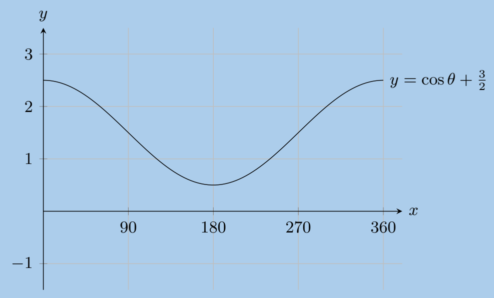
43. For which values of θ is the function increasing, in the interval shown?



Solution:

$0° < \theta < 90°$ and $270° < \theta < 360°$

44. For which values of θ is the function positive, in the interval shown?

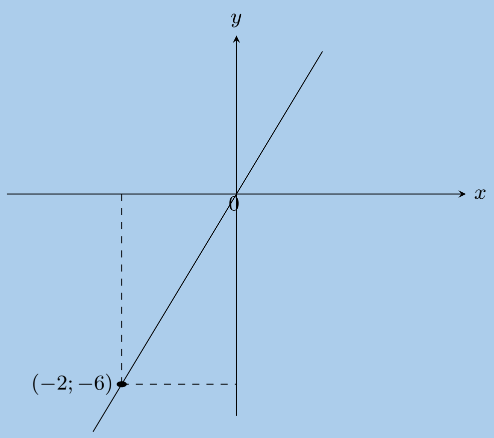


Solution:

$0° < \theta < 360°$

45. Given the general equations $y = mx + c$, $y = ax^2 + q$, $y = \frac{a}{x} + q$, $y = a \,.\, b^x + q$, $y = a\sin\theta + q$, $y = a\cos\theta + q$ and $y = a\tan\theta$, determine the specific equations for each of the following graphs.

a)



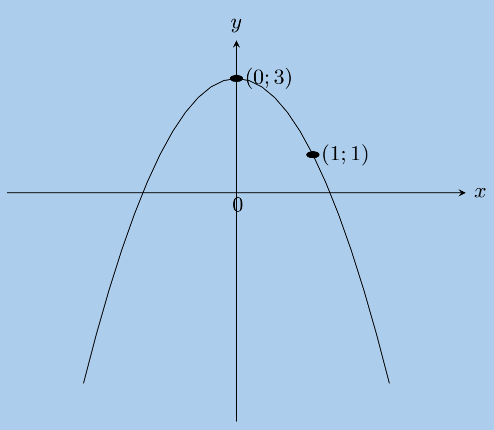
Solution:

This is a straight line graph and so the general equation is $y = mx + c$. The y-intercept is at $(0; 0)$ and so $c = 0$. To find m we substitute in the given point into the equation and solve for m:

$$\begin{aligned} y &= mx \\ -6 &= -2m \\ m &= 3 \end{aligned}$$

Therefore the equation is $y = 3x$.

b)



Solution:

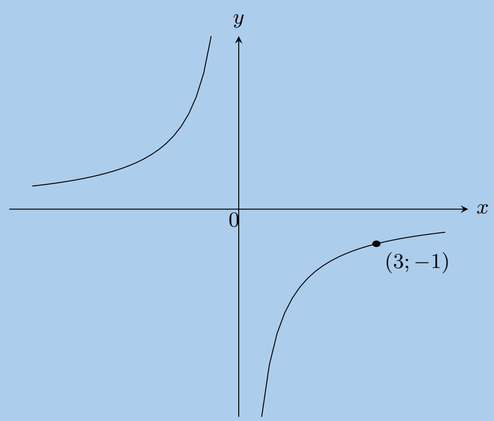
This is a parabola and so we use $y = ax^2 + q$. The y-intercept is at $(0; 3)$ and so $q = 3$.

We substitute the point $(1; 1)$ into the equation and solve for a:

$$\begin{aligned} y &= ax^2 + 3 \\ 1 &= a(1)^2 + 3 \\ -2 &= a \end{aligned}$$

Therefore the equation is $y = -2x^2 + 3$.

c)



Solution:

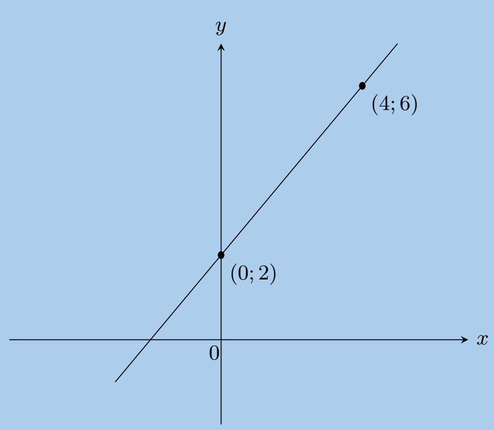
This is a hyperbola and so we use $y = \frac{a}{x} + q$. There is no x-intercept and so the graph has not been shifted upwards or downwards. Therefore $q = 0$.

We substitute the point $(3; -1)$ into the equation and solve for a:

$$\begin{aligned} y &= \frac{a}{x} \\ -1 &= \frac{a}{3} \\ -3 &= a \end{aligned}$$

Therefore the equation is $y = \frac{-3}{x}$.

d)



Solution:

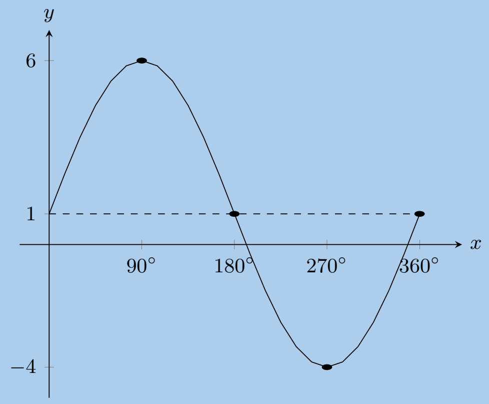
This is a straight line graph and so the general equation is $y = mx + c$. The y-intercept is at $(0; 2)$ and so $c = 2$.

To find m we substitute the point $(4; 6)$ into the equation and solve for m:

$$y = mx + 2$$
$$6 = 4m + 2$$
$$m = 1$$

Therefore the equation is $y = x + 2$.

e)



Solution:

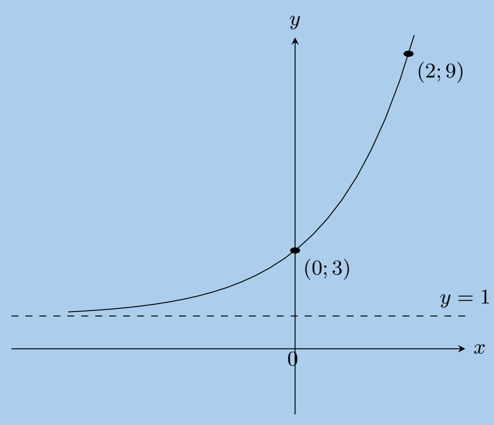
This is a sine graph and so the general equation is $y = a \sin \theta + q$.

To find a we note that the y-value at the bottom is -4, while the y-value at the top is 6. We can find the amplitude by working out the distance from the bottom of the graph to the top of the graph and dividing this value by 2: $\frac{6-(-4)}{2} = 5$. Therefore $a = 5$.

To find q we note that q shifts the graph up or down. To determine q we can look at any point on the graph. For instance we can see that when $x = 180°$, $y = 1$. For an unshifted sine graph with the same a value (i.e. $5 \sin \theta$) this point would be at $(180°; 0)$. For this graph we see that this point has been shifted upwards by 1 unit. Therefore $q = 1$.

The complete equation for the graph shown in this question is $y = 5 \sin \theta + 1$.

f)



Solution:

This is an exponential graph and so we use $y = a \,.\, b^x + q$. We see that the asymptote is at $y = 1$ and so $q = 1$. To find a we substitute the point $(0; 3)$ into the equation:

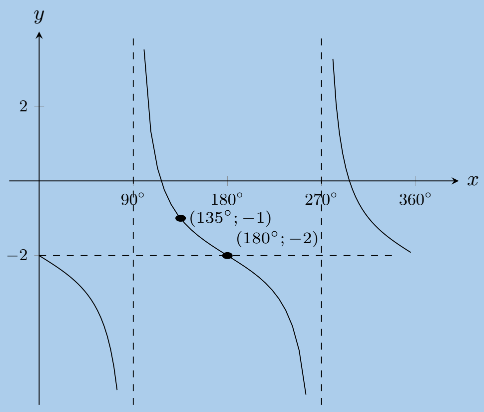
$$y = a \,.\, b^x + 1$$
$$3 = a \,.\, b^0 + 1$$
$$a = 2$$

To find b we substitute the point $(2; 9)$ into the equation:

$$
\begin{aligned}
y &= 2 \,.\, b^x + 1 \\
9 &= 2 \,.\, b^2 + 1 \\
4 &= b^2 \\
2^2 &= b^2 \\
b &= 2
\end{aligned}
$$

Therefore the equation is $y = 2 \times 2^x + 1$.

g)



Solution:

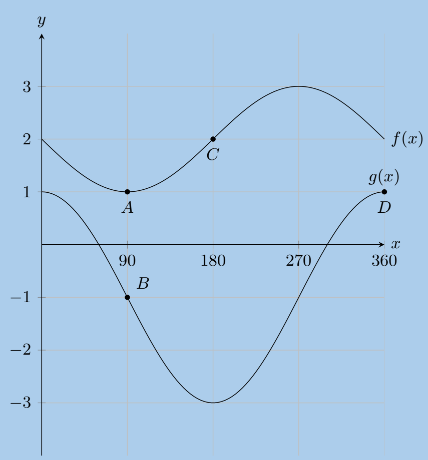
This is a tangent graph and so we use $y = a \tan \theta + q$. To find q we note that the graph has been shifted down by 2 units (the point $(180°; -2)$ is given). Therefore $q = -2$.

To find a we substitute $(135°; -1)$ into the equation:

$$
\begin{aligned}
y &= a \tan \theta - 2 \\
-1 &= a \tan(135°) - 2 \\
1 &= -a \\
a &= -1
\end{aligned}
$$

Therefore the equation is $y = -\tan \theta - 2$.

46.



a) State the coordinates at A, B, C and D.

Solution:

$A(90°; 1)$, $B(90°; -1)$, $C(180°; 2)$ and $D(360°; 1)$

b) How many times in this interval does $f(x)$ intersect $g(x)$.

Solution:

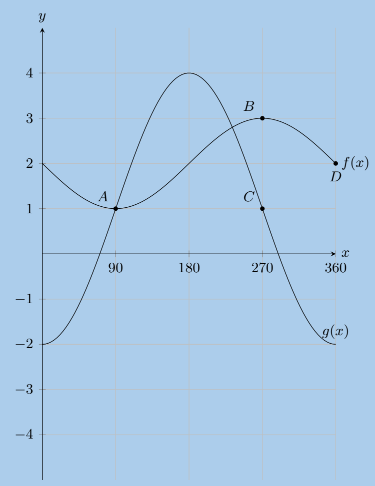
0

c) What is the amplitude of $f(x)$.
Solution:
1

d) Evaluate: $f(180°) - g(180°)$.
Solution:

$$\begin{aligned} f(180°) - g(180°) &= 2 - (-3) \\ &= 5 \end{aligned}$$

47.



a) State the coordinates at $A,\ B,\ C$ and D.
Solution:
$A(90°; 1),\ B(270°; 3),\ C(270°; 1)$ and $D(360°; 2)$

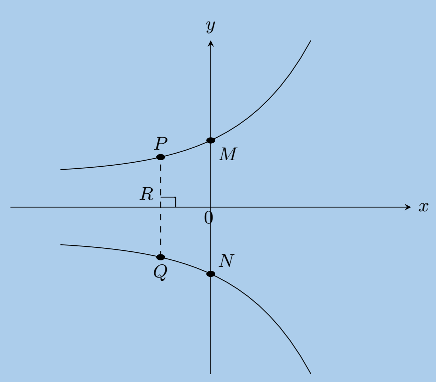
b) How many times in this interval does $f(x)$ intersect $g(x)$.
Solution:
2

c) What is the amplitude of $g(x)$.
Solution:
3

d) Evaluate: $g(180°) - f(180°)$.
Solution:

$$\begin{aligned} f(180°) - g(180°) &= 4 - 2 \\ &= 2 \end{aligned}$$

48. $y = 2^x$ and $y = -2^x$ are sketched below. Answer the questions that follow.



a) Calculate the coordinates of M and N.

Solution:

M is the y-intercept of $y = 2^x$ and so $y = 2^0 = 1$. Therefore the coordinates of M are $(0; 1)$.

N is the y-intercept of $y = -2^x$ and so $y = -(2^0) = -1$. Therefore the coordinates of N are $(0; -1)$.

Therefore $M(0; 1)$ and $N(0; -1)$.

b) Calculate the length of MN.

Solution:

M and N both lie on the y-axis and so they both lie on a straight line.

Therefore $MN = 1 + 1 = 2$.

c) Calculate the length of PQ if $OR = 1$ unit.

Solution:

At P, $x = -1$, therefore $y = 2^{-1} = \frac{1}{2}$.

At Q, $x = -1$, therefore $y = -(2^{-1}) = -\frac{1}{2}$.

Therefore length $PQ = \frac{1}{2} + \frac{1}{2} = 1$.

d) Give the equation of $y = 2^x$ reflected about the y-axis.

Solution:

$y = 2^{-x}$

e) Give the range of both graphs.

Solution:

Range $y = 2^x$: $(0; \infty)$

Range $y = -2^x$: $(-\infty; 0)$

49. Plot the following functions on the same set of axes and clearly label all points of intersection.

a) $y = -2x^2 + 3$
$y = 2x + 4$

Solution:

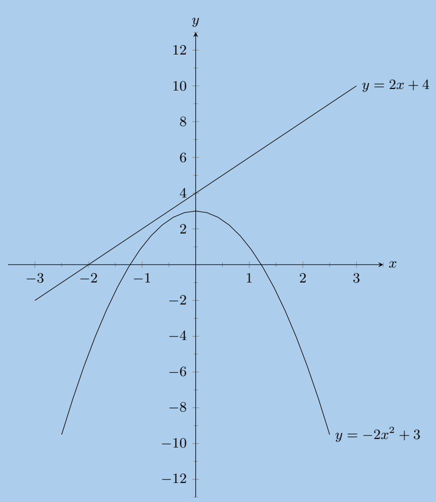
For $y = -2x^2 + 3$:

The y-intercept is at $(0; 3)$. The x-intercepts are at $\left(\sqrt{\frac{3}{2}}; 0\right)$ and $\left(-\sqrt{\frac{3}{2}}; 0\right)$.

For $y = 2x + 4$:

The y-intercept is at $(0; 4)$. The x-intercept is at $(-2; 0)$.

There are no points of intersection.



b) $y = x^2 - 4$
$y = 3x$

Solution:

For $y = x^2 - 4$:

The y-intercept is at $(0; -4)$. The x-intercepts are at $(2; 0)$ and $(-2; 0)$.

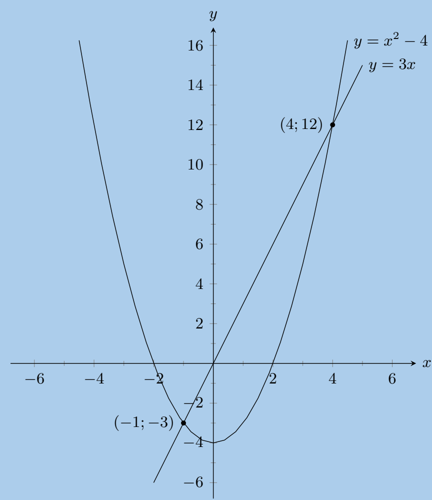
For $y = 3x$:

The y-intercept is at $(0; 0)$. The x-intercept is at $(0; 0)$.

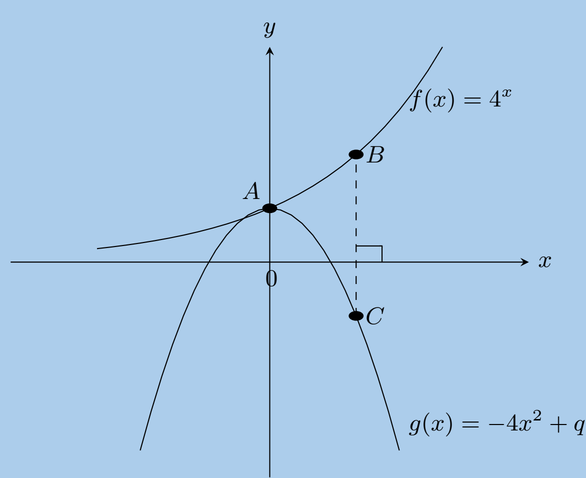
To find the point of intersection we equate the two functions:

$$
\begin{aligned}
x^2 - 4 &= 3x \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x &= 4 \text{ or } x = -1 \\
y = 3(4) &\text{ or } y = 3(-1) \\
y = 12 &\text{ or } y = -3
\end{aligned}
$$

Therefore the graphs intersect at $(4; 12)$ and $(-1; -3)$.



50. $f(x) = 4^x$ and $g(x) = -4x^2 + q$ are sketched below. The points $A(0; 1)$ and $B(1; 4)$ are given. Answer the questions that follow.



a) Determine the value of q.

Solution:

Point A is the y-intercept of $g(x)$ and so $q = 1$.

b) Calculate the length of BC.

Solution:

B is at $(1; 4)$ and so C is at $(1; y)$. To find y we substitute point C into $g(x)$:

$$
\begin{aligned}
g(x) &= -4x^2 + 1 \\
y &= -4(1)^2 + 1 \\
&= -3
\end{aligned}
$$

Therefore $BC = 3 + 4 = 7$ units.

c) Give the equation of $f(x)$ reflected about the x-axis.
Solution:
$y = -4^x$

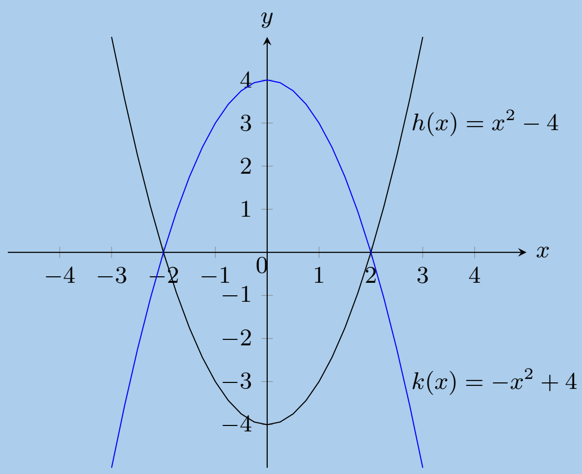
d) Give the equation of $f(x)$ shifted vertically upwards by 1 unit.
Solution:
$y = 4^x + 1$

e) Give the equation of the asymptote of $f(x)$.
Solution:
$y = 0$

f) Give the ranges of $f(x)$ and $g(x)$.
Solution:
Range $f(x)$: $(0; \infty)$, Range $g(x)$: $(-\infty; 1]$

51. Given $h(x) = x^2 - 4$ and $k(x) = -x^2 + 4$. Answer the questions that follow.

a) Sketch both graphs on the same set of axes.
Solution:
For $h(x)$ the y-intercept is at $(0; 4)$. The x-intercepts are at $(2; 0)$ and $(-2; 0)$.
For $k(x)$ the y-intercept is at $(0; -4)$. The x-intercepts are at $(2; 0)$ and $(-2; 0)$.



b) Describe the relationship between h and k.
Solution:

$$\begin{aligned} h(x) &= x^2 - 4 \\ k(x) &= -x^2 + 4 \\ &= -(x^2 - 4) \\ &= -h(x) \end{aligned}$$

$k(x)$ is therefore the reflection of $h(x)$ about the x-axis.

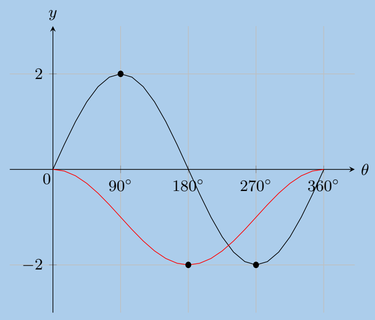
c) Give the equation of $k(x)$ reflected about the line $y = 4$.
Solution:
$y = x^2 + 4$

d) Give the domain and range of h.
Solution:
Domain h: $(-\infty; \infty)$. Range h: $[-4; \infty)$.

52. Sketch the graphs of $f(\theta) = 2\sin\theta$ and $g(\theta) = \cos\theta - 1$ on the same set of axes. Use your sketch to determine:

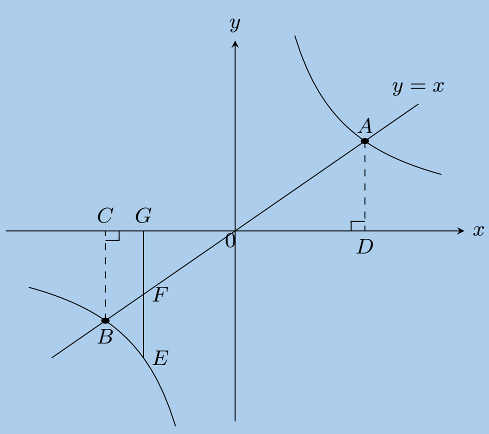
a) $f(180°)$
b) $g(180°)$
c) $g(270°) - f(270°)$
d) The domain and range of g.
e) The amplitude and period of f.

Solution:



a) $f(180°) = 0$
b) $g(180°) = -2$
c) $g(270°) - f(270°) = -1 - (-2) = -1 + 2 = 1$
d) Domain: $[0°; 360°]$. Range: $[-2; 0]$
e) Amplitude: 2. Period: 360° .

53. The graphs of $y = x$ and $y = \frac{8}{x}$ are shown in the following diagram.



Calculate:

a) The coordinates of points A and B.
Solution:
A and B are the points of intersection of the two functions. Therefore:

$$
\begin{aligned}
x &= \frac{8}{x} \\
x^2 &= 8 \\
\therefore x &= \pm\sqrt{8}
\end{aligned}
$$

Since the equation of the straight line is $y = x$ these are also the y-values of the points of intersection.
Therefore $A(\sqrt{8}; \sqrt{8})$ and $B(-\sqrt{8}; -\sqrt{8})$

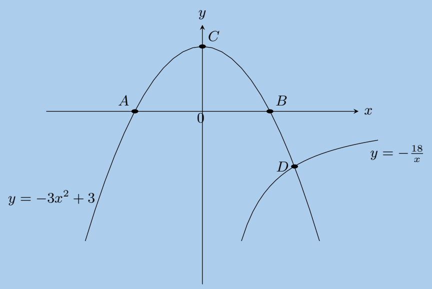
b) The length of CD.
Solution:
C has the same x value as A and D has the same x value as B.
Therefore $C(-\sqrt{8}; 0)$ and $D(\sqrt{8}; 0)$.
$CD = \sqrt{8} + \sqrt{8} = 2\sqrt{8}$.

c) The length of AB.
Solution:
Using Pythagoras:

$$OD = \sqrt{8} \text{ units and } AD = \sqrt{8} \text{ units}$$
$$\begin{aligned} AO^2 &= OD^2 + AD^2 \\ &= (\sqrt{8})^2 + (\sqrt{8})^2 \\ &= 8 + 8 \\ &= 16 \\ \therefore AO &= 4 \text{ units} \\ \text{Similarly, } OB &= 4 \text{ units} \\ \therefore AB &= 8 \text{ units} \end{aligned}$$

d) The length of EF, given $G(-2;0)$.
Solution:
F and E have the same x value as point G. F lies on $y = x$ and so $F(-2;-2)$. E lies on $y = \frac{8}{x}$ and so $E(-2;-4)$.
Therefore length $EF = 2 + 4 = 2$ units.

54. Given the diagram with $y = -3x^2 + 3$ and $y = -\dfrac{18}{x}$.



a) Calculate the coordinates of A, B and C.
Solution:
A and B are the x-intercepts of $y = -3x^2 + 3$. C is the y-intercept of $y = -3x^2 + 3$.
Therefore point C is at $(0;3)$.
Points B and A are at $(1;0)$ and $(-1;0)$ respectively.
Therefore $A(-1;0)$, $B(1;0)$, $C(0;3)$

b) Describe in words what happens at point D.
Solution:
The parabola and the hyperbola intersect at point D which lies in the fourth quadrant.

c) Calculate the coordinates of D.
Solution:

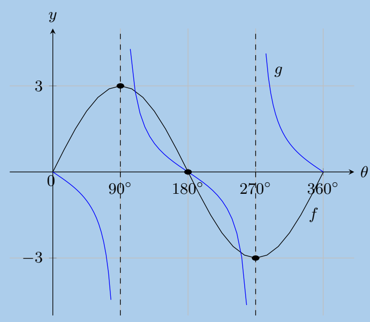
$$\begin{aligned} -\frac{18}{x} &= -3x^2 + 3 \\ -18 &= -3x^3 + 3x \\ 0 &= -3x^3 + 3x + 18 \\ 0 &= x^3 - x - 6 \\ 0 &= (x-2)(x^2 + 2x + 3) \\ x &= 2 \\ f(2) &= (2)^3 - 2 - 6 = 0 \\ \text{when } x = 2,\ y &= -3(2)^2 + 3 = -9 \\ \therefore D(2;-9) \end{aligned}$$

d) Determine the equation of the straight line that would pass through points C and D.
Solution:
Determine gradient $D(2;-9)$ and $C(0;3)$:

$$m = \frac{-9-3}{2-0}$$
$$= -6$$

C is the y-intercept and so $c = 3$.
Therefore $y = -6x + 3$.

55. The diagram shows the graphs of $f(\theta) = 3\sin\theta$ and $g(\theta) = -\tan\theta$



a) Give the domain of g.
Solution:
Domain: $\{\theta : 0° \leq \theta \leq 360°, \theta \neq 90°; 270°\}$

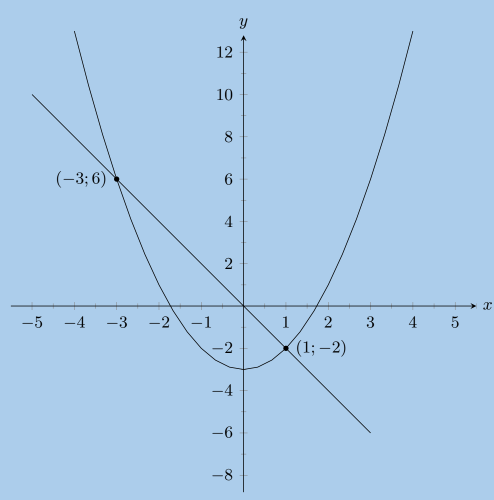
b) What is the amplitude of f?
Solution:
Amplitude: 3

c) Determine for which values of θ:
   i. $f(\theta) = 0 = g(\theta)$
   ii. $f(\theta) \times g(\theta) < 0$
   iii. $\frac{g(\theta)}{f(\theta)} > 0$
   iv. $f(\theta)$ is increasing

Solution:
   i. $\{0°;\ 180°;\ 360°\}$
   ii. $(0°;\ 90°) \cup (270°;\ 360°)$
   iii. $\{\theta : 90° < \theta < 270°, \theta \neq 180°\}$
   iv. $(0°;\ 90°) \cup (270°;\ 360°)$

56. Determine the equations for the graphs given below.

a)



Solution:
For the straight line:

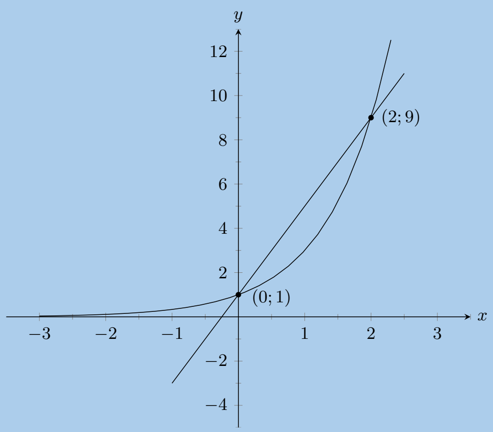
$$
\begin{aligned}
y &= mx + c \\
c &= 0 \\
m &= \frac{6-(-2)}{-3-1} \\
m &= -2 \\
y &= -2x
\end{aligned}
$$

For the parabola:

$$
\begin{aligned}
y &= ax^2 + q \\
q &= -3 \\
y &= ax^2 - 3 \\
6 &= a(-3)^2 - 3 \\
9 &= 9a \\
a &= 1 \\
y &= x^2 - 3
\end{aligned}
$$

Therefore the equations are: $y = -2x$ en $y = x^2 - 3$.

b)



Solution:
For the straight line:

$$
\begin{aligned}
y &= mx + c \\
c &= 1 \\
m &= \frac{9-1}{2-0} \\
m &= 4 \\
y &= 4x + 1
\end{aligned}
$$

For the exponential graph:

$$
\begin{aligned}
y &= a \,.\, b^x + q \\
q &= 0 \\
y &= a \,.\, b^x \\
1 &= a(b^0) \\
1 &= a
\end{aligned}
$$

$$y = b^x$$
$$9 = b^2$$
$$3^2 = b^2$$
$$b = 3$$
$$y = 3^x$$

Therefore the equations are $y = 4x + 1$ and $y = 3^x$.

57. Choose the correct answer:

a) Which of the following does not have a gradient of 3?

i. $y = 3x + 6$
ii. $3y = 9x - 1$
iii. $\frac{1}{3}(y - 1) = x$
iv. $\frac{1}{2}(y - 3) = 6x$

Solution:

(iv)

b) The asymptote of $xy = 3 + x$ is:

i. 3
ii. 1
iii. -3
iv. -1

Solution:

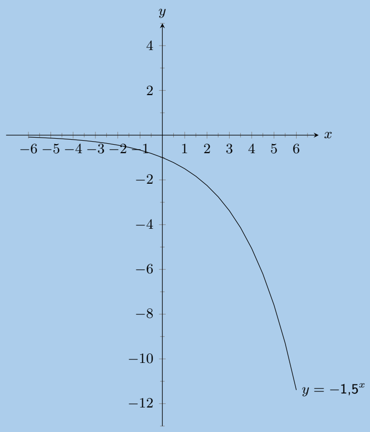
(ii)

58. Sketch the following

a) $y = -1{,}5^x$

Solution:

The asymptote is at $y = 0$. The y-intercept is at $(0; -1)$. There is no x-intercept.



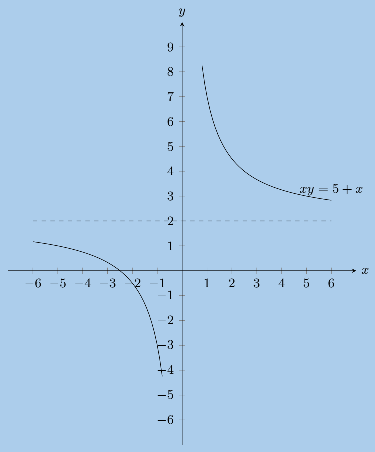
b) $xy = 5 + 2x$

Solution:

First rewrite the equation in standard form:

$$xy = 5 + 2x$$
$$y = \frac{5}{x} + 2$$

There is no y-intercept. The x-intercept is at $(-2{,}5; 0)$. The asymptote is at $y = 2$.

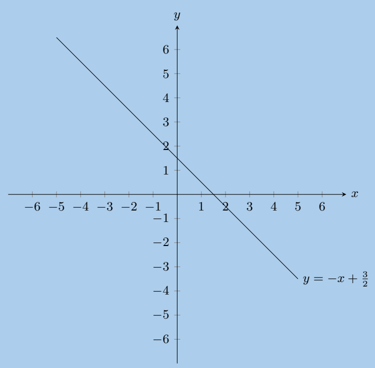


c) $2y + 2x = 3$

Solution:

First write the equation in standard form:

$$2y + 2x = 3$$
$$y = -x + \frac{3}{2}$$



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1a. 2G2M	1b. 2G2N	2a. 2G2P	2b. 2G2Q	3a. 2G2R	3b. 2G2S
4a. 2G2T	4b. 2G2V	5. 2G2W	6. 2G2X	7a. 2G2Y	7b. 2G2Z
7c. 2G32	8a. 2G33	8b. 2G34	9a. 2G35	9b. 2G36	9c. 2G37
10. 2G38	11. 2G39	12. 2G3B	13. 2G3C	14. 2G3D	15. 2G3F
16a. 2G3G	16b. 2G3H	16c. 2G3J	17. 2G3K	18. 2G3M	19. 2G3N
20. 2G3P	21a. 2G3Q	21b. 2G3R	21c. 2G3S	22a. 2G3T	22b. 2G3V
22c. 2G3W	22d. 2G3X	23. 2G3Y	24a. 2G3Z	24b. 2G42	24c. 2G43
25. 2G44	26. 2G45	27. 2G46	28a. 2G47	28b. 2G48	28c. 2G49
28d. 2G4B	28e. 2G4C	28f. 2G4D	29. 2G4F	30. 2G4G	31. 2G4H
32. 2G4J	33. 2G4K	34. 2G4M	35. 2G4N	36. 2G4P	37. 2G4Q
38. 2G4R	39a. 2G4S	39b. 2G4T	39c. 2G4V	39d. 2G4W	39e. 2G4X
40a. 2G4Y	40b. 2G4Z	41. 2G52	42. 2G53	43. 2G54	44. 2G55
45. 2G56	46. 2G57	47. 2G58	48. 2G59	49a. 2G5B	49b. 2G5C
50. 2G5D	51. 2G5F	52. 2G5G	53. 2G5H	54. 2G5J	55. 2G5K
56a. 2G5M	56b. 2G5N	57a. 2G5P	57b. 2G5Q	58a. 2G5R	58b. 2G5S
58c. 2G5T					

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CHAPTER 7

Euclidean geometry

- Content covered in this chapter includes revision of lines, angles and triangles. The mid-point theorem is introduced. Kites, parallelograms, rectangle, rhombus, square and trapezium are investigated.
- Solving problems and proving riders is only covered later in the year. The focus of this chapter is on introducing the special quadrilaterals and revising content from earlier grades.
- Revision of triangles should focus on similar and congruent triangles.
- Sketches are valuable and important tools. Encourage learners to draw accurate diagrams to solve problems.
- It is important to stress to learners that proportion gives no indication of actual length. It only indicates the ratio between lengths.
- Notation - emphasise to learners the importance of the correct ordering of letters, as this indicates which angles are equal and which sides are in the same proportion.

GeoGebra is a useful tool to use for sketching out the worked examples and activities.

7.1 Introduction

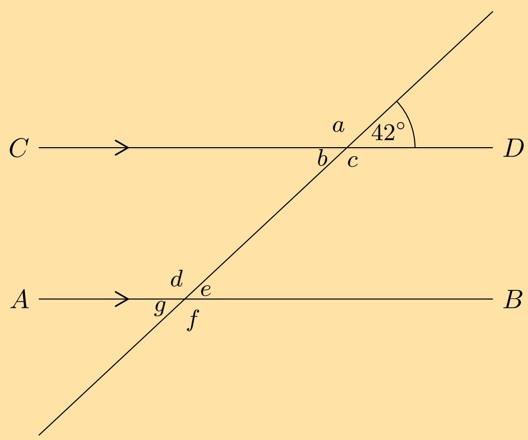
Angles

Properties and notation

Parallel lines and transversal lines

Exercise 7 – 1:

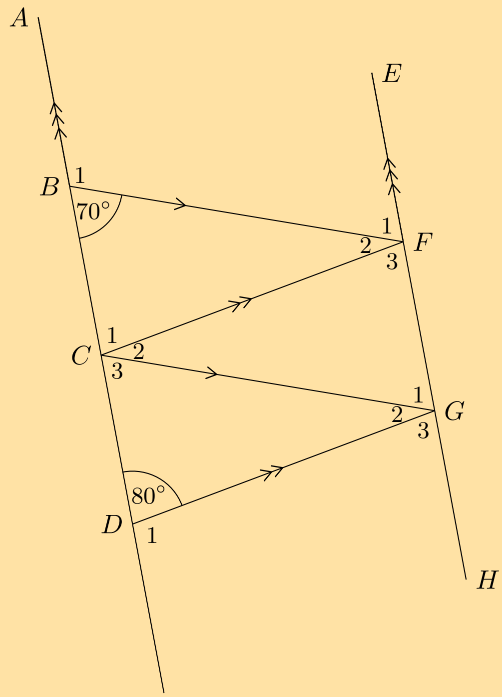
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labelled with letters in the diagram:



Solution:
You can redraw the diagram and fill in the angles as you find them.

$$\begin{aligned}
a &= 180° - 42° = 138° && (\angle\text{s on a str line})\\
b &= 42° && (\text{vert opp } \angle\text{s } =)\\
c &= 138° && (\text{vert opp } \angle\text{s } =)\\
d &= 138° && (\text{co-int } \angle\text{s; } AB \parallel CD)\\
e &= 180° - 138° = 42° && (\angle\text{s on a str line})\\
f &= 138° && (\text{vert opp } \angle\text{s } =)\\
g &= 42° && (\text{vert opp } \angle\text{s } =)
\end{aligned}$$

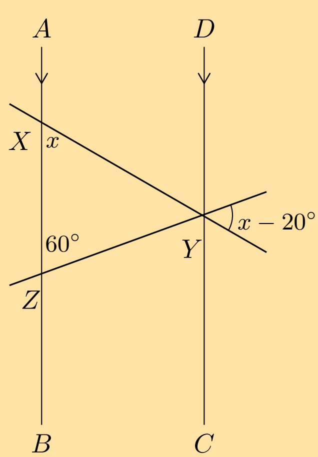
2. Find all the unknown angles in the figure:



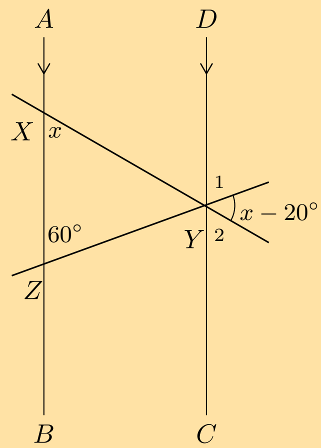
Solution:

$$\begin{aligned}
\hat{B}_1 &= 180° - 70° = 110° && \text{(}\angle\text{s on a str line)}\\
\hat{D}_1 &= 180° - 80° = 100° && \text{(}\angle\text{s on a str line)}\\
\hat{F}_1 &= 70° && \text{(co-int }\angle\text{s; } AD \parallel EH\text{)}\\
\hat{G}_3 &= 80° && \text{(co-int }\angle\text{s; } AD \parallel EH\text{)}\\
\hat{C}_3 &= 70° && \text{(corresp }\angle\text{s; } BF \parallel CG\text{)}\\
\hat{G}_1 &= 70° && \text{(corresp }\angle\text{s; } BF \parallel CG\text{)}\\
\hat{G}_2 &= 180° - 70° - 80° = 30° && \text{(}\angle\text{s on a str line)}\\
\hat{C}_2 &= 30° && \text{(alt }\angle\text{s; } CF \parallel DG\text{)}\\
\hat{F}_2 &= 30° && \text{(alt }\angle\text{s; } BF \parallel CG\text{)}\\
\hat{F}_3 &= 80° && \text{(sum of }\angle\text{'s str. line)}\\
\hat{C}_1 &= 80° && \text{(}\angle\text{s on a str line)}
\end{aligned}$$

3. Find the value of x in the figure:



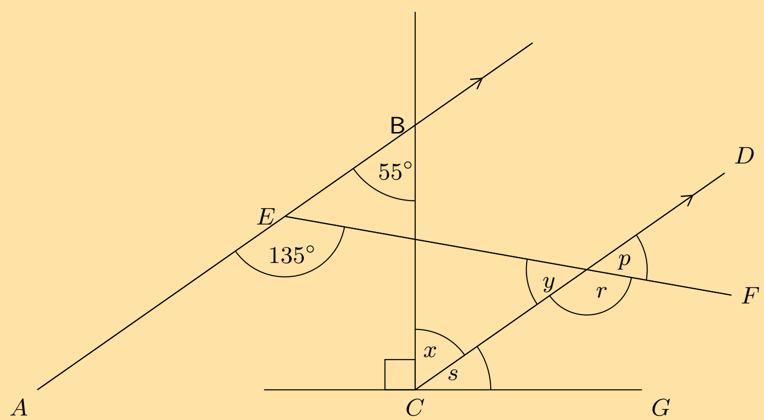
Solution:



$\hat{Y}_1 = 60°$ (corresp $\angle$s; $AB \parallel DC$).
$\hat{Y}_2 = x$ (corresp $\angle$s; $AB \parallel DC$).

$$\begin{aligned} \therefore x + 60° + (x - 20°) &= 180° \qquad (\angle\text{s on a str line}) \\ 2x &= 180° - 40° \\ 2x &= 140° \\ \therefore x &= 70° \end{aligned}$$

4. Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.



a) $\hat{x}$
Solution:
$\hat{x}$ and $A\hat{B}C$ are alternate interior angles on transversal BC. Therefore, they must be equal in size since $AB \parallel CD$.
Therefore $\hat{x} = 55°$.

b) $\hat{s}$
Solution:
We have just found that $\hat{x} = 55°$. $\hat{x} + \hat{s} + 90° = 180°$ ($\angle$s on a str line)

$$\begin{aligned} \hat{s} &= 90° - 55° \\ &= 35° \end{aligned}$$

c) $\hat{r}$
Solution:
$\angle A\hat{E}F$ and $\hat{r}$ are corresponding angles ($AB \parallel CD$).
Therefore: $\hat{r} = 135°$.

d) $\hat{y}$

Solution:

$\hat{r} + \hat{y} = 180°$ ($\angle$s on a str line)

$$\begin{aligned}\hat{y} &= 180° - 135° \\ &= 45°\end{aligned}$$

e) $\hat{p}$

Solution:

$\hat{p} = \hat{y}$ (vert opp $\angle$s $=$)

Therefore: $\hat{p} = 45°$.

f) Based on the results for the angles above, is $EF \parallel CG$?

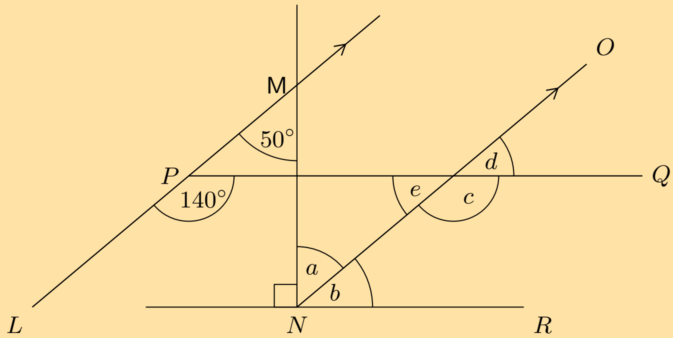
Solution:

To prove $EF \parallel CG$ we need to show that one of the following is true:

- $\hat{s} = \hat{p}$ (corresp $\angle$s)
- $\hat{s} = \hat{y}$ (alt $\angle$s)
- $\hat{s} + \hat{r} = 180°$ (co-int $\angle$s)

However $\hat{s} \neq \hat{p}$, therefore EF is not parallel to CG.

5. Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.



a) $\hat{a}$

Solution:

$\hat{a}$ and $L\hat{M}N$ are alternate interior angles on transversal MN. Since $LM \parallel NO$ they must be equal in size.

Therefore $\hat{a} = 50°$.

b) $\hat{b}$

Solution:

We have just found that $\hat{a} = 50°$. $\hat{a} + \hat{b} + 90° = 180°$ ($\angle$s on a str line)

$$\begin{aligned}\hat{b} &= 90° - 50° \\ &= 40°\end{aligned}$$

c) $\hat{c}$

Solution:

$\angle L\hat{P}Q$ and $\hat{c}$ are corresponding angles ($LM \parallel NO$).

Therefore: $\hat{c} = 140°$.

d) $\hat{e}$

Solution:

$\hat{c} + \hat{e} = 180°$ ($\angle$s on a str line)

$$\begin{aligned}\hat{e} &= 180° - 140° \\ &= 40°\end{aligned}$$

e) $\hat{d}$

Solution:

$\hat{d} = \hat{e}$ (vert opp $\angle$s $=$)

Therefore: $\hat{d} = 40°$.

f) Based on the results for the angles above, is $PQ \parallel NR$?

Solution:

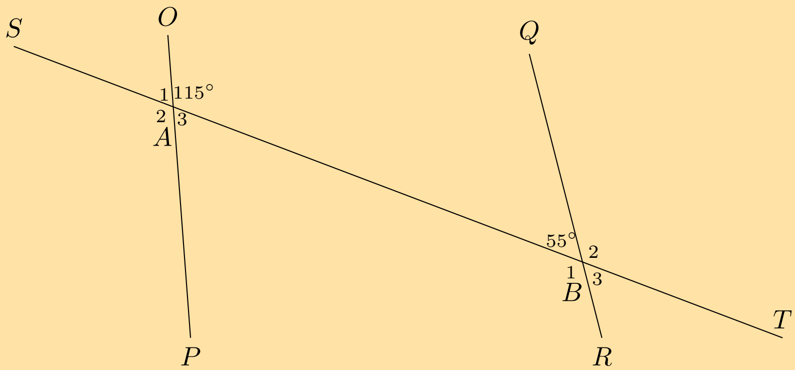
To prove $PQ \parallel NR$ we need to show that one of the following is true:

- $\hat{b} = \hat{d}$ (corresp $\angle$s)
- $\hat{b} = \hat{e}$ (alt $\angle$s)
- $\hat{b} + \hat{c} = 180°$ (co-int $\angle$s)

$\hat{b} = \hat{d}$ (corresp $\angle$s), therefore $PQ \parallel NR$. We also note that $\hat{b} = \hat{e}$ and $\hat{b} + \hat{c} = 180°$.

6. Determine whether the pairs of lines in the following figures are parallel:

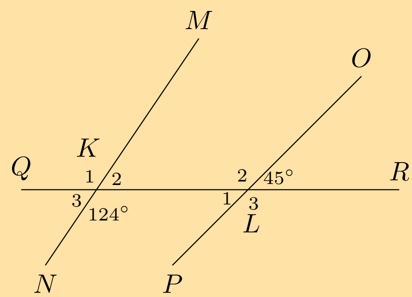
a)



Solution:

If $OP \parallel QR$ then $O\hat{A}B + Q\hat{B}A = 180°$ (co-int $\angle$s). But $O\hat{A}B + Q\hat{B}A = 115° + 55° = 170°$. Therefore there are no parallel lines, OP is not parallel to QR. Note that we do not consider ST as this is a transversal.

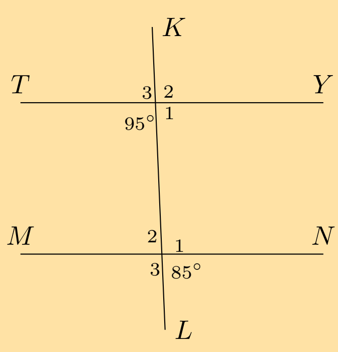
b)



Solution:

$K_2 = 180° - 124° = 56°$ ($\angle$s on a str line). If $MN \parallel OP$ then $\hat{K}_2$ would be equal to $\hat{L}$, $\therefore$ MN is not parallel to OP. Note that QR is a transversal.

c)

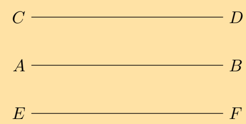


Solution:
Let U be point of intersection of lines KL and TY and V be the point of intersection of lines KL and MN.

$$\begin{aligned}
\hat{U}_4 &= 95° \\
\hat{U}_1 &= 180° - 95° \qquad (\angle\text{s on a str line}) \\
\hat{U}_1 &= 85° \\
\hat{V}_4 &= 85° \qquad (\text{given}) \\
\therefore \hat{V}_4 &= \hat{U}_1
\end{aligned}$$

These are corresponding angles $\therefore TY \parallel MN$.

7. If AB is parallel to CD and AB is parallel to EF, explain why CD must be parallel to EF.



Solution:
If $a = 2$ and $b = a$ then we know that $b = 2$.
Similarly if $AB \parallel CD$ and $EF \parallel AB$ then we know that $EF \parallel CD$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2G5Y 2. 2G5Z 3. 2G62 4. 2G63 5. 2G64 6a. 2G65 6b. 2G66 6c. 2G67 7. 2G68

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7.2 Triangles

Classification of triangles

Congruency

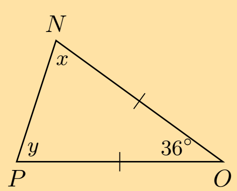
Similarity

The theorem of Pythagoras

Exercise 7 – 2:

1. Calculate the unknown variables in each of the following figures.

a)

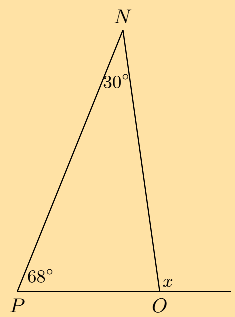


Solution:

The triangle is isosceles therefore $x = y$ ($\angle$s opp equal sides).

$$\begin{aligned} 180° &= 36° - 2x \qquad \text{(sum of } \angle\text{s in } \triangle) \\ 2x &= 144° \\ \therefore x &= 72° = y \end{aligned}$$

b)

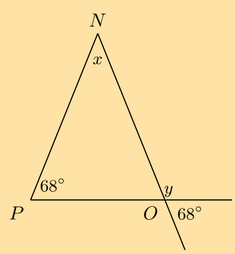


Solution:

x is an exterior angle, therefore $P\hat{N}O + O\hat{P}N = x$ (ext $\angle$ of $\triangle$).

$$\begin{aligned} x &= 30° + 68° \\ &= 98° \end{aligned}$$

c)



Solution:

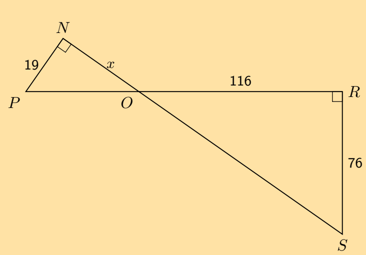
First find y. $y + 68° = 180°$ ($\angle$s on a str line). Therefore $y = 112°$.

y is an exterior angle, therefore $P\hat{N}O + O\hat{P}N = y$ (ext $\angle$ of $\triangle$).

$$\begin{aligned} 112° &= x + 68° \\ x &= 112° - 68° \\ &= 44° \end{aligned}$$

Therefore $y = 112°$ and $x = 44°$.

d)



Solution:

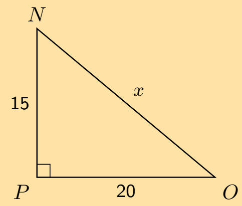
$$\begin{aligned} N\hat{P}O &= 180° - P\hat{N}O - N\hat{O}P \qquad \text{(sum of } \angle\text{s in } \triangle) \\ &= 180° - 90° - N\hat{O}P \\ &= 90° - N\hat{O}P \end{aligned}$$

$$\begin{aligned} R\hat{S}O &= 180° - O\hat{R}S - R\hat{O}S \qquad \text{(sum of } \angle\text{s in } \triangle) \\ &= 180° - 90° - R\hat{O}S \\ &= 90° - R\hat{O}S \end{aligned}$$

$N\hat{O}P = R\hat{O}S$ (vert opp $\angle$s).
$\therefore N\hat{P}O = R\hat{S}O$.
Therefore $\triangle NPO$ and $\triangle ROS$ are similar because they have the same angles.
Similar triangles have proportional sides:

$$\begin{aligned} \therefore \frac{NP}{RS} &= \frac{NO}{OR} \\ \frac{19}{76} &= \frac{x}{116} \\ \therefore x &= 29 \end{aligned}$$

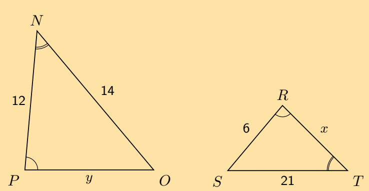
e)



Solution:
From the theorem of Pythagoras we have:

$$\begin{aligned} x^2 &= 15^2 + 20^2 \\ \therefore x &= \sqrt{625} \\ &= 25 \end{aligned}$$

f)



Solution:
We note that:

$$\begin{aligned} N\hat{P}O &= S\hat{R}T \qquad \text{(given)} \\ P\hat{N}O &= R\hat{T}S \qquad \text{(given)} \\ \therefore P\hat{N}O &= R\hat{T}S \qquad \text{(sum of} \angle\text{s in } \triangle) \\ \therefore \triangle NPO &\ ||| \ \triangle TSR \text{ (AAA)} \end{aligned}$$

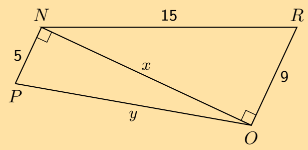
Now we can use the fact that the sides are in proportion to find x and y:

$$\begin{aligned}\frac{NO}{OP} &= \frac{TS}{TR} \\ \frac{14}{12} &= \frac{21}{x} \\ x &= \frac{21 \times 12}{14} \\ &= 18\end{aligned}$$

$$\begin{aligned}\frac{OP}{NP} &= \frac{SR}{TR} \\ \frac{y}{12} &= \frac{6}{18} \\ 18y &= 72 \\ y &= 4\end{aligned}$$

Therefore $x = 18$ and $y = 4$.

g)



Solution:
From the theorem of Pythagoras:

$$\begin{aligned}x^2 &= 15^2 - 9^2 \\ x &= \sqrt{144} \\ &= 12\end{aligned}$$

$$\begin{aligned}y^2 &= x^2 + 5^2 \\ y^2 &= 144 + 25 \\ y &= \sqrt{169} \\ y &= 13\end{aligned}$$

Therefore $x = 12$ and $y = 13$.

2. Given the following diagrams:
Diagram A

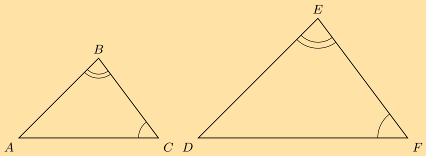
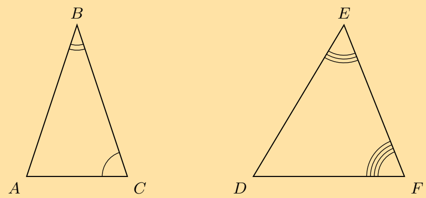


Diagram B



Which diagram correctly gives a pair of similar triangles?

Solution:

Diagram A shows a pair of triangles with all pairs of corresponding angles equal (the same three angle markers are shown in both triangles). Diagram B shows a pair of triangles with different angles in each triangle. All six angles are different and there are no pairs of corresponding angles that are equal.

Therefore diagram A gives a pair of triangles that are similar.

3. Given the following diagrams:

   Diagram A

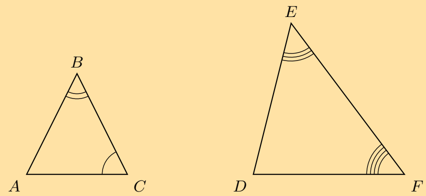
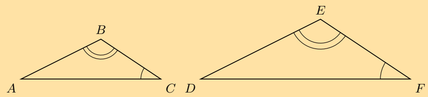


   Diagram B



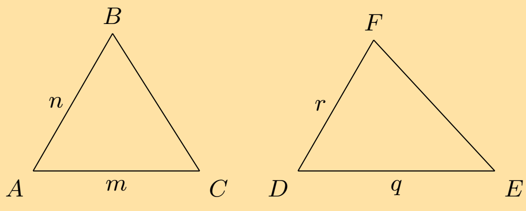
   Which diagram correctly gives a pair of similar triangles?

   Solution:

   Diagram A shows a pair of triangles with different angles in each triangle. All six angles are different and there are no pairs of corresponding angles that are equal. Diagram B shows a pair of triangles with all pairs of corresponding angles equal (the same two angle markers are shown in both triangles and the third angle in each triangle must be equal).

   Therefore diagram B gives a pair of triangles that are similar.

4. Have a look at the following triangles, which are drawn to scale:

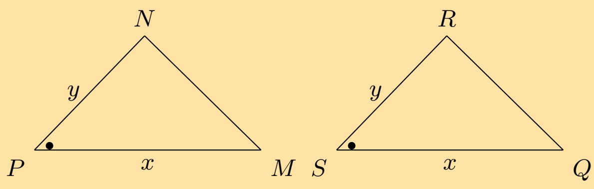


   Are the two triangles congruent? If so state the reason and use the correct notation to state that they are congruent.

   Solution:

   We are not told if $n = r$ and $m = q$ or $n = q$ and $m = r$ therefore we cannot say that the sides are the same length. Also we are not given any information about the angles of the two triangles. Therefore we cannot say if the two triangles are congruent.

5. Have a look at the following triangles, which are drawn to scale:



Are the two triangles congruent? If so state the reason and use the correct notation to state that they are congruent.

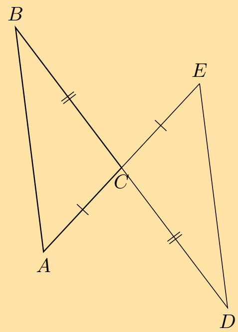
Solution:

Note that the two pairs of sides are equal, as indicated by the x and y. In addition, the angle between those two sides are marked as equal (this is the included angle).

Therefore, these two triangles are congruent. $\triangle PNM \equiv \triangle QSR$, reason: SAS.

6. State whether the following pairs of triangles are congruent or not. Give reasons for your answers. If there is not enough information to make a decision, explain why.

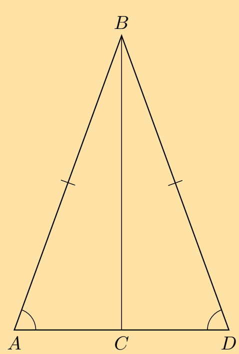
a)



Solution:

$$\begin{aligned} AC &= CE \qquad \text{(given)} \\ BC &= CD \qquad \text{(given)} \\ A\hat{C}B &= D\hat{C}E \qquad \text{(vert opp } \angle\text{s} = \text{)} \\ \therefore \triangle ABC &\equiv \triangle EDC \qquad \text{SAS} \end{aligned}$$

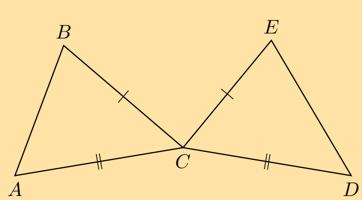
b)



Solution:

We have two equal sides ($AB = BD$ and BC is common to both triangles) and one equal angle ($\hat{A} = \hat{D}$) but the sides do not include the known angle. The triangles therefore do not have a SAS and are therefore not congruent. (Note: $A\hat{C}B$ is not necessarily equal to $D\hat{C}B$ because it is not given that $BC \perp AD$).

c)

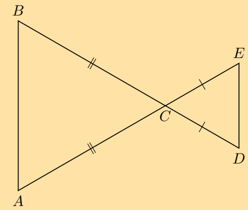


Solution:

There is not enough information given. We need at least three facts about the triangles and in this example we only know two sides in each triangle.

Note that BCD and ECA are not straight lines and so we cannot use vertically opposite angles.

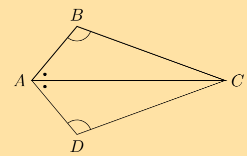
d)



Solution:

There is not enough information given. Although we can work out which angles are equal we are not given any sides as equal. All we know is that we have two isosceles triangles. Note how this question differs from part a). In part a) we were given equal sides in both triangles, in this question we are only given that sides in the same triangle are equal.

e)



Solution:

$$\begin{aligned} AC &= AC && \text{(common side)} \\ B\hat{A}C &= D\hat{A}C && \text{(given)} \\ A\hat{B}C &= A\hat{D}C && \text{(given)} \\ \therefore \triangle ABC &\equiv \triangle ADC && \text{AAS} \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2G6G 1b. 2G6H 1c. 2G6J 1d. 2G6K 1e. 2G6M 1f. 2G6N 1g. 2G6P 2. 2G6Q
3. 2G6R 4. 2G6S 5. 2G6T 6a. 2G6V 6b. 2G6W 6c. 2G6X 6d. 2G6Y 6e. 2G6Z

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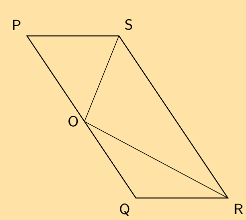
7.3 Quadrilaterals

Mathopenref has some useful simulations on different types of quadrilaterals. Clicking on any of the named quadrilaterals will take you to a page specific to that quadrilateral.

Parallelogram

Exercise 7 – 3:

1. $PQRS$ is a parallelogram. $PS = OS$ and $QO = QR$. $S\hat{O}R = 96°$ and $Q\hat{O}R = x$.



a) Find with reasons, two other angles equal to x.

Solution:
$S\hat{R}O = Q\hat{O}R = x$ (alt $\angle$s; $SR \parallel OQ$).
$O\hat{R}Q = Q\hat{O}R = x$ ($\angle$s opp equal sides).
Therefore $S\hat{R}O$ and $O\hat{R}Q$ are both equal to x.

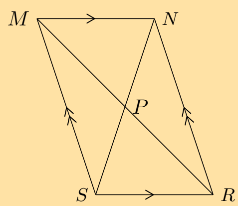
b) Write $\hat{P}$ in terms of x.
Solution:

$$\begin{aligned} \hat{P} &= Q\hat{R}S \qquad (\text{ opp } \angle\text{s of } \parallel \text{m}) \\ &= S\hat{R}O + O\hat{R}Q \\ \therefore \hat{P} &= 2x \end{aligned}$$

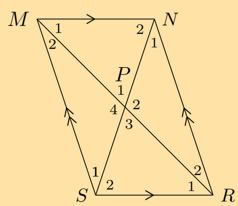
c) Calculate the value of x.
Solution:

$$\begin{aligned} S\hat{O}R &= 96^\circ \qquad (\text{given}) \\ S\hat{O}P &= \hat{P} \qquad (\angle\text{s opp equal sides}) \\ 180^\circ &= \hat{P} + 96^\circ + Q\hat{O}R \qquad (\text{sum of } \angle\text{s on a str line}) \\ 84^\circ &= 2x + x \\ 3x &= 84^\circ \\ \therefore x &= 28^\circ \end{aligned}$$

2. Prove that the diagonals of parallelogram $MNRS$ bisect one another at P.



Hint: Use congruency.
Solution:
First number each angle on the given diagram:



In $\triangle MNP$ and $\triangle RSP$:

$$\begin{aligned} \hat{M}_1 &= \hat{R}_1 \quad (\text{alt } \angle\text{s; } MN \parallel SR) \\ \hat{P}_1 &= \hat{P}_3 \quad (\text{vert opp } \angle\text{s } =) \\ MN &= RS \text{ (opp sides of } \parallel \text{m}) \end{aligned}$$

Therefore $\triangle MNP \equiv \triangle RSP$ (AAS).
Now we know that $MP = RP$ and therefore P is the mid-point of MR.
Similarly, in $\triangle MSP$ and $\triangle RNP$:

$$\hat{M}_2 = \hat{R}_2 \quad \text{(alt } \angle\text{s; } MS \parallel NR\text{)}$$
$$\hat{P}_4 = \hat{P}_2 \quad \text{(vert opp } \angle\text{s } =\text{)}$$
$$MS = RN \text{ (opp sides of } \parallel \text{m)}$$

Therefore $\triangle MSP \equiv \triangle RNP$ (AAS).
Now we know that $NP = SP$ and therefore P is the mid-point of NS.
Therefore the diagonals of a parallelogram bisect each other.

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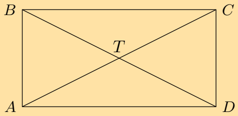
Rectangle

Exercise 7 – 4:

1. $ABCD$ is a quadrilateral. Diagonals AC and BD intersect at T. $AC = BD$, $AT = TC$, $DT = TB$. Prove that:



a) $ABCD$ is a parallelogram
Solution:
$AT = TC$ (given)
$\therefore DB$ bisects AC at T
and $DT = TB$ (given)
$\therefore AC$ bisects DB at T
therefore quadrilateral $ABCD$ is a parallelogram (diag of $\parallel$m)

b) $ABCD$ is a rectangle
Solution:
$AC = BD$ (given).
Therefore $ABCD$ is a rectangle (diags of rectangle).

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Rhombus

Square

Trapezium

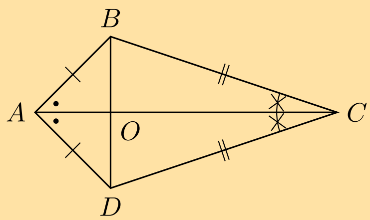
In British English a trapezium is used to indicate a quadrilateral with one pair of opposite sides parallel while in American English a trapezium is a quadrilateral with no pairs of opposite sides parallel. We will use the British English definition of trapezium in this book.

In British English a trapezoid is used to indicate a quadrilateral with no pairs of opposite sides parallel while in American English a trapezoid is a quadrilateral with one pair of opposite sides parallel.

Kite

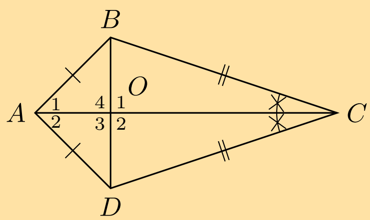
Exercise 7 – 5:

1. Use the sketch of quadrilateral $ABCD$ to prove the diagonals of a kite are perpendicular to each other.



Solution:

First number the angles:



In $\triangle ADO$ and $\triangle ABO$:

$$\begin{aligned} AD &= AB && \text{(given)} \\ AO & && \text{(common side)} \\ B\hat{A}O &= D\hat{A}O && \text{(given)} \\ \therefore \triangle ADO &\equiv \triangle ABO && \text{(SAS)} \\ \therefore A\hat{B}O &= A\hat{D}O \end{aligned}$$

In $\triangle ADB$:

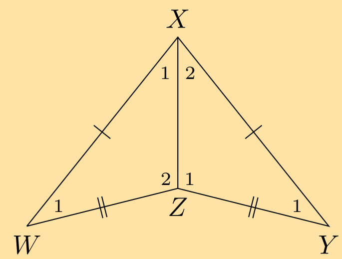
$$\begin{aligned} \text{let } \hat{A}_1 = \hat{A}_2 &= t \\ \text{and let } A\hat{D}O = A\hat{B}O &= p \\ 2t + 2p &= 180° \quad (\text{sum of } \angle\text{s in } \triangle) \\ \therefore t + p &= 90° \end{aligned}$$

Next we note that:

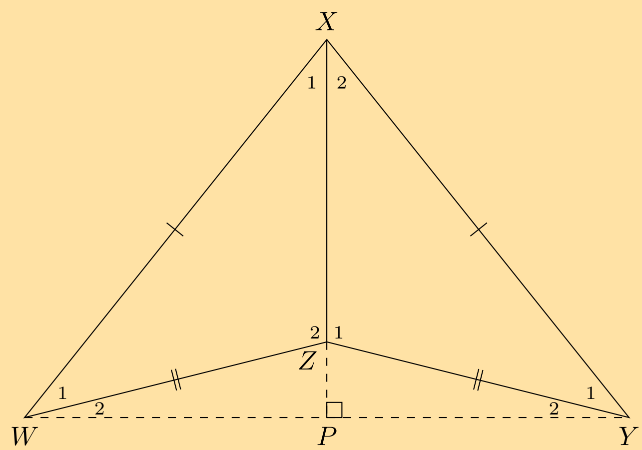
$$\begin{aligned} \hat{O}_1 &= A\hat{B}O + \hat{A}_1 \text{ (ext } \angle \text{ of } \triangle) \\ \hat{O}_1 &= p + t \\ &= 90° \\ \therefore AC &\perp BD \end{aligned}$$

Therefore the diagonals of a kite are perpendicular to each other.

2. Explain why quadrilateral $WXYZ$ is a kite. Write down all the properties of quadrilateral $WXYZ$.



Solution:



Quadrilateral $WXYZ$ is a kite because is has two pairs of adjacent sides that are equal in length.

- Diagonal between equal sides bisects the other diagonal: $WP = PY$.
- One pair of opposite angles are equal: $\hat{W}_1 = \hat{Y}_1$.
- Diagonal between equal sides bisects the interior angles and is an axis of symmetry: $\hat{X}_1 = \hat{X}_2$.
- Diagonals intersect at $90°$: $WY \perp PX$.

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Exercise 7 – 6:

1. The following shape is drawn **to scale** :

Give the most specific name for the shape.

Solution:

We start by counting the number of sides. There are four sides in this figure and so it is either just a quadrilateral or one of the special types of quadrilateral.

Next we ask ourselves if there are any parallel lines in the figure. You can look at the figure to see if any of the lines look parallel or make a quick sketch of the image and see if any pairs of opposite lines meet at a point.

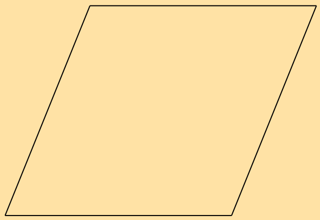
Both pairs of opposite sides are parallel. This means that the figure can only be one of the following: parallelogram, rectangle, rhombus or square.

Next we ask ourselves if all the interior angles are 90°. All the interior angles are 90° and so this must be a square or a rectangle. Finally we check to see if all the sides are equal in length. In this figure the sides are not equal in length and so it is a rectangle.

Therefore this is a rectangle.

The shape is also a parallelogram and a quadrilateral. This question, however, asked for the most specific name for the shape.

2. The following shape is drawn **to scale** :



Give the most specific name for the shape.

Solution:

We start by counting the number of sides. There are four sides in this figure and so it is either just a quadrilateral or one of the special types of quadrilateral.

Next we ask ourselves if there are any parallel lines in the figure. You can look at the figure to see if any of the lines look parallel or make a quick sketch of the image and see if any pairs of opposite lines meet at a point.

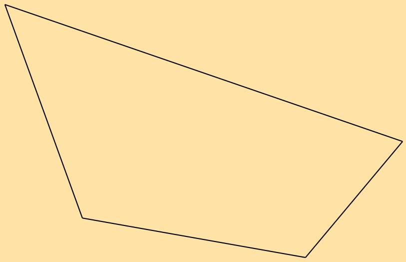
Both pairs of opposite sides are parallel. This means that the figure can only be one of the following: parallelogram, rectangle, rhombus or square.

Next we ask ourselves if all the interior angles are 90°. All the interior angles are not 90° and so this must be a parallelogram or a rhombus. Finally we check to see if all the sides are equal in length. In this figure the sides are equal in length and so it is a rhombus.

Therefore this is a rhombus.

The shape is also a parallelogram and a quadrilateral. This question, however, asked for the most specific name for the shape.

3. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale

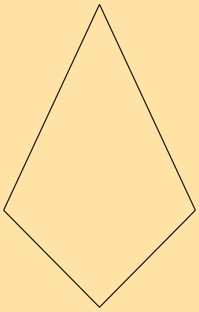


Solution:

Both pairs of opposite sides are not parallel. This means that the figure can only be some combination of the following: trapezium, kite, or quadrilateral.

The shape is definitely a quadrilateral because it has four sides. It does not have any special properties: it does not have parallel sides, or right angles, or sides which are equal in length. Therefore it is only a quadrilateral.

4. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale



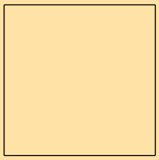
Solution:

Both pairs of opposite sides are not parallel. This means that the figure can only be some combination of the following: trapezium, kite, or quadrilateral.

The shape is definitely a quadrilateral because it has four sides. It is also a kite because it has two pairs of adjacent sides which are the same lengths. It cannot be a square or a rectangle because it does not have right angles. It cannot be a parallelogram or a trapezium because it does not have any parallel sides. And it is not a rhombus because the four sides are not all the same length.

Therefore the correct answer is: kite and quadrilateral.

5. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale



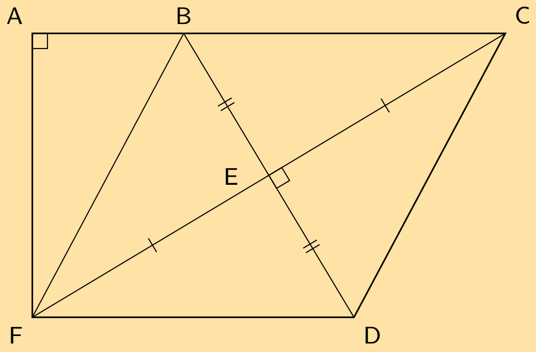
Solution:

Both pairs of opposite sides are parallel. That means that this shape can belong to one or more of these groups: square, rhombus, rectangle or parallelogram.

The given shape is a square. However, it is also a rectangle. A square is also a parallelogram, because it has parallel sides; and it is a rhombus as well, it just happens to have right angles. A square is also a kite, a trapezium and of course a quadrilateral.

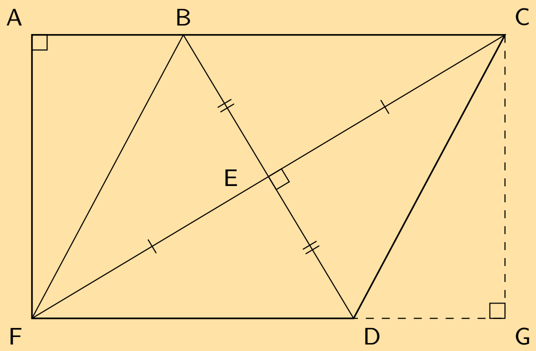
Therefore the correct answer is: square, rectangle, rhombus, parallelogram, kite, trapezium and quadrilateral.

6. Find the area of $ACDF$ if $AB = 8,\ BF = 17,\ FE = EC,\ BE = ED,\ \hat{A} = 90°,\ C\hat{E}D = 90°$



Solution:

Construct G such that $AC = FG$



$BCDF$ is a rhombus (diagonals bisect at right angles)

Since $BCDF$ is a rhombus $BC = DF$. We constructed G such that $AC = FG$. Therefore $AB = DG$.

In $\triangle ABF$ and $\triangle CGD$:

$$
\begin{aligned}
B\hat{A}F = C\hat{G}D &= 90^\circ \quad \text{(given and by construction)}\\
AB &= DG \quad \text{(by construction)}\\
BF &= CD \quad (BCDF \text{ is a rhombus})
\end{aligned}
$$

Therefore $ABF \equiv CGD$ (RHS)
Therefore $AF = CG$ and so $ACGF$ is a rectangle (both pairs of opposite sides equal in length and all interior angles are 90°).
We are given the length of AB and BF. Since $\triangle ABF$ is right-angled we can use the theorem of Pythagoras to find the length of AF:

$$
\begin{aligned}
BF^2 &= AB^2 + AF^2\\
(17)^2 &= (8)^2 + AF^2\\
AF^2 &= 225\\
AF &= 15
\end{aligned}
$$

We also know that $FD = BF = 17$ and so $AC = 17 + 8 = 25$.
Therefore the area of rectangle $ACGF$ is:

$$
\begin{aligned}
A_{\text{rectangle}} &= l \times b\\
&= (25)(15)\\
&= 375
\end{aligned}
$$

We are almost there. We now need to calculate the area of triangle CDG and subtract this from the area of the rectangle to get the area of $ACDF$.
The area of triangle CDG is:

$$
\begin{aligned}
A_{\text{triangle}} &= \frac{1}{2} DG \times CG\\
&= \frac{1}{2}(8 \times 15)\\
&= 60
\end{aligned}
$$

Therefore the area of $ACDF$ is $375 - 60 = 315$.

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1. 2G78 2. 2G79 3. 2G7B 4. 2G7C 5. 2G7D 6. 2G7F

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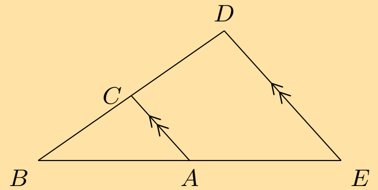
7.4 The mid-point theorem

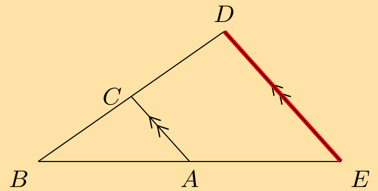
Exercise 7 – 7:

1. Points C and A are the mid-points on lines BD and BE. Study $\triangle EDB$ carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. Name the third side by its endpoints, e.g., FG.



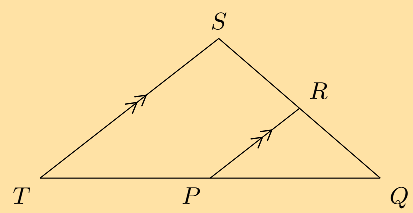
Solution:

The red line, ED or DE, indicates the third side of the triangle. According to the mid-point theorem, the line joining the mid-points of two sides of a triangle is parallel to the third side of the triangle.



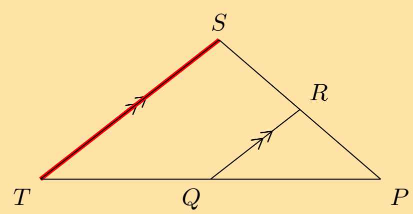
The third side is: ED or DE.

2. Points R and P are the mid-points on lines QS and QT. Study $\triangle TSQ$ carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. Name the third side by its endpoints, e.g., FG.



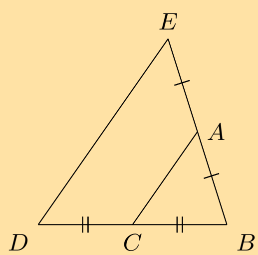
Solution:

The red line, TS or ST, indicates the third side of the triangle. According to the mid-point theorem, the line joining the mid-points of two sides of a triangle is parallel to the third side of the triangle.



The third side is: TS or ST.

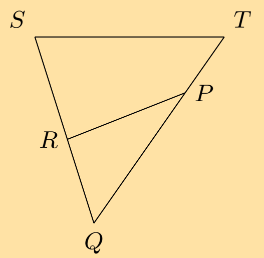
3. Points C and A are given on the lines BD and BE. Study the triangle carefully, then identify and name the parallel lines.



Solution:

The lines ED and AC are parallel according to the mid-point theorem because AC is bisecting the lines EB and DB.

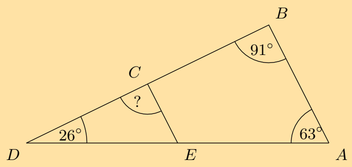
4. Points R and P are given on the lines QS and QT. Study the triangle carefully, then identify and name the parallel lines.



Solution:

The lines TS and PR are not parallel according to the mid-point theorem because line PR does not bisect TQ and SQ. Therefore there are no parallel lines in the triangle.

5. The figure below shows a large triangle with vertices A, B and D, and a smaller triangle with vertices at C, D and E. Point C is the mid-point of BD and point E is the mid-point of AD.

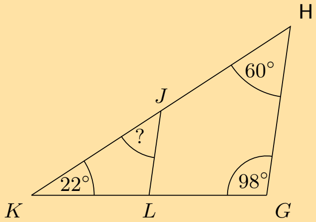


a) Three angles are given: $\hat{A} = 63°$, $\hat{B} = 91°$ and $\hat{D} = 26°$; determine the value of $D\hat{C}E$.
Solution:

$$\begin{aligned} AB &\parallel EC && \text{(Midpt Theorem)} \\ \therefore D\hat{C}E &= \hat{B} && \text{(corresp } \angle\text{s; } AB \parallel EC) \\ D\hat{C}E &= 91° \end{aligned}$$

b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
$\triangle DEC \ ||| \ \triangle ?$
Solution:
Angle D corresponds to angle D; angle E corresponds to angle A; and angle C corresponds to angle B. Therefore, $\triangle DEC \ ||| \ \triangle DAB$.

6. The figure below shows a large triangle with vertices G, H and K, and a smaller triangle with vertices at J, K and L. Point J is the mid-point of HK and point L is the mid-point of GK.



a) Three angles are given: $\hat{G} = 98°$, $\hat{H} = 60°$, and $\hat{K} = 22°$; determine the value of $K\hat{J}L$.
Solution:

$$\begin{aligned} GH &\parallel LJ && \text{(Midpt Theorem)} \\ \therefore K\hat{J}L &= \hat{H} && \text{(corresp } \angle\text{s; } GH \parallel LJ) \\ K\hat{J}L &= 60° \end{aligned}$$

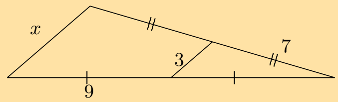
b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).

$\triangle HKG \; ||| \; \triangle ?$

Solution:

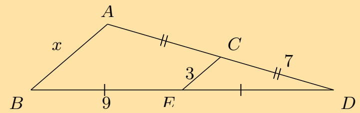
Angle H corresponds to angle J; angle K corresponds to angle K ; and angle G corresponds to angle L. Therefore, $\triangle HKG \; ||| \; \triangle JKL$.

7. Consider the triangle in the diagram below. There is a line crossing through a large triangle. Notice that some lines in the figure are marked as equal to each other. One side of the triangle has a given length of 3. Some information is also given about the lengths of other lines along the edges of the triangle.



Determine the value of x.

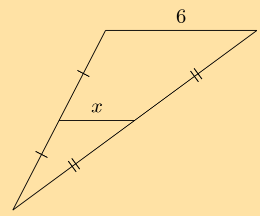
Solution:



From the mid-point theorem we know:

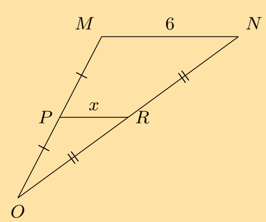
$$\begin{aligned} AB &= 2 \times CE \\ x &= 2(3) \\ &= 6 \end{aligned}$$

8. Consider the triangle in the diagram below. There is a line crossing through a large triangle. Notice that some lines in the figure are marked as equal to each other. One side of the triangle has a given length of 6.



Determine the value of x.

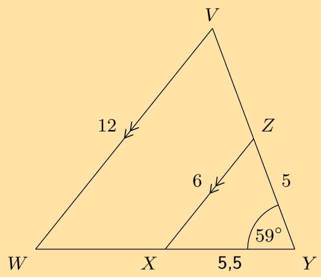
Solution:



From the mid-point theorem we know:

$$\begin{aligned} MN &= 2 \times PR \\ (6) &= 2x \\ \frac{1}{2}(6) &= x \\ 3 &= x \end{aligned}$$

9. In the figure below, $VW \parallel ZX$, as labelled. Furthermore, the following lengths and angles are given: $VW = 12$; $ZX = 6$; $XY = 5{,}5$; $YZ = 5$ and $\hat{V} = 59°$. The figure is drawn to scale.



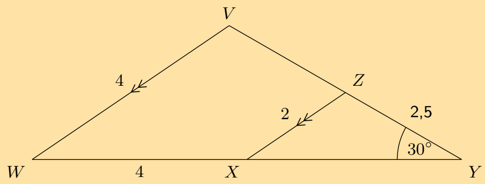
Determine the length of WY.

Solution:

X is the mid-point of WY and Z is the mid-point of VY ($VW \parallel ZX$, also it is given that $XZ = \frac{1}{2}VW$).

$$\begin{aligned} WY &= 2 \times XY \qquad \text{definition of mid-point} \\ &= 2(5{,}5) \\ &= 11 \end{aligned}$$

10. In the figure below, $VW \parallel ZX$, as labelled. Furthermore, the following lengths and angles are given: $VW = 4$; $ZX = 2$; $WX = 4$; $YZ = 3{,}5$ and $\hat{Y} = 30°$. The figure is drawn to scale.



Determine the length of XY.

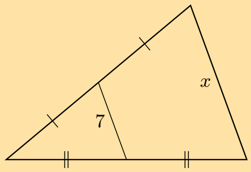
Solution:

X is the mid-point of WY and Z is the mid-point of VY ($VW \parallel ZX$, also it is given that $XZ = \frac{1}{2}VW$).

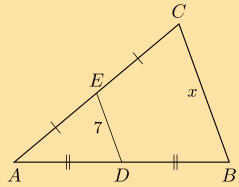
$$\begin{aligned} XY &= WX \qquad \text{definition of mid-point} \\ &= 4 \end{aligned}$$

11. Find x and y in the following:

a)



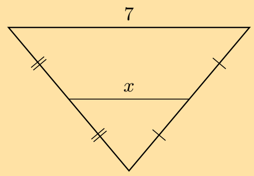
Solution:



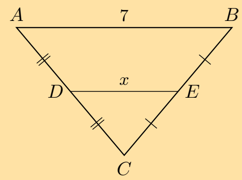
From the mid-point theorem we know:

$$\begin{aligned} BC &= 2 \times DE \\ x &= 2(7) \\ &= 14 \end{aligned}$$

b)



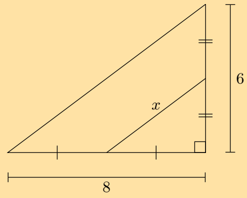
Solution:



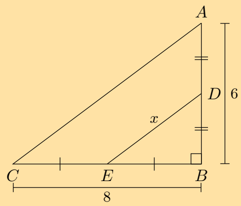
From the mid-point theorem we know:

$$\begin{aligned} AB &= 2 \times DE \\ 7 &= 2x \\ 3{,}5 &= x \end{aligned}$$

c)



Solution:



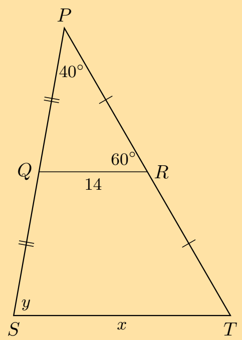
We can use the theorem of Pythagoras to find AC:

$$\begin{aligned} AC^2 &= BC^2 + AB^2 \\ &= (8)^2 + (6)^2 \\ &= 64 + 36 \\ &= 100 \\ AC &= 10 \end{aligned}$$

From the mid-point theorem we know:

$$AC = 2 \times DE$$
$$10 = 2x$$
$$5 = x$$

d)



Solution:

From the mid-point theorem we know:

$$\begin{aligned} ST &= 2 \times QR \\ x &= 2(14) \\ &= 28 \end{aligned}$$

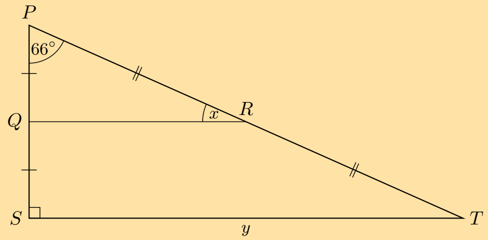
To find y we note the following:

- $P\hat{Q}R = 180° - 60° - 40° = 100°$ (sum of $\angle$s in $\triangle$).
- From the mid-point theorem we also know that $QR \parallel ST$.

Therefore $y = 100°$ (corresp $\angle$s; $QR \parallel ST$).

The final answer is: $x = 28$ units and $y = 100°$.

e) In the following diagram $PQ = 2{,}5$ and $RT = 6{,}5$.



Solution:

From the mid-point theorem we know that $QR \parallel ST$. Therefore $P\hat{Q}R = P\hat{S}T = 90°$ (corresp $\angle$s; $QR \parallel ST$).

Therefore $x = 180° - 90° - 66° = 24°$ (sum of $\angle$s in $\triangle$).

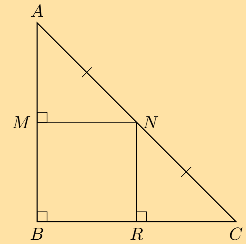
To find y we note that $PQ + QS = PS$ and $PQ = QS$, therefore $PS = 2PQ$. Similarly $PT = 2RT$.

We can use the theorem of Pythagoras to find ST:

$$\begin{aligned} ST^2 &= PS^2 + PT^2 \\ &= 2PQ + 2RT \\ &= (2(2{,}5))^2 + (2(6{,}5))^2 \\ &= 25 + 169 \\ &= 194 \\ ST &= 13{,}93 \end{aligned}$$

Therefore: $x = 24°$ and $y = 13{,}93$.

12. Show that M is the mid-point of AB and that $MN = RC$.



Solution:

We are given that $AN = NC$.

We are also given that $\hat{B} = \hat{M} = 90°$, therefore $MN \parallel BR$ ($\hat{B}$ and $\hat{M}$ are equal, corresponding angles).

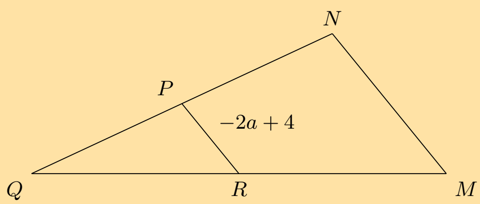
Therefore M is the mid-point of AB (converse of mid-point theorem).

Similarly we can show that R is the mid-point of BC.

We also know that $MN = BR$ ($MB \parallel NR$ and parallel lines are a constant distance apart).

But $BR = RC$ (R is the mid-point of BC), therefore $MN = RC$.

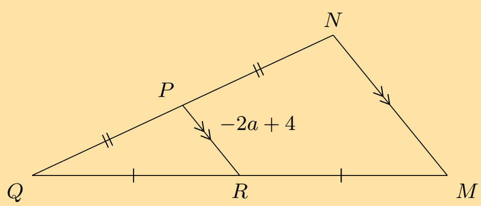
13. In the diagram below, P is the mid-point of NQ and R is the mid-point of MQ. The segment inside of the large triangle is labelled with a length of $-2a + 4$.



a) Calculate the value of MN in terms of a.

Solution:

Use the mid-point theorem to fill in known information on the diagram:



Remember that the mid-point theorem tells us that the segments MN and PR have a ratio of $2 : 1$ (MN is twice as long as PR).

$$
\begin{aligned}
MN &= 2 \times PR \\
&= 2(-2a + 4) \\
&= -4a + 8
\end{aligned}
$$

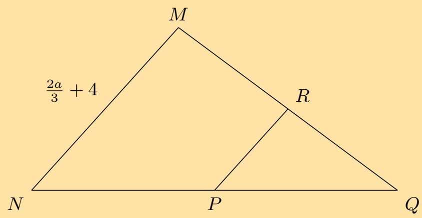
The final answer is $MN = -4a + 8$ (twice as long as PR).

b) You are now told that MN has a length of 18. What is the value of a? Give your answer as a fraction.

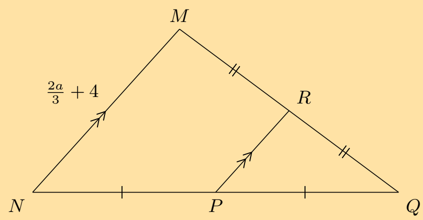
Solution:

$$
\begin{aligned}
-4a + 8 &= 18 \\
-4a &= 10 \\
\left(-\frac{1}{4}\right)(-4a) &= (10)\left(-\frac{1}{4}\right) \\
a &= -\frac{5}{2}
\end{aligned}
$$

14. In the diagram below, P is the mid-point of NQ and R is the mid-point of MQ. One side of the triangle has a given length of $\frac{2a}{3} + 4$.



a) Find the value of PR in terms of a.
Solution:
Use the mid-point theorem to fill in known information on the diagram:



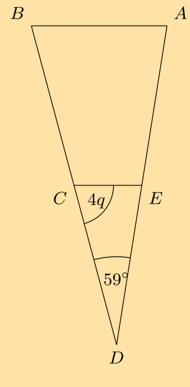
Remember that the mid-point theorem tells us that the segments MN and PR have a ratio of $2:1$ (MN is twice as long as PR).

$$\begin{aligned} MN &= 2 \times PR \\ \left(\frac{2a}{3} + 4\right) &= 2\,(PR) \\ \frac{1}{2}\left(\frac{2a}{3} + 4\right) &= PR \\ \frac{a}{3} + 2 &= PR \end{aligned}$$

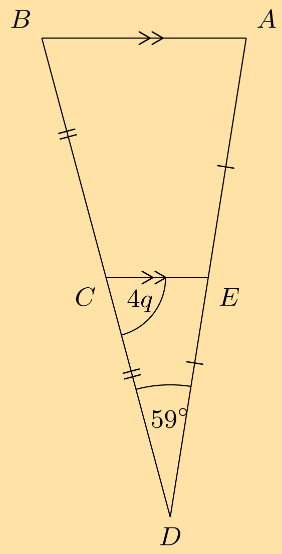
b) You are now told that PR has a length of 8. What is the value of a?
Solution:

$$\begin{aligned} \frac{a}{3} + 2 &= 8 \\ \frac{a}{3} &= 6 \\ (3)\left(\frac{a}{3}\right) &= (6)\,(3) \\ a &= 18 \end{aligned}$$

15. The figure below shows $\triangle ABD$ crossed by EC. Points C and E bisect their respective sides of the triangle.



a) The angles $\hat{D} = 59°$ and $E\hat{C}D = 4q$ are given; determine the value of $\hat{A}$ in terms of q.
Solution:
We note the following from the mid-point theorem:



Also $\hat{A} = D\hat{E}C$

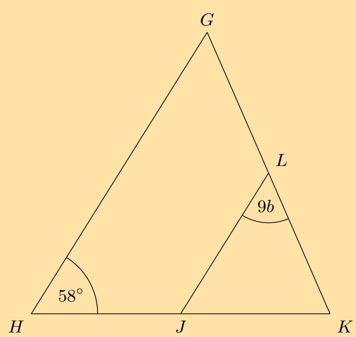
$$\begin{aligned} \hat{A} + 4q + 59° &= 180° \qquad \text{(sum of } \angle\text{s in } \triangle) \\ \hat{A} &= 180° - (4q + 59°) \\ &= -4q + 121° \end{aligned}$$

In terms of q, the answer is: $\hat{A} = -4q + 121°$.

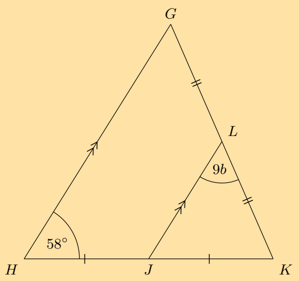
b) You are now told that $E\hat{C}D$ has a measure of $72°$. Calculate for the value of q.
Solution:

$$\begin{aligned} E\hat{C}D &= 72° \\ 4q &= 72° \\ q &= 18° \end{aligned}$$

16. The figure below shows $\triangle GHK$ crossed by LJ. Points J and L bisect their respective sides of the triangle.



a) Given the angles $\hat{H} = 58°$ and $K\hat{L}J = 9b$, determine the value of $\hat{K}$ in terms of b.
Solution:
Using the mid-point theorem we can add the following information to the diagram:



Also: $\hat{H} = K\hat{J}L = 58°$

$$\begin{aligned} \hat{K} + 9b + 58° &= 180° \qquad \text{(sum of } \angle\text{s in } \triangle) \\ \hat{K} &= 180° - (9b + 58°) \\ &= -9b + 122° \end{aligned}$$

In terms of b, the answer is: $\hat{K} \ = -9b + 122°$.

b) You are now told that $\hat{K}$ has a measure of $74°$. Solve for the value of b. Give your answer as a fraction.

Solution:

$$\begin{aligned} \hat{K} &= 74° \\ -9b + 122° &= 74° \\ b &= \frac{16}{3} \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2G7G	2. 2G7H	3. 2G7J	4. 2G7K	5. 2G7M	6. 2G7N	7. 2G7P	8. 2G7Q
9. 2G7R	10. 2G7S	11a. 2G7T	11b. 2G7V	11c. 2G7W	11d. 2G7X	11e. 2G7Y	12. 2G7Z
13. 2G82	14. 2G83	15. 2G84	16. 2G85				

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7.5 Chapter summary

End of chapter Exercise 7 – 8:

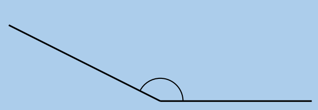
1. Identify the types of angles shown below:

a)

Solution:

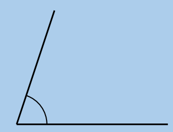
straight angle

b)



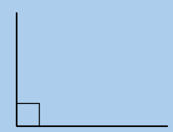
Solution:
obtuse angle

c)



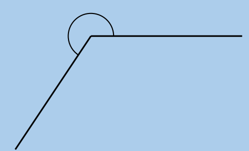
Solution:
acute angle

d)



Solution:
right angle

e)



Solution:
Reflex angle

f) An angle of 91°
Solution:
obtuse angle

g) An angle of 180°
Solution:
straight angle

h) An angle of 210°
Solution:
reflex angle

2. Assess whether the following statements are true or false. If the statement is false, explain why:
   a) A trapezium is a quadrilateral with two pairs of opposite sides that are parallel.
   Solution:
   False, a trapezium only has one pair of opposite parallel sides.
   b) Both diagonals of a parallelogram bisect each other.
   Solution:
   True
   c) A rectangle is a parallelogram that has all interior angles equal to 90°.
   Solution:
   True
   d) Two adjacent sides of a rhombus have different lengths.
   Solution:
   False, two adjacent sides of a rhombus are equal in length.
   e) The diagonals of a kite intersect at right angles.
   Solution:
   True
   f) All squares are parallelograms.
   Solution:
   True

g) A rhombus is a kite with a pair of equal, opposite sides.
Solution:
True

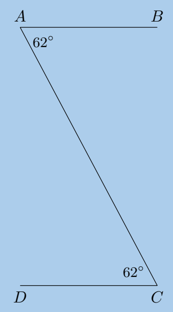
h) The diagonals of a parallelogram are axes of symmetry.
Solution:
True

i) The diagonals of a rhombus are equal in length.
Solution:
False, the diagonals of a rhombus are not equal in length.

j) Both diagonals of a kite bisect the interior angles.
Solution:
False, only one diagonal of a kite bisects one pair of interior angles.

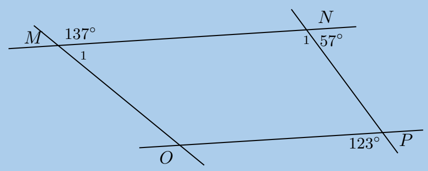
3. Find all pairs of parallel lines in the following figures, giving reasons in each case.

a)



Solution:
$AB \parallel CD$ (alt $\angle$s equal)

b)

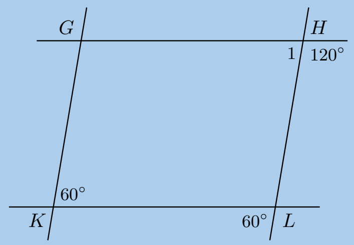


Solution:
Using the sum of angles on a straight line we can state the following:

- $\hat{M}_1 = 180° - 137° = 43°$
- $\hat{N}_1 = 180° - 57° = 123°$

NP not $\parallel$ MO (corresp $\angle$s not equal).
$MN \parallel OP$ (corresp $\angle$s equal).

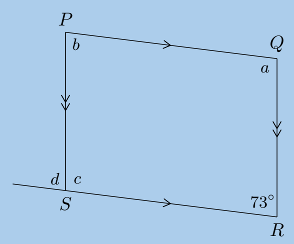
c)



Solution:
$\hat{H}_1 = 180° - 120° = 60°$ ($\angle$s on str line).
Therefore $GH \parallel KL$ (corresp $\angle$s equal).
And $GK \parallel HL$ (alt $\angle$s equal).
The pairs of parallel lines are $GH \parallel KL$ and $GK \parallel HL$.

4. Find angles a, b, c and d in each case, giving reasons:

a)

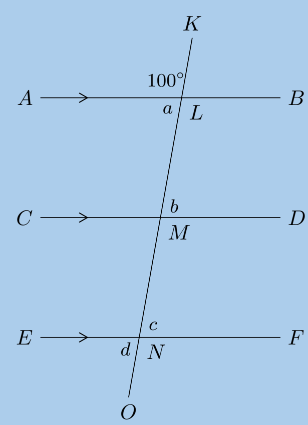


Solution:

$$a = 180° - 73° = 107° \qquad \text{(co-int } \angle\text{s; } PQ \parallel SR\text{)}$$
$$b = 180° - 107° = 73° \qquad \text{(co-int } \angle\text{s; } PS \parallel QR\text{)}$$
$$c = 180° - 73° = 107° \qquad \text{(co-int } \angle\text{s; } PQ \parallel SR\text{)}$$
$$d = 73° \qquad \text{(corresp } \angle\text{s; } PS \parallel QR\text{)}$$

Therefore $a = 107°, b = 73°, c = 107°, d = 73°$.

b)

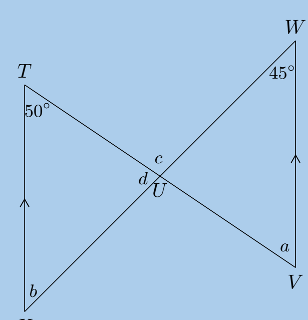


Solution:

$$a = 80° \qquad \text{(sum of } \angle\text{s on str line)}$$
$$b = 80° \qquad \text{(alt } \angle\text{s; } AB \parallel CD\text{)}$$
$$c = 80° \qquad \text{(corresp } \angle\text{s; } CD \parallel EF\text{)}$$
$$d = 80° \qquad \text{(vert opp } \angle\text{s } =\text{)}$$

Therefore $a = b = c = d = 80°$.

c)

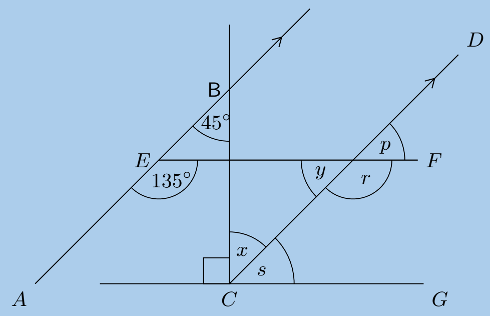


Solution:

$$\begin{aligned}
a &= 50° && \text{(alt } \angle\text{s; } TX \parallel WV) \\
b &= 45° && \text{(alt } \angle\text{s; } TX \parallel WV) \\
c &= 95° && \text{(ext } \angle \text{ of } \triangle) \\
d &= 85° && \text{(sum of } \angle\text{'s in } \triangle)
\end{aligned}$$

Therefore $a = 50°, b = 45°, c = 95°, d = 85°$.

5. Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.



a) $\hat{x}$
Solution:
$\hat{x}$ and $A\hat{B}C$ are alternate interior angles on transversal BC. $AB \parallel CD$, therefore they must be equal in size. Therefore $\hat{x} = 45°$.

b) $\hat{s}$
Solution:

$$\begin{aligned}
\hat{s} &= 90^\circ - 45^\circ \\
&= 45^\circ
\end{aligned}$$

c) $\hat{r}$
Solution:
$A\hat{E}F$ corresponds to (matches) $\hat{r}$; and corresponding angles are equal in size since $AB \parallel CD$.
Therefore: $\hat{r} = 135°$.

d) $\hat{y}$
Solution:
$\hat{r} + \hat{y} = 180^\circ$ ($\angle$s on str line):

$$\begin{aligned}
\hat{y} &= 180^\circ - 135^\circ \\
&= 45^\circ
\end{aligned}$$

e) $\hat{p}$
Solution:
$\hat{p}$ and $\hat{y}$ are vertically opposite angles and vertically opposite angles have the same measure (equal sizes).
Therefore: $\hat{p} = 45°$.

f) Based on the results for the angles above, is $EF \parallel CG$?
Solution:
If EF is parallel to CG, then the following things must all be true:

- $\hat{s} = \hat{p}$ (corresponding angles)
- $\hat{s} = \hat{y}$ (alternate interior angles)
- $\hat{s} + \hat{r} = 180^\circ$ (co-interior angles)

All the above is true, therefore the lines are parallel.

6. Given the following diagrams:
Diagram A

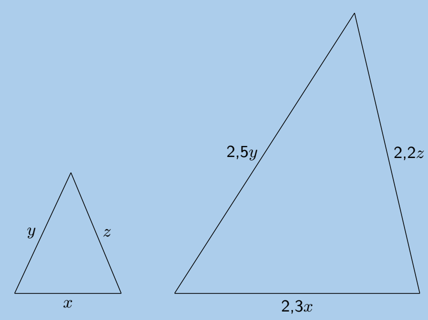
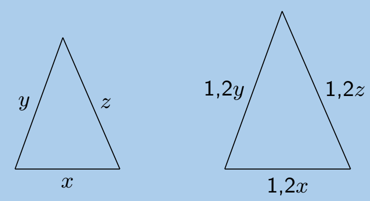


Diagram B



Which diagram correctly gives a pair of similar triangles?

Solution:

We look at the side labels. In diagram A we note that the three pairs of corresponding sides are in different proportions. In diagram B we note the three pairs of corresponding sides are in proportion.

Therefore diagram B gives a pair of triangles that are similar.

7. Given the following diagrams:

Diagram A

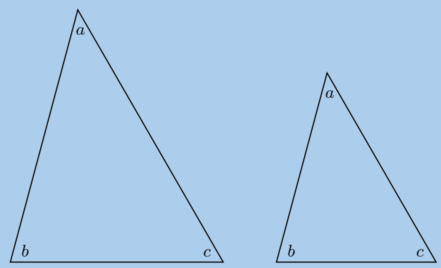
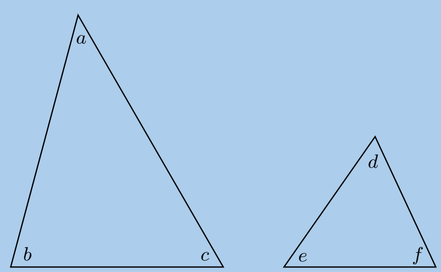


Diagram B



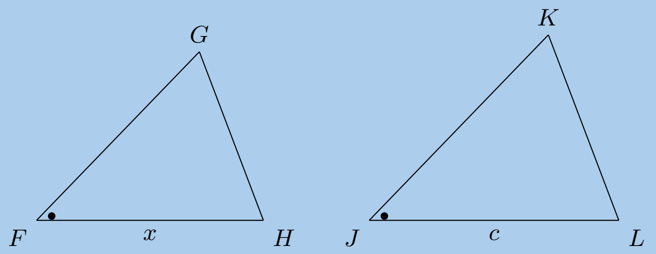
Which diagram correctly gives a pair of similar triangles?

Solution:

Diagram A shows a pair of triangles with all pairs of corresponding angles equal (the same three angle markers are shown in both triangles). Diagram B shows a pair of triangles with different angles in each triangle. All six angles are different and there are no pairs of corresponding angles that are equal.

Therefore diagram A gives a pair of triangles that are similar.

8. Have a look at the following triangles, which are drawn to scale:



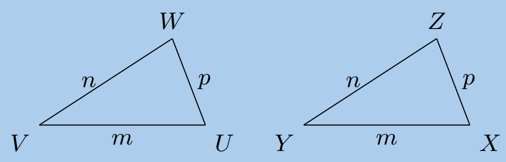
Are the triangles congruent? If so state the reason and use correct notation to state that they are congruent.

Solution:

We are given one angle that is equal. We are not given any equal sides (we do not know if $x = c$). To determine if two triangles are congruent we need to have three pieces of information (recall that the reasons for congruent triangles are: SSS, SAS, AAS and RHS). Therefore we cannot state whether or not the triangles are congruent.

Therefore, there is not enough information to determine if the two triangles are congruent.

9. Have a look at the following triangles, which are drawn to scale:



Are the triangles congruent? If so state the reason and use correct notation to state that they are congruent.

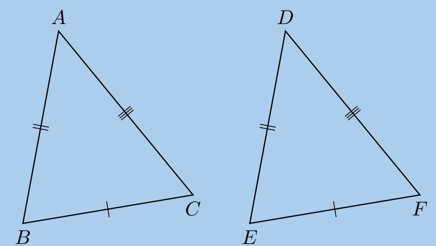
Solution:

The sides of both triangles are labelled with m, n and p. This means that there are three pairs of corresponding and equal sides.

Therefore, these two triangles are congruent ($\triangle VWU \equiv \triangle YZX$), reason: SSS.

10. Say which of the following pairs of triangles are congruent with reasons.

a)

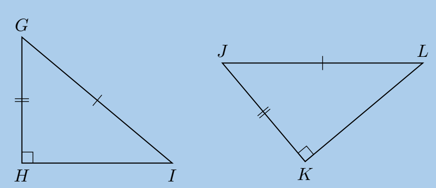


Solution:

We are given $CB = FE$, $AB = DE$ and $AC = DF$.

Therefore $\triangle ABC \equiv \triangle DEF$ by SSS.

b)

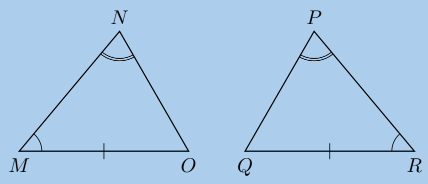


Solution:

We are given $GI = JL$, $GH = JK$ and $G\hat{H}I = J\hat{K}L = 90°$.

Therefore $\triangle GHI \equiv \triangle JKL$ by RHS.

c)

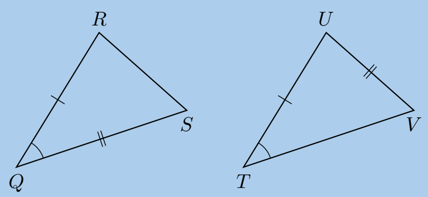


Solution:

We are given $MO = QR$, $O\hat{M}N = P\hat{R}Q$ and $M\hat{N}O = Q\hat{P}R$.

Therefore $\triangle MNO \equiv \triangle RPQ$ by AAS.

d)



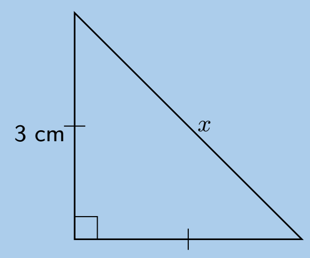
Solution:

We are given $QR = TU$, $QS = UV$ and $S\hat{Q}R = V\hat{T}U$. But $V\hat{T}U$ is not the included angle between sides UV and TU.

Therefore $\triangle QRS$ not congruent $\triangle TUV$.

11. Using the theorem of Pythagoras, calculate the length x:

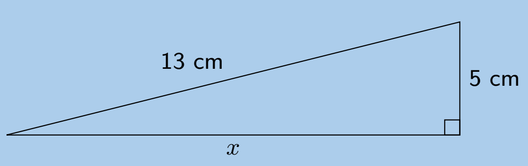
a)



Solution:

$$\begin{aligned} x^2 &= (3)^2 + (3)^2 \\ &= 18 \\ x &= \sqrt{18} \\ &= 4{,}24 \text{ cm} \end{aligned}$$

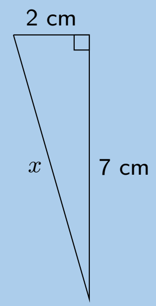
b)



Solution:

$$\begin{aligned} x^2 &= (13)^2 - (5)^2 \\ &= 144 \\ x &= 12 \text{ cm} \end{aligned}$$

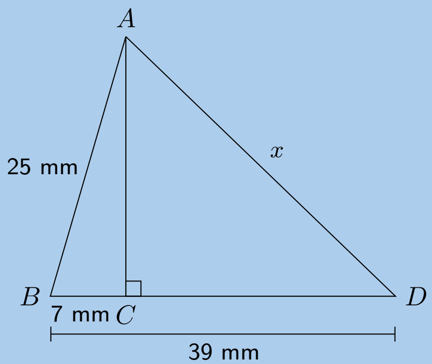
c)



Solution:

$$\begin{aligned} x^2 &= (2)^2 + (7)^2 \\ &= 53 \\ x &= \sqrt{53} \\ &= 7{,}28 \text{ cm} \end{aligned}$$

d)



Solution:
First find AC:

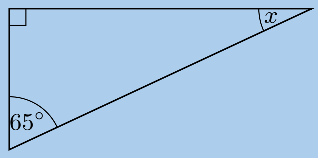
$$\begin{aligned} AC^2 &= (25)^2 - (7)^2 \\ &= 576 \\ AC &= \sqrt{576} \end{aligned}$$

Now we note that $CD = 39 - 7 = 32$ and then we find x:

$$\begin{aligned} x^2 &= (\sqrt{576})^2 + (32)^2 \\ x^2 &= 1600 \\ x &= 40 \text{ mm} \end{aligned}$$

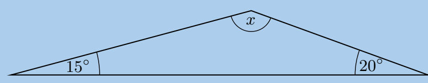
12. Calculate x and y in the diagrams below:

a)



Solution:
$x = 180° - 90° - 65° = 25°$ (sum of $\angle$s in $\triangle$).

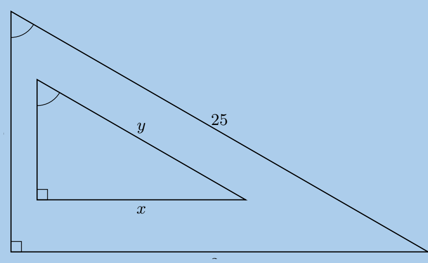
b)



Solution:

$x = 180° - 20° - 15° = 145°$ (sum of $\angle$s in $\triangle$).

c)



Solution:

We can find x using the theorem of Pythagoras:

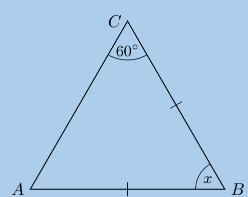
$$\begin{aligned} 25^2 &= 15^2 + (2x)^2 \\ 4x^2 &= 400 \\ x^2 &= 100 \\ \therefore x &= 10 \end{aligned}$$

We note that the triangles are similar by AAA. Therefore the sides must be in proportion. Therefore y is:

$$\begin{aligned} \frac{x}{2x} &= \frac{y}{25} \\ \therefore y &= 12{,}5 \end{aligned}$$

Therefore $x = 10$ and $y = 12{,}5$.

d)

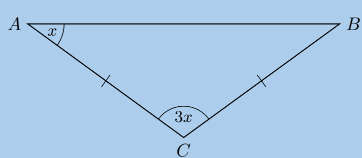


Solution:

This is an isosceles triangle so $\hat{C} = \hat{A} = 60°$.

Therefore $x = 180° - 60° - 60° = 60°$ (sum of $\angle$s in $triangle$).

e)

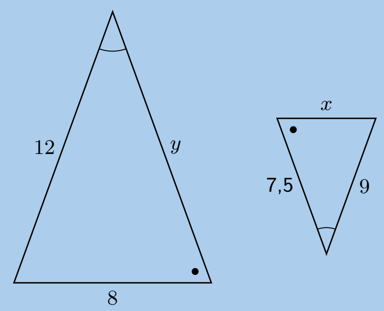


Solution:

This is an isosceles triangle so $\hat{A} = \hat{B} = x$.

$$\begin{aligned} x + x + 3x &= 180° \qquad \text{(sum of } \angle\text{s in } \triangle) \\ \therefore 5x &= 180° \\ x &= 36° \end{aligned}$$

f)



Solution:

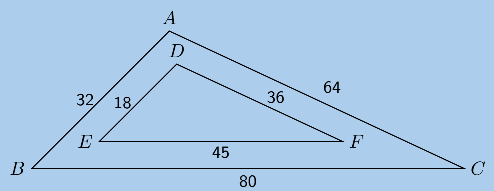
The two triangles are similar by AAA. Therefore the sides are in proportion.

$$\frac{x}{9} = \frac{8}{12}$$
$$\therefore x = 6$$

$$\frac{y}{12} = \frac{7{,}5}{9}$$
$$\therefore y = 10$$

Therefore $x = 6$ and $y = 10$.

13. Consider the diagram below. Is $\triangle ABC \ ||| \ \triangle DEF$? Give reasons for your answer.

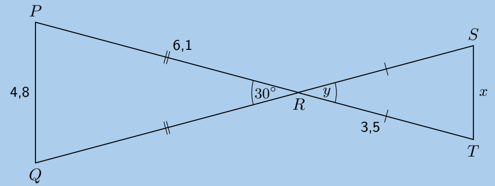


Solution:

$$\frac{ED}{BA} = \frac{18}{32} = \frac{9}{16}$$
$$\frac{DF}{AC} = \frac{32}{64} = \frac{9}{16}$$
$$\frac{EF}{BC} = \frac{45}{80} = \frac{9}{16}$$

All three pairs of sides are in proportion, $\therefore \triangle ABC \ ||| \ \triangle DEF$.

14. Explain why $\triangle PQR$ is similar to $\triangle TSR$ and calculate the values of x and y.



Solution:

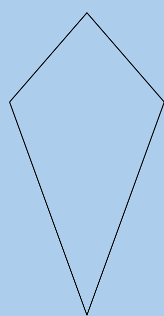
$$y = 30° \qquad \text{(vert opp } \angle\text{s} = \text{)}$$
$$\hat{P} = \hat{Q} \qquad (\angle\text{s opp equal sides})$$
$$\text{and } \hat{S} = \hat{T} \qquad (\angle\text{s opp equal sides})$$

However $\hat{P} + \hat{Q} + 30° = 180°$ (sum of $\angle$s in $\triangle$). Therefore $\hat{P} + \hat{Q} = 150°$.
Similarly $\hat{S} + \hat{T} = 150°$.
But $\hat{P} = \hat{Q}$ so $2\hat{P} = 150°$ and $\hat{S} = \hat{T}$ so $2\hat{S} = 150°$. Therefore $\hat{P} = \hat{S}$.
Therefore $\triangle PQR \ ||| \ \triangle TRS$ (AAA).
Now we can use the fact that the sides are in proportion to find x:

$$\frac{x}{4{,}8} = \frac{3{,}5}{6{,}1}$$
$$\therefore x = 2{,}75$$

Therefore $x = 2{,}75$ and $y = 30°$.

15. The following shape is drawn to scale:



Give the most specific name for the shape.

Solution:

We start by counting the number of sides. There are four sides in this figure and so it is either just a quadrilateral or one of the special types of quadrilateral.

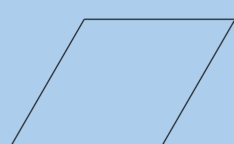
Next we ask ourselves if there are any parallel lines in the figure. You can look at the figure to see if any of the lines look parallel or make a quick sketch of the image and see if any pairs of opposite lines meet at a point.

Both pairs of opposite sides are not parallel. This means that the figure can only be one of the following: trapezium, kite or quadrilateral.

Next we ask ourselves if one of the pairs of opposite sides is parallel, while the other is not. Neither of the two pairs of opposite sides is parallel so we must now look to see if both pairs of adjacent sides are equal in length. Both pairs of adjacent sides are equal in length. So this is a kite.

Therefore this is a kite.

16. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



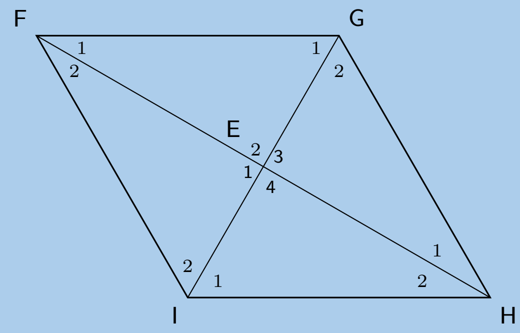
Solution:

Both pairs of opposite sides are parallel. That means that this shape can belong to one or more of these groups: square, rhombus, rectangle, and/or parallelogram.

The shape shown is a rhombus. It is certainly a quadrilateral (because it has four sides). It is also a parallelogram, because the opposite sides are parallel to each other. The rhombus is not a rectangle or a square because it does not have right angles. However, the rhombus is a kite, because it has two pairs of adjacent sides which are equal in length. And finally, it is a trapezium because it has a pair of opposite sides which are parallel.

Therefore the correct answer is: rhombus, parallelogram, kite, trapezium and quadrilateral.

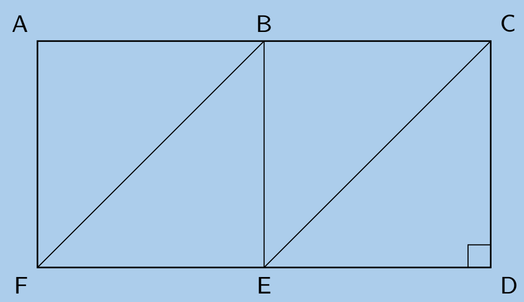
17. $FGHI$ is a rhombus. $\hat{F}_1 = 3x + 20°$; $\hat{G}_1 = x + 10°$. Determine the value of x.



Solution:
$\hat{E}_2 = 90°$ (diagonals of a rhombus bisect at right angles)

$$\begin{aligned} \hat{F}_1 + \hat{G}_1 + 90° &= 180° \qquad \text{(sum of } \angle\text{s in } \triangle) \\ 3x + 20° + x + 10° &= 90° \\ 4x &= 60° \\ \therefore x &= 15° \end{aligned}$$

18. In the diagram below, $AB = BC = CD = DE = EF = FA = BE$.



Name:

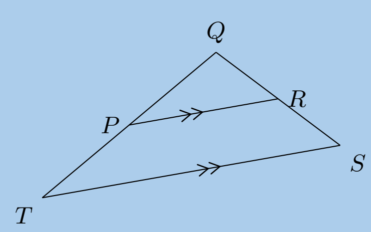
a) 3 rectangles
Solution:
$ACDF$, $ABEF$ and $BCDE$

b) 4 parallelograms
Solution:
$ACDF$, $ABEF$, $BCDE$ and $BCEF$

c) 2 trapeziums
Solution:
$ACEF$ and $BCDF$

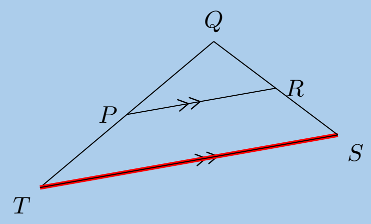
d) 2 rhombi
Solution:
$ABEF$ and $BCDE$

19. Points R and P are the mid-points on lines QS and QT. Study $\triangle TSQ$ carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. (Name the third side by its endpoints, e.g., FG.)



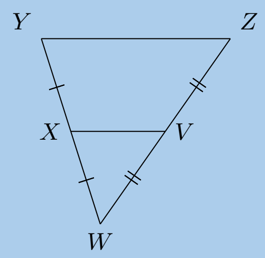
Solution:

The red line, TS or ST, indicates the third side of the triangle. According to the mid-point theorem, the line joining the mid-points of two sides of a triangle is parallel to the third side of the triangle.



The third side is: TS or ST.

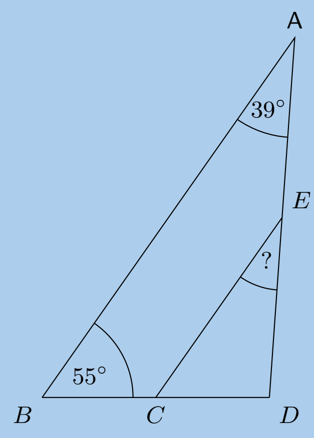
20. Points X and V are given on the segments WY and WZ. Study the triangle carefully, then identify and name the parallel line segments.



Solution:

The line segments YZ and VX are parallel according to the mid-point theorem because segment VX is bisecting the line segments WZ and WY.

21. The figure below shows a large triangle with vertices A, B and D, and a smaller triangle with vertices at C, D and E. Point C is the mid-point of BD and point E is the mid-point of AD.



a) The angles $\hat{A} = 39°$ and $\hat{B} = 55°$ are given; determine the value of $D\hat{E}C$.

Solution:

$$AB \parallel EC \quad \text{(Midpt Theorem)}$$
$$D\hat{E}C = \hat{A} \quad (\text{corresp } \angle\text{s; } AB \parallel EC)$$
$$D\hat{E}C = 39°$$

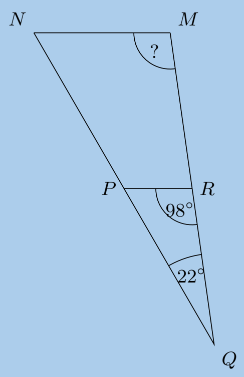
b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).

$\triangle DEC \ ||| \ \triangle ?$

Solution:

Angle D corresponds to angle D; angle E corresponds to angle A ; and angle C corresponds to angle B. Therefore, $\triangle DEC \ ||| \ \triangle DAB$.

22. The figure below shows a large triangle with vertices M, N and Q, and a smaller triangle with vertices at P, Q and R. Point P is the mid-point of NQ and point R is the mid-point of MQ.

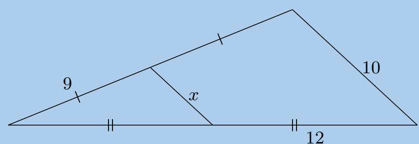


a) With the two angles given, $\hat{Q} = 22°$ and $Q\hat{R}P = 98°$, determine the value of $\hat{M}$.
Solution:

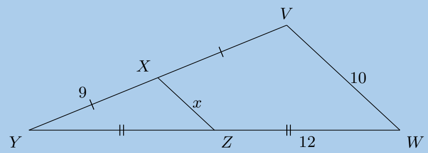
$$\begin{aligned} MN &\parallel RP && \text{(Midpt Theorem)} \\ \hat{M} &= Q\hat{R}P && \text{(corresp } \angle\text{s; } MN \parallel PR) \\ \hat{M} &= 98° \end{aligned}$$

b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
$\triangle QMN \ ||| \ \triangle ?$
Solution:
Angle Q corresponds to angle Q; angle M corresponds to angle R ; and angle N corresponds to angle P. Therefore, $\triangle QMN \ ||| \ \triangle QRP$.

23. Consider the triangle in the diagram below. There is a line segment crossing through a large triangle. Notice that some segments in the figure are marked as equal to each other. One side of the triangle has a given length of 10. Some information is also given about the lengths of other segments along the edges of the triangle.



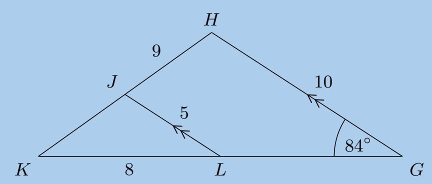
Determine the value of x.
Solution:



From the mid-point theorem we know:

$$\begin{aligned} VW &= 2 \times XZ \\ (10) &= 2x \\ \frac{1}{2}(10) &= x \\ 5 &= x \end{aligned}$$

24. In the figure below, $GH \parallel LJ$, as labelled. Furthermore, the following lengths and angles are given: $GH = 10$; $LJ = 5$; $HJ = 9$; $KL = 8$ and $\hat{G} = 84°$. The figure is drawn to scale.

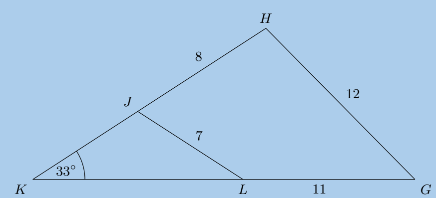


Calculate the length of JK.

Solution:

We are given that $GH \parallel LJ$. The length of JL is 5 and the length of GH is 10, therefore $JL = \frac{1}{2}GH$.
Therefore we know from the mid-point theorem that L is the mid-point of GK and J is the mid-point of HK.
Therefore $HJ = JK = 9$.

25. The figure below shows triangle GHK with the smaller triangle JKL sitting inside of it. Furthermore, the following lengths and angles are given: $GH = 12$; $LJ = 7$; $HJ = 8$; $LG = 11$; $\hat{K} = 33°$. The figure is drawn to scale.

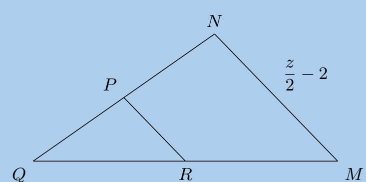


Find the length of KL.

Solution:

You can see in the figure that the segment LJ is not parallel to GH. This means that the mid-point theorem cannot apply to this triangle. There are no other options to use either: this question cannot be solved. There is no solution.

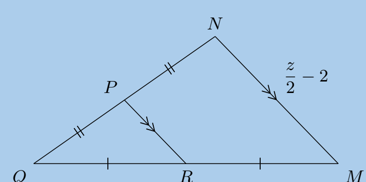
26. In the diagram below, P is the mid-point of NQ and R is the mid-point of MQ. One side of the triangle has a given length of $\frac{z}{2} - 2$.



a) Determine the value of PR in terms of z.

Solution:

Fill in information on the diagram using the mid-point theorem:



Remember that the mid-point theorem tells us that the segments MN and PR have a ratio of $2 : 1$ (MN is twice as long as PR).

$$\begin{aligned} MN &= 2 \times PR \\ \left(\frac{z}{2} - 2\right) &= 2\,(PR) \\ \frac{1}{2}\left(\frac{z}{2} - 2\right) &= PR \\ \frac{z}{4} - 1 &= PR \end{aligned}$$

The final answer is $PR = \frac{z}{4} - 1$ (half the size as MN).

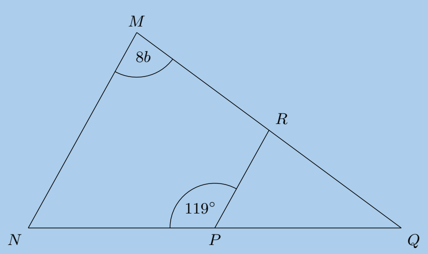
b) You are now told that PR has a length of 2. What is the value of z?

Solution:

$$\begin{aligned} \frac{z}{4} - 1 &= 2 \\ \frac{z}{4} &= 3 \\ (4)\left(\frac{z}{4}\right) &= (3)(4) \\ z &= 12 \end{aligned}$$

The final answer is $z = 12$.

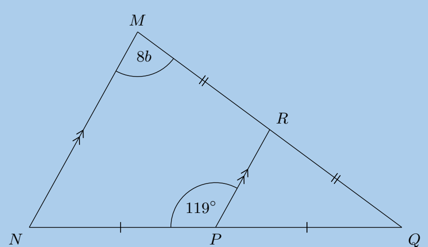
27. The figure below shows $\triangle MNQ$ crossed by RP. Points P and R bisect their respective sides of the triangle.



a) With the two angles given, $\hat{M} = 8b$ and $\angle NPR = 119°$, determine the value of $\hat{Q}$ in terms of b.

Solution:

Redraw the diagram and fill in the known information using the mid-point theorem:



$Q\hat{P}R = 180° - R\hat{P}N = 180° - 119° = 61°$ ($\angle$s on str line).

$Q\hat{R}P = 8b$ (corresp $\angle$s; $MN \parallel RP$).

Therefore:

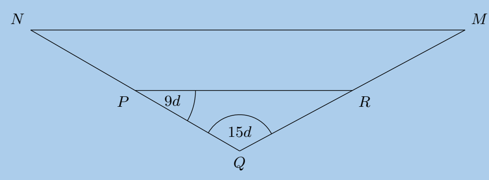
$$\begin{aligned} \hat{Q} + 8b + 61° &= 180° \qquad \text{(sum of } \angle\text{s in } \triangle) \\ \hat{Q} &= 180° - (8b + 61°) \\ &= -8b + 119° \end{aligned}$$

b) You are now told that $\hat{M}$ has a measure of $76°$. Determine for the value of b. Give your answer as an exact fractional value.

Solution:

$$\begin{aligned} \hat{M} &= 76° \\ 8b &= 76° \\ b &= \frac{19}{2} \end{aligned}$$

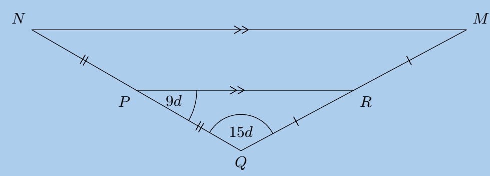
28. The figure below shows $\triangle MNQ$ crossed by RP. Points P and R bisect their respective sides of the triangle.



a) The angles $\hat{Q} = 15d$ and $R\hat{P}Q = 9d$ are given in the large triangle; determine the value of $\hat{M}$ in terms of d.

Solution:

Redraw the diagram and fill in known information using the mid-point theorem:



$P\hat{R}Q = \hat{M}$ (corresp $\angle$s; $MN \parallel RP$).

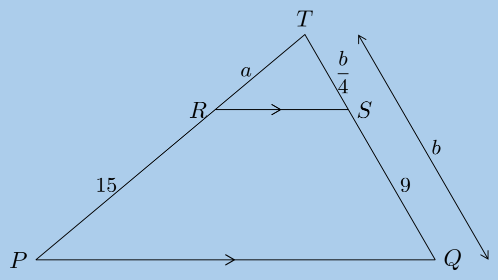
$$\begin{aligned} \hat{M} + 9d + 15d &= 180^\circ \qquad \text{(sum of } \angle\text{s in } \triangle) \\ \hat{M} &= 180^\circ - (9d + 15d) \\ &= -24d + 180^\circ \end{aligned}$$

b) You are now told that $R\hat{P}Q$ has a measure of 60°. Solve for the value of d. Give your answer as an exact fractional value.

Solution:

$$\begin{aligned} R\hat{P}Q &= 60^\circ \\ 9d &= 60^\circ \\ d &= \frac{20}{3} \end{aligned}$$

29. Calculate a and b:



Solution:

In $\triangle TRS$ and $\triangle PTQ$:

$$\begin{aligned} \hat{T} &= \hat{T} && \text{(common } \angle) \\ T\hat{R}S &= \hat{P} && \text{(corresp } \angle\text{s; } RS \parallel PQ) \\ T\hat{S}R &= \hat{Q} && \text{(corresp } \angle\text{s; } RS \parallel PQ) \end{aligned}$$

Therefore $\triangle TRS \; ||| \; \triangle PTQ$ (AAA).

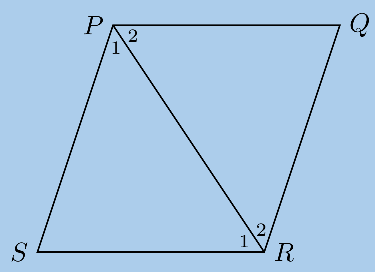
Therefore the sides are in proportion.

$$\frac{TR}{TP} = \frac{TS}{TQ}$$
$$\frac{a}{a+15} = \frac{\frac{b}{4}}{b}$$
$$\frac{a}{a+15} = \frac{1}{4}$$
$$a = (a+15)\left(\frac{1}{4}\right)$$
$$4a = a + 15$$
$$3a = 15$$
$$\therefore a = 5$$

$$b = \frac{b}{4} + 9$$
$$4b = b + 36$$
$$3b = 36$$
$$\therefore b = 12$$

Therefore: $a = 5$ and $b = 12$.

30. $\triangle PQR$ and $\triangle PSR$ are equilateral triangles. Prove that $PQRS$ is a rhombus.



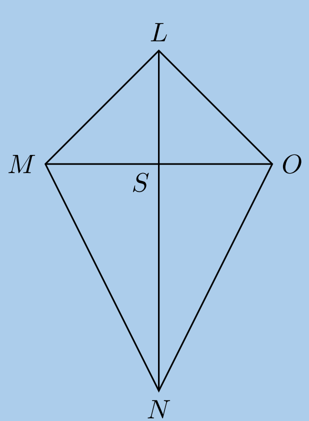
Solution:

We are given two equilateral triangles, therefore in $\triangle PSR$: $PS = SR = PR$ and in $\triangle PQR$: $PQ = QR = PR$. But PR is a common side and so $PR = PS = SR = PQ = QR$.

Also in each triangle all the interior angles are equal to 60°. Therefore $\hat{P}_1 = \hat{R}_2$ and $\hat{P}_2 = \hat{R}_2$. Therefore $PQ \parallel SR$ and $PS \parallel QR$ (alt. int. $\angle$'s equal).

$\therefore PQRS$ is a rhombus (all sides are equal in length, both pairs of opposite sides parallel).

31. $LMNO$ is a quadrilateral with $LM = LO$ and diagonals that intersect at S such that $MS = SO$. Prove that:



a) $M\hat{L}S = S\hat{L}O$

Solution:

$$\begin{aligned}
\text{In } \triangle LMS &\text{ and } \triangle LOS \\
LM &= LO \text{ (given)} \\
MS &= SO \text{ (given)} \\
LS &\text{ is a common side} \\
\therefore \triangle LMS &\equiv \triangle LOS \text{ (SSS)} \\
\therefore M\hat{L}S &= S\hat{L}O
\end{aligned}$$

b) $\triangle LON \equiv \triangle LMN$

Solution:

$$\begin{aligned}
\text{In } \triangle LON &\text{ and } \triangle LMN \\
LO &= LM \text{ (given)} \\
M\hat{L}S &= S\hat{L}O \text{ (proved above)} \\
LN &\text{ is a common side} \\
\therefore \triangle LON &\equiv \triangle LMN \text{ (SAS)}
\end{aligned}$$

c) $MO \perp LN$

Solution:

We need to show that one of $L\hat{S}M$ or $L\hat{S}O$ or $M\hat{S}N$ or $O\hat{S}N$ is equal to 90°.

We have already proved that $M\hat{L}S = O\hat{L}S$ and that $L\hat{M}S = L\hat{O}S$ (using congruent triangles).

We also note that $M\hat{L}O = M\hat{L}S + O\hat{L}S$.

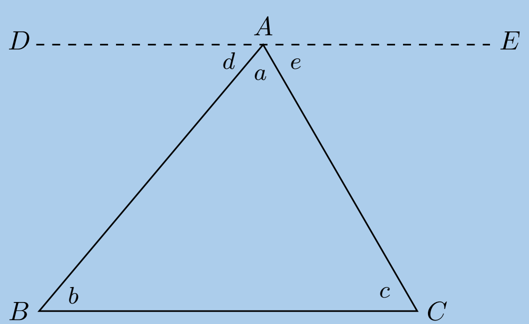
Next we note that:

$$\begin{aligned}
M\hat{L}S + O\hat{L}S + L\hat{M}S = L\hat{O}S &= 180° \text{ (sum of } \angle\text{s in } \triangle) \\
\therefore 2(M\hat{L}S) + 2(L\hat{M}S) &= 180° \\
2(M\hat{L}S + L\hat{M}S) &= 180° \\
M\hat{L}S + L\hat{M}S &= 90°
\end{aligned}$$

Now we note that:

$$\begin{aligned}
L\hat{S}O &= M\hat{L}S + L\hat{M}S \text{ (ext } \angle \text{ of } \triangle) \\
\therefore L\hat{S}O &= 90° \\
\therefore MO &\perp LN
\end{aligned}$$

32. Using the figure below, show that the sum of the three angles in a triangle is 180°. Line DE is parallel to BC.



Solution:

$DE \parallel BC$ (given).

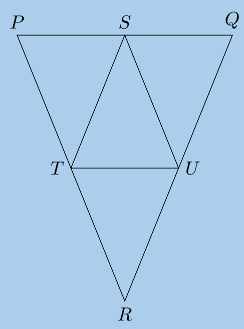
$e = c$ (alt $\angle$s; $DE \parallel BC$).

$d = b$ (alt $\angle$s; $DE \parallel BC$).

$d + a + e = 180°$ ($\angle$s on str line).

And we have shown that $e = c$ and $d = b$ therefore we can replace d with b and e with c to get:
$a + b + c = 180°$.
Therefore the angles in a triangle add up to 180°.

33. PQR is an isosceles triangle with $PR = QR$. S is the mid-point of PQ, T is the mid-point of PR and U is the mid-point of RQ.



a) Prove $\triangle STU$ is also isosceles.
Solution:

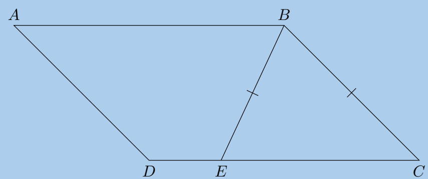
$$
\begin{aligned}
PT &= \frac{1}{2}PR \text{ (given)}\\
&S \text{ mid-point of } PQ\\
&U \text{ mid-point of } RQ\\
SU &= \frac{1}{2}PR\\
\therefore SU &= PT\\
&S \text{ mid-point of } PQ\\
&T \text{ mid-point of } PR\\
\therefore ST &= \frac{1}{2}QR = QU\\
\text{But } PR &= QR \text{ (given)}\\
\therefore SU &= ST\\
\therefore \triangle STU & \text{ is isosceles.}
\end{aligned}
$$

b) What type of quadrilateral is $STRU$? Motivate your answer.
Solution:
$STRU$ is a rhombus. It is a parallelogram since $SU \parallel TR$ and $ST \parallel UR$ (from the mid-point theorem) with four equal sides: $US = ST = TR = RU$ (given and proved above).

c) If $R\hat{T}U = 68°$ calculate, with reasons, the size of $T\hat{S}U$.
Solution:

$$
\begin{aligned}
R\hat{T}U &= 68°\\
\therefore T\hat{U}S &= 68° \qquad \text{(alt } \angle\text{s; } TR \parallel SU)\\
\therefore S\hat{T}U &= 68° \qquad (\angle\text{s opp equal sides}\\
\therefore T\hat{S}U &= 180° - 2(68)° \qquad \text{(sum of } \angle\text{s in } \triangle)\\
\therefore T\hat{S}U &= 180° - 136°\\
&= 44°
\end{aligned}
$$

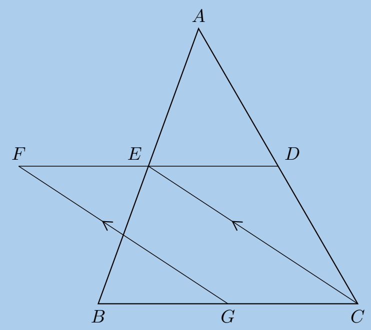
34. $ABCD$ is a parallelogram. $BE = BC$. Prove that $A\hat{B}E = B\hat{C}D$.



Solution:

$$\begin{aligned} B\hat{C}D &= B\hat{E}C \qquad (\angle\text{s opp equal sides} \\ A\hat{B}E &= B\hat{E}C \text{ (alt } \angle\text{s; } AB \parallel DC) \\ \therefore A\hat{B}E &= B\hat{C}D \end{aligned}$$

35. In the diagram below, D, E and G are the mid-points of AC, AB and BC respectively. $EC \parallel FG$.



a) Prove that $FECG$ is a parallelogram.
Solution:

$$\begin{aligned} &AE = EB \text{ (} E \text{ is mid-point)} \\ &AD = DC \text{ (} D \text{ is mid-point)} \\ &FD \parallel BC \text{ (Midpt Theorem)} \\ &EC \parallel FG \text{ (given)} \\ \therefore\ &FECG \text{ is a parallelogram (opp sides of quad are } \parallel \text{)} \end{aligned}$$

b) Prove that $FE = ED$.
Solution:

$$\begin{aligned} ED &= \frac{1}{2}BC \text{ (Midpt Theorem)} \\ GC &= \frac{1}{2}BC \text{(definition of mid-point)} \\ \therefore ED &= GC \\ FE &= GC \text{ (opp sides of } \parallel \text{m)} \\ \therefore ED &= FE \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2G87	1b. 2G88	1c. 2G89	1d. 2G8B	1e. 2G8C	1f. 2G8D
1g. 2G8F	1h. 2G8G	2a. 2G8H	2b. 2G8J	2c. 2G8K	2d. 2G8M
2e. 2G8N	2f. 2G8P	2g. 2G8Q	2h. 2G8R	2i. 2G8S	2j. 2G8T
3a. 2G8V	3b. 2G8W	3c. 2G8X	4a. 2G8Y	4b. 2G8Z	4c. 2G92
5. 2G93	6. 2G94	7. 2G95	8. 2G96	9. 2G97	10a. 2G98
10b. 2G99	10c. 2G9B	10d. 2G9C	11a. 2G9D	11b. 2G9F	11c. 2G9G
11d. 2G9H	12a. 2G9J	12b. 2G9K	12c. 2G9M	12d. 2G9N	12e. 2G9P
12f. 2G9Q	13. 2G9R	14. 2G9S	15. 2G9T	16. 2G9V	17. 2G9W
18. 2G9X	19. 2G9Y	20. 2G9Z	21. 2GB2	22. 2GB3	23. 2GB4
24. 2GB5	25. 2GB6	26. 2GB7	27. 2GB8	28. 2GB9	29. 2GBB
30. 2GBC	31. 2GBD	32. 2GBF	33. 2GBG	34. 2GBH	35. 2GBJ

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CHAPTER 8

Analytical geometry

- This chapter covers representing geometric figures on the Cartesian co-ordinate system. Also covered are the distance formula, gradient of a line and mid-point of a line.
- Distance formulae, gradient of a line and mid-point of a line should first be derived and then applied to solving problems.
- Integrate Euclidean geometry knowledge with analytical geometry. It may be helpful to have learners write down the properties of the special quadrilaterals and keep this handy while working through analytical geometry.
- Emphasise the value and importance of making sketches.
- Emphasise the importance of writing coordinates consistently for the distance formula and gradient.
- This chapter also draws strongly on the equation of a straight line. Ensure learners are comfortable working with the equation of a straight line.

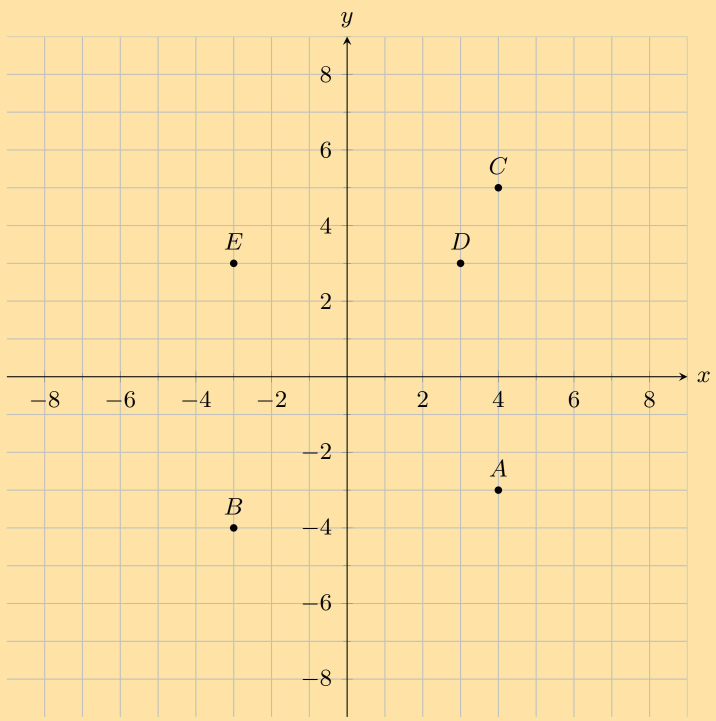
mathopenref.com has many interactive elements that you can use while teaching analytical geometry.

8.1 Drawing figures on the Cartesian plane

We use a semi-colon (;) to separate the x and y values but the internationally accepted method is to use a comma (,). If a comma is used than it becomes unclear as to whether the comma is separating the x and y values or one of the values is a decimal. For example the point $(5,5,5)$ is ambiguous. Is the x value 5,5 or is the y value 5,5?

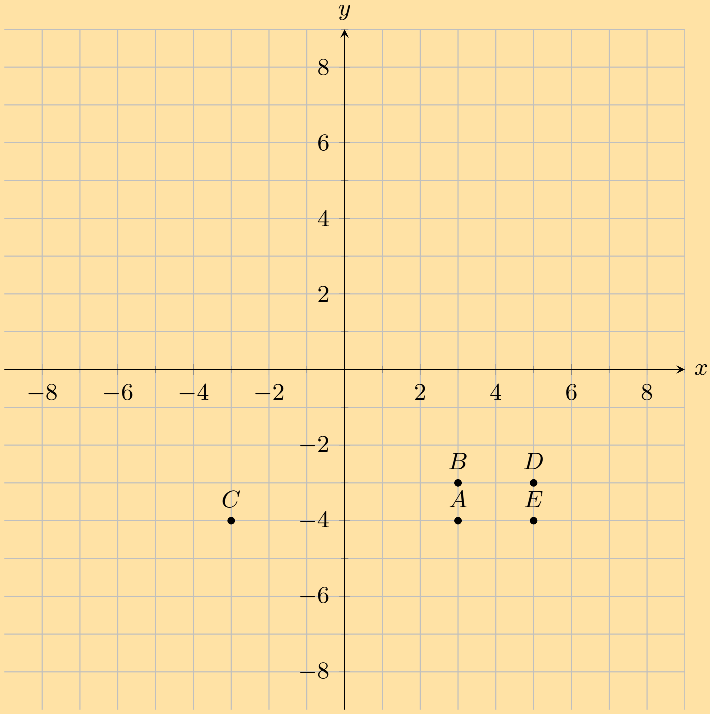
Exercise 8 – 1:

1. You are given the following diagram, with various points shown:



   Find the coordinates of point D.
   Solution:
   For this question, we are only interested in point D. From the graph we can read off the x and y values.
   Point D has the following coordinates: $(3;3)$.

2. You are given the following diagram, with various points shown:



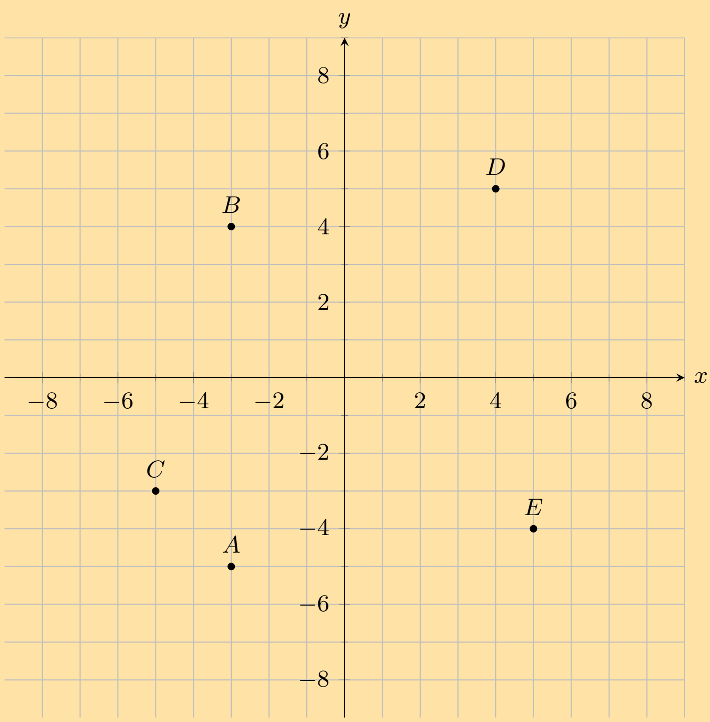
Find the coordinates of all the labelled points.

Solution:

From the graph we can read off the x and y values for each point.

$A(3;-4)$, $B(3;-3)$, $C(-3;-4)$, $D(5;-3)$ and $E(5;-4)$.

3. You are given the following diagram, with various points shown:



Which point lies at the coordinates $(5;-4)$?

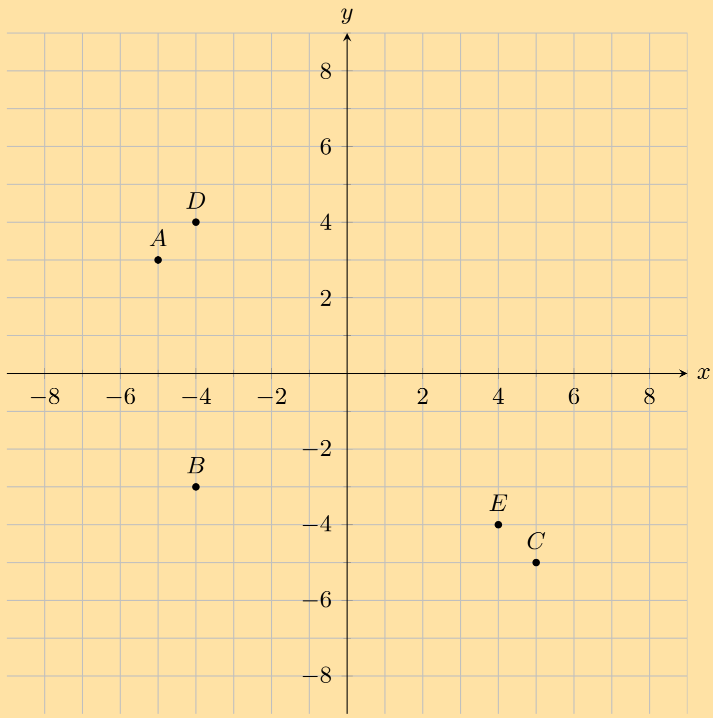
Solution:

For this question, we are must find point $(5;-4)$.

On the graph we can trace the x and y values to find which point lies at the coordinates $(5;-4)$.

Doing so we find that point E lies at the coordinates $(5; -4)$.

4. You are given the following diagram, with various points shown:



Which point lies at the coordinates $(-4; -3)$?

Solution:

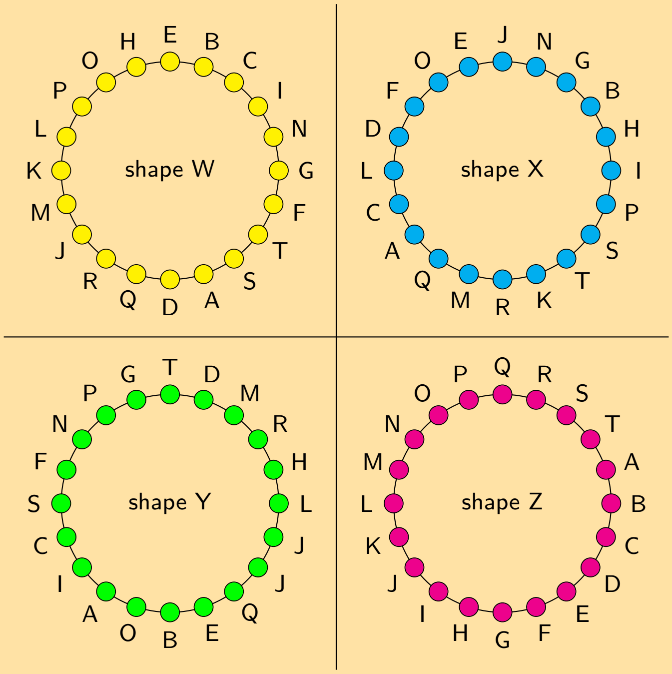
For this question, we are must find point $(-4; -3)$.

On the graph we can trace the x and y values to find which point lies at the coordinates $(-4; -3)$.

Doing so we find that point B lies at the coordinates $(-4; -3)$.

5. You are given the following diagram, with 4 shapes drawn.

All the shapes are identical, but each shape uses a different naming convention:

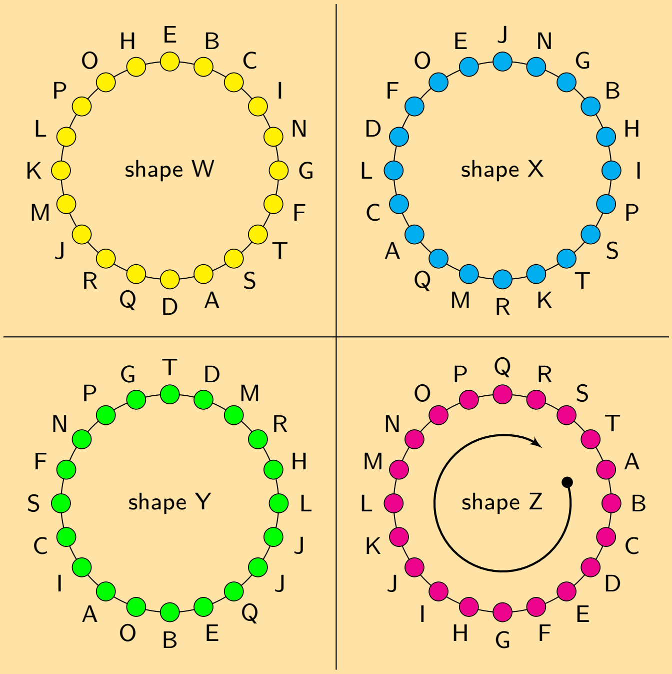


Which shape uses the correct naming convention?

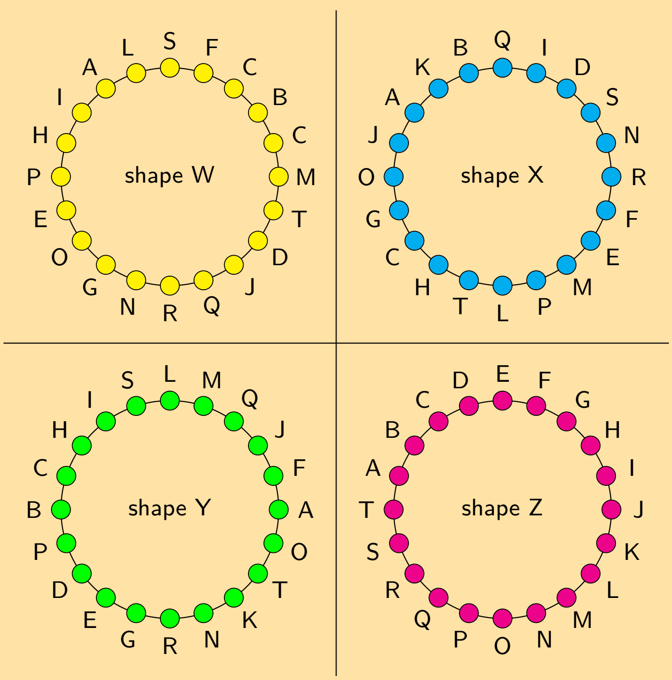
Solution:

We recall that the correct naming convention for a shape is in **alphabetical order**, either clockwise or anti-clockwise around the shape.

From the diagram, we can see that only **shape Z** sticks to this naming convention.



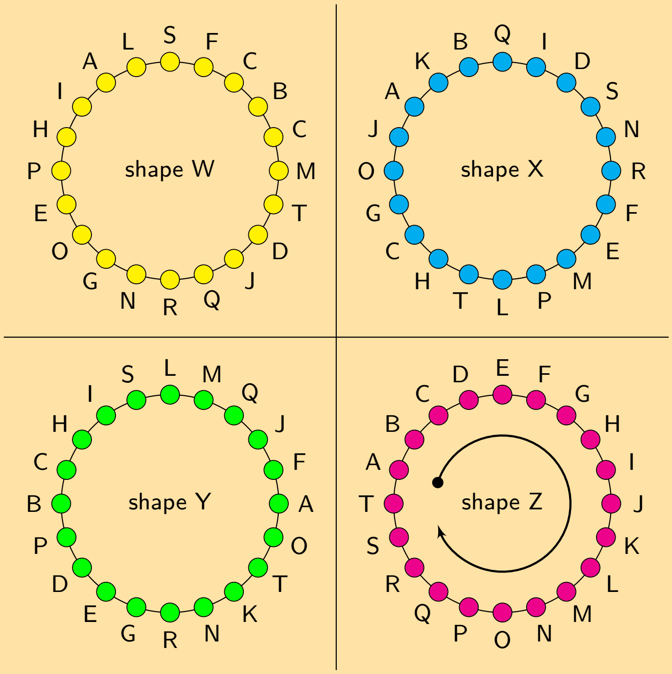
6. You are given the following diagram, with 4 shapes drawn.
   All the shapes are identical, but each shape uses a different naming convention:
   Which shape uses the correct naming convention?



Solution:

We recall that the correct naming convention for a shape is in **alphabetical order**, either clockwise or anti-clockwise around the shape.

From the diagram, we can see that only **shape Z** sticks to this naming convention.



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1. 2GBK 2. 2GBM 3. 2GBN 4. 2GBP 5. 2GBQ 6. 2GBR

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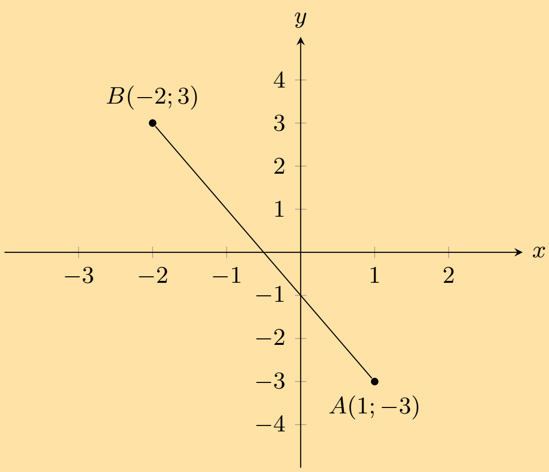
8.2 Distance between two points

Exercise 8 – 2:

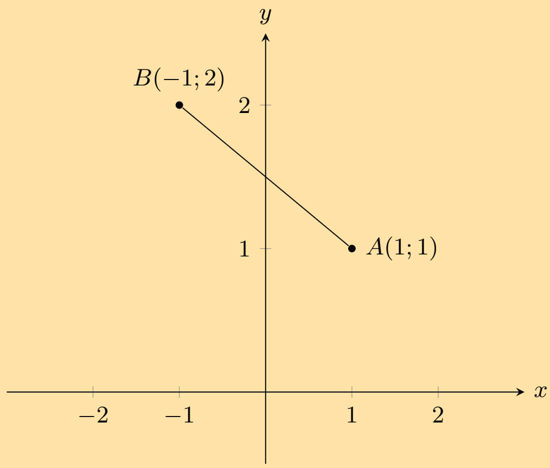
1. You are given the following diagram:



Calculate the length of line AB, correct to 2 decimal places.
Solution:
First we recall the equation for distance:

$$\begin{aligned}
d_{AB} &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\
&= \sqrt{(1 - (-2))^2 + (-3 - (3))^2} \\
&= \sqrt{(1+2)^2 + (-3 - 3)^2} \\
&= \sqrt{(3)^2 + (-6)^2} \\
&= \sqrt{9 + 36} \\
&= \sqrt{45} \\
&\approx 6{,}71
\end{aligned}$$

2. You are given the following diagram:



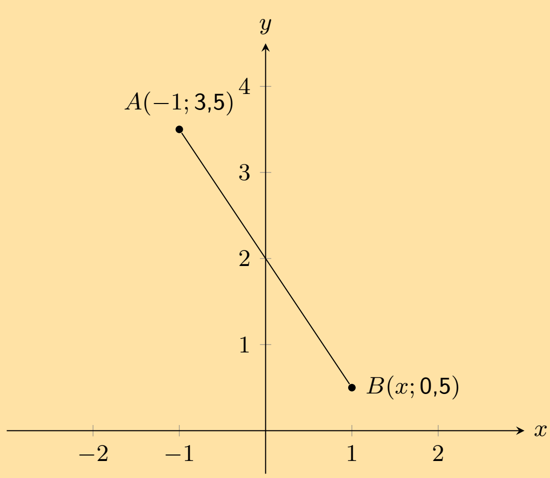
Calculate the length of line AB, correct to 2 decimal places.

Solution:

First we recall the equation for distance:

$$\begin{aligned}
d_{AB} &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\
&= \sqrt{(1 - (-1))^2 + (1 - (2))^2} \\
&= \sqrt{(1+1)^2 + (1 - 2)^2} \\
&= \sqrt{(2)^2 + (-1)^2} \\
&= \sqrt{4 + 1} \\
&= \sqrt{5} \\
&\approx 2{,}24
\end{aligned}$$

3. The following picture shows two points on the Cartesian plane, A and B.



The distance between the points is 3,6056. Calculate the missing coordinate of point B.
Solution:
First we recall the equation for distance:

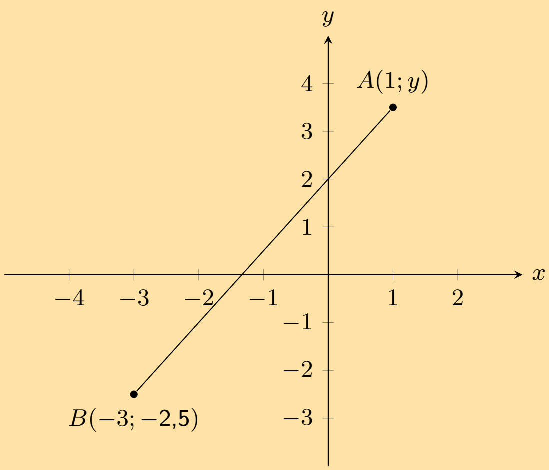
$$\begin{aligned} d_{AB} &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ 3{,}6056 &= \sqrt{(x - (-1))^2 + (0{,}5 - (3{,}5))^2} \\ &= \sqrt{(x+1)^2 + (0{,}5 - 3{,}5)^2} \end{aligned}$$

Now we re-arrange, and solve for the value of x:

$$\begin{aligned} (3{,}6056)^2 &= (x+1)^2 + (0{,}5 - 3{,}5)^2 \\ 13 &= (x+1)^2 + (0{,}5 - 3{,}5)^2 \\ 13 &= (x+1)^2 + 9 \\ (x+1)^2 &= 4 \\ x + 1 &= \pm\sqrt{4} \\ x &= \pm 2 - 1 \\ x &= 1 \text{ or } -3 \end{aligned}$$

We now have a choice between 2 values for x. From the diagram we can see that the appropriate value for this question is $x = 1$.
Note that in this case we can use the diagram to check that our answer is valid but we can also calculate the distance of line AB using our answer.

4. The following picture shows two points on the Cartesian plane, A and B.



The line AB has a length of 7,2111. Calculate the missing coordinate of point B. Round your answer to one decimal place.
Solution:
First we recall the equation for distance:

$$\begin{aligned} d_{AB} &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ 7{,}2111 &= \sqrt{(1 - (-3))^2 + (y - (-2{,}5))^2} \\ 7{,}2111 &= \sqrt{(1+3)^2 + (y + 2{,}5)^2} \end{aligned}$$

Now we re-arrange, and solve for the value of y:

$$\begin{aligned}(7{,}2111)^2 &= (1+3)^2 + (y+2{,}5)^2\\ 52 &= (1+3)^2 + (y+2{,}5)^2\\ 52 &= (y+2{,}5)^2 + 16\\ (y+2{,}5)^2 &= 36\\ y+2{,}5 &= \pm\sqrt{36}\\ y &= \pm 6 - 2{,}5\\ y &= 3{,}5 \text{ or } -8{,}5\end{aligned}$$

We now have a choice between 2 values for y. From the diagram we can see that the appropriate value for this question is $y = 3{,}5$.

Note that in this case we can use the diagram to check that our answer is valid but we can also calculate the distance of line AB using our answer.

5. Find the length of AB for each of the following. Leave your answer in surd form.

   a) $A(2;7)$ and $B(-3;5)$

   Solution:

$$\begin{aligned}d_{AB} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\ &= \sqrt{(2-(-3))^2+(7-5)^2}\\ &= \sqrt{(5)^2+(2)^2}\\ &= \sqrt{29}\end{aligned}$$

   b) $A(-3;5)$ and $B(-9;1)$

   Solution:

$$\begin{aligned}d_{AB} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\ &= \sqrt{(-3-(-9))^2+(5-1)^2}\\ &= \sqrt{(6)^2+(4)^2}\\ &= \sqrt{52}\end{aligned}$$

   c) $A(x;y)$ and $B(x+4;y-1)$

   Solution:

$$\begin{aligned}d_{AB} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\ &= \sqrt{(x-(x+4))^2+(y-(y-1))^2}\\ &= \sqrt{(x-x-4)^2+(y-y+1)^2}\\ &= \sqrt{(-4)^2+(1)^2}\\ &= \sqrt{17}\end{aligned}$$

6. The length of $CD = 5$. Find the missing coordinate if:

   a) $C(6;-2)$ and $D(x;2)$.

   Solution:

$$\begin{aligned}d_{CD} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\ 5 &= \sqrt{(6-x)^2+(-2-2)^2}\\ 5^2 &= 36-12x+x^2+16\\ 0 &= x^2-12x+36-25+16\\ &= x^2-12x+27\\ &= (x-3)(x-9)\end{aligned}$$

Therefore $x = 3$ or $x = 9$.
Check solution for $x = 3$:

$$\begin{aligned} d_{CD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(6 - 3)^2 + (-2 - 2)^2} \\ &= \sqrt{(3)^2 + (-4)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Solution is valid.
Check solution for $x = 9$:

$$\begin{aligned} d_{CD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(6 - 9)^2 + (-2 - 2)^2} \\ &= \sqrt{(-3)^2 + (-4)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Solution is valid.

b) $C(4; y)$ and $D(1; -1)$.
Solution:

$$\begin{aligned} d_{CD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ 5 &= \sqrt{(4 - 1)^2 + (y + 1)^2} \\ 5^2 &= 9 + y^2 + 2y + 1 \\ 0 &= y^2 + 2y + 1 + 9 - 25 \\ &= y^2 + 2y - 15 \\ &= (y - 3)(y + 5) \end{aligned}$$

Therefore $y = 3$ or $y = -5$.
Check solution for $y = 3$:

$$\begin{aligned} d_{CD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(4 - 1)^2 + (3 + 1)^2} \\ &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Solution is valid.
Check solution for $y = -5$:

$$\begin{aligned} d_{CD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(4 - 1)^2 + (-5 + 1)^2} \\ &= \sqrt{(3)^2 + (-4)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Solution is valid.

7. If the distance between $C(0; -3)$ and $F(8; p)$ is 10 units, find the possible values of p.
Solution:

$$
\begin{aligned}
10 &= \sqrt{(8-0)^2+(p+3)^2} \\
&= \sqrt{8^2+(p+3)^2} \\
100 &= 8^2+(p+3)^2 \\
36 &= p^2+6p+9 \\
0 &= p^2+6p-27 \\
&= (p-3)(p+9) \\
\therefore p &= 3 \text{ or } p=-9
\end{aligned}
$$

Check $p=3$:

$$
\begin{aligned}
d &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\
&= \sqrt{(8-0)^2+(3+3)^2} \\
&= \sqrt{64+36} \\
&= \sqrt{100} \\
&= 10
\end{aligned}
$$

Solution is valid.
Check $p=-9$:

$$
\begin{aligned}
d &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\
&= \sqrt{(8-0)^2+(-9+3)^2} \\
&= \sqrt{64+36} \\
&= \sqrt{100} \\
&= 10
\end{aligned}
$$

Solution is valid.
Therefore $p=3$ or $p=-9$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GBT 2. 2GBV 3. 2GBW 4. 2GBX 5a. 2GBY 5b. 2GBZ
5c. 2GC2 6a. 2GC3 6b. 2GC4 7. 2GC5

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8.3 Gradient of a line

Exercise 8 – 3:

1. Find the gradient of AB if:

   a) $A(7;10)$ and $B(-4;1)$
   Solution:
   Let the coordinates of A be $(x_1;y_1)$ and the coordinates of B be $(x_2;y_2)$

$$x_1 = 7 \quad y_1 = 10 \quad x_2 = -4 \quad y_2 = 1$$

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 10}{-4 - 7} \\ &= \frac{9}{11} \end{aligned}$$

b) $A(-5; -9)$ and $B(3; 2)$

Solution:

Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$

$$x_1 = -5 \quad y_1 = -9 \quad x_2 = 3 \quad y_2 = 2$$

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - (-9)}{3 - (-5)} \\ &= \frac{11}{8} \end{aligned}$$

c) $A(x - 3; y)$ and $B(x; y + 4)$

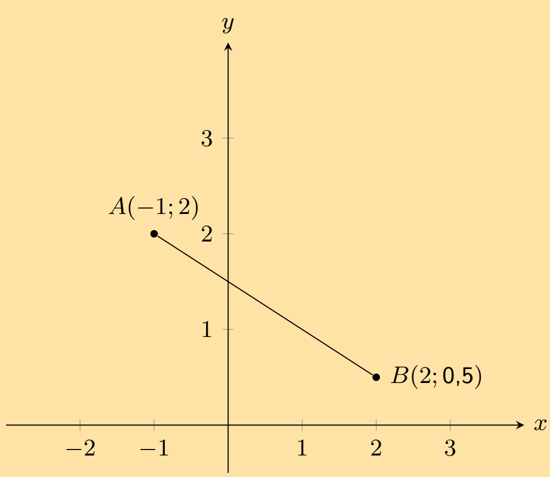
Solution:

Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$

$$x_1 = x - 3 \quad y_1 = y \quad x_2 = x \quad y_2 = y + 4$$

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{y + 4 - y}{x - (x - 3)} \\ &= \frac{4}{3} \end{aligned}$$

2. You are given the following diagram:



Calculate the gradient (m) of line AB.

Solution:

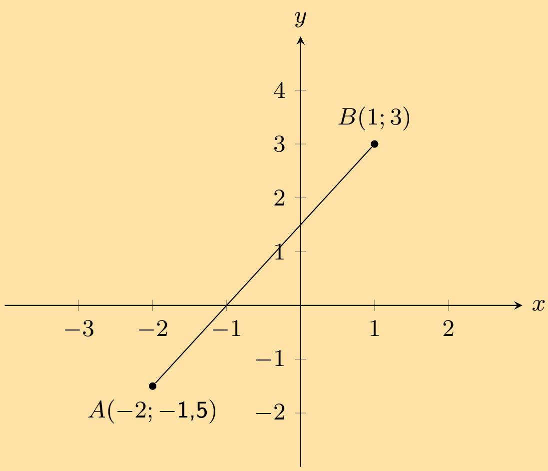
Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$

$$x_1 = -1 \quad y_1 = 2 \quad x_2 = 2 \quad y_2 = 0{,}5$$

$$
\begin{aligned}
m &= \frac{y_B - y_A}{x_B - x_A} \\
&= \frac{(0{,}5) - (2)}{(2) - (-1)} \\
&= \frac{-1{,}5}{3} \\
&= -0{,}5
\end{aligned}
$$

Therefore the gradient m of the line AB is $-0{,}5$.

3. You are given the following diagram:



Calculate the gradient (m) of line AB.

Solution:

Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$

$$
x_1 = -2 \quad y_1 = -1{,}5 \quad x_2 = 1 \quad y_2 = 3
$$

$$
\begin{aligned}
m &= \frac{y_B - y_A}{x_B - x_A} \\
&= \frac{(3) - (-1{,}5)}{(1) - (-2)} \\
&= \frac{4{,}5}{3} \\
&= 1{,}5
\end{aligned}
$$

Therefore the gradient m of the line AB is $1{,}5$.

4. If the gradient of $CD = \frac{2}{3}$, find p given:

a) $C(16; 2)$ and $D(8; p)$.

Solution:

Let the coordinates of C be $(x_1; y_1)$ and the coordinates of D be $(x_2; y_2)$

$$
x_1 = 16 \quad y_1 = 2 \quad x_2 = 8 \quad y_2 = p
$$

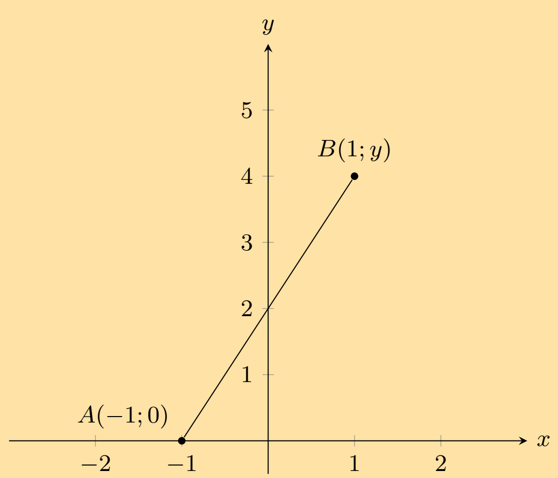
$$\begin{aligned}
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\frac{2}{3} &= \frac{p - 2}{8 - 16} \\
\frac{2}{3} \times (-8) &= p - 2 \\
\frac{-16}{3} + 2 &= p \\
\frac{-16 + 6}{3} &= p \\
\frac{-10}{3} &= p
\end{aligned}$$

b) $C(3; 2p)$ and $D(9; 14)$.
Solution:
Let the coordinates of C be $(x_1; y_1)$ and the coordinates of D be $(x_2; y_2)$

$$x_1 = 3 \quad y_1 = 2p \quad x_2 = 9 \quad y_2 = 14$$

$$\begin{aligned}
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\frac{2}{3} &= \frac{14 - 2p}{9 - 3} \\
\frac{2}{3} \times (6) &= 14 - 2p \\
4 &= 14 - 2p \\
2p &= 14 - 4 \\
p &= \frac{10}{2} \\
&= 5
\end{aligned}$$

5. You are given the following diagram:

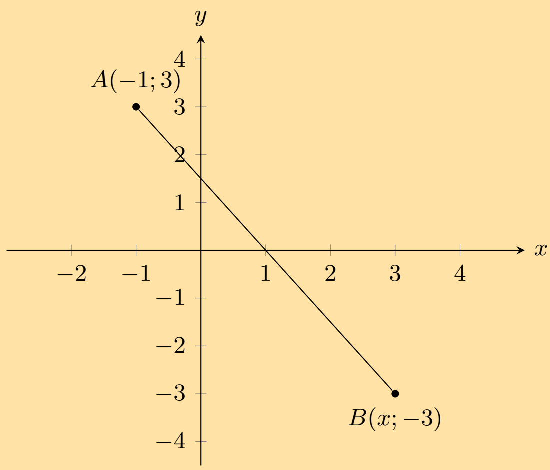


You are also told that line AB has a gradient (m) of 2.
Calculate the missing co-ordinate of point B.
Solution:
Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$

$$x_1 = -1 \quad y_1 = 0 \quad x_2 = 1 \quad y_2 = y$$

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ 2 &= \frac{y - 0}{1 - (-1)} \\ 2 &= \frac{y}{2} \\ 4 &= y \end{aligned}$$

6. You are given the following diagram:



You are also told that line AB has a gradient (m) of $-1{,}5$.
Calculate the missing co-ordinate of point B.
Solution:
Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$

$$x_1 = -1 \quad y_1 = 3 \quad x_2 = x \quad y_2 = -3$$

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ -1{,}5 &= \frac{-3 - 3}{x - (-1)} \\ -1{,}5 &= \frac{-6}{x + 1} \\ -1{,}5(x + 1) &= -6 \\ -1{,}5x - 1{,}5 &= -6 \\ 1{,}5x &= 4{,}5 \\ x &= 3 \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1a. 2GC7 1b. 2GC8 1c. 2GC9 2. 2GCB 3. 2GCC 4a. 2GCD
4b. 2GCF 5. 2GCG 6. 2GCH

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Straight lines

The worked example below shows one method of determining the equation of a straight line. The equation of a straight line can also be determined by first finding the gradient and then substituting one of the given points into $y = mx + c$.

Parallel and perpendicular lines

Horizontal and vertical lines

Points on a line

Exercise 8 – 4:

1. Determine whether AB and CD are parallel, perpendicular or neither if:

a) $A(3;-4)$, $B(5;2)$, $C(-1;-1)$, $D(7;23)$

Solution:

We need to calculate the gradients of lines AB and CD. Then we can compare the gradients to determine if the lines are parallel (the gradients are equal), perpendicular (the gradients are the negative inverses of each other) or neither.

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-(-4)}{5-3} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

And:

$$\begin{aligned} m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{23-(-1)}{7-(-1)} \\ &= \frac{24}{8} \\ &= 3 \end{aligned}$$

Therefore:

$$\begin{aligned} m_{AB} &= m_{CD} \\ \therefore AB &\parallel CD \end{aligned}$$

Lines AB and CD are parallel.

b) $A(3;-4)$, $B(5;2)$, $C(-1;-1)$, $D(0;-4)$

Solution:

We need to calculate the gradients of lines AB and CD. Then we can compare the gradients to determine if the lines are parallel (the gradients are equal), perpendicular (the gradients are the negative inverses of each other) or neither.

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-(-4)}{5-3} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

And:

$$
\begin{aligned}
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-4 - (-1)}{0 - (-1)} \\
&= \frac{-3}{1} \\
&= -3
\end{aligned}
$$

So $m_{AB} \neq m_{CD}$. Therefore AB is not parallel to CD.
And $m_{AB} \times \frac{1}{m_{CD}} \neq -1$. Therefore AB and CD are not perpendicular.

Lines AB and CD are neither parallel nor perpendicular.

c) $A(3;-4)$, $B(5;2)$, $C(-1;3)$, $D(-2;2)$

Solution:

We need to calculate the gradients of lines AB and CD. Then we can compare the gradients to determine if the lines are parallel (the gradients are equal), perpendicular (the gradients are the negative inverses of each other) or neither.

$$
\begin{aligned}
m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - (-4)}{5 - 3} \\
&= \frac{6}{2} \\
&= 3
\end{aligned}
$$

And:

$$
\begin{aligned}
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - 3}{1 - (-2)} \\
&= \frac{-1}{3}
\end{aligned}
$$

So $m_{AB} \neq m_{CD}$. Therefore AB is not parallel to CD.
And $m_{AB} \times \frac{1}{m_{CD}} = -1$. Therefore AB and CD are perpendicular.

2. Determine whether the following points lie on the same straight line:

a) $E(0;3)$, $F(-2;5)$, $G(2;1)$

Solution:

$$
\begin{aligned}
m_{EF} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5 - 3}{-2 - 0} \\
&= \frac{2}{-2} \\
&= -1
\end{aligned}
$$

And,

$$
\begin{aligned}
m_{FG} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1 - 5}{2 - (-2)} \\
&= \frac{-4}{4} \\
&= -1
\end{aligned}
$$

So $m_{EF} = m_{FG}$ and F is a common point. Therefore E, F and G are collinear (they lie on the same line).

b) $H(-3;-5)$, $I(0;0)$, $J(6;10)$

Solution:

$$\begin{aligned} m_{HI} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - (-5)}{0 - (-3)} \\ &= \frac{5}{3} \end{aligned}$$

And,

$$\begin{aligned} m_{IJ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{10 - 0}{6 - 0} \\ &= \frac{10}{6} \\ &= \frac{5}{3} \end{aligned}$$

So $m_{HI} = m_{IJ}$ and I is a common point. Therefore H, I and J are collinear (they lie on the same line).

c) $K(-6;2)$, $L(-3;1)$, $M(1;-1)$

Solution:

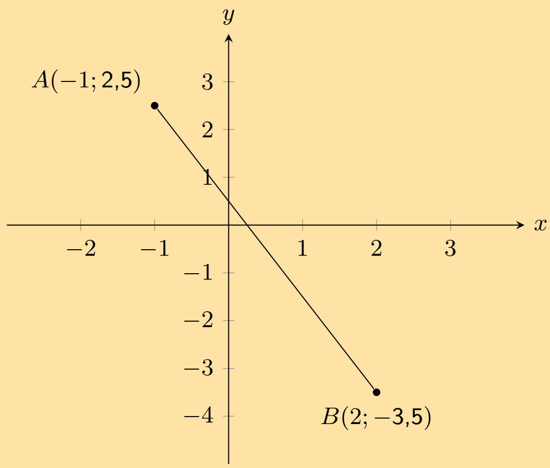
$$\begin{aligned} m_{KL} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 2)}{-3 - (-6)} \\ &= -\frac{1}{3} \end{aligned}$$

And,

$$\begin{aligned} m_{LM} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 1}{1 - (-3)} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2} \end{aligned}$$

So $m_{KL} \neq m_{LM}$. Therefore K, L and M are not collinear (they do not lie on the same line).

3. You are given the following diagram:



Calculate the equation of the line AB.

Solution:

To calculate the equation of the straight line, we first calculate the gradient (m) of the line AB:

$$m = \frac{y_B - y_A}{x_B - x_A}$$

$$m = \frac{(-3{,}5) - (2{,}5)}{(2) - (-1)}$$

$$m = -2$$

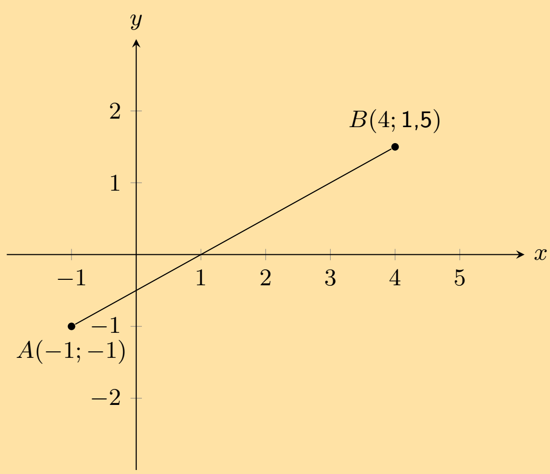
Secondly, we calculate the value of the y-intercept (c) of the line AB. We do this by substituting either point A or B into the general form for a straight line. We will use point A.

$$\begin{aligned} y &= mx + c \\ (2{,}5) &= (-2) \times (-1) + c \\ c &= 0{,}5 \end{aligned}$$

Therefore, the equation of the line AB is as follows:

$$y = -2x + 0{,}5$$

4. You are given the following diagram:



Calculate the equation of the line AB.

Solution:

To calculate the equation of the straight line, we first calculate the gradient (m) of the line AB:

$$m = \frac{y_B - y_A}{x_B - x_A}$$

$$m = \frac{(1{,}5) - (-1)}{(4) - (-1)}$$

$$m = 0{,}5$$

Secondly, we calculate the value of the y-intercept (c) of the line AB. We do this by substituting either point A or B into the general form for a straight line. We will use point A.

$$\begin{aligned} y &= mx + c \\ (-1) &= (0{,}5) \times (-1) + c \\ c &= -0{,}5 \end{aligned}$$

Therefore, the equation of the line AB is as follows:

$y = 0{,}5x - 0{,}5$

5. Points $P(-6;2)$, $Q(2;-2)$ and $R(-3;r)$ lie on a straight line. Find the value of r.

   Solution:

   Since the three points lie on a straight line we can use the fact that the gradient of PQ is equal to the gradient of QR to find r.

   The gradient of PQ is:

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-2}{2-(-6)} \\ &= \frac{-4}{8} \\ &= -\frac{1}{2} \end{aligned}$$

   And the gradient of QR in terms of r is:

$$\begin{aligned} m_{QR} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{r-(-2)}{-3-2} \\ &= \frac{r+2}{-5} \end{aligned}$$

   Now we let $m_{PR} = m_{QR} = -\frac{1}{2}$ and solve for r:

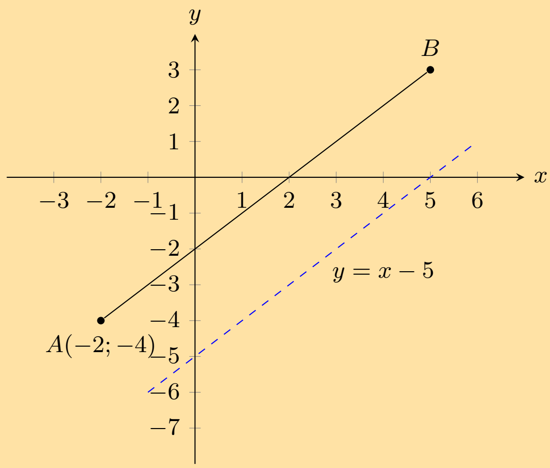
$$\begin{aligned} -\frac{1}{2} &= \frac{r+2}{-5} \\ (-1) \times (-5) &= 2(r+2) \\ 5 &= 2r + 4 \\ 5 - 4 &= 2r \\ 1 &= 2r \\ r &= \frac{1}{2} \end{aligned}$$

6. Line PQ with $P(-1;-7)$ and $Q(q;0)$ has a gradient of 1. Find q.

   Solution:

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ 1 &= \frac{0-(-7)}{q-(-1)} \\ q + 1 &= 7 \\ q &= 6 \end{aligned}$$

7. You are given the following diagram:



You are also told that line AB runs parallel to the following line: $y = x - 5$. Point A is at $(-2; -4)$. Find the equation of the line AB.

Solution:

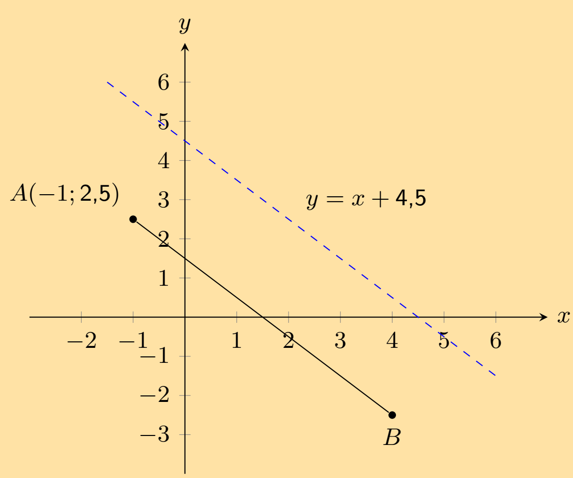
We are told that line AB is parallel to $y = x - 5$, so the gradient of line AB is equal to the gradient of $y = x - 5$. The gradient of $y = x - 5$ is 1.

Now we can use point A and the gradient of the line to find the y-intercept of the line:

$$\begin{aligned} y &= mx + c \\ (-4) &= (1)(-2) + c \\ c &= -2 \end{aligned}$$

Therefore, the equation of the line AB is: $y = x - 2$.

8. You are given the following diagram:



You are also told that line AB runs parallel to the following line: $y = -x + 4{,}5$. Point A is at $(-1; 2{,}5)$. Find the equation of the line AB.

Solution:

We are told that line AB is parallel to $y = -x + 4{,}5$, so the gradient of line AB is equal to the gradient of $y = -x + 4{,}5$. The gradient of $y = -x + 4{,}5$ is -1.

Now we can use point A and the gradient of the line to find the y-intercept of the line:

$$\begin{aligned} y &= mx + c \\ (2{,}5) &= (-1) \times (-1) + c \\ c &= 1{,}5 \end{aligned}$$

Therefore, the equation of the line AB is: $y = -x + 1{,}5$.

9. Given line AB which runs parallel to $y = 0{,}5x - 6$. Points $A(-1; -2{,}5)$ and $B(x; 0)$ are also given. Calculate the missing co-ordinate of point B.

Solution:

We are told that line AB is parallel to $y = 0{,}5x - 6$, so the gradient of line AB is equal to the gradient of $y = 0{,}5x - 6$. The gradient of $y = 0{,}5x - 6$ is 0,5.

Now we can use point A and the gradient of the line to find the y-intercept of the line:

$$\begin{aligned} y &= mx + c \\ (-2{,}5) &= (0{,}5)(-1) + c \\ c &= -2 \end{aligned}$$

Therefore, the equation of the line AB is: $y = 0{,}5x - 2$.

Now we can substitute point B into the equation of line AB to solve for x:

$$\begin{aligned} y &= 0{,}5x - 2 \\ 0 &= 0{,}5x - 2 \\ 4 &= x \end{aligned}$$

Therefore the coordinates of point B are $(0; 4)$.

10. Given line AB which runs parallel to $y = -1{,}5x + 4$. Points $A(-2; 4)$ and $B(2; y)$ are also given. Calculate the missing co-ordinate of point $B(2; y)$.

Solution:

We are told that line AB is parallel to $y = -1{,}5x + 4$, so the gradient of line AB is equal to the gradient of $y = -1{,}5x + 4$. The gradient of $y = -1{,}5x + 4$ is $-1{,}5$.

Now we can use point A and the gradient of the line to find the y-intercept of the line:

$$\begin{aligned} y &= mx + c \\ (4) &= (-1{,}5)(-2) + c \\ c &= 1 \end{aligned}$$

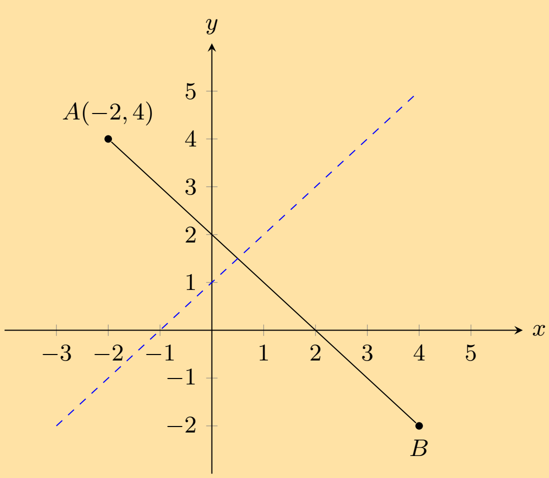
Therefore, the equation of the line AB is: $y = -1{,}5x + 1$.

Now we can substitute point B into the equation of line AB to solve for x:

$$\begin{aligned} y &= -1{,}5x + 1 \\ y &= -1{,}5(2) + 1 \\ y &= -2 \end{aligned}$$

Therefore the coordinates of point B are $(2; -2)$.

11. The graph here shows line AB. The blue dashed line is perpendicular to AB.



The equation of the blue dashed line is $y = x + 1$. Point A is at $(-2; 4)$. Determine the equation of line AB.

Solution:

The general form of a straight line is: $y = mx + c$.

Line AB is perpendicular to the blue dashed line and so $m_{AB} = \frac{-1}{m_{\text{blue line}}}$.

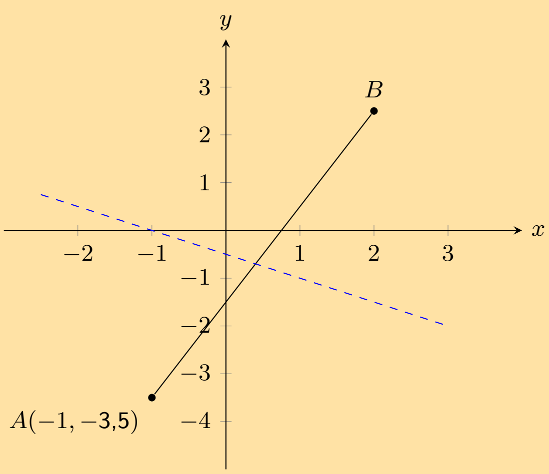
$$\begin{aligned} y &= mx + c \\ y &= \left(\frac{-1}{m_{\text{blue line}}}\right) x + c \\ y &= \left(\frac{-1}{1}\right) x + c \\ y &= -x + c \end{aligned}$$

Now we can substitute in the coordinates of point A to find the y-intercept:

$$\begin{aligned} y &= mx + c \\ (4) &= (-1)(-2) + c \\ c &= 2 \end{aligned}$$

Therefore, the equation of the line AB is: $y = -x + 2$.

12. The graph here shows line AB. The blue dashed line is perpendicular to AB.



The equation of the blue dashed line is $y = -0{,}5x - 0{,}5$. Point A is at $(-1; -3{,}5)$. Determine the equation of line AB.

Solution:

The general form of a straight line is: $y = mx + c$.

Line AB is perpendicular to the blue dashed line and so $m_{AB} = \frac{-1}{m_{\text{blue line}}}$.

$$\begin{aligned} y &= mx + c \\ y &= \left(\frac{-1}{m_{\text{blue line}}}\right) x + c \\ y &= \left(\frac{-1}{-0{,}5}\right) x + c \\ y &= 2x + c \end{aligned}$$

Now we can substitute in the coordinates of point A to find the y-intercept:

$$\begin{aligned} y &= mx + c \\ (-3{,}5) &= (2)(-1) + c \\ c &= -1{,}5 \end{aligned}$$

Therefore, the equation of the line AB is: $y = 2x - 1{,}5$.

13. Given line AB which runs perpendicular to line CD with equation $y = -2x + 1$. Points $A(-5; -1)$ and $B(3; a)$ are also given.
    Calculate the missing co-ordinate of point B.
    Solution:
    The general form of a straight line is: $y = mx + c$.
    Line AB is perpendicular to line CD and so $m_{AB} = \frac{-1}{m_{CD}}$.

$$\begin{aligned} y &= mx + c \\ y &= \left(\frac{-1}{m_{CD}}\right) x + c \\ y &= \left(\frac{-1}{-2}\right) x + c \\ y &= 0{,}5x + c \end{aligned}$$

    Now we can substitute point A into the equation to find the y-intercept:

$$\begin{aligned} y &= 0{,}5x + c \\ -1 &= (0{,}5)(-5) + c \\ c &= 1{,}5 \end{aligned}$$

    Next we can substitute in point B to find the missing coordinate:

$$\begin{aligned} y &= 0{,}5x + 1{,}5 \\ a &= (0{,}5)(3) + 1{,}5 \\ &= 3 \end{aligned}$$

    Therefore the missing coordinate is $B(3; 3)$.

14. Given line AB which runs perpendicular to line CD with equation $y = 2x - 0{,}75$. Points $A(-5; 1)$ and $B(a; -2{,}5)$ are also given.
    Calculate the missing co-ordinate of point B.
    Solution:
    The general form of a straight line is: $y = mx + c$.
    Line AB is perpendicular to line CD and so $m_{AB} = \frac{-1}{m_{CD}}$.

$$\begin{aligned} y &= mx + c \\ y &= \left(\frac{-1}{m_{CD}}\right) x + c \\ y &= \left(\frac{-1}{2}\right) x + c \\ y &= -0{,}5x + c \end{aligned}$$

    Now we can substitute point A into the equation to find the y-intercept:

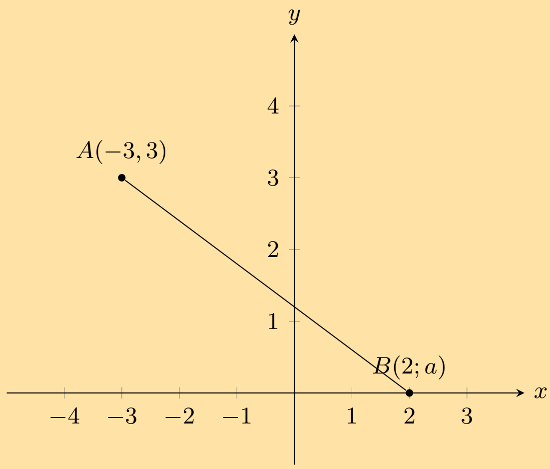
$$\begin{aligned} y &= -0{,}5x + c \\ 1 &= (0{,}5)(-5) + c \\ c &= -1{,}5 \end{aligned}$$

Next we can substitute in point B to find the missing coordinate:

$$\begin{aligned} y &= 0{,}5x - 1{,}5 \\ -2{,}5 &= 0{,}5a - 1{,}5 \\ a &= -2 \end{aligned}$$

Therefore the missing coordinate is $B(-2; -2{,}5)$.

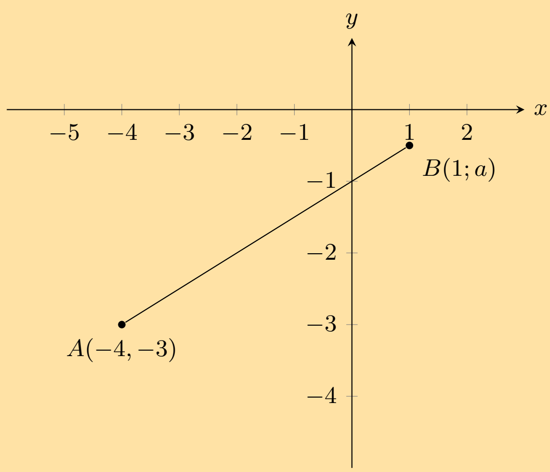
15. You are given the following diagram:



You are also told that line AB has the following equation: $y = -0{,}5x + 1{,}5$.
Calculate the missing co-ordinate of point B.
Solution:
We substitute the known value for point B into the equation and solve for the unknown value:

$$\begin{aligned} y &= (-0{,}5)x + 1{,}5 \\ a &= (-0{,}5)(2) + 1{,}5 \\ &= 0{,}5 \end{aligned}$$

16. You are given the following diagram:



You are also told that line AB has the following equation: $y = 0{,}5x - 1$.
Calculate the missing coordinate of point B.
Solution:
We substitute the known value for point B into the equation and solve for the unknown value:

$$\begin{aligned} y &= (0{,}5)x - 1 \\ a &= (0{,}5)(1) - 1 \\ &= -0{,}5 \end{aligned}$$

17. A is the point $(-3;-5)$ and B is the point $(n;-11)$. AB is perpendicular to line CD with equation $y = \frac{3}{2}x - 5$. Find the value of n.

Solution:

Line AB is perpendicular to line CD and so $m_{AB} = \frac{-1}{m_{CD}}$.

$$\begin{aligned} m_{AB} &= -1 \div \frac{3}{2} \\ &= \frac{-2}{3} \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{-2}{3} &= \frac{-11-(-5)}{n-(-3)} \\ \frac{-2}{3} &= \frac{-6}{n+3} \\ \frac{-2}{3}(n+3) &= -6 \\ -2(n+3) &= -18 \\ n+3 &= 9 \\ n &= 6 \end{aligned}$$

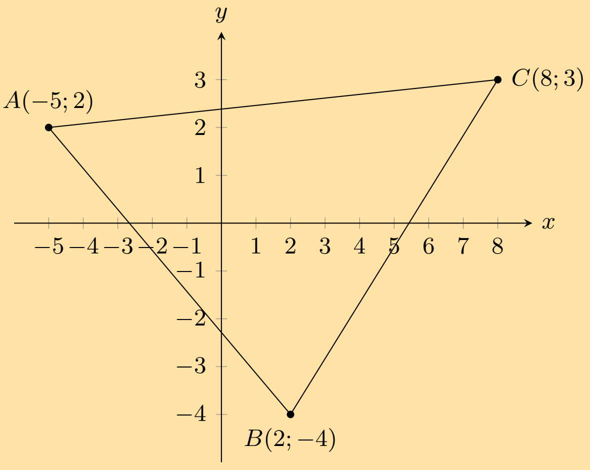
18. The points $A(4;-3)$, $B(-5;0)$ and $C(-3;p)$ are given. Determine the value of p if A, B and C are collinear.

Solution:

We are told that A, B and C are collinear, therefore $m_{AB} = m_{BC}$.

$$\begin{aligned} \frac{0+3}{-5-4} &= \frac{p}{-3+5} \\ \frac{3}{-9} &= \frac{p}{2} \\ \therefore p &= \frac{6}{-9} \\ &= \frac{-2}{3} \end{aligned}$$

19. Refer to the diagram below:



a) Show that $\triangle ABC$ is right-angled. Show your working.

Solution:

To show that $\triangle ABC$ is right angled we need to show that $AB \perp AC$ or that $AC \perp BC$ or $AB \perp BC$. We can do this by calculating the gradients of AB, AC and BC and then seeing if any of these gradients is the negative inverse of any of the other two gradients.

$$
\begin{aligned}
m_{AB} &= \frac{-4-2}{2-(-5)} \\
&= \frac{-6}{7}
\end{aligned}
$$

$$
\begin{aligned}
m_{BC} &= \frac{-4-3}{2-8} \\
&= \frac{-7}{-6} \\
&= \frac{7}{6}
\end{aligned}
$$

$$
\begin{aligned}
m_{AC} &= \frac{3-2}{8-(-5)} \\
&= \frac{1}{13}
\end{aligned}
$$

Now we note that:

$$
\begin{aligned}
m_{AB} \times m_{BC} &= \frac{-6}{7} \times \frac{7}{6} = -1 \\
\therefore AB &\perp BC \\
\therefore \triangle ABC &\text{ is right-angled}
\end{aligned}
$$

b) Find the area of $\triangle ABC$.

Solution:

This is a right-angled triangle (with right-angle $A\hat{B}C$) and so the perpendicular height is the length of one of the sides. We will use side BC as the perpendicular height and side AB as the base.

Note that we cannot use side AC for either the base or the height as we would then need to construct a new perpendicular height from A to B and calculate the length of that line.

The length of BC is:

$$
\begin{aligned}
d &= \sqrt{(-5-2)^2+(2+4)^2} \\
&= \sqrt{49+36} \\
&= \sqrt{85}
\end{aligned}
$$

The length of AB is:

$$
\begin{aligned}
d &= \sqrt{(8-2)^2+(3+4)^2} \\
&= \sqrt{36+49} \\
&= \sqrt{85}
\end{aligned}
$$

Therefore the area of $\triangle ABC$ is:

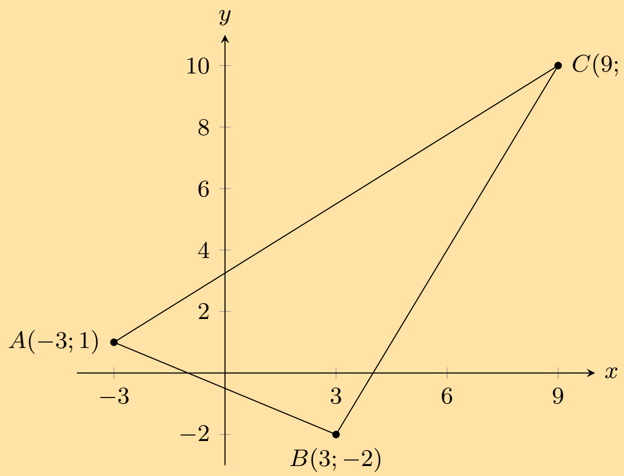
$$
\begin{aligned}
A &= \frac{1}{2}bh \\
&= \frac{1}{2}(AB)(BC) \\
&= \frac{1}{2}\left(\sqrt{85}\right)\left(\sqrt{85}\right) \\
&= \frac{1}{2}(85) \\
&= 42{,}5
\end{aligned}
$$

20. The points $A(-3;1)$, $B(3;-2)$ and $C(9;10)$ are given.

a) Prove that triangle ABC is a right-angled triangle.

Solution:

We first draw a sketch:



To show that $\triangle ABC$ is right angled we need to show that $AB \perp AC$ or that $AC \perp BC$ or $AB \perp BC$. We can do this by calculating the gradients of AB, AC and BC and then seeing if any of these gradients is the negative inverse of any of the other two gradients.

$$\begin{aligned} m_{AB} &= \frac{-2-1}{3-(-3)} \\ &= \frac{-3}{6} \\ &= \frac{-1}{2} \end{aligned}$$

$$\begin{aligned} m_{BC} &= \frac{10-(-2)}{9-3} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

$$\begin{aligned} m_{AC} &= \frac{10-1}{9-(-3)} \\ &= \frac{9}{12} \\ &= \frac{3}{4} \end{aligned}$$

Now we note that:

$$\begin{aligned} m_{AB} \times m_{BC} &= \frac{-1}{2} \times 2 = -1 \\ \therefore AB &\perp BC \\ \therefore \triangle ABC &\text{ is right-angled} \end{aligned}$$

b) Find the coordinates of D, if $ABCD$ is a parallelogram.

Solution:

A parallelogram has both sides equal in length and parallel. Therefore $CD \parallel AB$ and $AD \parallel BC$. Also $CD = AB$ and $AD = BC$.

Let the coordinates of D be $(x; y)$.

Since $AD \parallel BC$, $m_{AD} = m_{BC}$.

From the previous question we know that $m_{BC} = 2$. Therefore the gradient of AD is:

$$
\begin{aligned}
m_{AD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
2 &= \frac{y - 1}{x - (-3)} \\
2(x + 3) &= y - 1 \\
2x + 6 &= y - 1 \\
2x + 7 &= y
\end{aligned}
$$

Since $CD \parallel AB$, $m_{CD} = m_{AB}$.
From the previous question we know that $m_{AB} = \frac{-1}{2}$. Therefore the gradient of CD is:

$$
\begin{aligned}
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\frac{-1}{2} &= \frac{y - 10}{x - 9} \\
\frac{-1}{2}(x - 9) &= y - 10 \\
-x + 9 &= 2y - 20 \\
-x + 29 &= 2y
\end{aligned}
$$

Now we have two equations with two unknowns. We can equate the two equations and solve for x:

$$
\begin{aligned}
4x + 14 &= -x + 29 \\
5x &= 15 \\
x &= 3
\end{aligned}
$$

Now we can solve for y:

$$
\begin{aligned}
2x + 7 &= y \\
2(3) + 7 &= y \\
13 &= y
\end{aligned}
$$

Therefore the coordinates of D are $(3; 13)$.

c) Find the equation of a line parallel to the line BC, which passes through the point A.

Solution:

We know from the first question that the gradient of line BC is 2. We also know that point A is at $(-3; 1)$. The line parallel to BC will have the same gradient as BC so we can write the equation of this line as: $y = 2x+c$. Now we can use point A to find the y-intercept of the line:

$$
\begin{aligned}
2 &= 2(-5) + c \\
2 &= -10 + c \\
12 &= c
\end{aligned}
$$

Therefore the equation of the line parallel to BC and passing through A is $y = 2x + 12$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2GCK	1b. 2GCM	1c. 2GCN	2a. 2GCP	2b. 2GCQ	2c. 2GCR
3. 2GCS	4. 2GCT	5. 2GCV	6. 2GCW	7. 2GCX	8. 2GCY
9. 2GCZ	10. 2GD2	11. 2GD3	12. 2GD4	13. 2GD5	14. 2GD6
15. 2GD7	16. 2GD8	17. 2GD9	18. 2GDB	19. 2GDC	20. 2GDD

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8.4 Mid-point of a line

For worked example 10 (calculating the mid-point) learners can check their answer using the distance formula.

Using the distance formula, we can confirm that the distances from the mid-point to each end point are equal:

$$\begin{aligned} PS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0 - 2)^2 + (-0{,}5 - 1)^2} \\ &= \sqrt{(-2)^2 + (-1{,}5)^2} \\ &= \sqrt{4 + 2{,}25} \\ &= \sqrt{6{,}25} \end{aligned}$$

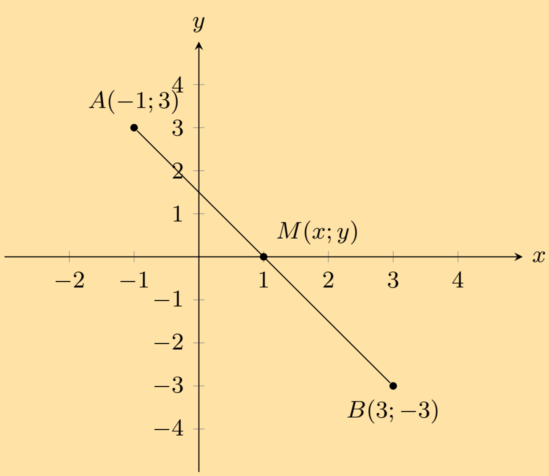
and

$$\begin{aligned} QS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0 - (-2))^2 + (-0{,}5 - (-2))^2} \\ &= \sqrt{(0 + 2)^2 + (-0{,}5 + 2)^2} \\ &= \sqrt{(2)^2 + (-1{,}5)^2} \\ &= \sqrt{4 + 2{,}25} \\ &= \sqrt{6{,}25} \end{aligned}$$

As expected, $PS = QS$, therefore F is the mid-point.

Exercise 8 – 5:

1. You are given the following diagram:



Calculate the coordinates of the mid-point (M) between point $A(-1; 3)$ and point $B(3; -3)$.
Solution:
Let the coordinates of A be $(x_1; y_1)$ and the coordinates of B be $(x_2; y_2)$.

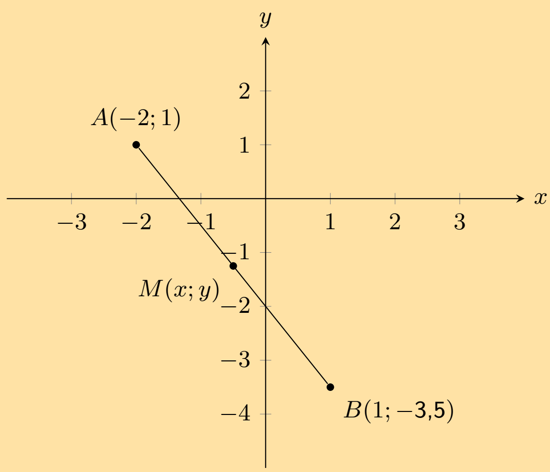
$$x_1 = -1 \quad y_1 = 3 \quad x_2 = 3 \quad y_2 = -3$$

Substitute values into the mid-point formula:

$$
\begin{aligned}
M\left(x;y\right) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)\\
x &= \frac{x_1+x_2}{2}\\
&= \frac{-1+3}{2}\\
&= 1\\
y &= \frac{y_1+y_2}{2}\\
&= \frac{3+(-3)}{2}\\
&= 0
\end{aligned}
$$

The mid-point is at $M\left(1;0\right)$.

2. You are given the following diagram:



Calculate the coordinates of the mid-point (M) between point $A(-2;1)$ and point $B(1;-3{,}5)$.
Solution:
Let the coordinates of A be $(x_1;y_1)$ and the coordinates of B be $(x_2;y_2)$.

$$x_1=-2 \quad y_1=1 \quad x_2=1 \quad y_2=-3{,}5$$

Substitute values into the mid-point formula:

$$
\begin{aligned}
M\left(x;y\right) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)\\
x &= \frac{x_1+x_2}{2}\\
&= \frac{-2+1}{2}\\
&= -0{,}5\\
y &= \frac{y_1+y_2}{2}\\
&= \frac{1+(-3{,}5)}{2}\\
&= -1{,}25
\end{aligned}
$$

The mid-point is at $M\left(-0{,}5;-1{,}25\right)$.

3. Find the mid-points of the following lines:

a) $A(2;5)$, $B(-4;7)$
Solution:

$$
\begin{aligned}
M_{AB} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{2 - 4}{2}; \frac{5 + 7}{2}\right) \\
&= \left(\frac{-2}{2}; \frac{12}{2}\right) \\
&= (-1; 6)
\end{aligned}
$$

b) $C(5; 9)$, $D(23; 55)$
Solution:

$$
\begin{aligned}
M_{CD} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{5 + 23}{2}; \frac{9 + 55}{2}\right) \\
&= \left(\frac{28}{2}; \frac{64}{2}\right) \\
&= (14; 32)
\end{aligned}
$$

c) $E(x + 2; y - 1)$, $F(x - 5; y - 4)$
Solution:

$$
\begin{aligned}
M_{EF} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{x + 2 + x - 5}{2}; \frac{y - 1 + y - 4}{2}\right) \\
&= \left(\frac{2x - 3}{2}; \frac{2y - 5}{2}\right)
\end{aligned}
$$

4. The mid-point M of PQ is $(3; 9)$. Find P if Q is $(-2; 5)$.
Solution:
The mid-point formula is:

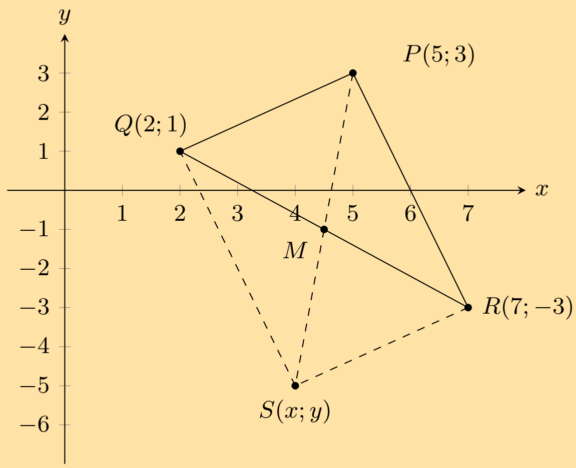
$$
M\,(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)
$$

Substituting values and solving for x_2 and y_2 gives:

$$
\begin{aligned}
3 &= \frac{-2 + x_2}{2} & \qquad 9 &= \frac{5 + y_2}{2} \\
6 &= -2 + x_2 & 18 &= 5 + y_2 \\
x_2 &= 6 + 2 & y_2 &= 18 - 5 \\
x_2 &= 8 & y_2 &= 13
\end{aligned}
$$

The coordinates of point P are $(8; 13)$.

5. $PQRS$ is a parallelogram with the points $P(5; 3)$, $Q(2; 1)$ and $R(7; -3)$. Find point S.
Solution:
Draw a sketch:



The diagonals of a parallelogram bisect each other, therefore the mid-point of QR will be the same as the mid-point of PS. We must first find the mid-point of QR. We can then use it to determine the coordinates of point H.

$$\begin{aligned} M_{QR} &= \left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{2+7}{2}; \frac{1-3}{2}\right) \\ &= \left(\frac{9}{2}; \frac{-2}{2}\right) \\ &= \left(\frac{9}{2}; -1\right) \end{aligned}$$

Use mid-point M to find the coordinates of S:

$$\begin{aligned} M_{QR} &= (\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}) \\ \left(\frac{9}{2}; -1\right) &= \left(\frac{x+5}{2}; \frac{y+3}{2}\right) \end{aligned}$$

Solve for x:

$$\begin{aligned} \frac{9}{2} &= \frac{x+5}{2} \\ 9 &= x+5 \\ x &= 4 \end{aligned}$$

Solve for y:

$$\begin{aligned} -1 &= \frac{y+3}{2} \\ -2 &= y+3 \\ y &= -5 \end{aligned}$$

Therefore $S(4;-5)$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GDG 2. 2GDH 3a. 2GDJ 3b. 2GDK 3c. 2GDM 4. 2GDN
5. 2GDP

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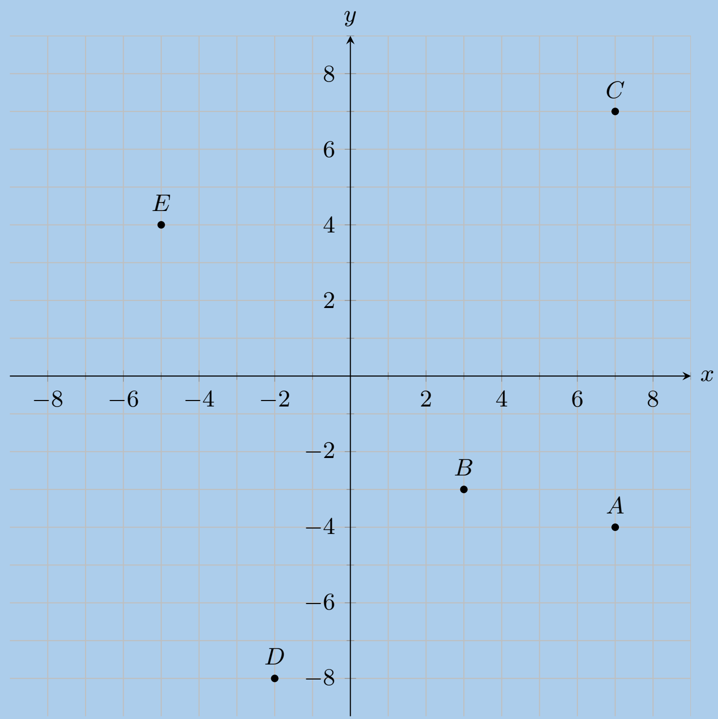
End of chapter Exercise 8 – 6:

1. You are given the following diagram, with various points shown:



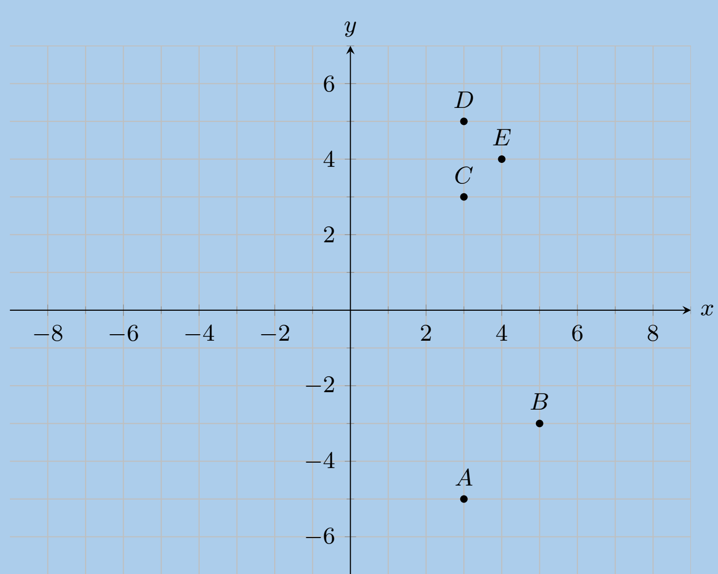
Find the coordinates of points A, B, C, D and E.

Solution:

From the graph we can read off the x and y values for each point.

$A(7;-4)$, $B(3;-3)$, $C(7;7)$, $D(-2;-8)$ and $E(-5;4)$

2. You are given the following diagram, with various points shown:



Which point lies at the coordinates $(3; -5)$?

Solution:

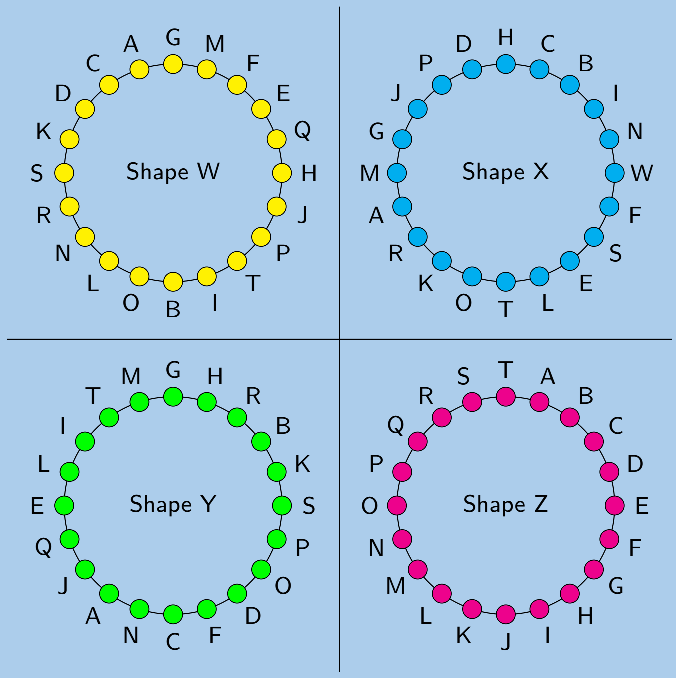
For this question, we are must find point $(3; -5)$.

Therefore, on the graph we can trace the x and y values.

Therefore, point A lies at the coordinates: $(3; -5)$

3. You are given the following diagram, with 4 shapes drawn.

   All the shapes are identical, but each shape uses a different naming convention:

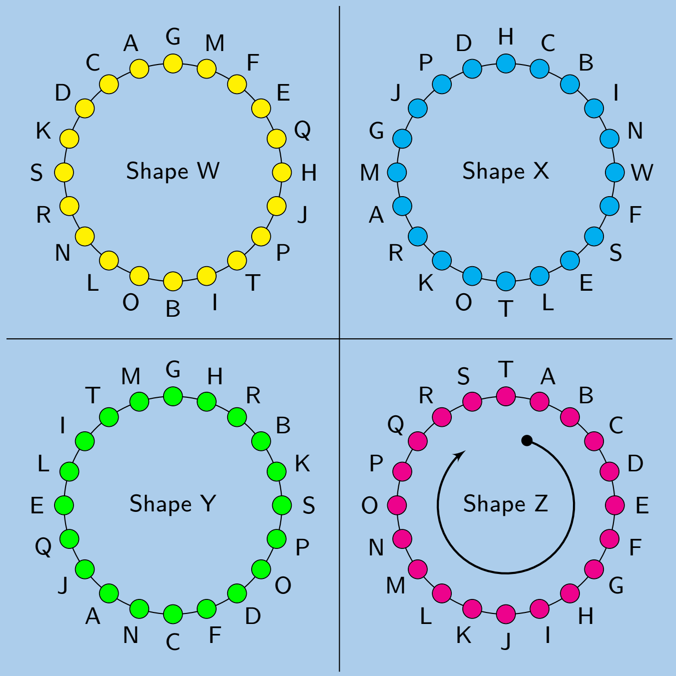


Which shape uses the correct naming convention?

Solution:

We recall that the correct naming convention for a shape is in **alphabetical order**, either clockwise or anti-clockwise around the shape.

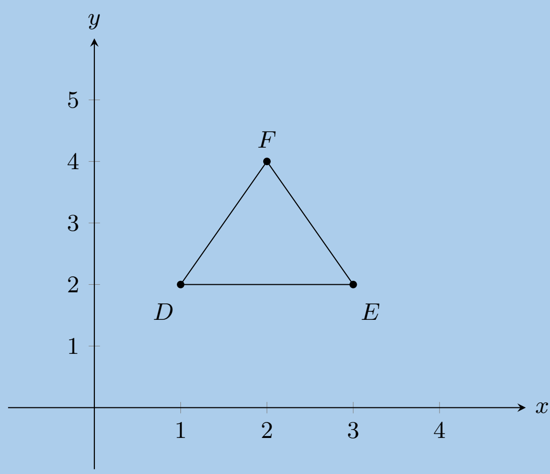
From the diagram, we can see that only **Shape Z** sticks to this naming convention.



4. Represent the following figures in the Cartesian plane:

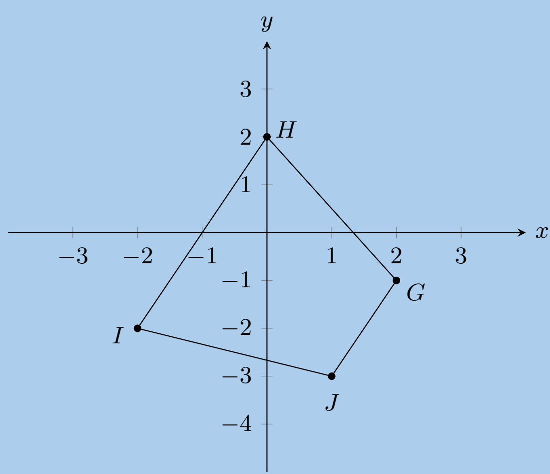
a) Triangle DEF with $D(1;2)$, $E(3;2)$ and $F(2;4)$.

Solution:



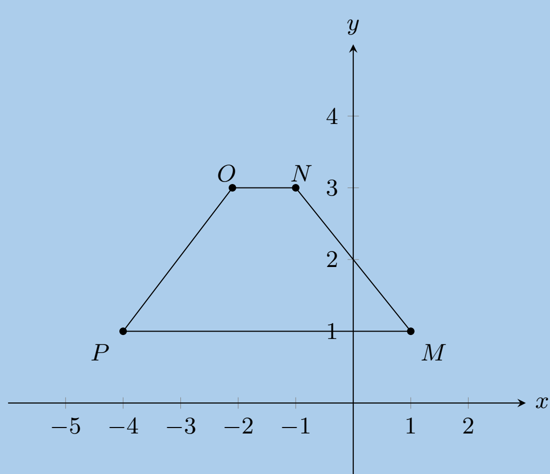
b) Quadrilateral $GHIJ$ with $G(2;-1)$, $H(0;2)$, $I(-2;-2)$ and $J(1;-3)$.

Solution:



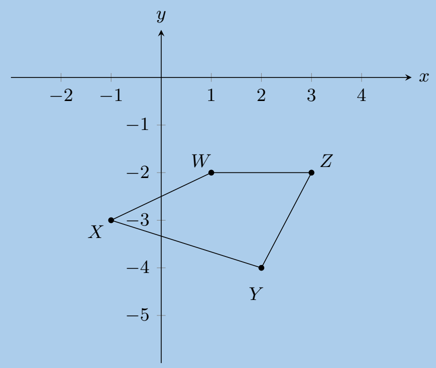
c) Quadrilateral $MNOP$ with $M(1;1)$, $N(-1;3)$, $O(-2;3)$ and $P(-4;1)$.

Solution:

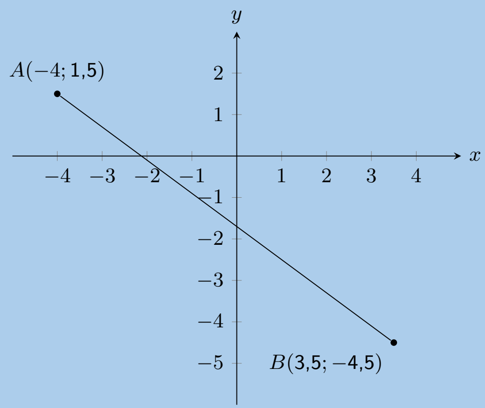


d) Quadrilateral $WXYZ$ with $W(1;-2)$, $X(-1;-3)$, $Y(2;-4)$ and $Z(3;-2)$.

Solution:



5. You are given the following diagram:



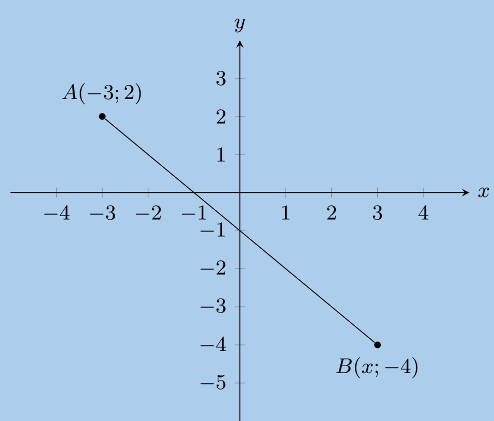
Calculate the length of line AB, correct to 2 decimal places.

Solution:

The equation for distance is $d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$. We substitute in $A(-4; 1{,}5)$ and $B(3{,}5; -4{,}5)$ and solve:

$$\begin{aligned} d_{AB} &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ &= \sqrt{(3{,}5 - (-4))^2 + (-4{,}5 - (1{,}5))^2} \\ &= \sqrt{(3{,}5 + 4)^2 + (-4{,}5 - 1{,}5)^2} \\ &= \sqrt{(7{,}5)^2 + (-6)^2} \\ &= \sqrt{(56{,}25) + (36)} \\ &= \sqrt{(92{,}25)} \\ &= 9{,}81 \end{aligned}$$

6. The following picture shows two points on the Cartesian plane, A and B.



The distance between the points is 8,4853. Calculate the missing coordinate of point B.
Solution:
The equation for distance is $d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$. We substitute in $A(-3;2)$ and $B(x;-4)$:

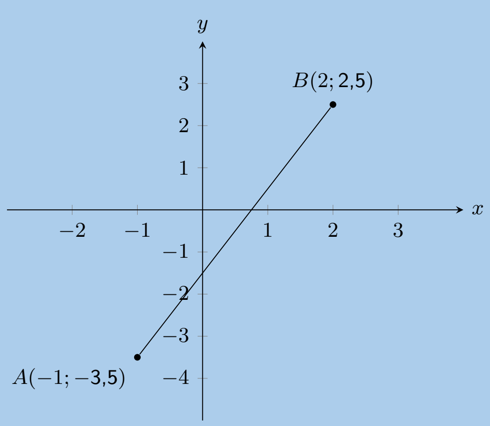
$$\begin{aligned} d_{AB} &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ 8{,}4853 &= \sqrt{(x-(-3))^2 + (-4-(2))^2} \\ 8{,}4853 &= \sqrt{(x+3)^2 + (-4-2)^2} \end{aligned}$$

Now we re-arrange, and solve for the value of x:

$$\begin{aligned} (8{,}4853)^2 &= (x+3)^2 + (-4-2)^2 \\ 72 &= (x+3)^2 + (-4-2)^2 \\ 72 &= (x+3)^2 + 36 \\ (x+3)^2 &= 36 \\ x+3 &= \pm\sqrt{36} \\ x &= \pm 6 - 3 \\ x &= (3) \text{ or } (-9) \end{aligned}$$

We now have a choice between 2 values for x. From the diagram we can see that the appropriate value for this question is 3.

7. You are given the following diagram:

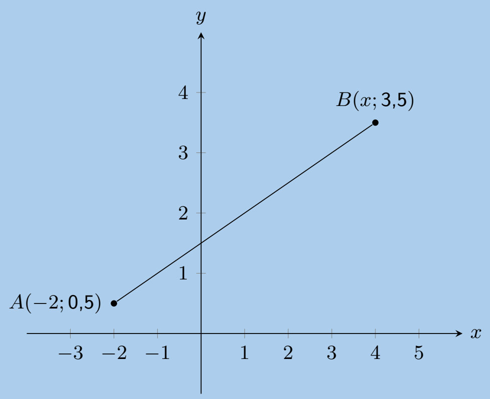


Calculate the gradient (m) of line AB. The coordinates are $A(-1;-3{,}5)$ and $B(2;2{,}5)$ respectively.
Solution:

$$\begin{aligned} m &= \left(\frac{y_B - y_A}{x_B - x_A}\right) \\ &= \left(\frac{(2{,}5) - (-3{,}5)}{(2) - (-1)}\right) \\ &= 2 \end{aligned}$$

Therefore the gradient, m, of the line AB is 2.

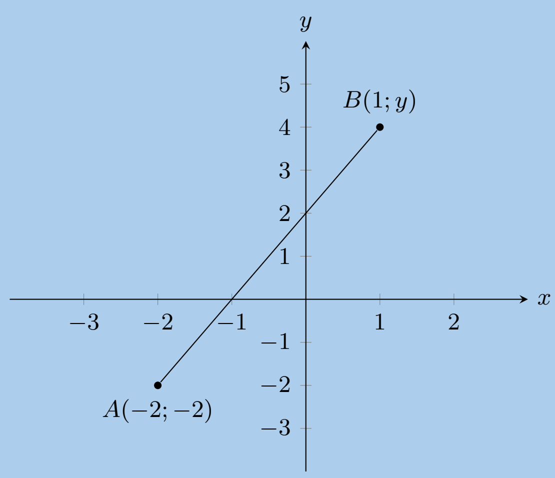
8. You are given the following diagram:



You are also told that line AB has a gradient, m, of 0,5.
Calculate the missing co-ordinate of point B.
Solution:

$$\begin{aligned}
m &= \left(\frac{y_B - y_A}{x_B - x_A}\right)\\
0{,}5 &= \left(\frac{(3{,}5) - (0{,}5)}{x - (-2)}\right)\\
0{,}5(x+2) &= 3\\
x + 2 &= 6\\
x &= 4
\end{aligned}$$

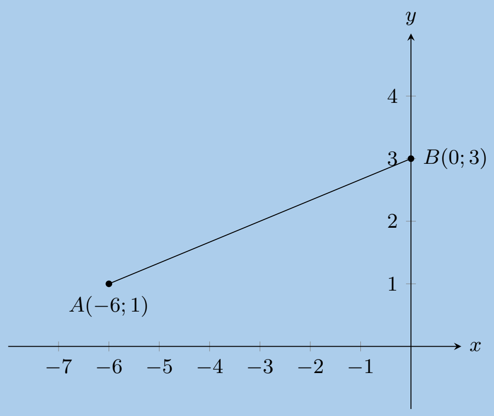
9. You are given the following diagram:



You are also told that line AB has a gradient, m, of 2.
Calculate the missing co-ordinate of point B.
Solution:

$$\begin{aligned}
m &= \left(\frac{y_B - y_A}{x_B - x_A}\right)\\
2 &= \left(\frac{y - (-2)}{(1) - (-2)}\right)\\
2(3) &= y + 2\\
6 &= y + 2\\
4 &= y
\end{aligned}$$

10. In the diagram, A is the point $(-6;1)$ and B is the point $(0;3)$.



a) Find the equation of line AB.
Solution:
We first find the gradient of the line:

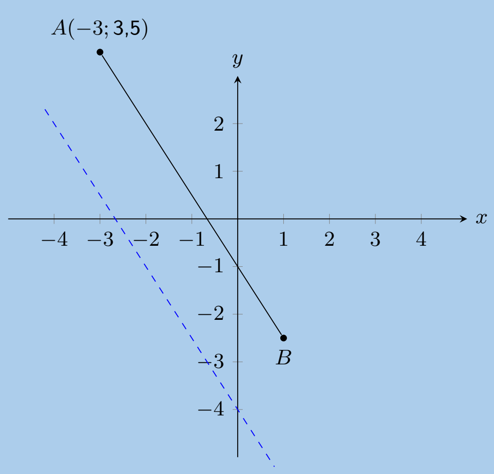
$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-3}{-6-0} \\ &= \frac{-2}{-6} \\ &= \frac{1}{3} \end{aligned}$$

Next we note that point B lies on the y-axis and so is the y-intercept. Therefore $c = 3$.
Therefore the equation of the line AB is: $y = \frac{1}{3}x + 3$.

b) Calculate the length of AB.
Solution:

$$\begin{aligned} d_{AB} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-6+0)^2 + (1-3)^2} \\ &= \sqrt{(-6)^2 + (-2)^2} \\ &= \sqrt{40} \end{aligned}$$

11. You are given the following diagram:



You are also told that line AB runs parallel to the following line: $y = -1{,}5x - 4$. Point A is at $(-3;3{,}5)$.
Find the equation of the line AB.

Solution:

Let $y = -1{,}5x - 4$ be line CD.

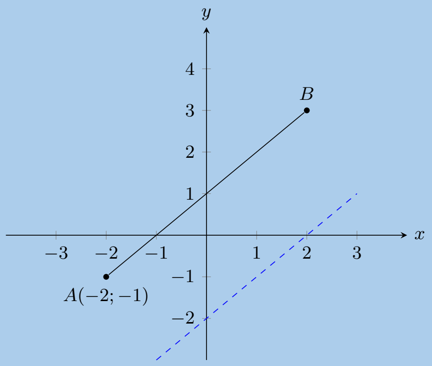
Since line AB is parallel to line CD, $m_{AB} = m_{CD} = -1{,}5$.

Now we can substitute in the known point (A) and find the y-intercept of the line:

$$\begin{aligned} y &= -1{,}5x + c \\ (3{,}5) &= (-1{,}5)(-3) + c \\ c &= -1 \end{aligned}$$

Therefore, the equation of the line AB is: $y = -1{,}5x - 1$

12. You are given the following diagram:



You are also told that line AB runs parallel to the following line: $y = x - 2$.

Calculate the missing co-ordinate of point $B(x; 3)$.

Solution:

Let $y = x - 2$ be line CD.

Since line AB is parallel to line CD, $m_{AB} = m_{CD} = 1$.

Now we can substitute in the known point (A) and find the y-intercept of the line:

$$\begin{aligned} y &= x + c \\ -1 &= -2 + c \\ c &= 1 \end{aligned}$$

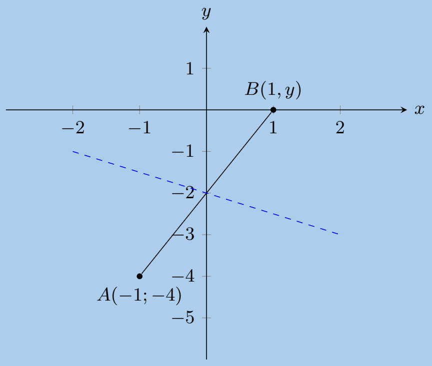
Therefore, the equation of the line AB is: $y = x + 1$.

Finally, we substitute the known value for point B into the equation for line AB:

$$\begin{aligned} y &= x + 1 \\ 3 &= x + 1 \\ 2 &= x \end{aligned}$$

Therefore point B is at $(2; 3)$.

13. You are given the following diagram:



You are also told that line AB runs perpendicular to the following line: $y = -0{,}5x - 2$.
Calculate the missing co-ordinate of point B.

Solution:

The general form of a straight line is: $y = mx + c$.
Let line CD be $y = -0{,}5x - 2$.
Line AB is perpendicular to line CD and so $m_{AB} = \frac{-1}{m_{CD}}$.

$$\begin{aligned} y &= mx + c \\ y &= \left(\frac{-1}{m_{CD}}\right)x + c \\ y &= \left(\frac{-1}{-0{,}5}\right)x + c \\ y &= 2x + c \end{aligned}$$

Now we can substitute point A into the equation to find the y-intercept:

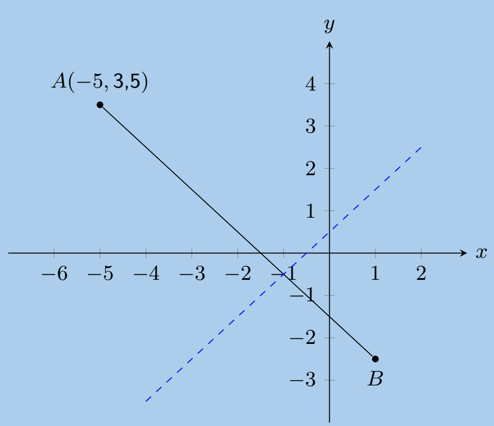
$$\begin{aligned} y &= 2x + c \\ -4 &= (2)(-1) + c \\ c &= -2 \end{aligned}$$

Next we can substitute in point B to find the missing coordinate:

$$\begin{aligned} y &= 2x - 2 \\ y &= (2)(1) - 2 \\ y &= 0 \end{aligned}$$

Therefore the missing coordinate is $B(1;0)$.

14. The graph here shows line, AB. The blue dashed line is perpendicular to AB.



The equation of the blue dashed line is $y = x + 0{,}5$. Point A is at $(-5; 3{,}5)$.
Determine the equation of line AB.

Solution:

Let line CD be the blue dashed line.
Line AB is perpendicular to line CD and so $m_{AB} = \frac{-1}{m_{CD}}$.

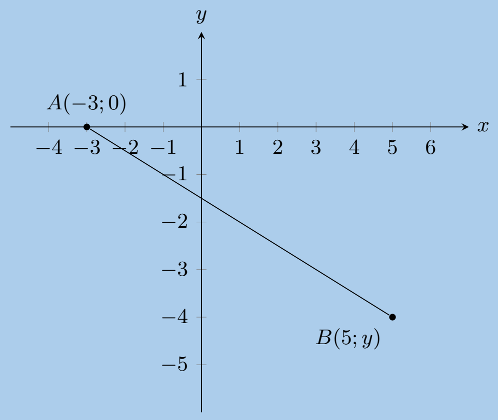
$$\begin{aligned} y &= mx + c \\ y &= \left(\frac{-1}{m_{CD}}\right)x + c \\ y &= \left(\frac{-1}{1}\right)x + c \\ y &= -x + c \end{aligned}$$

Now we can substitute point A into the equation to find the y-intercept:

$$\begin{aligned} y &= -x + c \\ 3{,}5 &= (-1)(-5) + c \\ c &= -1{,}5 \end{aligned}$$

Therefore the equation of line AB is: $y = -x - 1{,}5$.

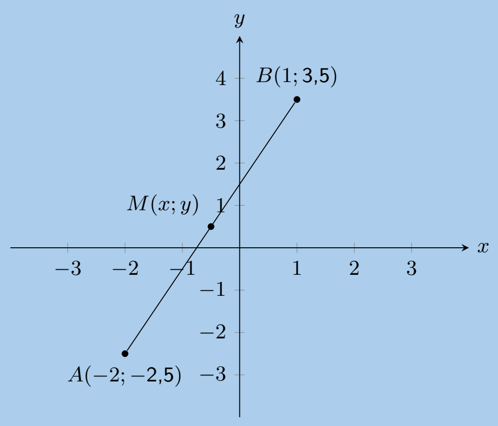
15. You are given the following diagram:



You are also told that line AB has the following equation: $y = -0{,}5x - 1{,}5$.
Calculate the missing co-ordinate of point B.
Solution:
We can substitute the known value for point B into the given equation for line AB:

$$\begin{aligned} y &= (-0{,}5)x - 1{,}5 \\ y &= (-0{,}5)(5) - 1{,}5 \\ y &= -4 \end{aligned}$$

16. You are given the following diagram:



Calculate the coordinates of the mid-point (M) between point $A(-2; -2{,}5)$ and point $B(1; 3{,}5)$ correct to 1 decimal place.
Solution:

$$\begin{aligned} M(x; y) &= \left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right) \\ &= \left(\frac{(-2) + (1)}{2}; \frac{(-2{,}5) + (3{,}5)}{2}\right) \\ M(x; y) &= (-0{,}5; 0{,}5) \end{aligned}$$

17. $A(-2;3)$and $B(2;6)$ are points in the Cartesian plane. $C(a;b)$ is the mid-point of AB. Find the values of a and b.
Solution:

$$\begin{aligned} M_{AB} &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-2-2}{2};\frac{3+6}{2}\right) \\ (a;b) &= \left(0;\frac{9}{2}\right) \\ \therefore a = 0 \text{ and } b &= \frac{9}{2} \end{aligned}$$

18. Determine the equations of the following straight lines:

a) passing through $P(5;5)$ and $Q(-2;12)$.
Solution:

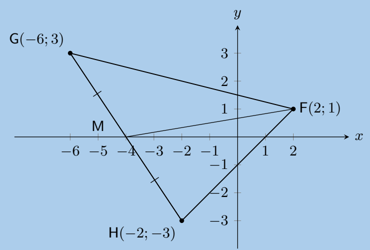
$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ m_{PQ} &= \frac{12-5}{-2-5} \\ &= \frac{7}{-7} \\ m_{PQ} &= -1 \end{aligned}$$

$$\begin{aligned} y &= mx+c \\ (5) &= (-1)(5)+c \\ c &= 5+5 \\ c &= 10 \\ \therefore y &= -x+10 \end{aligned}$$

b) parallel to $y = 3x+4$ and passing through $(4;0)$.
Solution:

$$\begin{aligned} m &= 3 \text{ (parallel lines)} \\ y &= mx+c \\ (0) &= (3)(4)+c \\ c &= 12 \\ \therefore y &= 3x+12 \end{aligned}$$

c) passing through $F(2;1)$ and the mid-point of GH where $G(-6;3)$ and $H(-2;-3)$.



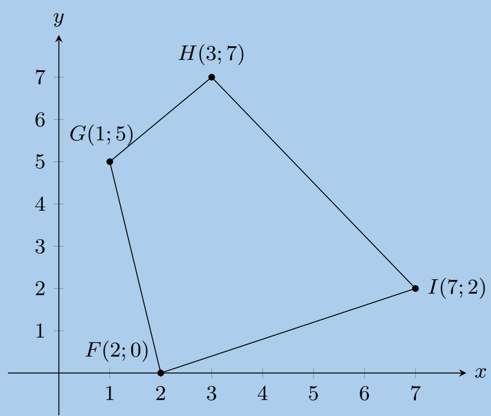
Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ M_{GH} &= \left(\frac{-6+(-2)}{2};\frac{3+(-3)}{2}\right) \\ M_{GH} &= (-4;0) \end{aligned}$$

$$(y - y_1) = m(x - x_1)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$(y - 0) = \frac{1 - 0}{2 - (-4)}(x - (-4))$$
$$\therefore y = \frac{1}{6}(x + 4)$$
$$\therefore y = \frac{1}{6}x + \frac{2}{3}$$

19. In the diagram below, the vertices of the quadrilateral are $F(2;0)$, $G(1;5)$, $H(3;7)$ and $I(7;2)$.



a) Calculate the lengths of the sides of $FGHI$.

Solution:

To calculate the lengths of the sides we need to use the distance formula. The four sides are FG, GH, HI and FI

$$\begin{aligned} d_{FG} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(1 - 2)^2 + (5 - 0)^2} \\ &= \sqrt{(-1)^2 + (5)^2} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} d_{GH} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3 - 1)^2 + (7 - 5)^2} \\ &= \sqrt{(2)^2 + (2)^2} \\ &= \sqrt{8} \end{aligned}$$

$$\begin{aligned} d_{HI} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(7 - 3)^2 + (2 - 7)^2} \\ &= \sqrt{(4)^2 + (-5)^2} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} d_{FI} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2 - 7)^2 + (0 - 2)^2} \\ &= \sqrt{(-5)^2 + (-2)^2} \\ &= \sqrt{29} \end{aligned}$$

b) Are the opposite sides of $FGHI$ parallel?

Solution:

We want to know if $GH \parallel FI$ and $FG \parallel HI$. We can calculate the gradient of each of the sides and then compare the gradients.

$$\begin{aligned} m_{FG} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0-5}{2-1} \\ &= \frac{-5}{1} \\ &= -5 \end{aligned}$$

$$\begin{aligned} m_{HI} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-7}{7-3} \\ &= \frac{-5}{4} \\ &= -\frac{5}{4} \end{aligned}$$

$$\begin{aligned} m_{GH} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5-7}{1-3} \\ &= \frac{-2}{-2} \\ &= 1 \end{aligned}$$

$$\begin{aligned} m_{FI} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0-2}{2-7} \\ &= \frac{-2}{-5} \\ &= \frac{2}{5} \end{aligned}$$

We note that $m_{FG} \neq m_{HI}$ and $m_{GH} \neq m_{FI}$ therefore the opposite sides are not parallel.

c) Do the diagonals of $FGHI$ bisect each other?

Solution:

To determine if the diagonals bisect each other we need to find the mid-point of FH and GI.

$$\begin{aligned} M_{GI} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1+7}{2}; \frac{5+2}{2}\right) \\ &= \left(\frac{8}{2}; \frac{7}{2}\right) \\ &= \left(4; \frac{7}{2}\right) \end{aligned}$$

$$\begin{aligned} M_{FH} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3+2}{2}; \frac{7+0}{2}\right) \\ &= \left(\frac{5}{2}; \frac{7}{2}\right) \end{aligned}$$

Therefore $M_{GI} \neq M_{FH}$ and the diagonals do not bisect each other.

d) Can you state what type of quadrilateral $FGHI$ is? Give reasons for your answer.

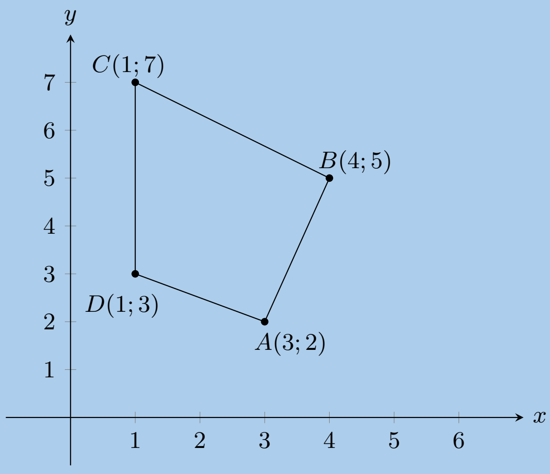
Solution:

This is an ordinary quadrilateral. The opposite sides are not parallel, the diagonals do not bisect each other and none of the sides are equal in length.

20. Consider a quadrilateral $ABCD$ with vertices $A(3;2)$, $B(4;5)$, $C(1;7)$ and $D(1;3)$.

a) Draw the quadrilateral.

Solution:



b) Find the lengths of the sides of the quadrilateral.

Solution:

To calculate the lengths of the sides we need to use the distance formula. The four sides are AB, BC, CD and AD

$$\begin{aligned} d_{AB} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3-4)^2 + (2-5)^2} \\ &= \sqrt{(-1)^2 + (-3)^2} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned} d_{BC} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(4-1)^2 + (5-7)^2} \\ &= \sqrt{(3)^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} d_{CD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(1-1)^2 + (7-3)^2} \\ &= \sqrt{0 + (4)^2} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

$$\begin{aligned} d_{AD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3-1)^2 + (2-3)^2} \\ &= \sqrt{(2)^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned}$$

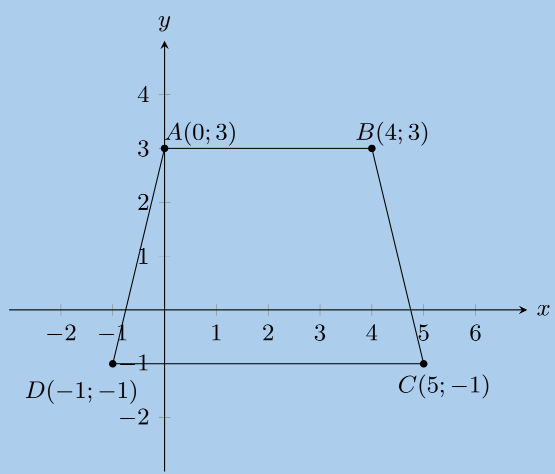
21. $ABCD$ is a quadrilateral with vertices $A(0;3)$, $B(4;3)$, $C(5;-1)$ and $D(-1;-1)$.

a) Show by calculation that:

i. $AD = BC$

Solution:

First draw a sketch of the quadrilateral:



To show that $AD = BC$ we need to use the distance formula to find the length of AD and BC.

$$\begin{aligned}
d_{AD} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(0 - (-1))^2 + (3 - (-1))^2} \\
&= \sqrt{(1)^2 + (4)^2} \\
&= \sqrt{17}
\end{aligned}$$

$$\begin{aligned}
d_{BC} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(4 - 5)^2 + (3 - (-1))^2} \\
&= \sqrt{(-1)^2 + (4)^2} \\
&= \sqrt{17}
\end{aligned}$$

Therefore sides AD and BC are equal.

ii. $AB \parallel DC$

Solution:

To show that $AB \parallel DC$ we need to show that $m_{AB} = m_{DC}$.

$$\begin{aligned}
m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3 - 3}{0 - 4} \\
&= \frac{0}{-4} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
m_{DC} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 + 1}{-1 - 5} \\
&= \frac{0}{-6} \\
&= 0
\end{aligned}$$

The gradients are equal, therefore $AB \parallel DC$.

b) What type of quadrilateral is $ABCD$?

Solution:

An isosceles trapezium; one pair of opposite sides equal in length and one pair of opposite sides parallel.

c) Show that the diagonals AC and BD do not bisect each other.

Solution:

To show this we need to find the mid-points of AC and BD.

$$\begin{aligned} M_{AC} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{0 + 5}{2}; \frac{3 - 1}{2}\right) \\ &= \left(\frac{5}{2}; \frac{2}{2}\right) \\ &= \left(\frac{5}{2}; 1\right) \end{aligned}$$

$$\begin{aligned} M_{BD} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{4 - 1}{2}; \frac{3 - 1}{2}\right) \\ &= \left(\frac{3}{2}; \frac{2}{2}\right) \\ &= \left(\frac{3}{2}; 1\right) \end{aligned}$$

$M_{AC} \neq M_{BD}$, therefore the diagonals do not bisect each other.

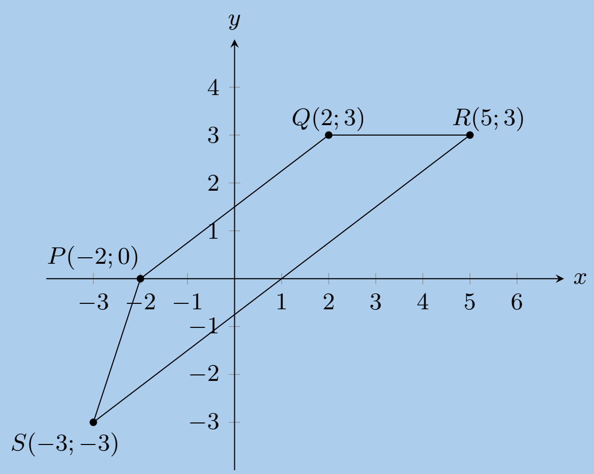
22. P, Q, R and S are the points $(-2; 0)$, $(2; 3)$, $(5; 3)$ and $(-3; -3)$ respectively.

a) Show that:

i. $SR = 2PQ$

Solution:

First draw a sketch:



We can use the distance formula to show that $SR = 2PQ$.

$$
\begin{aligned}
d_{PQ} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(-2 - 2)^2 + (0 - 3)^2} \\
&= \sqrt{(-4)^2 + (-3)^2} \\
&= \sqrt{25} \\
&= 5
\end{aligned}
$$

$$
\begin{aligned}
d_{SR} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(-3 - 5)^2 + (-3 - 3)^2} \\
&= \sqrt{(-8)^2 + (-6)^2} \\
&= \sqrt{100} \\
&= 10 \\
\therefore SR &= 2PQ
\end{aligned}
$$

ii. $SR \parallel PQ$

Solution:

$$
\begin{aligned}
m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3 - 0}{2 - (-2)} \\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
m_{SR} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3 - 3}{-3 - 5} \\
&= \frac{-6}{-8} \\
&= \frac{3}{4} \\
\therefore m_{PQ} &= m_{SR}
\end{aligned}
$$

Since the gradients are equal $SR \parallel PQ$.

b) Calculate:

i. PS

Solution:

We need to calculate the length of PS:

$$
\begin{aligned}
d_{PS} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(-2 - (-3))^2 + (0 - (-3))^2} \\
&= \sqrt{(1)^2 + (3)^2} \\
&= \sqrt{10}
\end{aligned}
$$

ii. QR

Solution:

We need to calculate the length of QR:

$$
\begin{aligned}
d_{QR} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(2 - 5)^2 + (3 - 3)^2} \\
&= \sqrt{(-3)^2 + 0} \\
&= \sqrt{9} \\
&= 3
\end{aligned}
$$

c) What kind of quadrilateral is $PQRS$? Give reasons for your answer.

Solution:

Trapezium. One pair of opposite sides parallel.

23. $EFGH$ is a parallelogram with vertices $E(-1;2)$, $F(-2;-1)$ and $G(2;0)$. Find the coordinates of H by using the fact that the diagonals of a parallelogram bisect each other.

Solution:

Since the diagonal bisect each other the mid-point of EG is equal to the mid-point of FH. We can first calculate the mid-point of EG since we have the co-ordinates of both E and G. We can then use that mid-point to help us find the co-ordinates of H.

$$\begin{aligned} M_{EG} &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-1+2}{2};\frac{2+0}{2}\right) \\ &= \left(\frac{1}{2};1\right) \end{aligned}$$

$$\begin{aligned} M_{FH} &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-2+x}{2};\frac{-1+y}{2}\right) \\ \left(\frac{1}{2};1\right) &= \left(\frac{-2+x}{2};\frac{-1+y}{2}\right) \end{aligned}$$

Solve for x:

$$\begin{aligned} \frac{1}{2} &= \frac{-2+x}{2} \\ 1 &= -2+x \\ x &= 3 \end{aligned}$$

Solve for y:

$$\begin{aligned} 1 &= \frac{-1+y}{2} \\ 2 &= -1+y \\ y &= 3 \end{aligned}$$

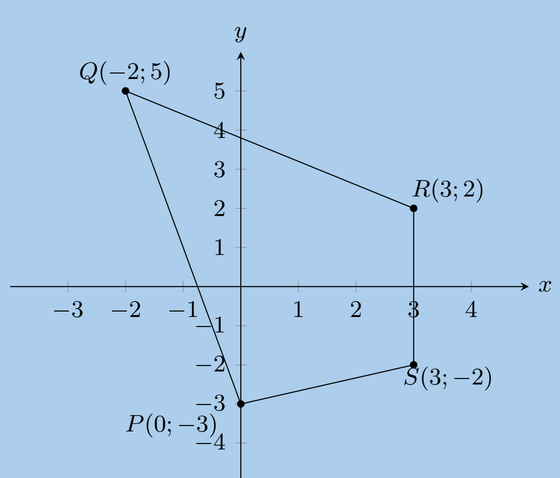
Therefore $H(3;3)$.

24. $PQRS$ is a quadrilateral with points $P(0;-3)$, $Q(-2;5)$, $R(3;2)$ and $S(3;-2)$ in the Cartesian plane.

a) Find the length of QR.

Solution:

First draw a sketch of the quadrilateral:



$$
\begin{aligned}
d_{QR} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(-2 - 3)^2 + (5 - 2)^2} \\
&= \sqrt{(-5)^2 + (3)^2} \\
&= \sqrt{34}
\end{aligned}
$$

b) Find the gradient of PS.
Solution:

$$
\begin{aligned}
m_{PS} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3 + 2}{0 - 3} \\
&= \frac{-1}{-3} \\
&= \frac{1}{3}
\end{aligned}
$$

c) Find the mid-point of PR.
Solution:

$$
\begin{aligned}
M_{QR} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{0 + 3}{2}; \frac{-3 + 2}{2}\right) \\
&= \left(\frac{3}{2}; \frac{-1}{2}\right)
\end{aligned}
$$

d) Is $PQRS$ a parallelogram? Give reasons for your answer.
Solution:
We need to calculate the gradients of each of the sides to see if opposite sides are parallel. We have calculated the gradient of PS so we only need to check the gradients of the other three sides. However, looking at our sketch PS is not parallel to QR (you can check this by calculating the gradient of QR).

$$
\begin{aligned}
m_{RS} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-2 - 2}{3 - 3} \\
&= \frac{-4}{0} \\
&= \text{undefined}
\end{aligned}
$$

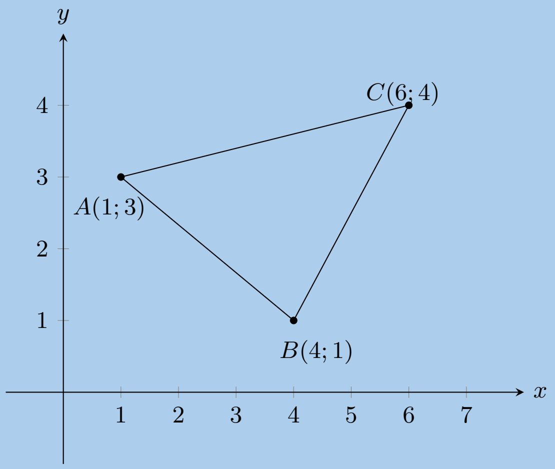
And,

$$
\begin{aligned}
m_{QR} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - 5}{3 - (-2)} \\
&= \frac{-3}{5}
\end{aligned}
$$

Therefore $PQRS$ is not a parallelogram. Opposite sides are not parallel.

25. Consider triangle ABC with vertices $A(1; 3)$, $B(4; 1)$ and $C(6; 4)$.

a) Sketch triangle ABC on the Cartesian plane.
Solution:



b) Show that ABC is an isosceles triangle.

Solution:

We need to show that two sides are equal in length. We therefore calculate the length of each of the sides of the triangle.

$$\begin{aligned} d_{AB} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(1-4)^2 + (3-1)^2} \\ &= \sqrt{(-3)^2 + (2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} d_{BC} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(4-6)^2 + (1-4)^2} \\ &= \sqrt{(-2)^2 + (-3)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} d_{AC} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(1-6)^2 + (3-4)^2} \\ &= \sqrt{(-5)^2 + (-1)^2} \\ &= \sqrt{26} \end{aligned}$$

Two sides of the triangle are equal in length, therefore $\triangle ABC$ is isosceles.

c) Determine the coordinates of M, the mid-point of AC.

Solution:

$$\begin{aligned} M_{AC} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1+6}{2}; \frac{3+4}{2}\right) \\ &= \left(\frac{7}{2}; \frac{7}{2}\right) \end{aligned}$$

d) Determine the gradient of AB.

Solution:

$$
\begin{aligned}
m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1 - 3}{4 - 1} \\
&= \frac{-2}{3}
\end{aligned}
$$

e) Show that $D(7; -1)$ lies on the line that goes through A and B.

Solution:

We have just calculated the gradient of AB: $m_{AB} = \frac{-2}{3}$. We need to calculate the gradient of BD and AD:

$$
\begin{aligned}
m_{BD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 - 1}{7 - 4} \\
&= \frac{-2}{3}
\end{aligned}
$$

$$
\begin{aligned}
m_{AD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 - 3}{7 - 1} \\
&= \frac{-2}{3}
\end{aligned}
$$

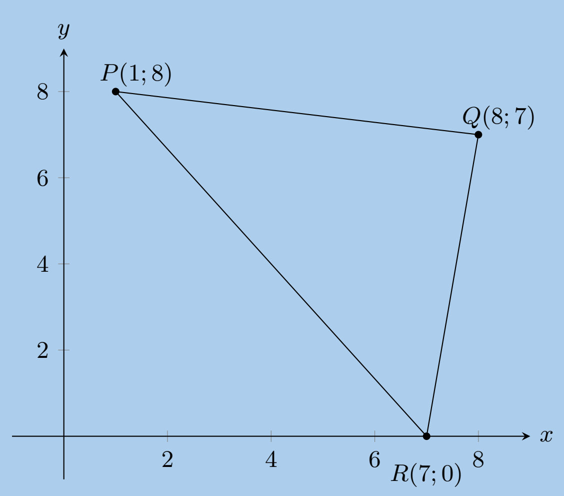
$$
m_{AB} = m_{BD} = m_{AD}
$$

Therefore A, B and D are collinear.
Therefore D lies on line AB.

26. $\triangle PQR$ has vertices $P(1; 8)$, $Q(8; 7)$ and $R(7; 0)$. Show through calculation that $\triangle PQR$ is a right angled isosceles triangle.

Solution:

First draw a sketch:



Next calculate the gradient of each of the three sides of the triangle:

$$
\begin{aligned}
m_{PQ} &= \frac{8-7}{1-8} \\
&= -\frac{1}{7}
\end{aligned}
$$

$$
\begin{aligned}
m_{QR} &= \frac{7-0}{8-7} \\
&= \frac{7}{1}
\end{aligned}
$$

$$
\begin{aligned}
m_{PR} &= \frac{8-0}{1-7} \\
&= -\frac{4}{3}
\end{aligned}
$$

Now we can check $m_{PQ} \times m_{QR}$, $m_{QR} \times m_{PR}$ and $m_{PQ} \times m_{PR}$. As soon as we find one of these values is equal to -1 then we have proved that the triangle is right-angled.

$$
\begin{aligned}
m_{PR} \times m_{QR} &= -\frac{1}{7} \times \frac{7}{1} \\
&= -1
\end{aligned}
$$

Therefore $\triangle PQR$ is right-angled, $PR \perp QR$. The right-angle is $P\hat{Q}R$.

Finally we calculate the lengths of sides PQ and RQ to show that the triangle is isosceles. We do not need to calculate PR as this is the hypotenuse of the triangle and must be longer than PQ and RQ.

$$
\begin{aligned}
PQ &= \sqrt{(1-8)^2 + (8-7)^2} \\
&= \sqrt{49+1} \\
&= \sqrt{50}
\end{aligned}
$$

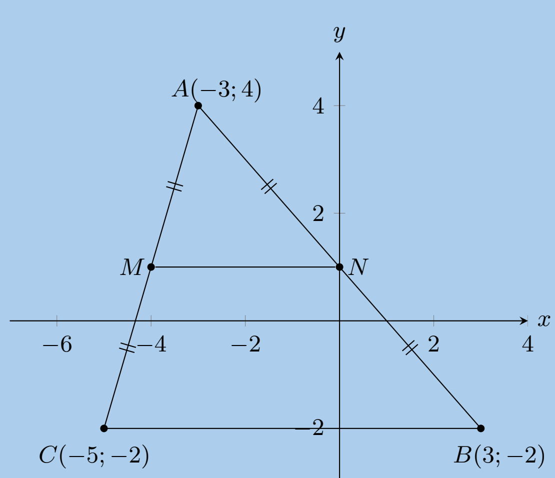
$$
\begin{aligned}
RQ &= \sqrt{(8-7)^2 + (7-0)^2} \\
&= \sqrt{1+49} \\
&= \sqrt{50}
\end{aligned}
$$

Therefore $PQ = RQ$ and therefore $\triangle PQR$ is a right-angled, isosceles triangle.

27. $\triangle ABC$ has vertices $A(-3; 4)$, $B(3; -2)$ and $C(-5; -2)$. M is the mid-point of AC and N is the mid-point of BC. Use $\triangle ABC$ to prove the mid-point theorem using analytical geometry methods.

Solution:

First draw a sketch:



The mid-point theorem states that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side. Therefore we need to show that $MN \parallel BC$ and that $MN = \frac{1}{2}BC$.
We need to calculate the co-ordinates of mid-points M and N:

$$\begin{aligned} M &= \left(\frac{-3-5}{2}; \frac{4-2}{2}\right) \\ &= (-4; 1) \\ N &= \left(\frac{-3+3}{2}; \frac{4-2}{2}\right) \\ &= (0; 1) \end{aligned}$$

Now we can show that $MN \parallel BC$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{aligned} m_{BC} &= \frac{-2-(-2)}{3-(-5)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} m_{MN} &= \frac{1-1}{-4-0} \\ &= 0 \end{aligned}$$

$$\therefore MN \parallel BC$$

Finally we can use the distance formula to show that $MN = \frac{1}{2}BC$:

$$\begin{aligned} d &= \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} \\ d_{MN} &= \sqrt{(1-1)^2 + (-4-0)^2} \\ &= 4 \\ d_{BC} &= \sqrt{(-2-(-2))^2 + (-5-3)^2} \\ &= 8 \\ \therefore MN &= \frac{1}{2}CB \end{aligned}$$

28. a) List two properties of a parallelogram.
Solution:
Any two of the following:
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

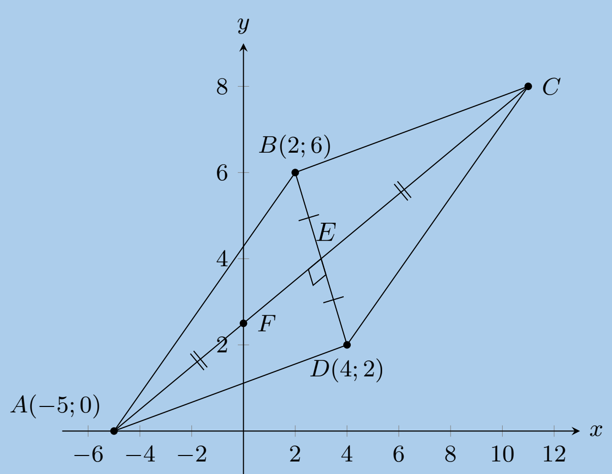
b) The points $A(-2; -4)$, $B(-4; 1)$, $C(2; 4)$ and $D(4; -1)$ are the vertices of a quadrilateral. Show that the quadrilateral is a parallelogram.
Solution:
We need to show that both pairs of opposite sides are parallel. So we need to calculate the gradient of each of the sides:

$$
\begin{aligned}
m_{AB} &= \frac{1+4}{-4+2} \\
&= \frac{5}{-2} \\
m_{CD} &= \frac{-1-4}{4-2} \\
&= \frac{-5}{2} \\
\therefore AB &\parallel CD
\end{aligned}
$$

$$
\begin{aligned}
m_{BC} &= \frac{4-1}{2+4} \\
&= \frac{3}{6} = \frac{1}{2} \\
m_{AD} &= \frac{-1+4}{4+2} \\
&= \frac{3}{6} = \frac{1}{2} \\
\therefore BC &\parallel AD
\end{aligned}
$$

Therefore $ABCD$ is a parallelogram (2 pairs opp. sides $\parallel$).

29. The diagram shows a quadrilateral. The points B and D have the coordinates $(2;6)$ and $(4;2)$ respectively. The diagonals of $ABCD$ bisect each other at right angles. F is the point of intersection of line AC with the y-axis.



a) Determine the gradient of AC.

Solution:

Since the diagonals bisect each other we know that $AC \perp BD$. Therefore:

$$
\begin{aligned}
m_{AC} \times m_{BD} &= -1 \\
m_{AC} \times \frac{6-2}{2-4} &= -1 \\
m_{AC} \times -2 &= -1 \\
m_{AC} &= \frac{1}{2}
\end{aligned}
$$

b) Show that the equation of AC is given by $2y = x + 5$.

Solution:

From above we have that $m = \frac{1}{2}$.

Substitute point A into the equation:

$$
\begin{aligned}
(0) &= \frac{1}{2}(-5) + c \\
0 &= \frac{-5}{2} + c \\
\therefore c &= \frac{5}{2} \\
\therefore y &= \frac{1}{2}x + \frac{5}{2} \\
\therefore 2y &= x + 5
\end{aligned}
$$

c) Determine the coordinates of C.
Solution:
We first calculate the coordinates of E:

$$
\begin{aligned}
E(\frac{2+4}{2}&; \frac{6+2}{2}) \\
&E(3; 4)
\end{aligned}
$$

Now we can use the coordinates of E to find the coordinates of C. Since E is the mid-point of AC we can use the mid-point formula to find C:

$$
\begin{aligned}
\frac{-5+x}{2} &= 3 \\
-5 + x &= 6 \\
x &= 11
\end{aligned}
$$

$$
\begin{aligned}
\frac{0+y}{2} &= 4 \\
y &= 8
\end{aligned}
$$

Therefore $C(11; 8)$.

30. $A(4; -1)$, $B(-6; -3)$ and $C(-2; 3)$ are the vertices of $\triangle ABC$.

a) Find the length of BC, correct to 1 decimal place.
Solution:

$$
\begin{aligned}
BC &= \sqrt{(-2+6)^2 + (3+3)^2} \\
&= \sqrt{4^2 + 9^2} \\
&= \sqrt{16 + 81} \\
&= 9{,}8
\end{aligned}
$$

b) Calculate the gradient of AC.
Solution:

$$
\begin{aligned}
m_{AC} &= \frac{3+1}{-2-4} \\
&= \frac{4}{-6} \\
&= \frac{-2}{3}
\end{aligned}
$$

c) If P has coordinates $(-26; 19)$, show that A, C and P are collinear.
Solution:

$$
\begin{aligned}
m_{AP} &= \frac{19+1}{-26-4} \\
&= \frac{20}{-30} \\
&= \frac{-2}{3}
\end{aligned}
$$

From the previous question we have that $m_{AC} = \frac{-2}{3}$, therefore $m_{AC} = m_{AP}$.
Therefore A, C and P are collinear.

d) Determine the equation of line BC.

Solution:

$$\begin{aligned} m_{BC} &= \frac{3+3}{-2+6} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \\ \therefore y &= \frac{3}{2}x + c \end{aligned}$$

Substitute $B(-6;-3)$:

$$\begin{aligned} (-3) &= \frac{3}{2}(-6) + c \\ -3 &= -9 + c \\ \therefore c &= 6 \\ \therefore y &= \frac{3}{2}x + 6 \end{aligned}$$

The equation of line BC is $y = \frac{3}{2}x + 6$.

e) Show that $\triangle ABC$ is right-angled.

Solution:

For a triangle to be right-angled the gradients of two of the sides must be perpendicular. We need to calculate m_{AC}, m_{BC} and m_{AB}.

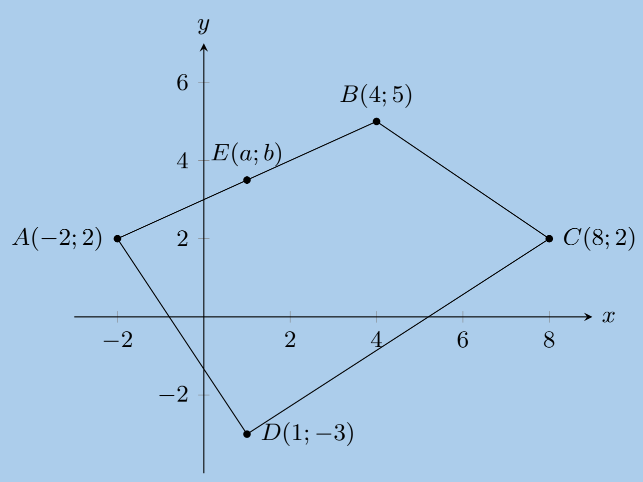
We will start with m_{AC} and m_{BC} since we have these values from previous questions: $m_{AC} = \frac{-2}{3}$ and $m_{BC} = \frac{3}{2}$.

We can check if these two gradients are perpendicular:

$$\begin{aligned} m_{AC} \times m_{BC} &= \frac{-2}{3} \times \frac{3}{2} \\ &= \frac{-6}{6} \\ &= -1 \end{aligned}$$

Since the product of these two gradients is -1, $AC \perp BC$.
Therefore $\triangle ABC$ is right-angled.

31. Given the following diagram:



a) If E is the mid-point of AB, find the values of a and b.
Solution:

$$\begin{aligned} M_{AB} &= \left(\frac{-2+4}{2}; \frac{2+5}{2}\right) \\ &= \left(1; \frac{7}{2}\right) \end{aligned}$$

Therefore $a = 1$ and $b = \frac{7}{2}$.

b) Find the equation of the line perpendicular to BC, which passes through the origin.
Solution:
Let the line be FG.
First calculate the gradient of BC:

$$\begin{aligned} m_{BC} &= \frac{2-5}{8-4} \\ m_{BC} &= \frac{-3}{4} \end{aligned}$$

Now we can calculate the gradient of FG:

$$\begin{aligned} -1 &= m_{BC} \times m_{FG} \\ -1 &= \left(\frac{-3}{4}\right) m_{FG} \\ m_{FG} &= \frac{4}{3} \end{aligned}$$

Therefore we have $y = \frac{4}{3}x - c$. Since the line passes through the origin the y-intercept is 0.

Therefore the equation of the line perpendicular to BC, which passes through the origin is: $y = \frac{4}{3}x$.

c) Find the coordinates of the mid-point of diagonal BD.
Solution:

$$\begin{aligned} M_{BD} &= \left(\frac{4+1}{2}; \frac{5-3}{2}\right) \\ &= \left(\frac{7}{2}; 1\right) \end{aligned}$$

d) Hence show that $ABCD$ is not a parallelogram.
Solution:

$$\begin{aligned} M_{AC} &= \left(\frac{-2+8}{2}; \frac{2+2}{2}\right) \\ &= (3; 2) \end{aligned}$$

The mid-point of BD is not the mid-point of AC. Therefore the diagonals of the quadrilateral do not bisect each other and $ABCD$ is not a parallelogram.

e) If C could be moved, give its new coordinates so that $ABCD$ would be a parallelogram.
Solution:
We use the mid-point of BD to find the new x and y coordinates of C:

$$\begin{aligned}
\frac{5}{2} &= \frac{-2+x}{2} \\
\frac{10}{2} &= -2+x \\
x &= 7
\end{aligned}$$

$$\begin{aligned}
1 &= \frac{2+y}{2} \\
2 &= 2+y \\
0 &= y
\end{aligned}$$

Therefore $C(7;0)$ would make $ABCD$ a parallelogram.

32. A triangle has vertices $A(-1;7)$, $B(8;4)$ and $C(5;-5)$.

a) Calculate the gradient of AB.

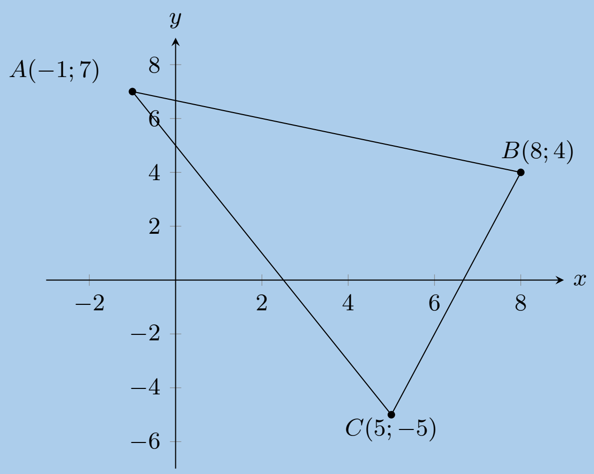
Solution:

$$\begin{aligned}
m_{AB} &= \frac{4-7}{8+1} \\
&= \frac{-3}{9} \\
&= -\frac{1}{3}
\end{aligned}$$

b) Prove that the triangle is right-angled at B.

Solution:

First draw a sketch:



For the triangle to be right-angled at B, $m_{AB} \times m_{BC} = -1$. We have m_{AB} from the previous question.

$$\begin{aligned}
m_{BC} &= \frac{-5-4}{5-8} \\
&= \frac{-9}{-3} \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\therefore m_{AB} \times m_{BC} &= 3 \times \frac{-1}{3} \\
&= -1
\end{aligned}$$

Therefore $BC \perp AB$ and $\triangle ABC$ is right-angled at B.

c) Determine the length of AB.

Solution:

$$\begin{aligned} AB &= \sqrt{(8-(-1))^2+(4-7)^2} \\ &= \sqrt{90} \\ &= 9{,}49 \end{aligned}$$

d) Determine the equation of the line from A to the mid-point of BC.
Solution:
First find the mid-point of BC:

$$\begin{aligned} M_{BC} &= \left(\frac{8+5}{2}; \frac{4-5}{2}\right) \\ &= \left(\frac{13}{2}; \frac{-1}{2}\right) \end{aligned}$$

Now we can calculate the gradient of the line:

$$\begin{aligned} m &= \frac{7-\frac{-1}{2}}{-1-\frac{13}{2}} \\ &= -1 \\ \therefore y &= -x+c \end{aligned}$$

Substitute $A(-2;2)$:

$$\begin{aligned} (2) &= -(-1)+c \\ c &= 6 \\ \therefore y &= -x+6 \end{aligned}$$

The equation of the the line from A to the mid-point of BC is $y=-x+6$.

e) Find the area of the triangle ABC.
Solution:
First find the length of BC:

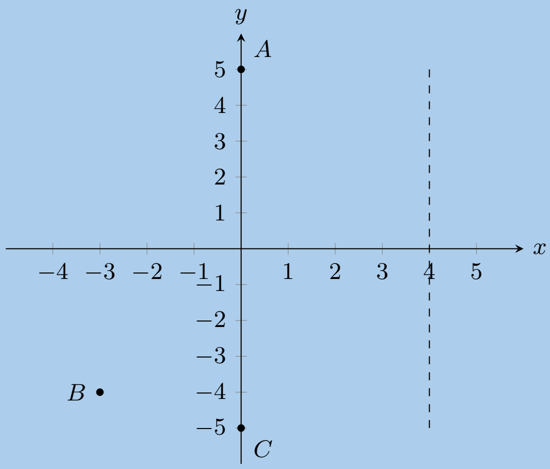
$$\begin{aligned} BC &= \sqrt{(8-5)^2+(4+5)^2} \\ &= 3\sqrt{10} \\ &= 9{,}5 \end{aligned}$$

Now we can find the area:

$$\begin{aligned} \text{Area } &= \frac{1}{2}bh \\ &= \frac{1}{2}(AB)(BC) \\ &= \frac{1}{2}(3\sqrt{10})(3\sqrt{10}) \\ &= 45 \end{aligned}$$

33. A quadrilateral has vertices $A(0;5)$, $B(-3;-4)$, $C(0;-5)$ and $D(4;k)$ where $k \geq 0$.

a) What should k be so that AD is parallel to CD?
Solution:
We first draw a sketch. We note that D lies somewhere on the line $x=4$.



For parallel lines $m_{AD} = m_{CD}$:

$$\begin{aligned}
\frac{k-5}{4-0} &= \frac{k+5}{4-0} \\
\frac{k-5}{4} &= \frac{k+5}{4} \\
\therefore k-5 &= k+5 \\
0 &= 10
\end{aligned}$$

Therefore there is no value of k such that AD will be parallel to DC.
We can see that this must be true since D is a common point on lines AD and CD so AD cannot be parallel to CD.

b) What should k be so that $CD = \sqrt{52}$?
Solution:

$$\begin{aligned}
\sqrt{52} &= \sqrt{(4-0)^2 + (k+5)^2} \\
\sqrt{52} &= \sqrt{16 + (k+5)^2} \\
\therefore 52 &= 16 + k^2 + 10k + 25 \\
0 &= k^2 + 10k - 11 \\
&= (k+11)(k-1) \\
\therefore k = -11 &\text{ or } k = 1
\end{aligned}$$

But $k \geq 0$, therefore $k = 1$

34. On the Cartesian plane, the three points $P(-3;4)$, $Q(7;-1)$ and $R(3;b)$ are collinear.

a) Find the length of PQ.
Solution:

$$\begin{aligned}
PQ &= \sqrt{(-3-7)^2 + (-1-4)^2} \\
&= 5\sqrt{5}
\end{aligned}$$

b) Find the gradient of PQ.
Solution:

$$\begin{aligned}
m_{PQ} &= \frac{-1-4}{7+3} \\
&= \frac{-5}{10} \\
&= -\frac{1}{2}
\end{aligned}$$

c) Find the equation of PQ.

Solution:

$y = -\frac{1}{2}x + c$. Substitute $P(-3;4)$:

$$\begin{aligned}(4) &= -\frac{1}{2}(-3) + c \\ c &= \frac{5}{2} \\ \therefore y &= -\frac{1}{2}x + \frac{5}{2}\end{aligned}$$

The equation of the PQ is $y = -\frac{1}{2} + \frac{5}{2}$.

d) Find the value of b.

Solution:

$$\begin{aligned}m_{QR} &= m_{PQ} \\ \frac{b+1}{3-7} &= -\frac{1}{2} \\ b+1 &= 2 \\ b &= 1\end{aligned}$$

35. Given $A(4;9)$ and $B(-2;-3)$.

a) Find the mid-point M of AB.

Solution:

$$\begin{aligned}M\left(\frac{4+(-2)}{2}; \frac{9-3}{2}\right) \\ M(1;3)\end{aligned}$$

b) Find the gradient of AB.

Solution:

$$\begin{aligned}m_{AB} &= \frac{-3-9}{-2-4} \\ &= \frac{-12}{-6} \\ &= 2\end{aligned}$$

c) Find the gradient of the line perpendicular to AB.

Solution:

$$\begin{aligned}m &= -1 \div m_{AB} \\ &= -1 \div 2 \\ &= -\frac{1}{2}\end{aligned}$$

d) Find the equation of the perpendicular bisector of AB.

Solution:

From the previous question we have that $y = -\frac{1}{2}x + c$. Substitute M:

$$\begin{aligned}3 &= -\frac{1}{2}(1) + c \\ c &= \frac{7}{2} \\ \therefore y &= -\frac{1}{2}x + \frac{7}{2}\end{aligned}$$

The equation of the perpendicular bisector of AB is $y = -\frac{1}{2}x + \frac{7}{2}$

e) Find the equation of the line parallel to AB, passing through $(0;6)$.

Solution:

Since the line is parallel to AB the gradient is the same and so we have: $y = 2x + c$. Substitute: $(0;6)$:

$$\begin{aligned} (6) &= 2(0) + c \\ c &= 6 \\ \therefore y &= 2x + 6 \end{aligned}$$

The equation of the line parallel to AB and passing through $(0;6)$ is $y = 2x + 6$.

36. $L(-1;-1)$, $M(-2;4)$, $N(x;y)$ and $P(4;0)$ are the vertices of parallelogram $LMNP$.

a) Determine the coordinates of N.

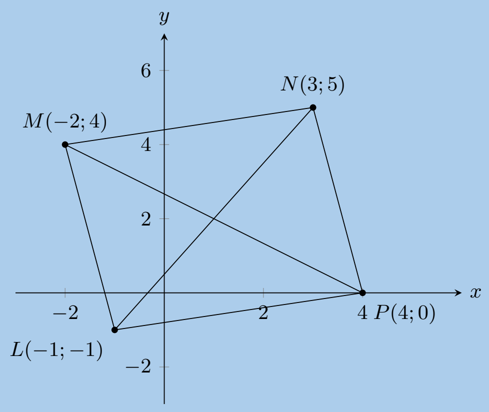
Solution:

Since $LMNP$ is a parallelogram we know that $LM \parallel NP$. Therefore:

$$\begin{aligned} m_{LM} &= m_{NP} \\ \frac{4+1}{-2+1} &= \frac{y-0}{x-4} \\ \frac{5}{-1} &= \frac{y}{x-4} \\ \therefore y &= 5 \\ -1 &= x - 4 \\ 3 &= x \end{aligned}$$

Therefore $N(3;5)$.

We can now sketch the parallelogram:



b) Show that MP is perpendicular to LN and state what type of quadrilateral $LMNP$ is, other than a parallelogram.

Solution:

$$\begin{aligned} m_{MP} &= \frac{0-4}{4+2} \\ &= \frac{-4}{6} \\ &= \frac{-2}{3} \end{aligned}$$

$$\begin{aligned} m_{LN} &= \frac{5+1}{3+1} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

$$
\begin{aligned}
m_{MP} \times m_{LN} &= \frac{-2}{3} \times \frac{3}{2} \\
&= -1 \\
\therefore MP \perp LN
\end{aligned}
$$

$LMNP$ is a rhombus since the diagonals intersect at right-angles.

c) Show that $LMNP$ is a square.

Solution:

We can calculate the length of each side and show that all four lengths are the same:

$$
\begin{aligned}
MN &= \sqrt{(-2-3)^2 + (4-5)^2} \\
&= \sqrt{26}
\end{aligned}
$$

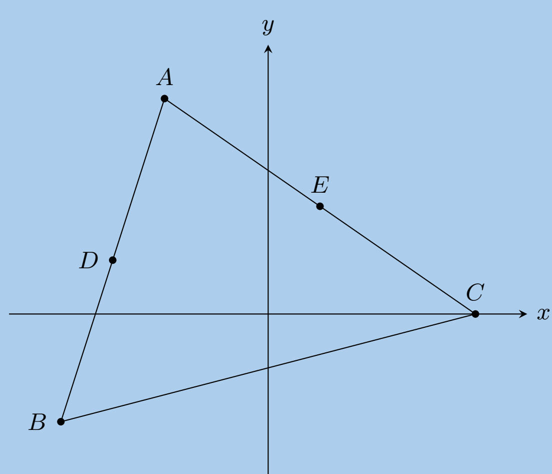
$$
\begin{aligned}
NP &= \sqrt{(3-4)^2 + (5-0)^2} \\
&= \sqrt{26}
\end{aligned}
$$

$$
\begin{aligned}
LP &= \sqrt{(-1-4)^2 + (-1-0)^2} \\
&= \sqrt{26}
\end{aligned}
$$

$$
\begin{aligned}
ML &= \sqrt{(-2+1)^2 + (4+1)^2} \\
&= \sqrt{26}
\end{aligned}
$$

Therefore $LMNP$ is a square, all the sides are equal in length.

37. $A(-2;4)$, $B(-4;-2)$ and $C(4;0)$ are the vertices of $\triangle ABC$. D and $E(1;2)$ are the mid-points of AB and AC respectively.



a) Find the gradient of BC.

Solution:

$$
\begin{aligned}
m_{BC} &= \frac{0+2}{4+4} \\
&= \frac{2}{8} \\
&= \frac{1}{4}
\end{aligned}
$$

b) Show that the coordinates of D, the mid-point of AB are $(-3;1)$.

Solution:

$$
\begin{aligned}
D &= \left(\frac{-2-4}{2}; \frac{-2+4}{2}\right) \\
&= (-3; 1)
\end{aligned}
$$

c) Find the length of DE.
Solution:

$$
\begin{aligned}
DE &= \sqrt{(1-(-3))^2 + (2-1)^2} \\
&= \sqrt{17} \\
&= 4{,}1
\end{aligned}
$$

d) Find the gradient of DE. Make a conjecture regarding lines BC and DE.
Solution:

$$
\begin{aligned}
m_{DE} &= \frac{1-2}{-3-1} \\
&= \frac{1}{4}
\end{aligned}
$$

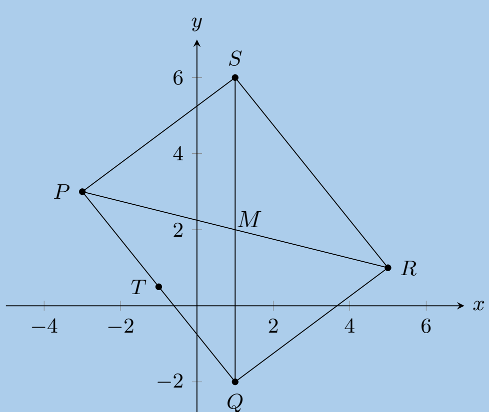
A conjecture regarding lines BC and DE is $DE \parallel BC$.

e) Determine the equation of BC.
Solution:
From earlier we have the gradient of BC, so the equation of BC is: $y = \frac{1}{4}x + c$. Substitute B:

$$
\begin{aligned}
(-2) &= \frac{1}{4}(-4) + c \\
c &= -1 \\
y &= \frac{1}{4}x - 1
\end{aligned}
$$

The equation of BC is $y = -\frac{1}{2}x - 4$.

38. In the diagram points $P(-3;3)$, $Q(1;-2)$, $R(5;1)$ and $S(x;y)$ are the vertices of a parallelogram.



a) Calculate the length of PQ.
Solution:

$$
\begin{aligned}
PQ &= \sqrt{(-3-1)^2 + (3+2)^2} \\
&= \sqrt{41}
\end{aligned}
$$

b) Find the coordinates of M where the diagonals meet.
Solution:

M is the mid-point of both PR and QS ($PQRS$ is a parallelogram). Since we do not know the coordinates of S we will use PR to find M.

$$
\begin{aligned}
&M\left(\frac{-3+5}{2};\frac{3+1}{2}\right)\\
&\therefore M(1;2)
\end{aligned}
$$

c) Find T, the mid-point of PQ.
Solution:

$$
\begin{aligned}
&T\left(\frac{-3+1}{2};\frac{3-2}{2}\right)\\
&\therefore T\left(-1;\frac{1}{2}\right)
\end{aligned}
$$

d) Show that $MT \parallel QR$.
Solution:

$$
\begin{aligned}
m_{MT} &= \frac{2-\frac{1}{2}}{1+1}\\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
m_{QR} &= \frac{-2-1}{1-4}\\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
\therefore m_{MT} &= m_{QR}\\
\therefore MT &\parallel QR
\end{aligned}
$$

e) Calculate the coordinates of S.
Solution:
We can use the coordinates of M to find the coordinates of S:

$$M(1;2) = S\left(\frac{x+1}{2};\frac{y-2}{2}\right)$$

Solve for x:

$$
\begin{aligned}
1 &= \frac{x+1}{2}\\
2 &= x+1\\
x &= 1
\end{aligned}
$$

Solve for y:

$$
\begin{aligned}
2 &= \frac{y-2}{2}\\
4 &= y-2\\
y &= 6
\end{aligned}
$$

Therefore $S(1;6)$.

39. The coordinates of $\triangle PQR$ are as follows: $P(5;1)$, $Q(1;3)$ and $R(1;-2)$.

a) Through calculation, determine whether the triangle is equilateral, isosceles or scalene. Be sure to show all your working.
Solution:

$$PQ = \sqrt{(5-1)^2 + (1-3)^2}$$
$$= 2\sqrt{5}$$

$$QR = \sqrt{(1-1)^2 + (3+2)^2}$$
$$= 5$$

$$PR = \sqrt{(5-1)^2 + (1+2)^2}$$
$$= 5$$

$\therefore QR = PR$ triangle is isosceles

$\triangle PQR$ is an isosceles triangle.

b) Find the coordinates of points S and T, the mid-points of PQ and QR.

Solution:

$$S\left(\frac{5+1}{2}; \frac{1+3}{2}\right)$$
$$\therefore S(3;2)$$

$$T\left(\frac{1+1}{2}; \frac{3-2}{2}\right)$$
$$\therefore T\left(1; \frac{1}{2}\right)$$

c) Determine the gradient of the line ST.

Solution:

$$m_{ST} = \frac{2-\frac{1}{2}}{3-1}$$
$$= \frac{\frac{3}{2}}{2}$$
$$= \frac{3}{4}$$

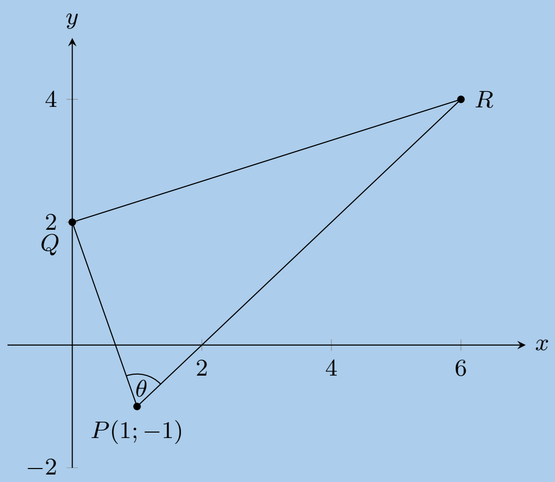
d) Prove that $ST \parallel PR$.

Solution:

$$m_{PR} = \frac{1+2}{5-1}$$
$$= \frac{3}{4}$$

$\therefore ST \parallel PR$ (equal gradients)

40. The following diagram shows $\triangle PQR$ with $P(-1;1)$. The equation of QR is $x - 3y = -6$ and the equation of PR is $x - y - 2 = 0$. $R\hat{P}Q = \theta$.



a) Write down the coordinates of Q.
Solution:
Q lies on the y-axis so we have $Q(0;y)$. Using the equation of QR we get:

$$\begin{aligned} -3y &= -6 \\ \therefore y &= 2 \\ \therefore Q(0;2) \end{aligned}$$

b) Prove that $PQ \perp QR$.
Solution:

$$\begin{aligned} y &= \frac{1}{3}x + 2 \\ \therefore m_{QR} &= \frac{1}{3} \\ \\ m_{QP} &= -3 \\ \\ m_{QP} \times m_{QR} &= \frac{1}{3} \times -3 \\ &= -1 \end{aligned}$$

Therefore $PQ \perp QR$.

c) Write down the gradient of PR.
Solution:
The equation of PR is $x - y - 2 = 0$. In standard form this is $y = x - 2$.
Therefore $m_{PR} = 1$.

d) If the y-coordinate of R is 4, calculate the x-coordinate.
Solution:
We use the equation of PR and substitute in $y = 4$:

$$\begin{aligned} x - 4 - 2 &= 0 \\ \therefore x &= 6 \end{aligned}$$

e) Find the equation of the line from P to S (the mid-point of QR).
Solution:
We first calculate S:

$$\begin{aligned} S\left(\frac{0+6}{2}; \frac{2+4}{2}\right) \\ S\,(3;3) \end{aligned}$$

Now we can find the equation of the line SP:

$$
\begin{aligned}
m_{SP} &= \frac{-1-3}{1-3} \\
&= \frac{-4}{-2} \\
&= 2 \\
\therefore y &= 2x + c \\
-1 &= 2(1) + c \\
y &= 2x - 3
\end{aligned}
$$

The equation of the line from P to S is $y = 2x - 3$.

41. The points $E(-3;0)$, $L(3;5)$ and $S(t+1, 2{,}5)$ are collinear.

a) Determine the value of t.

Solution:

Since E, L and S are collinear, $m_{EL} = m_{LS}$. Therefore:

$$
\begin{aligned}
m_{EL} &= \frac{5}{6} \\
\therefore \frac{5-2{,}5}{3-(t+1)} &= \frac{5}{6} \\
\frac{2{,}5}{2-t} &= \frac{5}{6} \\
6(2{,}5) &= 5(2-t) \\
15 &= 10 - 5t \\
5t &= -5 \\
t &= -1
\end{aligned}
$$

b) Determine the values of a and b if the equation of the line passing through E, L and S is $\frac{x}{a} + \frac{y}{b} = 1$.

Solution:

$$
\begin{aligned}
y &= \frac{5}{6}x + c \\
\therefore 0 &= \frac{5}{6}(-3) + c \\
c &= \frac{5}{2} \\
y &= \frac{5}{6}x + \frac{5}{2}
\end{aligned}
$$

Now we can rearrange the equation:

$$
\begin{aligned}
\frac{5}{2} &= y - \frac{5}{6}x \\
1 &= \frac{2}{5}y - \frac{1}{3}x \\
&= \frac{2y}{5} - \frac{x}{3}
\end{aligned}
$$

We have $\frac{x}{a} = -\frac{x}{3}$, therefore $a = -3$.

We have $\frac{y}{b} = \frac{2y}{5}$, therefore $b = \frac{5}{2}$.

42. Given: $A(-3;-4), B(-1;-7), C(2;-5)$ and $D(0;-2)$.

a) Calculate the distance AC and the distance BD. Leave your answers in surd form.

Solution:

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ AC &= \sqrt{(2 - (-3))^2 + (-5 - (-4))^2} \\ &= \sqrt{5^2 (-1)^2} \\ &= \sqrt{25 + 1} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ BD &= \sqrt{(0 - (-1))^2 + (-2 - (-7))^2} \\ &= \sqrt{1^2 + 5^2} \\ &= \sqrt{26} \end{aligned}$$

Therefore $d_{AC} = d_{BD} = \sqrt{26}$.

b) Determine the coordinates of M, the mid-point of BD.

Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-1 + 0}{2}; \frac{-7 + (-2)}{2}\right) \\ &= \left(\frac{-1}{2}; \frac{-9}{2}\right) \\ M &= (-0{,}5; -4{,}5) \end{aligned}$$

c) Prove that $AM \perp BD$.

Solution:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m_{AM} &= \frac{-4 - (-4{,}5)}{-3 - (-0{,}5)} \\ &= \frac{0{,}5}{-2{,}5} \\ &= -\frac{1}{5} \end{aligned}$$

$$\begin{aligned} m_{BD} &= \frac{-7 - (-2)}{-1 - (0)} \\ &= \frac{-5}{-1} \\ &= 5 \end{aligned}$$

$$\begin{aligned} m_{BD} \times m_{AM} &= -\frac{1}{5} \times 5 \\ &= -1 \end{aligned}$$

Therefore $AM \perp BD$.

d) Prove that A, M and C are collinear.

Solution:

$$\begin{aligned} m_{MC} &= \frac{-5 - (-4{,}5)}{-2 - (-0{,}5)} \\ &= \frac{-0{,}5}{2{,}5} \\ &= -\frac{1}{5} \end{aligned}$$

From earlier we know that $m_{AM} = -\frac{1}{5}$.

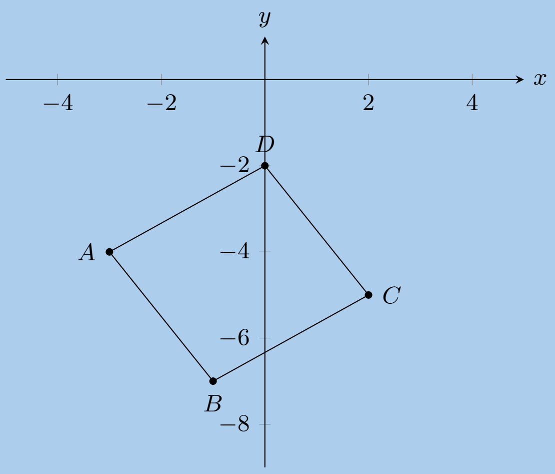
$m_{MC} = m_{AM}$ and M is a common point.

Therefore A, M and C are collinear.

e) What type of quadrilateral is $ABCD$?

Solution:

First draw a sketch:



From earlier we found that $d_{AC} = d_{BD} = \sqrt{26}$. Therefore the diagonals are equal in length.

Since the diagonals are equal in length we know that the quadrilateral must be a square.

We can confirm this by showing that all four sides are equal in length and that $AB \perp BC$, $BC \perp CD$, $CD \perp AD$ and $AD \perp AB$.

43. $M(2;-2)$ is the mid-point of AB with point $A(3;1)$. Determine:

a) the coordinates of B.

Solution:

$$M(x;y) = (\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2})$$
$$(2;-2) = (\frac{3 + x_B}{2}; \frac{1 + y_B}{2})$$

$$2 = \frac{3 + x_B}{2}$$
$$4 = 3 + x_B$$
$$x_B = 1$$

$$-2 = \frac{1 + y_B}{2}$$
$$-4 = 1 + y_B$$
$$y_B = -5$$

$$\therefore B(x;y) = (1;-5)$$

b) the gradient of AM.

Solution:

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
m_{AM} &= \frac{-2 - 1}{2 - 3} \\
&= \frac{-3}{-1} \\
&= 3
\end{aligned}
$$

c) the equation of the line AM.

Solution:

$$
\begin{aligned}
y &= mx + c \\
(-2) &= (3)(2) + c \\
c &= -2 - 6 \\
c &= -8 \\
\\
\therefore y &= 3x - 8
\end{aligned}
$$

d) the perpendicular bisector of AB.

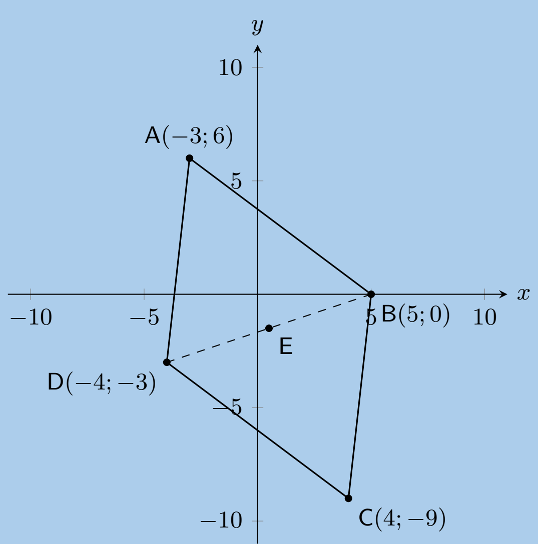
Solution:

The perpendicular bisector of AB passes through M and has a gradient $-\frac{1}{m_{AB}}$.

$$
\begin{aligned}
m &= -\frac{1}{m_{AB}} \\
m &= -\frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
y &= mx + c \\
(-2) &= (-\frac{1}{3})(2) + c \\
c &= \frac{2}{3} - 2 \\
c &= -\frac{4}{3} \\
\therefore y &= -\frac{1}{3}x - \frac{4}{3}
\end{aligned}
$$

44. ABCD is a quadrilateral with $A(-3; 6),\ B(5; 0),\ C(4; -9),\ D(-4; -3)$.



a) Determine the coordinates of E, the mid-point of BD.
Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ E &= \left(\frac{5+(-4)}{2};\frac{0+(-3)}{2}\right) \\ \therefore E &= \left(\frac{1}{2};-\frac{3}{2}\right) \end{aligned}$$

b) Prove that $ABCD$ is a parallelogram.
Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ M_{AC} &= \left(\frac{-3+4}{2};\frac{6+(-9)}{2}\right) \\ &= \left(\frac{1}{2};-\frac{3}{2}\right) \\ \therefore M_{AC} &= E \end{aligned}$$

$ABCD$ is a parallelogram (Diagonals bisect at E).

c) Find the equation of diagonal BD.
Solution:

$$\begin{aligned} (y-y_1) &= m(x-x_1) \\ m &= \frac{y_2-y_1}{x_2-x_1} \end{aligned}$$

$$\begin{aligned} (y-0) &= \frac{-3-0}{-4-(5)}(x-(5)) \\ \therefore y &= \frac{-3}{-9}(x-5) \\ \therefore y &= \frac{1}{3}x-\frac{5}{3} \end{aligned}$$

d) Determine the equation of the perpendicular bisector of BD.
Solution:
The perpendicular bisector of BD passes through E and has a gradient $-\frac{1}{m_{BD}}$

$$\begin{aligned} m &= -\frac{1}{m_{BD}} \\ m &= \frac{1}{-\frac{1}{3}} \\ m &= -3 \end{aligned}$$

$$\begin{aligned} y &= mx+c \\ -\frac{3}{2} &= (3)(\frac{1}{2})+c \\ c &= -\frac{3}{2}+\frac{3}{2} \\ c &= 0 \\ \therefore y &= -3x \end{aligned}$$

e) Determine the gradient of AC.
Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{6 - (-9)}{-3 - 4}$$
$$m_{AC} = -\frac{15}{7}$$

f) Is $ABCD$ a rhombus? Explain why or why not.
Solution:

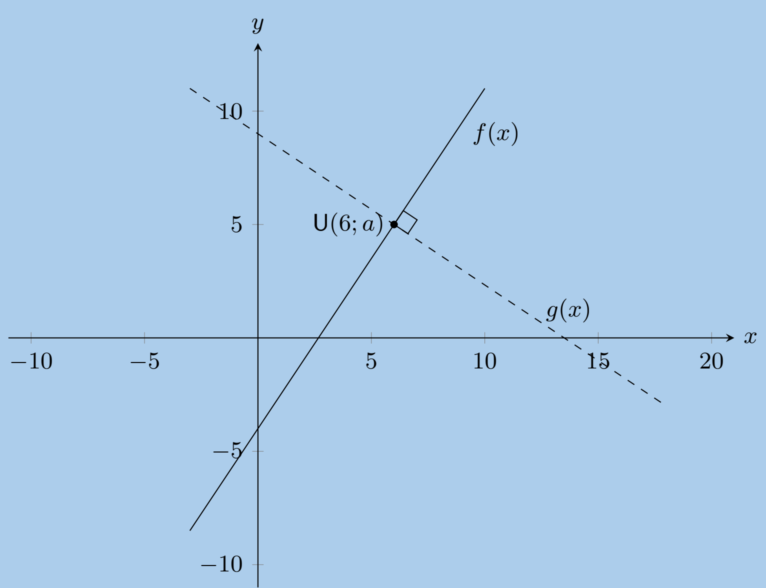
$$m_{AC} \times m_{BD} = -\frac{15}{7} \times \frac{1}{3}$$
$$= -\frac{15}{21} \neq -1$$

No, $ABCD$ is not a rhombus because the diagonals do not intersect at right angles.

g) Find the length of AB.
Solution:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$AB = \sqrt{(-3 - 5))^2 + (6 - 0)^2}$$
$$= \sqrt{8^2 + 6^2}$$
$$= \sqrt{64 + 36}$$
$$= \sqrt{100}$$
$$= 10$$

45. In the diagram below, $f(x) = \frac{3}{2}x - 4$ is sketched with $U(6; a)$ on $f(x)$.



a) Determine the value of a in $U(6; a)$.
Solution:
We can substitute U into $f(x)$ to find a:

$$f(x) = \frac{3}{2}x - 4$$
$$a = \frac{3}{2}(6) - 4$$
$$= 9 - 5$$
$$\therefore a = 5$$

b) A line, $g(x)$, passing through U, is perpendicular to $f(x)$. $V(b;4)$ lies on $g(x)$. Determine the value of b.
Solution:
We know that $m_{g(x)} \times m_{f(x)} = -1$, therefore $m_{g(x)} = -\frac{2}{3}$.

$$\begin{aligned} g(x) &= -\frac{2}{3}x + c \\ 5 &= -\frac{2}{3}(6) + c \\ c &= 5 + 4 \\ c &= 9 \end{aligned}$$

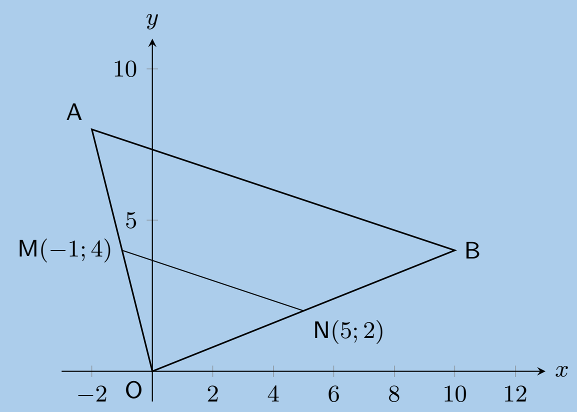
Therefore $g(x) = -\frac{2}{3}x + 9$. Substitute in V to solve for b:

$$\begin{aligned} 4 &= -\frac{2}{3}b + 9 \\ -5 &= -\frac{2}{3}b \\ b &= \frac{15}{2} \\ \therefore b &= 7\frac{1}{2} \end{aligned}$$

c) If $U(6;5)$, $V(7\frac{1}{2}, 4)$ and $W(1;c)$ are collinear, determine the value of c.
Solution:
U, V and W are collinear and the equation for the line is $g(x) = -\frac{2}{3}x + 9$. We can substitute W into the equation for the line to solve for c:

$$\begin{aligned} c &= -\frac{2}{3}(1) + 9 \\ \therefore c &= 8\frac{1}{3} \end{aligned}$$

46. In the diagram below, M and N are the mid-points of OA and OB respectively.



a) Calculate the gradient of MN.
Solution:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 2}{-1 - 5} \\ m_{MN} &= \frac{2}{-6} \\ m_{MN} &= -\frac{1}{3} \end{aligned}$$

b) Find the equation of the line through M and N in the form $y = mx + c$.
Solution:

$$\begin{aligned} y &= mx + c \\ 4 &= -\frac{1}{3}(-1) + c \\ c &= 4 - \frac{1}{3} \\ c &= 3\frac{2}{3} \\ y &= -\frac{1}{3}x + 3\frac{2}{3} \end{aligned}$$

c) Show that $AB \parallel MN$.
Solution:
First find the coordinates of A:

$$\begin{aligned} M_{OA} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ (-1, 4) &= \left(\frac{x_A + 0}{2}; \frac{y_A + 0}{2}\right) \end{aligned}$$

$$\begin{aligned} -1 &= \frac{x_A}{2} \\ x_A &= -2 \\ 4 &= \frac{y_A}{2} \\ y_A &= 8 \\ \therefore A(x; y) &= (-2; 8) \end{aligned}$$

Next find the coordinates of B:

$$\begin{aligned} M_{OB} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ (5, 2) &= \left(\frac{x_B}{2}; \frac{y_B}{2}\right) \end{aligned}$$

$$\begin{aligned} 5 &= \frac{x_B}{2} \\ x_B &= 10 \\ 2 &= \frac{y_B}{2} \\ y_B &= 4 \\ \therefore B(x; y) &= (10; 4) \end{aligned}$$

Now we can calculate the gradient of AB and compare it to the gradient of MN:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m_{AB} &= \frac{8 - 4}{-2 - 10} \\ &= -\frac{4}{12} \\ m_{AB} &= -\frac{1}{3} = m_{MN} \\ &\therefore AB \parallel MN \end{aligned}$$

d) Write down the value of the ratio: $\dfrac{\text{area } \triangle OAB}{\text{area } \triangle OMN}$.

Solution:
We note the following:

- $OA = 2OM$ and $OB = 2ON$ (Mid-points)
- $AB = 2MN$ (Mid-point theorem)

From this we can see that $\triangle OAB$ is twice the size of $\triangle OMN$ (each side of $\triangle OAB$ is twice the size of the same side in $\triangle OMN$). Therefore the area of $\triangle OAB$ is twice the area of $\triangle OMN$:

$$\begin{aligned}\frac{\text{area } \triangle OAB}{\text{area } \triangle OMN} &= \frac{2\text{area } \triangle OMN}{\text{area } \triangle OMN} \\ &= 2\end{aligned}$$

e) Write down the coordinates of P such that $OAPB$ is a parallelogram.
Solution:
We can use the fact that the diagonals of a parallelogram bisect each other to find P. The mid-point of AB must be the same as the mid-point of OP.

$$\begin{aligned}M_{AB} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ M_{AB} &= \left(\frac{-2 + 10}{2}; \frac{8 + 4}{2}\right) \\ M_{AB} &= (4; 6)\end{aligned}$$

$$\begin{aligned}(4; 6) &= \left(\frac{x_p + 0}{2}; \frac{y_p + 0}{2}\right) \\ (8; 12) &= \left(\frac{x_p}{2}; \frac{y_p}{2}\right) \\ \therefore P(x; y) &= (8; 12)\end{aligned}$$

47. $A(6; -4)$, $B(8; 2)$, $C(3; a)$ and $D(b; c)$ are points on the Cartesian plane. Determine the value of:

a) a if A, B and C are collinear.
Solution:
First find the gradient of AB:

$$\begin{aligned}m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - 2}{6 - 8} \\ &= \frac{-6}{-2} \\ &= 3\end{aligned}$$

Therefore we have $y = 3x + c$. Now substitute in A:

$$\begin{aligned}-4 &= 3(6) + c \\ c &= -22\end{aligned}$$

Now we can substitute in C to solve for a:

$$\begin{aligned}a &= 3(3) - 22 \\ \therefore a &= -13\end{aligned}$$

b) b and c if B is the mid-point of A and D.
Solution:

$$
\begin{aligned}
M_{AD} &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
(8; 2) &= \left(\frac{6 + b}{2}; \frac{-4 + c}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
8 &= \frac{6 + b}{2} \\
16 &= 6 + b \\
b &= 10
\end{aligned}
$$

$$
\begin{aligned}
2 &= \frac{-4 + c}{2} \\
4 &= -4 + c \\
c &= 8 \\
B(x; y) &= (8; 12)
\end{aligned}
$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GDR	2. 2GDS	3. 2GDT	4a. 2GDV	4b. 2GDW	4c. 2GDX
4d. 2GDY	5. 2GDZ	6. 2GF2	7. 2GF3	8. 2GF4	9. 2GF5
10. 2GF6	11. 2GF7	12. 2GF8	13. 2GF9	14. 2GFB	15. 2GFC
16. 2GFD	17. 2GFF	18. 2GFG	19. 2GFH	20. 2GFJ	21. 2GFK
22. 2GFM	23. 2GFN	24. 2GFP	25. 2GFQ	26. 2GFR	27. 2GFS
28. 2GFT	29. 2GFV	30. 2GFW	31. 2GFX	32. 2GFY	33. 2GFZ
34. 2GG2	35. 2GG3	36. 2GG4	37. 2GG5	38. 2GG6	39. 2GG7
40. 2GG8	41. 2GG9	42. 2GGB	43. 2GGC	44. 2GGD	45. 2GGF
46. 2GGG	47. 2GGH				

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CHAPTER 9

Finance and growth

9.1 Introduction

- Content covered in this chapter includes the simple and compound interest formulae. These formulae are then applied to hire purchase, inflation and population growth. A short introduction to exchange rates is included.
- For compound interest learners are not expected to solve for n.
- Discuss terminology relating to simple and compound interest such as principle amount, accumulated amount, etc.
- It is very important to emphasise not rounding off calculations until final answer as this affects accuracy.
- Learners should do calculations in one step using the memory function on their calculators.

Some of the videos for this chapter use dollars instead of rands in the examples. Because both rands and dollars are decimal currencies you can simply change the currency symbol and the calculations will work out the same.

9.2 Simple interest

Exercise 9 – 1:

1. An amount of R 3500 is invested in a savings account which pays simple interest at a rate of 7,5% per annum. Calculate the balance accumulated by the end of 2 years.
   Solution:

$$\begin{aligned} P &= 3500 \\ i &= 0{,}075 \\ n &= 2 \\ A &= ? \end{aligned}$$

$$\begin{aligned} A &= P(1 + in) \\ A &= 3500(1 + (0{,}075)(2)) \\ A &= 3500(1{,}15) \\ A &= \text{R } 4025 \end{aligned}$$

2. An amount of R 4090 is invested in a savings account which pays simple interest at a rate of 8% per annum. Calculate the balance accumulated by the end of 4 years.
   Solution:
   Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= 4090 \\ n &= 4 \\ i &= \frac{8}{100} = 0{,}08 \end{aligned}$$

$$\begin{aligned} A &= P(1 + in) \\ &= \text{R } 4090\,(1 + (0{,}08) \times 4) \\ &= \text{R } 5398{,}80 \end{aligned}$$

3. An amount of R 1250 is invested in a savings account which pays simple interest at a rate of 6% per annum. Calculate the balance accumulated by the end of 6 years.
   Solution:

Read the question carefully and write down the given information:

$$\begin{aligned}A &= ?\\ P &= 1250\\ n &= 6\\ i &= \frac{6}{100} = 0{,}06\end{aligned}$$

Simple interest formula:

$$\begin{aligned}A &= P(1 + in)\\ &= \text{R } 1250\,(1 + (0{,}06) \times 6)\\ &= \text{R } 1700{,}00\end{aligned}$$

4. An amount of R 5670 is invested in a savings account which pays simple interest at a rate of 8% per annum. Calculate the balance accumulated by the end of 3 years.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned}A &= ?\\ P &= 5670\\ n &= 3\\ i &= \frac{8}{100} = 0{,}08\end{aligned}$$

Simple interest formula:

$$\begin{aligned}A &= P(1 + in)\\ &= \text{R } 5670{,}00\,(1 + (0{,}08) \times 3)\\ &= \text{R } 7030{,}80\end{aligned}$$

5. Calculate the accumulated amount in the following situations:
   a) A loan of R 300 at a rate of 8% for 1 year.
   Solution:

$$\begin{aligned}P &= 300\\ i &= 0{,}08\\ n &= 1\\ A &= ?\\ \\ A &= P(1 + in)\\ A &= 300(1 + (0{,}08)(1))\\ A &= 300(1{,}08)\\ A &= \text{R } 324\end{aligned}$$

   b) An investment of R 2250 at a rate of 12,5% p.a. for 6 years.
   Solution:

$$\begin{aligned}P &= 2250\\ i &= 0{,}125\\ n &= 6\\ A &= ?\\ \\ A &= P(1 + in)\\ A &= 2250(1 + (0{,}125)(6))\\ A &= 2250(1{,}75)\\ A &= \text{R } 3937{,}50\end{aligned}$$

6. A bank offers a savings account which pays simple interest at a rate of 6% per annum. If you want to accumulate R 15 000 in 5 years, how much should you invest now?

Solution:

Read the question carefully and write down the given information:

$$
\begin{aligned}
A &= \text{R } 15\ 000 \\
P &= ? \\
i &= \frac{6}{100} = 0{,}06 \\
n &= 5
\end{aligned}
$$

Simple interest formula:

$$
\begin{aligned}
A &= P(1 + in) \\
\text{R } 15\ 000 &= P\,(1 + (0{,}06) \times 5) \\
P &= \frac{\text{R } 15\ 000}{1{,}3} \\
&= \text{R } 11\ 538{,}46
\end{aligned}
$$

7. Sally wanted to calculate the number of years she needed to invest R 1000 for in order to accumulate R 2500. She has been offered a simple interest rate of 8,2% p.a. How many years will it take for the money to grow to R 2500?

Solution:

$$
\begin{aligned}
A &= 2500 \\
P &= 1000 \\
i &= 0{,}082 \\
n &= ?
\end{aligned}
$$

$$
\begin{aligned}
A &= P(1 + in) \\
2500 &= 1000(1 + (0{,}082)(n)) \\
\frac{2500}{1000} &= 1 + 0{,}082n \\
\frac{2500}{1000} - 1 &= 0{,}082n \\
\left(\frac{2500}{1000} - 1\right) \div 0{,}082 &= n \\
n &= 18{,}3
\end{aligned}
$$

It would take 19 years for R 1000 to become R 2500 at 8,2% p.a.

8. Joseph deposited R 5000 into a savings account on his son's fifth birthday. When his son turned 21, the balance in the account had grown to R 18 000. If simple interest was used, calculate the rate at which the money was invested.

Solution:

$$
\begin{aligned}
A &= 18\ 000 \\
P &= 5000 \\
i &= ? \\
n &= 21 - 5 = 16
\end{aligned}
$$

$$
\begin{aligned}
A &= P(1 + in) \\
18\ 000 &= 5000(1 + (i)(16)) \\
\frac{18\ 000}{5000} &= 1 + 16i \\
\frac{18\ 000}{5000} - 1 &= 16i \\
\left(\frac{18\ 000}{5000} - 1\right) \div 16 &= i \\
i &= 0{,}1625
\end{aligned}
$$

The interest rate at which the money was invested was 16,25%.

9. When his son was 6 years old, Methuli made a deposit of R 6610 in the bank. The investment grew at a simple interest rate and when Methuli's son was 18 years old, the value of the investment was R 11 131,24.

   At what rate was the money invested? Give the answer correct to one decimal place.

   Solution:

   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 11\ 131{,}24 \\ P &= \text{R } 6610 \\ i &= ? \\ n &= 18 - 6 = 12 \end{aligned}$$

   The question says that the investment "grew at a simple interest rate", so we must use the simple interest formula. To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1 + in) \\ \frac{A}{P} &= 1 + in \\ \frac{A}{P} - 1 &= in \\ \frac{\frac{A}{P} - 1}{n} &= i \\ \text{Therefore } i &= \frac{\left(\frac{11\ 131{,}24}{6610}\right) - 1}{12} \\ &= 0{,}057 \\ &= 5{,}7\% \text{ per annum} \end{aligned}$$

10. When his son was 6 years old, Phillip made a deposit of R 5040 in the bank. The investment grew at a simple interest rate and when Phillip's son was 18 years old, the value of the investment was R 7338,24.

   At what rate was the money invested? Give your answer correct to one decimal place.

   Solution:

   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 7338{,}24 \\ P &= \text{R } 5040 \\ i &= ? \\ n &= 18 - 6 = 12 \end{aligned}$$

   The question says that the investment "grew at a simple interest rate", so we must use the simple interest formula. To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1 + in) \\ \frac{A}{P} &= 1 + in \\ \frac{A}{P} - 1 &= in \\ \frac{\frac{A}{P} - 1}{n} &= i \\ \text{Therefore } i &= \frac{\left(\frac{7338{,}24}{5040}\right) - 1}{12} \\ &= 0{,}038 \\ &= 3{,}8\% \text{ per annum} \end{aligned}$$

11. When his son was 10 years old, Lefu made a deposit of R 2580 in the bank. The investment grew at a simple interest rate and when Lefu's son was 20 years old, the value of the investment was R 3689,40.

   At what rate was the money invested? Give your answer correct to one decimal place.

   Solution:

   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 3689{,}40 \\ P &= \text{R } 2580 \\ i &= ? \\ n &= 20 - 10 = 10 \end{aligned}$$

The question says that the investment "grew at a simple interest rate", so we must use the simple interest formula. To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1+in) \\ \frac{A}{P} &= 1+in \\ \frac{A}{P}-1 &= in \\ \frac{\frac{A}{P}-1}{n} &= i \\ \text{Therefore } i &= \frac{\left(\frac{3689{,}40}{2580}\right)-1}{10} \\ &= 0{,}043 \\ &= 4{,}3\% \text{ per annum} \end{aligned}$$

12. Abdoul wants to invest R 1080 at a simple interest rate of 10,9% p.a.
    How many years will it take for the money to grow to R 3348? Round **up** your answer to the nearest year.
    Solution:
    Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 3348 \\ P &= \text{R } 1080 \\ i &= \frac{10{,}9}{100} = 0{,}109 \\ n &= ? \end{aligned}$$

    To calculate the number of years, we need to make n the subject of the formula:

$$\begin{aligned} A &= P(1+in) \\ \frac{A}{P} &= 1+in \\ \frac{A}{P}-1 &= in \\ \frac{\frac{A}{P}-1}{i} &= n \\ \text{Therefore } n &= \frac{\left(\frac{3348}{1080}\right)-1}{0{,}109} \\ &= 19{,}3 \\ &= 20 \text{ years} \quad \Leftarrow \text{round UP to the nearest integer} \end{aligned}$$

13. Andrew wants to invest R 3010 at a simple interest rate of 11,9% p.a.
    How many years will it take for the money to grow to R 14 448? Round **up** your answer to the nearest year.
    Solution:
    Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 14\,448 \\ P &= \text{R } 3010 \\ i &= \frac{11{,}9}{100} = 0{,}119 \\ n &= ? \end{aligned}$$

To calculate the number of years, we need to make n the subject of the formula:

$$\begin{aligned}
A &= P(1+in) \\
\frac{A}{P} &= 1 + in \\
\frac{A}{P} - 1 &= in \\
\frac{\frac{A}{P} - 1}{i} &= n \\
\text{Therefore } n &= \frac{\left(\frac{14\,448}{3010}\right) - 1}{0{,}119} \\
&= 31{,}9 \\
&= 32 \text{ years} \quad \Leftarrow \text{round UP to the nearest integer}
\end{aligned}$$

Rounding up to the nearest year, it will take 32 years to reach the goal of saving R 14 448.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GGJ 2. 2GGK 3. 2GGM 4. 2GGN 5a. 2GGP 5b. 2GGQ 6. 2GGR 7. 2GGS
8. 2GGT 9. 2GGV 10. 2GGW 11. 2GGX 12. 2GGY 13. 2GGZ

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9.3 Compound interest

The power of compound interest

Exercise 9 – 2:

1. An amount of R 3500 is invested in a savings account which pays a compound interest rate of 7,5% p.a. Calculate the balance accumulated by the end of 2 years.
Solution:

$$\begin{aligned}
P &= 3500 \\
i &= 0{,}075 \\
n &= 2 \\
A &= ?
\end{aligned}$$

$$\begin{aligned}
A &= P(1+i)^n \\
&= 3500(1+0{,}075)^2 \\
&= \text{R } 4044{,}69
\end{aligned}$$

2. An amount of R 3070 is invested in a savings account which pays a compound interest rate of 11,6% p.a. Calculate the balance accumulated by the end of 6 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned}
P &= 3070 \\
i &= \frac{11{,}6}{100} = 0{,}116 \\
n &= 6 \\
A &= ?
\end{aligned}$$

The accumulated amount is:

$$\begin{aligned} A &= P(1+i)^n \\ &= 3070\,(1+0{,}116)^6 \\ &= \text{R } 5930{,}94 \end{aligned}$$

3. An amount of R 6970 is invested in a savings account which pays a compound interest rate of 10,2% p.a. Calculate the balance accumulated by the end of 3 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.
   Solution:
   Read the question carefully and write down the given information:

$$\begin{aligned} P &= 6970 \\ i &= \frac{10{,}2}{100} = 0{,}102 \\ n &= 3 \\ A &= ? \end{aligned}$$

   The accumulated amount is:

$$\begin{aligned} A &= P(1+i)^n \\ &= 6970\,(1+0{,}102)^3 \\ &= \text{R } 9327{,}76 \end{aligned}$$

4. Nicola wants to invest some money at a compound interest rate of 11% p.a. How much money (to the nearest rand) should be invested if she wants to reach a sum of R 100 000 in five years time?
   Solution:

$$\begin{aligned} A &= 100\ 000 \\ P &= ? \\ i &= 0{,}11 \\ n &= 5 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ 100\ 000 &= P(1+0{,}11)^5 \\ \frac{100\ 000}{(1{,}11)^5} &= P \\ P &= \text{R } 59\ 345{,}13 \end{aligned}$$

5. Thobeka wants to invest some money at a compound interest rate of 11,8% p.a.
   How much money should be invested if she wants to reach a sum of R 30 000 in 2 years' time? Round up your answer to the nearest rand.
   Solution:
   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 30\ 000 \\ P &= ? \\ i &= \frac{11{,}8}{100} = 0{,}118 \\ n &= 2 \end{aligned}$$

   To determine the amount she must invest, we need to make P the subject of the formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{(1+i)^n} &= P \\ \frac{\text{R } 30\ 000}{(1+0{,}118)^2} &= P \\ P &= \text{R } 24\ 001{,}46 \end{aligned}$$

She must invest R 24 002,00.

6. Likengkeng wants to invest some money at a compound interest rate of 11,4% p.a.

   How much money should be invested if she wants to reach a sum of R 38 200 in 7 years' time? Round up your answer to the nearest rand.

   Solution:

   Read the question carefully and write down the given information:

$$\begin{aligned} A &= 38\ 200 \\ i &= \frac{11{,}4}{100} = 0{,}114 \\ n &= 7 \\ P &= ? \end{aligned}$$

   To determine the amount she must invest, we need to make P the subject of the formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{(1+i)^n} &= P \\ \frac{38\ 200}{(1+\frac{11{,}4}{100})^7} &= P \\ P &= \text{R } 17\ 941{,}84 \end{aligned}$$

   Therefore, the answer is: R 17 942,00

7. Morgan invests R 5000 into an account which pays out a lump sum at the end of 5 years. If he gets R 7500 at the end of the period, what compound interest rate did the bank offer him?

   Solution:

$$\begin{aligned} A &= 7500 \\ P &= 5000 \\ i &= ? \\ n &= 5 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ 7500 &= 5000(1+i)^5 \\ \frac{7500}{5000} &= (1+i)^5 \\ \sqrt[5]{\frac{7500}{5000}} &= (1+i) \\ \sqrt[5]{\frac{7500}{5000}} - 1 &= i \\ i &= 0{,}0844717712 \end{aligned}$$

   The interest rate is 8,45% p.a

8. Kabir invests R 1790 into an account which pays out a lump sum at the end of 9 years.

   If he gets R 2613,40 at the end of the period, what compound interest rate did the bank offer him? Give the answer correct to one decimal place.

   Solution:

   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 2613{,}40 \\ P &= \text{R } 1790 \\ i &= ? \\ n &= 9 \end{aligned}$$

To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \\ \left(\frac{A}{P}\right)^{\frac{1}{n}} &= 1+i \\ \left(\frac{A}{P}\right)^{\frac{1}{n}} - 1 &= i \\ \text{Therefore } i &= \left(\frac{2613{,}40}{1790}\right)^{\frac{1}{9}} - 1 \\ &= 0{,}043 \\ &= 4{,}3\% \text{ per annum} \end{aligned}$$

9. Bongani invests R 6110 into an account which pays out a lump sum at the end of 7 years. If he gets R 6904,30 at the end of the period, what compound interest rate did the bank offer him? Give the answer correct to one decimal place.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 6904{,}30 \\ P &= \text{R } 6110 \\ i &= ? \\ n &= 7 \end{aligned}$$

To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \\ \left(\frac{A}{P}\right)^{\frac{1}{n}} &= 1+i \\ \left(\frac{A}{P}\right)^{\frac{1}{n}} - 1 &= i \\ \text{Therefore } i &= \left(\frac{6904{,}30}{6110}\right)^{\frac{1}{7}} - 1 \\ &= 0{,}018 \\ &= 1{,}8\% \text{ per annum} \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GH4 2. 2GH5 3. 2GH6 4. 2GH7 5. 2GH8 6. 2GH9 7. 2GHB 8. 2GHC 9. 2GHD

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9.4 Calculations using simple and compound interest

Hire purchase

Exercise 9 – 3:

1. Angelique wants to buy a microwave on a hire purchase agreement. The cash price of the microwave is R 4400. She is required to pay a deposit of 10% and pay the remaining loan amount off over 12 months at an interest rate of 9%

p.a.

a) What is the principal loan amount?

Solution:

First calculate the amount for the deposit:

$$\begin{aligned} \text{deposit} &= 4400 \times \frac{10}{100} \\ &= 440 \end{aligned}$$

To determine the principal loan amount, we must subtract the deposit amount from the cash price:

$$\begin{aligned} P &= \text{cash price} - \text{deposit} \\ &= 4400 - 440 \\ &= \text{R } 3960{,}00 \end{aligned}$$

b) What is the accumulated loan amount?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= \text{R } 3960{,}00 \\ i &= \frac{9}{100} = 0{,}09 \\ n &= 1 \end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned} A &= P(1 + in) \\ &= \text{R } 3960{,}00\,(1 + 0{,}09 \times 1) \\ &= \text{R } 4316{,}40 \end{aligned}$$

c) What are Angelique's monthly repayments?

Solution:

To determine the monthly payment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned} \text{Monthly repayment} &= \frac{A}{\text{no. of months}} \\ &= \frac{\text{R } 4316{,}40}{12} \\ &= \text{R } 359{,}70 \end{aligned}$$

d) What is the total amount she has paid for the microwave?

Solution:

To determine the total amount paid, we add the accumulated loan amount and the deposit:

$$\begin{aligned} \text{Total amount} &= A + \text{deposit amount} \\ &= \text{R } 4316{,}40 + 440 \\ &= \text{R } 4756{,}40 \end{aligned}$$

2. Nyakallo wants to buy a television on a hire purchase agreement. The cash price of the television is R 5600. She is required to pay a deposit of 15% and pay the remaining loan amount off over 24 months at an interest rate of 14% p.a.

a) What is the principal loan amount?

Solution:

First calculate the amount for the deposit:

$$\begin{aligned} \text{deposit} &= 5600 \times \frac{15}{100} \\ &= 840 \end{aligned}$$

To determine the principal loan amount, we must subtract the deposit amount from the cash price:

$$\begin{aligned} P &= \text{cash price} - \text{deposit} \\ &= 5600 - 840 \\ &= \text{R } 4760{,}00 \end{aligned}$$

b) What is the accumulated loan amount?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= \text{R } 4760{,}00 \\ i &= \frac{14}{100} = 0{,}14 \\ n &= 2 \end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned} A &= P(1 + in) \\ &= \text{R } 4760{,}00\,(1 + 0{,}14 \times 2) \\ &= \text{R } 6092{,}80 \end{aligned}$$

c) What are Nyakallo's monthly repayments?

Solution:

To determine the monthly payment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned} \text{Monthly repayment} &= \frac{A}{\text{no. of months}} \\ &= \frac{\text{R } 6092{,}80}{24} \\ &= \text{R } 253{,}87 \end{aligned}$$

d) What is the total amount she has paid for the television?

Solution:

To determine the total amount paid we add the accumulated loan amount and the deposit:

$$\begin{aligned} \text{Total amount} &= A + \text{deposit amount} \\ &= \text{R } 6092{,}80 + 840 \\ &= \text{R } 6932{,}80 \end{aligned}$$

3. A company wants to purchase a printer. The cash price of the printer is R 4500. A deposit of 15% is required on the printer. The remaining loan amount will be paid off over 24 months at an interest rate of 12% p.a.

a) What is the principal loan amount?

Solution:

To calculate the principal loan amount, we first calculate the amount for the deposit and then subtract the deposit amount from the cash price:

$$\begin{aligned} P &= 4500 - (4500 \times 0{,}15) \\ &= 4500 - 675 \\ &= \text{R } 3825 \end{aligned}$$

b) What is the accumulated loan amount?

Solution:

Remember that hire purchase uses simple interest. We write down the given information and then substitute these values into the simple interest formula.

$$
\begin{aligned}
P &= \text{R } 3825 \\
i &= 0{,}12 \\
n &= \frac{24}{12} = 2
\end{aligned}
$$

$$
\begin{aligned}
A &= P(1 + in) \\
A &= 3825(1 + (0{,}12)(2)) \\
A &= \text{R } 4743
\end{aligned}
$$

c) How much will the company pay each month?

Solution:

To determine the monthly payment amount (how much the company pays each month), we divide the accumulated amount A by the total number of months:

$\frac{4743}{24} = \text{R } 197{,}63$

d) What is the total amount the company paid for the printer?

Solution:

To determine the total amount paid we add the accumulated loan amount and the deposit:

$675 + 4743 = \text{R } 5418$

4. Sandile buys a dining room table costing R 8500 on a hire purchase agreement. He is charged an interest rate of 17,5% p.a. over 3 years.

a) How much will Sandile pay in total?

Solution:

The question does not mention a deposit so we assume Sandile did not pay one. We write down the given information and then use the simple interest formula to calculate the accumulated amount.

$$
\begin{aligned}
A &= ? \\
P &= 8500 \\
i &= 0{,}175 \\
n &= 3
\end{aligned}
$$

$$
\begin{aligned}
A &= P(1 + in) \\
A &= 8500(1 + (0{,}175)(3)) \\
A &= \text{R } 12\ 962{,}50
\end{aligned}
$$

b) How much interest does he pay?

Solution:

To calculate the total interest paid we subtract the cash price from the accumulated amount.

$12\ 962{,}50 - 8500 = \text{R } 4462{,}50$

c) What is his monthly instalment?

Solution:

To determine the monthly payment amount, we divide the accumulated amount A by the total number of months:

$\frac{12\ 962{,}50}{36} = \text{R } 360{,}07$

5. Mike buys a table costing R 6400 on a hire purchase agreement. He is charged an interest rate of 15% p.a. over 4 years.

a) How much will Mike pay in total?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned}
A &= ? \\
P &= \text{R } 6400 \\
i &= \frac{15}{100} = 0{,}15 \\
n &= 4
\end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned}
A &= P(1 + in) \\
&= 6400(1 + 0{,}15 \times 4) \\
&= \text{R } 10\,240
\end{aligned}$$

b) How much interest does he pay?

Solution:

To determine the interest amount, we subtract the principal amount from the accumulated amount:

$$\begin{aligned}
\text{Interest amount} &= A - P \\
&= 10\,240 - 6400 \\
&= \text{R } 3840
\end{aligned}$$

c) What is his monthly instalment?

Solution:

To determine the monthly instalment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned}
\text{Monthly instalment} &= \frac{A}{\text{no. of months}} \\
&= \frac{10\,240}{4 \times 12} \\
&= \text{R } 213{,}33
\end{aligned}$$

6. Talwar buys a cupboard costing R 5100 on a hire purchase agreement. He is charged an interest rate of 12% p.a. over 2 years.

a) How much will Talwar pay in total?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned}
A &= ? \\
P &= \text{R } 5100 \\
i &= \frac{12}{100} = 0{,}12 \\
n &= 2
\end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned}
A &= P(1 + in) \\
&= 5100(1 + 0{,}12 \times 2) \\
&= \text{R } 6324
\end{aligned}$$

b) How much interest does he pay?

Solution:

To determine the interest amount, we subtract the principal amount from the accumulated amount:

$$\begin{aligned}
\text{Interest amount} &= A - P \\
&= 6324 - 5100 \\
&= \text{R } 1224
\end{aligned}$$

c) What is his monthly instalment?

Solution:

To determine the monthly instalment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned}\text{Monthly instalment} &= \frac{A}{\text{no. of months}} \\ &= \frac{6324}{2 \times 12} \\ &= \text{R } 263{,}50\end{aligned}$$

7. A lounge suite is advertised for sale on TV, to be paid off over 36 months at R 150 per month.

   a) Assuming that no deposit is needed, how much will the buyer pay for the lounge suite once it has been paid off?

   Solution:

   $36 \times 150 = \text{R } 5400$

   b) If the interest rate is 9% p.a., what is the cash price of the suite?

   Solution:

$$\begin{aligned}A &= 5400 \\ P &= ? \\ i &= 0{,}09 \\ n &= 3\end{aligned}$$

$$\begin{aligned}A &= P(1 + in) \\ 5400 &= P(1 + (0{,}09)(3)) \\ \frac{5400}{1{,}27} &= P \\ P &= \text{R } 4251{,}97\end{aligned}$$

8. Two stores are offering a fridge and washing machine combo package. Store A offers a monthly payment of R 350 over 24 months. Store B offers a monthly payment of R 175 over 48 months.

   If both stores offer 7,5% interest, which store should you purchase the fridge and washing machine from if you want to pay the least amount of interest?

   Solution:

   To calculate the interest paid at each store we need to first find the cash price of the fridge and washing machine.

   Store A:

$$\begin{aligned}A &= 350 \times 24 = 8400 \\ P &= ? \\ i &= 0{,}075 \\ n &= 2\end{aligned}$$

$$\begin{aligned}A &= P(1 + in) \\ 8400 &= P(1 + (0{,}075)(2)) \\ \frac{8400}{2{,}15} &= P \\ P &= \text{R } 3906{,}98\end{aligned}$$

   Therefore the interest is R 8400 − R 3906,98 = R 4493,02

   Store B:

$$
\begin{aligned}
A &= 175 \times 48 = 8400 \\
P &= ? \\
i &= 0{,}075 \\
n &= 4
\end{aligned}
$$

$$
\begin{aligned}
A &= P(1 + in) \\
8400 &= P(1 + (0{,}075)(4)) \\
\frac{8400}{4{,}3} &= P \\
P &= \text{R } 1953{,}49
\end{aligned}
$$

Therefore the interest is R 8400 − R 1953,49 = R 6446,51

If you want to pay the least amount in interest you should purchase the fridge and washing machine from store A.

9. Tlali wants to buy a new computer and decides to buy one on a hire purchase agreement. The computers cash price is R 4250. He will pay it off over 30 months at an interest rate of 9,5% p.a. An insurance premium of R 10,75 is added to every monthly payment. How much are his monthly payments?

Solution:

$$
\begin{aligned}
P &= 4250 \\
i &= 0{,}095 \\
n &= \frac{30}{12} = 2{,}5
\end{aligned}
$$

The question does not mention a deposit, therefore we assume that Tlali did not pay one.

$$
\begin{aligned}
A &= P\,(1 + in) \\
A &= 4250\,(1 + 0{,}095 \times 2{,}5) \\
&= 5259{,}38
\end{aligned}
$$

The monthly payment is:

$$
\begin{aligned}
\text{Monthly payment } &= \frac{5259{,}38}{36} \\
&= 146{,}09
\end{aligned}
$$

Add the insurance premium: R 146,09 + R 10,75 = R 156,84

10. Richard is planning to buy a new stove on hire purchase. The cash price of the stove is R 6420. He has to pay a 10% deposit and then pay the remaining amount off over 36 months at an interest rate of 8% p.a. An insurance premium of R 11,20 is added to every monthly payment. Calculate Richard's monthly payments.

Solution:

$$
\begin{aligned}
P &= 6420 - (0{,}10)(6420) = 5778 \\
i &= 0{,}08 \\
n &= \frac{36}{12} = 3
\end{aligned}
$$

Calculate the accumulated amount:

$$
\begin{aligned}
A &= P\,(1 + in) \\
A &= 5778\,(1 + 0{,}08 \times 3) \\
&= 7164{,}72
\end{aligned}
$$

Calculate the monthly repayments on the hire purchase agreement:

$$
\begin{aligned}
\text{Monthly payment } &= \frac{7164{,}72}{36} \\
&= 199{,}02
\end{aligned}
$$

Add the insurance premium: R 199,02 + R 11,20 = R 210,22

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GHG 2. 2GHH 3. 2GHJ 4. 2GHK 5. 2GHM 6. 2GHN
7. 2GHP 8. 2GHQ 9. 2GHR 10. 2GHS

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Inflation

Exercise 9 – 4:

1. The price of a bag of apples is R 12. How much will it cost in 9 years time if the inflation rate is 12% p.a.?
   Solution:
   Read the question carefully and write down the given information:
   - $A = ?$
   - $P = \text{R } 12$
   - $n = 9$
   - $i = \frac{12}{100}$

   To determine the future cost, we use the compound interest formula:
   $$\begin{aligned} A &= P(1+i)^n \\ &= 12 \times \left(1 + \frac{12}{100}\right)^9 \\ &= \text{R } 33{,}28 \end{aligned}$$

2. The price of a bag of potatoes is R 15.
   How much will it cost in 6 years time if the inflation rate is 12% p.a.?
   Solution:
   Read the question carefully and write down the given information:
   - $A = ?$
   - $P = \text{R } 15$
   - $n = 6$
   - $i = \frac{12}{100}$

   To determine the future cost, we use the compound interest formula:
   $$\begin{aligned} A &= P(1+i)^n \\ &= 15 \times \left(1 + \frac{12}{100}\right)^6 \\ &= \text{R } 29{,}61 \end{aligned}$$

3. The price of a box of popcorn is R 15. How much will it cost in 4 years time if the inflation rate is 11% p.a.?
   Solution:
   Read the question carefully and write down the given information:
   - $A = ?$
   - $P = \text{R } 15$
   - $n = 4$
   - $i = \frac{11}{100}$

   To determine the future cost, we use the compound interest formula:
   $$\begin{aligned} A &= P(1+i)^n \\ &= 15 \times \left(1 + \frac{11}{100}\right)^4 \\ &= \text{R } 22{,}77 \end{aligned}$$

4. A box of raisins costs R 24 today. How much did it cost 4 years ago if the average rate of inflation was 13% p.a.? Round your answer to 2 decimal places.
Solution:
Read the question carefully and write down the given information:
   - $A =$ R 24
   - $P =?$
   - $i = \frac{13}{100}$
   - $n = 4$

   We use the compound interest formula and make P the subject:

$$\begin{aligned} A &= P(1+i)^n \\ P &= \frac{A}{(1+i)^n} \\ &= \frac{24}{\left(1+\frac{13}{100}\right)^4} \\ &= \text{R } 14{,}72 \end{aligned}$$

5. A box of biscuits costs R 24 today. How much did it cost 5 years ago if the average rate of inflation was 11% p.a.? Round your answer to 2 decimal places.
Solution:
Read the question carefully and write down the given information:
   - $A =$ R 24
   - $P =?$
   - $i = \frac{11}{100}$
   - $n = 5$

   We use the compound interest formula and make P the subject:

$$\begin{aligned} A &= P(1+i)^n \\ P &= \frac{A}{(1+i)^n} \\ &= \frac{24}{\left(1+\frac{11}{100}\right)^5} \\ &= \text{R } 14{,}24 \end{aligned}$$

6. If the average rate of inflation for the past few years was 7,3% p.a. and your water and electricity account is R 1425 on average, what would you expect to pay in 6 years time?
Solution:

$$\begin{aligned} A &=? \\ P &= 1425 \\ i &= 0{,}073 \\ n &= 6 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ A &= 1425(1+0{,}073)^6 \\ A &= \text{R } 2174{,}77 \end{aligned}$$

7. The price of popcorn and a cooldrink at the movies is now R 60. If the average rate of inflation is 9,2% p.a. what was the price of popcorn and cooldrink 5 years ago?
Solution:

$$
\begin{aligned}
A &= \text{R } 60 \\
P &= ? \\
i &= 0{,}092 \\
n &= 5 \\
\\
A &= P(1+i)^n \\
\\
60 &= P(1+0{,}092)^5 \\
\frac{60}{(1{,}092)^5} &= P \\
P &= \text{R } 38{,}64
\end{aligned}
$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GHT 2. 2GHV 3. 2GHW 4. 2GHX 5. 2GHY 6. 2GHZ 7. 2GJ2

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Population growth

Exercise 9 – 5:

1. The current population of Durban is 3 879 090 and the average rate of population growth in South Africa is 1,1% p.a.
   What can city planners expect the population of Durban to be in 6 years time? Round your answer to the nearest integer.
   Solution:
   Read the question carefully and write down the given information:
   - $A = ?$
   - $P = 3\ 879\ 090$
   - $i = \frac{1.1}{100}$
   - $n = 6$

   We use the following formula to determine the expected population for Durban:

$$
\begin{aligned}
A &= P\,(1+i)^n \\
&= 3\ 879\ 090\left(1+\frac{1{,}1}{100}\right)^6 \\
&= 4\ 142\ 255
\end{aligned}
$$

2. The current population of Polokwane is 3 878 970 and the average rate of population growth in South Africa is 0,7% p.a.
   What can city planners expect the population of Polokwane to be in 12 years time? Round your answer to the nearest integer.
   Solution:
   Read the question carefully and write down the given information:
   - $A = ?$
   - $P = 3\ 878\ 970$
   - $i = \frac{0.7}{100}$
   - $n = 12$

We use the following formula to determine the expected population for Polokwane:

$$\begin{aligned} A &= P(1+i)^n \\ &= 3\ 878\ 970 \left(1 + \frac{0,7}{100}\right)^{12} \\ &= 4\ 217\ 645 \end{aligned}$$

3. A small town in Ohio, USA is experiencing a huge increase in births. If the average growth rate of the population is 16% p.a., how many babies will be born to the 1600 residents in the next 2 years?
Solution:

$$\begin{aligned} A &= ? \\ P &= 1600 \\ i &= 0,16 \\ n &= 2 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ A &= 1600(1+0,16)^2 \\ A &= 2152,96 \\ 2153 - 1600 &= 553 \end{aligned}$$

There will be roughly 553 babies born in the next two years.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'. 1. 2GJ3 2. 2GJ4 3. 2GJ5

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9.5 Foreign exchange rates

Exercise 9 – 6:

1. Bridget wants to buy an iPod that costs £ 100, with the exchange rate currently at £ 1 = R 14. She estimates that the exchange rate will drop to R 12 in a month.
   a) How much will the iPod cost in rands, if she buys it now?
   Solution:
   Cost in rands = (cost in pounds) × exchange rate.
   $= 100 \times \frac{14}{1}$ = R 1400
   b) How much will she save if the exchange rate drops to R 12?
   Solution:
   Cost in rands $= 100 \times \frac{12}{1}$ = R 1200
   So she will save R 200 (Saving = R 1400 − R 1200)
   c) How much will she lose if the exchange rate moves to R 15?
   Solution:
   Cost in rands $= 100 \times \frac{15}{1}$ = R 1500
   So she will lose R 100 (Loss = R 1400 − R 1500)
2. Mthuli wants to buy a television that costs £ 130, with the exchange rate currently at £ 1 = R 11. He estimates that the exchange rate will drop to R 9 in a month.
   a) How much will the television cost in rands, if he buys it now?
   Solution:

$$\begin{aligned}\text{Cost} &= 130 \times \text{R } 11\\ &= \text{R } 1430\end{aligned}$$

b) How much will he save if the exchange rate drops to R 9?
Solution:

$$\begin{aligned}\text{Cost} &= 130 \times \text{R } 9\\ &= \text{R } 1170\end{aligned}$$

Therefore the amount he will have saved is:

$$\begin{aligned}\text{Saved} &= \text{R } 1430 - \text{R } 1170\\ &= \text{R } 260\end{aligned}$$

c) How much will he lose if the exchange rate moves to R 19?
Solution:

$$\begin{aligned}\text{Cost} &= 130 \times \text{R } 19\\ &= \text{R } 2470\end{aligned}$$

Therefore the amount he will lose is:

$$\begin{aligned}\text{Loss} &= \text{R } 2470 - \text{R } 1430\\ &= \text{R } 1040\end{aligned}$$

3. Nthabiseng wants to buy an iPad that costs £ 120, with the exchange rate currently at £ 1 = R 14. She estimates that the exchange rate will drop to R 9 in a month.

a) How much will the iPad cost, in rands, if she buys it now?
Solution:

$$\begin{aligned}\text{Cost} &= 120 \times \text{R } 14\\ &= \text{R } 1680\end{aligned}$$

b) How much will she save if the exchange rate drops to R 9?
Solution:

$$\begin{aligned}\text{Cost} &= 120 \times \text{R } 9\\ &= \text{R } 1080\end{aligned}$$

Therefore the amount she will have saved is:

$$\begin{aligned}\text{Saved} &= \text{R } 1680 - \text{R } 1080\\ &= \text{R } 600\end{aligned}$$

c) How much will she lose if the exchange rate moves to R 18?
Solution:

$$\begin{aligned}\text{Cost} &= 120 \times \text{R } 18\\ &= \text{R } 2160\end{aligned}$$

Therefore the amount she will lose is:

$$\begin{aligned}\text{Loss} &= \text{R } 2160 - \text{R } 1680\\ &= \text{R } 480\end{aligned}$$

4. Study the following exchange rate table:

Country	Currency	Exchange Rate
United Kingdom (UK)	Pounds (£)	R 14,13
United States (USA)	Dollars ($)	R 7,04

a) In South Africa the cost of a new Honda Civic is R 173 400. In England the same vehicle costs £ 12 200 and in the USA $21 900. In which country is the car the cheapest?

Solution:

To answer this question we work out the cost of the car in rand for each country and then compare the three answers to see which is the cheapest. Cost in rands = cost in currency times exchange rate.

Cost in UK: $12\ 200 \times \frac{14,13}{1} = \text{R } 172\ 386$

Cost in USA: $21\ 900 \times \frac{7,04}{1} = \text{R } 154\ 400$

Comparing the three costs we find that the car is the cheapest in the USA.

b) Sollie and Arinda are waiters in a South African restaurant attracting many tourists from abroad. Sollie gets a £ 6 tip from a tourist and Arinda gets $12. Who got the better tip?

Solution:

Sollie: $6 \times \frac{14,31}{1} = \text{R } 84,78$

Arinda: $12 \times \frac{7,04}{1} = \text{R } 84,48$.

Therefore Sollie got the better tip. He got 30 cents more than Arinda.

5. Yaseen wants to buy a book online. He finds a publisher in London selling the book for £ 7,19. This publisher is offering free shipping on the product.

   He then finds the same book from a publisher in New York for $8,49 with a shipping fee of $2.

   Next he looks up the exchange rates to see which publisher has the better deal. If $\$1 = \text{R } 11,48$ and $£\ 1 = \text{R } 17,36$, which publisher should he buy the book from?

   Solution:

   London publisher: $7,19 \times \frac{17,36}{1} = \text{R } 124,82$

   New York publisher: $(8,49 + 2) \times \frac{11,48}{1} = \text{R } 120,43$.

   Therefore Yaseen should buy the book from the New York publisher.

6. Mathe is saving up to go visit her friend in Germany. She estimates the total cost of her trip to be R 50 000. The exchange rate is currently $€\ 1 = \text{R } 13,22$.

   Her friend decides to help Mathe out by giving her € 1000. How much (in rand) does Mathe now need to save up?

   Solution:

   We first calculate how much Mathe's friend will give her in rands:

   $1000 \times \frac{13,22}{1} = \text{R } 13\ 220$.

   Therefore Mathe now needs to save up: $\text{R } 50\ 000 - \text{R } 13\ 220 = \text{R } 36\ 780$.

7. Lulamile and Jacob give tours over the weekends. They do not charge for these tours but instead accept tips from the group. The table below shows the total amount of tips they receive from various tour groups.

Group	Total tips
British tourists	£ 5,50
Japanese tourists	¥ 85,50
American tourists	$ 7,00
Dutch tourists	€ 9,70
Brazilian tourists	40,50 BRL
Australian tourists	9,20 AUD
South African tourists	R 55,00

The current exchange rates are:

$$\begin{aligned} £\ 1 &= \text{R } 17,12 \\ ¥\ 1 &= \text{R } 0,10 \\ \$\ 1 &= \text{R } 11,42 \\ €\ 1 &= \text{R } 12,97 \\ 1 \text{ BRL} &= \text{R } 4,43 \\ 1 \text{ AUD} &= \text{R } 9,12 \end{aligned}$$

a) Which group of tourists tipped the most? How much did they tip (give your answer in rand)?

Solution:

We need to calculate the value of each tip in rand:

Group	Total tips	Value of tip in rands
British tourists	£ 5,50	R 94,16
Japanese tourists	¥ 85,50	R 8,55
American tourists	$ 7,00	R 79,94
Dutch tourists	€ 9,70	R 125,81
Brazilian tourists	40,50 BRL	R 179,42
Australian tourists	9,20 AUD	R 83,90
South African tourists	R 55,00	R 55,00

The Brazilian tourists tipped the most. The rand value of their tip was R 179,42.

b) Which group of tourists tipped the least? How much did they tip (give your answer in rand)?
Solution:
The Japanese tourists tipped the least. The rand value of their tip was R 8,55.

8. Kayla is planning a trip to visit her family in Malawi followed by spending some time in Tanzania at the Serengeti. She will first need to convert her South African rands into the Malawian kwacha. After that she will convert her remaining Malawian kwacha into Tanzanian shilling.
She looks up the current exchange rates and finds the following information:

$$\text{R } 1 = 39{,}46 \text{ MWK}$$
$$1 \text{ MWK} = 4{,}01 \text{ TZS}$$

She starts off with R 5000 in South Africa. In Malawi she spends 65 000 MWK. When she converts the remaining Malawian kwacha to Tanzanian shilling, how much money does she have (in Tanzanian shilling)?
Solution:
We first convert from rands to Malawian kwacha: $5000 \times \frac{39{,}46}{1} = 197\ 300$ MWK
She spends 65 000 MWK of this and so she has 132 300 MWK to convert to Tanzanian shillings.
Now we can convert from Malawian kwacha to Tanzanian shillings: 132 300 MWK $\times \frac{4{,}01}{1} = 530\ 523$ TZS
So she will have 530 523 TZS to spend in Tanzania.

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1. 2GJ7 2. 2GJ8 3. 2GJ9 4. 2GJB 5. 2GJC 6. 2GJD 7. 2GJF 8. 2GJG

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9.6 Chapter summary

End of chapter Exercise 9 – 7:

1. An amount of R 6330 is invested in a savings account which pays simple interest at a rate of 11% p.a.. Calculate the balance accumulated by the end of 7 years.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= \text{R } 6330 \\ i &= \frac{11}{100} = 0{,}11 \\ n &= 7 \end{aligned}$$

Simple interest formula:

$$\begin{aligned} A &= P(1 + in) \\ &= \text{R } 6330\,(1 + (0{,}11) \times 7) \\ &= \text{R } 11\ 204{,}10 \end{aligned}$$

2. An amount of R 1740 is invested in a savings account which pays simple interest at a rate of 7% p.a.. Calculate the balance accumulated by the end of 6 years.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= \text{R } 1740 \\ i &= \frac{7}{100} = 0{,}07 \\ n &= 6 \end{aligned}$$

Simple interest formula:

$$\begin{aligned} A &= P(1+in) \\ &= \text{R } 1740\,(1 + (0{,}07) \times 6) \\ &= \text{R } 2470{,}80 \end{aligned}$$

3. Adam opens a savings account when he is 13. He would like to have R 50 000 by the time he is 18. If the savings account offers simple interest at a rate of 8,5% per annum, how much money should he invest now to reach his goal?
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 50\ 000 \\ P &= ? \\ i &= \frac{8{,}5}{100} = 0{,}085 \\ n &= 5 \end{aligned}$$

Simple interest formula:

$$\begin{aligned} A &= P(1+in) \\ \text{R } 50\ 000 &= P\,(1 + (0{,}085) \times 5) \\ P &= \frac{\text{R } 50\ 000}{1{,}425} \\ &= \text{R } 35\ 087{,}72 \end{aligned}$$

4. When his son was 4 years old, Dumile made a deposit of R 6700 in the bank. The investment grew at a simple interest rate and when Dumile's son was 24 years old, the value of the investment was R 11 524.
At what rate was the money invested? Give your answer correct to one decimal place.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 11\ 524 \\ P &= \text{R } 6700 \\ i &= ? \\ n &= 24 - 4 = 20 \end{aligned}$$

The question says that the investment "grew at a simple interest rate", so we must use the simple interest formula. To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1+in) \\ \frac{A}{P} &= 1 + in \\ \frac{A}{P} - 1 &= in \\ \frac{\frac{A}{P} - 1}{n} &= i \\ \text{Therefore } i &= \frac{\left(\frac{11\ 524{,}00}{6700}\right) - 1}{20} \\ &= 0{,}036 \\ &= 3{,}6\% \text{ per annum} \end{aligned}$$

5. When his son was 7 years old, Jared made a deposit of R 5850 in the bank. The investment grew at a simple interest rate and when Jared's son was 35 years old, the value of the investment was R 11 746,80.
   At what rate was the money invested? Give your answer correct to one decimal place.
   Solution:
   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 11\ 746{,}80 \\ P &= \text{R } 5850 \\ i &= ? \\ n &= 35 - 7 = 28 \end{aligned}$$

   The question says that the investment "grew at a simple interest rate", so we must use the simple interest formula. To calculate the interest rate, we need to make i the subject of the formula:

$$\begin{aligned} A &= P(1 + in) \\ \frac{A}{P} &= 1 + in \\ \frac{A}{P} - 1 &= in \\ \frac{\frac{A}{P} - 1}{n} &= i \\ \text{Therefore } i &= \frac{\left(\frac{11\ 746{,}80}{5850}\right) - 1}{28} \\ &= 0{,}036 \\ &= 3{,}6\% \text{ per annum} \end{aligned}$$

6. Sehlolo wants to invest R 6360 at a simple interest rate of 12,4% p.a.
   How many years will it take for the money to grow to R 26 075? Round **up** your answer to the nearest year.
   Solution:
   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 26\ 075 \\ P &= \text{R } 6360 \\ i &= \frac{12{,}4}{100} = 0{,}124 \\ n &= ? \end{aligned}$$

   To calculate the number of years, we need to make n the subject of the formula:

$$\begin{aligned} A &= P(1 + in) \\ \frac{A}{P} &= 1 + in \\ \frac{A}{P} - 1 &= in \\ \frac{\frac{A}{P} - 1}{i} &= n \\ \text{Therefore } n &= \frac{\left(\frac{26\ 075}{6360}\right) - 1}{0{,}124} \\ &= 24{,}9987\ldots \\ &= 25 \text{ years} \quad \Leftarrow \text{round UP to the nearest integer} \end{aligned}$$

   Rounding up to the nearest year, it will take 25 years to reach the goal of saving R 26 075.
7. Mphikeleli wants to invest R 5540 at a simple interest rate of 9,1% p.a.
   How many years will it take for the money to grow to R 16 620? Round **up** your answer to the nearest year.
   Solution:
   Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 16\ 620 \\ P &= \text{R } 5540 \\ i &= \frac{9{,}1}{100} = 0{,}091 \\ n &= ? \end{aligned}$$

To calculate the number of years, we need to make n the subject of the formula:

$$\begin{aligned} A &= P(1+in) \\ \frac{A}{P} &= 1+in \\ \frac{A}{P}-1 &= in \\ \frac{\frac{A}{P}-1}{i} &= n \\ \text{Therefore } n &= \frac{\left(\frac{16\,620}{5540}\right)-1}{0{,}091} \\ &= 21{,}9780... \\ &= 22 \text{ years} \quad \Leftarrow \text{round UP to the nearest integer} \end{aligned}$$

Rounding up to the nearest year, it will take 22 years to reach the goal of saving R 16 620.

8. An amount of R 3500 is invested in an account which pays simple interest at a rate of 6,7% per annum. Calculate the amount of interest accumulated at the end of 4 years.

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &=? \\ P &= \text{R } 3500 \\ i &= \frac{6{,}7}{100} = 0{,}067 \\ n &= 4 \end{aligned}$$

$$\begin{aligned} A &= P(1+in) \\ &= 3500\,(1+(0{,}067)\,4) \\ &= \text{R } 4438 \end{aligned}$$

Therefore the interest earned is R 4438 − R 3500 = R 938

9. An amount of R 3270 is invested in a savings account which pays a compound interest rate of 12,2% p.a. Calculate the balance accumulated by the end of 7 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &=? \\ P &= \text{R } 3270 \\ i &= \frac{12{,}2}{100} = 0{,}122 \\ n &= 7 \end{aligned}$$

The accumulated amount is:

$$\begin{aligned} A &= P(1+i)^n \\ &= 3270\,(1+0{,}122)^7 \\ &= \text{R } 7319{,}78 \end{aligned}$$

10. An amount of R 2380 is invested in a savings account which pays a compound interest rate of 8,3% p.a. Calculate the balance accumulated by the end of 7 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &=? \\ P &= \text{R } 2380 \\ i &= \frac{8{,}3}{100} = 0{,}083 \\ n &= 7 \end{aligned}$$

The accumulated amount is:

$$\begin{aligned} A &= P(1+i)^n \\ &= 2380\,(1+0{,}083)^7 \\ &= \text{R } 4158{,}88 \end{aligned}$$

11. Emma wants to invest some money at a compound interest rate of 8,2% p.a.
    How much money should be invested if she wants to reach a sum of R 61 500 in 4 years' time? Round up your answer to the nearest rand.
    Solution:
    Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 61\ 500 \\ P &= ? \\ i &= \frac{8{,}2}{100} = 0{,}082 \\ n &= 4 \end{aligned}$$

    To determine the amount she must invest, we need to make P the subject of the formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{(1+i)^n} &= P \\ \frac{61\ 500}{(1+0{,}082)^4} &= P \\ P &= \text{R } 44\ 871{,}03 \end{aligned}$$

    Therefore, the answer is: R 44 872

12. Limpho wants to invest some money at a compound interest rate of 13,9% p.a.
    How much money should be invested if she wants to reach a sum of R 24 300 in 2 years' time? Round up your answer to the nearest rand.
    Solution:
    Read the question carefully and write down the given information:

$$\begin{aligned} A &= \text{R } 24\ 300 \\ P &= ? \\ i &= \frac{13{,}9}{100} = 0{,}139 \\ n &= 2 \end{aligned}$$

    To determine the amount she must invest, we need to make P the subject of the formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{(1+i)^n} &= P \\ \frac{24\ 300}{(1+0{,}139)^2} &= P \\ P &= \text{R } 18\ 730{,}91 \end{aligned}$$

    Therefore, the answer is: R 18 731,00

13. Calculate the compound interest for the following problems.

    a) A R 2000 loan for 2 years at 5% p.a.
       Solution:

$$
\begin{aligned}
P &= 2000 \\
i &= 0{,}05 \\
n &= 2 \\
A &= ? \\
\\
A &= P(1+i)^n \\
\\
A &= 2000(1+0{,}05)^2 \\
A &= \text{R } 2205
\end{aligned}
$$

So the amount of interest is: $2205 - 2000 = \text{R } 205$

b) A R 1500 investment for 3 years at 6% p.a.

Solution:

$$
\begin{aligned}
P &= 1500 \\
i &= 0{,}06 \\
n &= 3 \\
A &= ? \\
\\
A &= P(1+i)^n \\
\\
A &= 1500(1+0{,}06)^3 \\
A &= \text{R } 1786{,}52
\end{aligned}
$$

So the amount of interest is: $1786{,}52 - 1500 = \text{R } 286{,}52$

c) A R 800 loan for 1 year at 16% p.a.

Solution:

$$
\begin{aligned}
P &= 800 \\
i &= 0{,}16 \\
n &= 1 \\
A &= ? \\
\\
A &= P(1+i)^n \\
\\
A &= 800(1+0{,}16)^1 \\
A &= \text{R } 928
\end{aligned}
$$

So the amount of interest is: $928 - 800 = \text{R } 128$

14. Ali invests R 1110 into an account which pays out a lump sum at the end of 12 years. If he gets R 1642,80 at the end of the period, what compound interest rate did the bank offer him? Give your answer correct to one decimal place.

Solution:

Read the question carefully and write down the given information:

$$
\begin{aligned}
A &= \text{R } 1642{,}80 \\
P &= \text{R } 1110 \\
i &= ? \\
n &= 12
\end{aligned}
$$

$$\begin{aligned}
A &= P(1+i)^n \\
\text{R } 1642{,}80 &= \text{R } 1110(1+i)^{12} \\
\text{R } 1{,}48 &= (1+i)^{12} \\
\sqrt[12]{\text{R } 1{,}48} &= (1+i) \\
i &= 1{,}033... - 1 \\
&= 0{,}033... \\
&\approx 3{,}3\% \text{ per annum}
\end{aligned}$$

15. Christopher invests R 4480 into an account which pays out a lump sum at the end of 7 years.
    If he gets R 6496,00 at the end of the period, what compound interest rate did the bank offer him? Give your answer correct to one decimal place.
    Solution:
    Read the question carefully and write down the given information:

$$\begin{aligned}
A &= \text{R } 6496 \\
P &= \text{R } 4480 \\
i &= ? \\
n &= 7
\end{aligned}$$

$$\begin{aligned}
A &= P(1+i)^n \\
\text{R } 6496 &= \text{R } 4480(1+i)^{7} \\
\text{R } 1{,}45 &= (1+i)^{7} \\
\sqrt[7]{\text{R } 1{,}45} &= (1+i) \\
i &= 1{,}0545... - 1 \\
&= 0{,}0545... \\
&\approx 5{,}5\% \text{ per annum}
\end{aligned}$$

16. Calculate how much you will earn if you invested R 500 for 1 year at the following interest rates:
    a) 6,85% simple interest
    Solution:

$$\begin{aligned}
P &= 500 \\
i &= 0{,}685 \\
n &= 1 \\
A &= ?
\end{aligned}$$

$$\begin{aligned}
A &= P(1+in) \\
A &= 500(1+(0{,}685)(1)) \\
A &= 500(1{,}685) \\
A &= \text{R } 534{,}25
\end{aligned}$$

    b) 4,00% compound interest
    Solution:

$$\begin{aligned}
P &= 500 \\
i &= 0{,}04 \\
n &= 1 \\
A &= ?
\end{aligned}$$

$$\begin{aligned}
A &= P(1+i)^n \\
A &= 500(1+0{,}04)^1 \\
A &= \text{R } 520
\end{aligned}$$

17. Bianca has R 1450 to invest for 3 years. Bank A offers a savings account which pays simple interest at a rate of 11% per annum, whereas Bank B offers a savings account paying compound interest at a rate of 10,5% per annum. Which account would leave Bianca with the highest accumulated balance at the end of the 3 year period?

Solution:

Bank A:

$$\begin{aligned} P &= 1450 \\ i &= 0{,}11 \\ n &= 3 \\ A &= ? \\ \\ A &= P(1 + in) \\ A &= 1450(1 + (0{,}11)(3)) \\ A &= 1450(1{,}33) \\ A &= \text{R } 1928{,}50 \end{aligned}$$

Bank B:

$$\begin{aligned} P &= 1450 \\ i &= 0{,}105 \\ n &= 3 \\ A &= ? \\ \\ A &= P(1 + i)^n \\ A &= 1450(1 + 0{,}105)^3 \\ A &= \text{R } 1956{,}39 \end{aligned}$$

She should choose Bank B as it will give her more money after 3 years.

18. Given:

A loan of R 2000 for a year at an interest rate of 10% p.a.

a) How much simple interest is payable on the loan?

Solution:

$$\begin{aligned} P &= 2000 \\ i &= 0{,}10 \\ n &= 1 \\ A &= ? \\ \\ A &= P(1 + in) \\ A &= 2000(1 + (0{,}10)(1)) \\ A &= 2000(1{,}10) \\ A &= \text{R } 2200 \end{aligned}$$

So the amount of interest is: $2200 - 2000 = \text{R } 200$

b) How much compound interest is payable on the loan?

Solution:

$$
\begin{aligned}
P &= 2000 \\
i &= 0{,}10 \\
n &= 1 \\
A &=? \\
\\
A &= P(1+i)^n \\
\\
A &= 2000(1+0{,}10)^1 \\
A &= \text{R } 2200
\end{aligned}
$$

So the amount of interest is: $2200 - 2000 = \text{R } 200$

19. R 2250 is invested at an interest rate of 5,25% per annum.
Complete the following table.

Number of years	Simple interest	Compound interest
1		
2		
3		
4		
20		

Solution:
We need to calculate the amount accumulated if the interest rate is simple interest. We use $A = P(1 + in)$ to do this.
We also need to calculate the amount accumulated if the interest rate is compound interest. We use $A = P(1+i)^n$ to do this.
For both cases we note that:

$$
\begin{aligned}
A &=? \\
P &= \text{R } 2250 \\
i &= 6{,}25\%
\end{aligned}
$$

Number of years	Simple interest	Compound interest
1	R 2390,63	R 2390,63
2	R 2531,25	R 2540,04
3	R 2671,88	R 2698,79
4	R 2812,50	R 2867,47
20	R 5062,50	R 7564,17

20. Discuss:
   a) Which type of interest would you like to use if you are the borrower?
   Solution:
   Simple interest. Interest is only calculated on the principal amount and not on the interest earned during prior periods. This will lead to the borrower paying less interest.
   b) Which type of interest would you like to use if you were the banker?
   Solution:
   Compound interest. Interest is calculated from the principal amount as well as interest earned from prior periods. This will lead to the banker getting more money for the bank.

21. Portia wants to buy a television on a hire purchase agreement. The cash price of the television is R 6000. She is required to pay a deposit of 20% and pay the remaining loan amount off over 12 months at an interest rate of 9% p.a.
   a) What is the principal loan amount?
   Solution:
   First calculate the amount for the deposit:

$$
\begin{aligned}
\text{deposit} &= 6000 \times \frac{20}{100} \\
&= 1200
\end{aligned}
$$

To determine the principal loan amount, we must subtract the deposit amount from the cash price:

$$\begin{aligned} P &= \text{cash price} - \text{deposit} \\ &= 6000 - 1200 \\ &= \text{R } 4800{,}00 \end{aligned}$$

b) What is the accumulated loan amount?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= 4800{,}00 \\ i &= \frac{9}{100} = 0{,}09 \\ n &= \frac{12}{12} \end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned} A &= P(1 + in) \\ &= 4800{,}00\left(1 + 0{,}09 \times \frac{12}{12}\right) \\ &= \text{R } 5232{,}00 \end{aligned}$$

c) What are Portia's monthly repayments?

Solution:

To determine the monthly payment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned} \text{Monthly repayment} &= \frac{A}{\text{no. of months}} \\ &= \frac{5232{,}00}{12} \\ &= \text{R } 436{,}00 \end{aligned}$$

d) What is the total amount she has paid for the television?

Solution:

To determine the total amount paid we add the accumulated loan amount and the deposit:

$$\begin{aligned} \text{Total amount} &= A + \text{deposit amount} \\ &= 5232{,}00 + 1200 \\ &= \text{R } 6432{,}00 \end{aligned}$$

22. Gabisile wants to buy a heater on a hire purchase agreement. The cash price of the heater is R 4800. She is required to pay a deposit of 10% and pay the remaining loan amount off over 12 months at an interest rate of 12% p.a.

a) What is the principal loan amount?

Solution:

First calculate the amount for the deposit:

$$\begin{aligned} \text{deposit} &= 4800 \times \frac{10}{100} \\ &= 480 \end{aligned}$$

To determine the principal loan amount, we must subtract the deposit amount from the cash price:

$$\begin{aligned} P &= \text{cash price} - \text{deposit} \\ &= 4800 - 480 \\ &= \text{R } 4320 \end{aligned}$$

b) What is the accumulated loan amount?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= \text{R } 4320 \\ i &= \frac{12}{100} = 0{,}12 \\ n &= \frac{12}{12} \end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned} A &= P(1 + in) \\ &= \text{R } 4320\left(1 + \frac{12}{100} \times \frac{12}{12}\right) \\ &= \text{R } 4838{,}40 \end{aligned}$$

c) What are Gabisile's monthly repayments?

Solution:

To determine the monthly payment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned} \text{Monthly repayment} &= \frac{A}{\text{no. of months}} \\ &= \frac{4838{,}40}{12} \\ &= \text{R } 403{,}20 \end{aligned}$$

d) What is the total amount she has paid for the heater?

Solution:

To determine the total amount paid we add the accumulated loan amount and the deposit:

$$\begin{aligned} \text{Total amount} &= A + \text{deposit amount} \\ &= 4838{,}40 + 480 \\ &= \text{R } 5318{,}40 \end{aligned}$$

23. Khayalethu buys a couch costing R 8000 on a hire purchase agreement. He is charged an interest rate of 12% p.a. over 3 years.

a) How much will Khayalethu pay in total?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned} A &= ? \\ P &= \text{R } 8000 \\ i &= \frac{12}{100} = 0{,}12 \\ n &= 3 \end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned} A &= P(1 + in) \\ &= 8000(1 + 0{,}12 \times 3) \\ &= \text{R } 10\ 880 \end{aligned}$$

b) How much interest does he pay?

Solution:

To determine the interest amount, we subtract the principal amount from the accumulated amount:

$$\begin{aligned} \text{Interest amount} &= A - P \\ &= 10\ 880 - 8000 \\ &= \text{R } 2880 \end{aligned}$$

c) What is his monthly instalment?

Solution:

To determine the monthly instalment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned}\text{Monthly instalment} &= \frac{A}{\text{no. of months}}\\ &= \frac{10\ 880}{3 \times 12}\\ &= \text{R } 302{,}22\end{aligned}$$

24. Jwayelani buys a sofa costing R 7700 on a hire purchase agreement. He is charged an interest rate of 16% p.a. over 5 years.

a) How much will Jwayelani pay in total?

Solution:

Read the question carefully and write down the given information:

$$\begin{aligned}A &= ?\\ P &= 7700{,}00\\ i &= \frac{16}{100} = 0{,}16\\ n &= 5\end{aligned}$$

To determine the accumulated loan amount, we use the simple interest formula:

$$\begin{aligned}A &= P(1 + in)\\ &= 7700(1 + 0{,}16 \times 5)\\ &= \text{R } 13\ 860\end{aligned}$$

b) How much interest does he pay?

Solution:

To determine the interest amount, we subtract the principal amount from the accumulated amount:

$$\begin{aligned}\text{Interest amount} &= A - P\\ &= 13\ 860 - 7700\\ &= \text{R } 6160\end{aligned}$$

c) What is his monthly instalment?

Solution:

To determine the monthly instalment amount, we divide the accumulated amount A by the total number of months:

$$\begin{aligned}\text{Monthly instalment} &= \frac{A}{\text{no. of months}}\\ &= \frac{13\ 860}{5 \times 12}\\ &= \text{R } 231{,}00\end{aligned}$$

25. Bonnie bought a stove for R 3750. After 3 years she had finished paying for it and the R 956,25 interest that was charged for hire purchase. Determine the rate of simple interest that was charged.

Solution:

$$
\begin{aligned}
\text{Total paid } &= 3750 + 956{,}25 = 4706{,}25 \\
P &= 3750 \\
i &= ? \\
n &= 3 \\
A &= 4706{,}25 \\
\\
A &= P(1 + in) \\
\\
4706{,}25 &= 3750(1 + i(3)) \\
1{,}255 &= (1 + 3i) \\
0{,}255 &= 3i \\
i &= 0{,}085
\end{aligned}
$$

So the interest rate is: 8,5%

26. A new furniture store has just opened in town and is offering the following special:

    Purchase a lounge suite, a bedroom suite and kitchen appliances (fridge, stove, washing machine) for just R 50 000 and receive a free microwave. No deposit required, 5 year payment plan available. Interest charged at just 6,5% p.a.

    Babelwa purchases all the items on hire purchase. She decides to pay a R 1500 deposit. The store adds in an insurance premium of R 35,00 per month.

    What is Babelwa's monthly payment on the items?

    Solution:

$$
\begin{aligned}
P &= 50\ 000 - 1500 = 48\ 500 \\
i &= 0{,}065 \\
n &= 5
\end{aligned}
$$

Calculate the accumulated amount:

$$
\begin{aligned}
A &= P\,(1 + in) \\
A &= 48\ 500\,(1 + 0{,}065 \times 5) \\
&= 64\ 262{,}50
\end{aligned}
$$

Calculate the monthly repayments on the hire purchase agreement:

$$
\begin{aligned}
\text{Monthly payment } &= \frac{64\ 262{,}50}{(5)(12)} \\
&= 1071{,}04
\end{aligned}
$$

Add the insurance premium: R 1071,04 + R 35,00 = R 1106,04

27. The price of 2 litres of milk is R 17. How much will it cost in 3 years time if the inflation rate is 13% p.a.?

    Solution:

    Read the question carefully and write down the given information:

    - $A = ?$
    - $P =$ R 17
    - $n = 3$
    - $i = \frac{13}{100} = 0{,}13$

    To determine the future cost, we use the compound interest formula:

$$
\begin{aligned}
A &= P\,(1 + i)^n \\
&= 17 \times (1 + 0{,}13)^3 \\
&= \text{R } 24{,}53
\end{aligned}
$$

28. The price of a 2 l bottle of juice is R 16. How much will the juice cost in 8 years time if the inflation rate is 7% p.a.?

    Solution:

    Read the question carefully and write down the given information:

- $A = ?$
- $P =$ R 16
- $n = 8$
- $i = \frac{7}{100} = 0{,}07$

To determine the future cost, we use the compound interest formula:

$$\begin{aligned} A &= P(1+i)^n \\ &= 16 \times (1 + 0{,}07)^8 \\ &= \text{R } 27{,}49 \end{aligned}$$

29. A box of fruity-chews costs R 27 today. How much did it cost 8 years ago if the average rate of inflation was 10% p.a.? Round your answer to 2 decimal places.
Solution:
Read the question carefully and write down the given information:
    - $A =$ R 27
    - $P = ?$
    - $i = \frac{10}{100} = 0{,}10$
    - $n = 8$

    We use the compound interest formula and make P the subject:

$$\begin{aligned} A &= P(1+i)^n \\ P &= \frac{A}{(1+i)^n} \\ &= \frac{27}{(1+0{,}10)^8} \\ &= \text{R } 12{,}60 \end{aligned}$$

30. A box of smarties costs R 23 today. How much did the same box cost 8 years ago if the average rate of inflation was 14% p.a.? Round your answer to 2 decimal places.
Solution:
Read the question carefully and write down the given information:
    - $A =$ R 23
    - $P = ?$
    - $i = \frac{14}{100} = 0{,}14$
    - $n = 8$

    We use the compound interest formula and make P the subject:

$$\begin{aligned} A &= P(1+i)^n \\ P &= \frac{A}{(1+i)^n} \\ &= \frac{23}{(1+0{,}14)^8} \\ &= \text{R } 8{,}06 \end{aligned}$$

31. According to the latest census, South Africa currently has a population of 57 000 000.
    a) If the annual growth rate is expected to be 0,9%, calculate how many South Africans there will be in 10 years time (correct to the nearest hundred thousand).
    Solution:

$$\begin{aligned} A &= 57\,000\,000(1 + \frac{0{,}9}{100})^{10} \\ &= 57\,000\,000(1{,}009)^{10} \\ &= 57\,000\,000(1{,}0937)^{10} \\ &= 62{,}3 \text{ million people} \end{aligned}$$

    b) If it is found after 10 years that the population has actually increased by 10 million to 67 million, what was the growth rate?

Solution:

$$\begin{aligned}
67 &= 57\left(1+\frac{i}{100}\right)^{10} \\
\sqrt[10]{\frac{67}{57}} &= 1+\frac{i}{100} \\
\frac{i}{100} &= 1{,}01629 - 1 \\
i &= 100(0{,}016) \\
i &= 1{,}69 \\
i &\approx 1{,}7
\end{aligned}$$

32. The current population of Cape Town is 3 875 190 and the average rate of population growth in South Africa is 0,4% p.a.
What can city planners expect the population of Cape Town to be in 12 years time?
Note: Round your answer to the nearest integer.
Solution:
Read the question carefully and write down the given information:

$$\begin{aligned}
A &= ? \\
P &= 3\ 875\ 190 \\
i &= \frac{0{,}4}{100} = 0{,}004 \\
n &= 12
\end{aligned}$$

We use the following formula to determine the expected population for Cape Town:

$$\begin{aligned}
A &= P(1+i)^n \\
&= 3\ 875\ 190\,(1+0{,}004)^{12} \\
&= 4\ 065\ 346
\end{aligned}$$

33. The current population of Pretoria is 3 888 420 and the average rate of population growth in South Africa is 0,7% p.a.
What will the population of Pretoria be in 7 years time?
Note: Round your answer to the nearest integer.
Solution:
Read the question carefully and write down the given information:
   - $A = ?$
   - $P = 3\ 888\ 420$
   - $i = \frac{0.7}{100}$
   - $n = 7$

We use the following formula to determine the expected population for Pretoria:

$$\begin{aligned}
A &= P(1+i)^n \\
&= 3\ 888\ 420\left(1+\frac{0{,}7}{100}\right)^7 \\
&= 4\ 083\ 001
\end{aligned}$$

34. Monique wants to buy an iPad that costs £ 140, with the exchange rate currently at £ 1 = R 15. She estimates that the exchange rate will drop to R 9 in a month.
   a) How much will the iPad cost in rands, if she buys it now?
   Solution:

$$\begin{aligned}
\text{Cost} &= 140 \times \text{R } 15 \\
&= \text{R } 2100
\end{aligned}$$

   b) How much will she save if the exchange rate drops to R 9?

Solution:

$$\begin{aligned}\text{Cost} &= 140 \times \text{R } 9 \\ &= \text{R } 1260\end{aligned}$$

Therefore the amount she will have saved is:

$$\begin{aligned}\text{Saved} &= \text{R } 2100 - \text{R } 1260 \\ &= \text{R } 840\end{aligned}$$

c) How much will she lose if the exchange rate moves to R 20?
Solution:

$$\begin{aligned}\text{Cost} &= 140 \times \text{R } 20 \\ &= \text{R } 2800\end{aligned}$$

Therefore the amount she will lose is:

$$\begin{aligned}\text{Loss} &= \text{R } 2800 - \text{R } 2100 \\ &= \text{R } 700\end{aligned}$$

35. Xolile wants to buy a CD player that costs £ 140, with the exchange rate currently at £ 1 = R 14. She estimates that the exchange rate will drop to R 10 in a month.

a) How much will the CD player cost in rands, if she buys it now?
Solution:

$$\begin{aligned}\text{Cost} &= 140 \times \text{R } 14 \\ &= \text{R } 1960\end{aligned}$$

b) How much will she save if the exchange rate drops to R 10?
Solution:

$$\begin{aligned}\text{Cost} &= 140 \times \text{R } 10 \\ &= \text{R } 1400\end{aligned}$$

Therefore the amount she will have saved is:

$$\begin{aligned}\text{Saved} &= \text{R } 1960 - \text{R } 1400 \\ &= \text{R } 560\end{aligned}$$

c) How much will she lose if the exchange rate moves to R 20?
Solution:

$$\begin{aligned}\text{Cost} &= 140 \times \text{R } 20 \\ &= \text{R } 2800\end{aligned}$$

Therefore the amount she will lose is:

$$\begin{aligned}\text{Loss} &= \text{R } 2800 - \text{R } 1960 \\ &= \text{R } 840\end{aligned}$$

36. Alison is going on holiday to Europe. Her hotel will cost € 200 per night. How much will she need, in rands, to cover her hotel bill, if the exchange rate is € 1 = R 9,20?
Solution:
Cost in rands = cost in Euros × exchange rate

$$\begin{aligned}\text{Cost in rands } &= 200 \times 9{,}20 \\ &= \text{R } 1840\end{aligned}$$

37. Jennifer is buying some books online. She finds a publisher in the UK selling the books for £ 16,99.

She then finds the same books from a publisher in the USA for \$23,50.

Next she looks up the exchange rates to see which publisher has the better deal. If \$1 = R 12,43 and £ 1 = R 16,89, which publisher should she buy the books from?

Solution:

UK publisher: $16{,}99 \times \frac{16{,}89}{1} = \text{R } 286{,}96$

USA publisher: $23{,}50 \times \frac{12{,}43}{1} = \text{R } 292{,}11$.

Therefore Jennifer should buy the books from the UK publisher.

38. Bonani won a trip to see Machu Picchu in Peru followed by a trip to Brazil for the carnival. He is given R 25 000 to spend while on the trip.

He then looks up the current exchange rates and finds the following information:

$$\text{R } 1 = 0{,}26 \text{ PEN}$$
$$1 \text{ BRL} = 1{,}17 \text{ PEN}$$

In Peru he spends 2380 PEN. When he converts the remaining Peruvian sol to Brazilian real, how much money does he have (in Brazilian real)?

Solution:

We first convert from rand to Peruvian sol: $25\ 000 \times \frac{0{,}26}{1} = 6500$ PEN

He spends 2380 PEN of this and so he has 4120 PEN to convert to Brazilian real.

Now we can convert from Peruvian sol to Brazilian real: $4120 \text{ PEN} \times \frac{1}{1{,}17} = 3521{,}37$ BRL

So he will have 3521,37 BRL to spend in Brazil.

39. If the exchange rate to the rand for the Japanese yen is ¥ 100 = R 6,23 and for the Australian dollar is 1 AUD = R 5,11, determine the exchange rate between the Australian dollar and the Japanese yen.

Solution:

$$\begin{aligned} \frac{\text{AUD}}{\text{Yen}} &= \frac{\text{ZAR}}{\text{Yen}} \times \frac{\text{AUD}}{\text{ZAR}} \\ &= \frac{6{,}2287}{100} \times \frac{1}{5{,}1094} \\ &= 0{,}012 \text{ AUD} \\ &= 1 \text{ Yen} \\ \text{or } 1 \text{ AUD} &= 82{,}02 \text{ Yen} \end{aligned}$$

40. Khetang has just been to Europe to work for a few months. He returns to South Africa with € 2850 to invest in a savings account.

His bank offers him a savings account which pays 5,3% compound interest per annum. The bank converts Khetang's Euros to rands at an exchange rate of € 1 = R 12,89.

If Khetang invests his money for 6 years, how much interest does he earn on his investment?

Solution:

We first convert from Euros to rands: $2850 \times \frac{12{,}89}{1} = \text{R } 36\ 736{,}50$.

Now we can calculate how much Khetang earns.

$$\begin{aligned} P &= 36\ 736{,}50 \\ i &= 0{,}053 \\ n &= 6 \\ \\ A &= P(1+i)^n \\ A &= 36\ 736{,}50(1+0{,}053)^6 \\ A &= \text{R } 50\ 080{,}42 \end{aligned}$$

Interest earned:

$$\text{R } 50\ 080{,}42 - \text{R } 36\ 736{,}50 = \text{R } 13\ 343{,}92$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GJJ	2. 2GJK	3. 2GJM	4. 2GJN	5. 2GJP	6. 2GJQ	7. 2GJR	8. 2GJS
9. 2GJT	10. 2GJV	11. 2GJW	12. 2GJX	13a. 2GJY	13b. 2GJZ	13c. 2GK2	14. 2GK3
15. 2GK4	16. 2GK5	17. 2GK6	18. 2GK7	19. 2GK8	20. 2GK9	21. 2GKB	22. 2GKC
23. 2GKD	24. 2GKF	25. 2GKG	26. 2GKH	27. 2GKJ	28. 2GKK	29. 2GKM	30. 2GKN
31. 2GKP	32. 2GKQ	33. 2GKR	34. 2GKS	35. 2GKT	36. 2GKV	37. 2GKW	38. 2GKX
39. 2GKY	40. 2GKZ						

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CHAPTER 10

Statistics

- This chapter covers revision of central tendency in ungrouped data and then extends this to measures of central tendency in grouped data. The range is revised and extended to include percentiles, quartiles, interquartile and semi interquartile range. The five number summary and box and whisker diagram is introduced here. Finally statistical summaries are applied to data to make meaningful comments on the context associated with the data.
- Intervals for grouped data should be given using inequalities ($0 \leq x < 20$) rather than 0 - 19.
- Discuss the misuse of statistics in the real world and encourage awareness.

You can find data sets and statistics relevant to South Africa from the statssa website.

10.1 Collecting data

Exercise 10 – 1:

1. The following data set of dreams that learners have was collected from Grade 12 learners just after their final exams.
   {"I want to build a bridge!"; "I want to help the sick."; "I want running water!"}
   Categorise the data set.
   Solution: This data set cannot be written as numbers and so must be qualitative. This data set is anecdotal since it takes the form of a story. Therefore the data set is qualitative anecdotal.
2. The following data set of sweets in a packet was collected from visitors to a sweet shop.
   $\{23; 25; 22; 26; 27; 25; 21; 28\}$
   Categorise the data set.
   Solution: This data set is a set of numbers and so must be quantitative. This data set is discrete since it can be represented by integers and is a count of the number of sweets. Therefore the data set is quantitative discrete.
3. The following data set of questions answered correctly was collected from a class of maths learners.
   $\{3; 5; 2; 6; 7; 5; 1; 2\}$
   Categorise the data set.
   Solution: This data set is a set of numbers and so must be quantitative. This data set is discrete since it can be represented by integers and is a count of the number of questions answered correctly. Therefore the data set is quantitative discrete.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'. 1. 2GM2 2. 2GM3 3. 2GM4

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10.2 Measures of central tendency

Mean

Median

Mode

Exercise 10 – 2:

1. Calculate the **mean** of the following data set:
   $\{9; 14; 9; 14; 8; 8; 9; 8; 9; 9\}$. Round your answer to 1 decimal place.
   Solution:

$$\text{mean} = \frac{9+14+9+14+8+8+9+8+9+9}{10}$$
$$= 9{,}7$$

The mean is: 9,7.

2. Calculate the **median** of the following data set:
   $\{4; 13; 10; 13; 13; 4; 2; 13; 13; 13\}$.
   Solution:
   We first need to order the data set:
   $\{2; 4; 4; 10; 13; 13; 13; 13; 13; 13\}$.
   Since there are an even number of values in this data set (10) the median lies between the fifth and sixth place:

$$\text{median} = \frac{13+13}{2}$$
$$= 13$$

   The median is: 13.

3. Calculate the **mode** of the following data set:
   $\{6; 10; 6; 6; 13; 12; 12; 7; 13; 6\}$
   Solution:
   We first sort the data set: $\{6; 6; 6; 6; 7; 10; 12; 12; 13; 13\}$. The mode is the value that occurs most often in the data set.
   Therefore the mode is: 6

4. Calculate the mean, median and mode of the following data sets:

   a) $\{2; 5; 8; 8; 11; 13; 22; 23; 27\}$
   Solution:
   The data set is already ordered.

$$\text{mean} = \frac{2+5+8+8+11+13+22+23+27}{9}$$
$$= 13{,}2$$

   Since there is an odd number of values in this data set the median lies at the fifth number: 11
   The mode is the value that occurs the most. In this data set the mode is 8.
   The mean, median and mode are: mean: 13,2; median: 11; mode: 8.

   b) $\{15; 17; 24; 24; 26; 28; 31; 43\}$
   Solution:
   The data set is already ordered.

$$\text{mean} = \frac{15+17+24+24+26+28+31+43}{8}$$
$$= 26$$

   Since there is an even number of values in this data set the median lies between the fourth and fifth numbers:

$$\text{median} = \frac{24+26}{2}$$
$$= 25$$

   The mode is the value that occurs the most. In this data set the mode is 24.
   The mean, median and mode are: mean: 26; median: 25; mode: 24.

   c) $\{4; 11; 3; 15; 11; 13; 25; 17; 2; 11\}$
   Solution:
   We first need to order the data set: $\{2; 3; 4; 11; 11; 11; 13; 15; 17; 25\}$.

$$\text{mean} = \frac{2+3+4+11+11+11+13+15+17+25}{10}$$
$$= 11{,}2$$

Since there is an even number of values in this data set the median lies between the fifth and sixth numbers:

$$\begin{aligned}\text{median} &= \frac{11+11}{2} \\ &= 11\end{aligned}$$

The mode is the value that occurs the most. In this data set the mode is 11.

Therefore the mean, median and mode are: mean: 11,2; median: 11; mode: 11.

d) $\{24; 35; 28; 41; 31; 49; 31\}$

Solution:

We first need to order the data set: $\{24; 28; 31; 31; 35; 41; 49\}$

$$\begin{aligned}\text{mean} &= \frac{24+28+31+31+35+41+49}{7} \\ &= 34{,}3\end{aligned}$$

Since there is an odd number of values in this data set the median lies at the fourth number: 31

The mode is the value that occurs the most. In this data set the mode is 31.

The mean, median and mode are: mean: 34,29; median: 31; mode: none.

5. The ages of 15 runners of the Comrades Marathon were recorded:

$$\{31; 42; 28; 38; 45; 51; 33; 29; 42; 26; 34; 56; 33; 46; 41\}$$

Calculate the mean, median and modal age.

Solution:

We first need to order the data set: $\{26; 28; 29; 31; 33; 33; 34; 38; 41; 42; 42; 45; 46; 51; 56\}$

$$\begin{aligned}\text{mean} &= \frac{26+28+29+31+33+33+34+38+41+42+42+45+46+51+56}{15} \\ &= 38{,}3\end{aligned}$$

Since there is an odd number of values in this data set the median lies at the eighth number: 38.

The mode is the value that occurs the most. In this data set there are two modes: 33 and 42.

Therefore the mean, median and modal ages are: mean: 38,3; median 38; mode 33 and 42.

6. A group of 10 friends each have some stones. They work out that the **mean** number of stones they have is 6. Then 7 friends leave with an unknown number (x) of stones. The remaining 3 friends work out that the **mean** number of stones they have left is 12,33.

When the 7 friends left, how many stones did they take with them?

Solution:

If the **mean** number of stones the group originally had was **6** then the total number of stones must have been:

$$\begin{aligned}\text{mean} &= \frac{\text{number of stones (before)}}{\text{group size}} \\ \text{number of stones (before)} &= \text{mean} \times \text{group size} \\ \text{number of stones (before)} &= (6) \times (10) \\ \text{number of stones (before)} &= 60\end{aligned}$$

We are then told that **7** friends leave and thereafter the **mean** number of stones left is **12,33**. Now we can work out the remaining number of stones.

$$\begin{aligned}\text{mean} &= \frac{\text{number of stones (after)}}{\text{group size}} \\ \text{number of stones (after)} &= \text{mean} \times \text{group size} \\ \text{number of stones (after)} &= (12{,}33) \times (3) \\ \text{number of stones (after)} &= 37\end{aligned}$$

Now we can calculate how many stones were taken by the 7 friends who left the group.

$$\begin{aligned}
\text{number of stones removed } (x) &= \text{items before} - \text{items after}\\
\text{number of stones removed } (x) &= (60) - (37)\\
\text{number of stones removed } (x) &= 23
\end{aligned}$$

7. A group of 9 friends each have some coins. They work out that the **mean** number of coins they have is 4. Then 5 friends leave with an unknown number (x) of coins. The remaining 4 friends work out that the **mean** number of coins they have left is 2,5.

   When the 5 friends left, how many coins did they take with them?

   Solution:

   If the **mean** number of coins the group originally had was **4** then the total number of coins must have been:

$$\begin{aligned}
\text{mean} &= \frac{\text{number of coins (before)}}{\text{group size}}\\
\text{number of coins (before)} &= \text{mean} \times \text{group size}\\
\text{number of coins (before)} &= (4) \times (9)\\
\text{number of coins (before)} &= 36
\end{aligned}$$

   We are then told that **5** friends leave and thereafter the **mean** number of coins left is **2,5**. Let us work out the remaining number of coins.

$$\begin{aligned}
\text{mean} &= \frac{\text{number of coins (after)}}{\text{group size}}\\
\text{number of coins (after)} &= \text{mean} \times \text{group size}\\
\text{number of coins (after)} &= (2{,}5) \times (4)\\
\text{number of coins (after)} &= 10
\end{aligned}$$

   Now we can calculate how many coins were taken by the 5 friends who left the group.

$$\begin{aligned}
\text{number of coins removed } (x) &= \text{items before} - \text{items after}\\
\text{number of coins removed } (x) &= (36) - (10)\\
\text{number of coins removed } (x) &= 26
\end{aligned}$$

8. A group of 9 friends each have some marbles. They work out that the **mean** number of marbles they have is 3. Then 3 friends leave with an unknown number (x) of marbles. The remaining 6 friends work out that the **mean** number of marbles they have left is 1,17.

   When the 3 friends left, how many marbles did they take with them?

   Solution:

   If the **mean** number of marbles the group originally had was **3** then the total number of marbles must have been:

$$\begin{aligned}
\text{mean} &= \frac{\text{number of marbles (before)}}{\text{group size}}\\
\text{number of marbles (before)} &= \text{mean} \times \text{group size}\\
\text{number of marbles (before)} &= (3) \times (9)\\
\text{number of marbles (before)} &= 27
\end{aligned}$$

   We are then told that **3** friends leave and thereafter the **mean** number of marbles left is **1,17**. Let us work out the remaining number of marbles.

$$\begin{aligned}
\text{mean} &= \frac{\text{number of marbles (after)}}{\text{group size}}\\
\text{number of marbles (after)} &= \text{mean} \times \text{group size}\\
\text{number of marbles (after)} &= (1{,}17) \times (6)\\
\text{number of marbles (after)} &= 7
\end{aligned}$$

   Now we can calculate how many marbles were taken by the 3 friends who left the group.

$$\text{number of marbles removed } (x) = \text{items before} - \text{items after}$$
$$\text{number of marbles removed } (x) = (27) - (7)$$
$$\text{number of marbles removed } (x) = 20$$

9. In the first of a series of jars, there is 1 sweet. In the second jar, there are 3 sweets. The mean number of sweets in the first two jars is 2.

   a) If the mean number of sweets in the first three jars is 3, how many sweets are there in the third jar?
   Solution:
   Let n_3 be the number of sweets in the third jar:

$$\frac{1+3+n_3}{3} = 3$$
$$1+3+n_3 = 9$$
$$n_3 = 5$$

   b) If the mean number of sweets in the first four jars is 4, how many sweets are there in the fourth jar?
   Solution:
   Let n_4 be the number of sweets in the fourth jar:

$$\frac{1+3+5+n_4}{4} = 4$$
$$9+n_4 = 16$$
$$n_4 = 7$$

10. Find a set of five ages for which the mean age is 5, the modal age is 2 and the median age is 3 years.
    Solution:
    Let the five different ages be $x_1,\ x_2,\ x_3,\ x_4$ and x_5. Therefore the mean is:

$$\frac{x_1+x_2+x_3+x_4+x_5}{5} = 5$$
$$x_1+x_2+x_3+x_4+x_5 = 25$$

The median value is at position 3, therefore $x_3 = 3$.
The mode is the age that occurs most often. We have 5 ages to work with and we know one of the ages is 3 (from the median). So the ordered data set is: $\{x_1; x_2; 3; x_4; x_5\}$ (remember that we always calculate mean, mode and median using the ordered data set). We are told that the mode is 2. Looking at the ordered data set we see that either x_1 or x_2 must be 2 (x_4 and x_5 cannot be 2 as that would make the data set unordered). However, if only one of these values is 2 then the mode will not be 2. Therefore $x_1 = x_2 = 2$.
So we can now update our calculation of the mean:

$$2+2+3+x_4+x_5 = 25$$
$$18 = x_4+x_5$$

x_4 and x_5 can be any numbers that add up to 18 and are not the same (if they were the same then the mode would not be 2), so 12 and 6 or 8 and 10 or 3 and 15, etc.
Possible data sets:

Data set 1: $\{2; 2; 3; 4; 14\}$
Data set 2: $\{2; 2; 3; 5; 13\}$
Data set 3: $\{2; 2; 3; 6; 12\}$
Data set 4: $\{2; 2; 3; 7; 11\}$
Data set 5: $\{2; 2; 3; 8; 10\}$

Note that the set of ages must be ordered, the median value must be 3 and there must be 2 ages of 2.

11. Four friends each have some marbles. They work out that the mean number of marbles they have is 10. One friend leaves with 4 marbles. How many marbles do the remaining friends have together?

Solution:

Let the number of marbles per friend be x_1, x_2, x_3 and x_4.

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = 10$$
$$x_1 + x_2 + x_3 + x_4 = 40$$

One friend leaves:

$$x_1 + x_2 + x_3 = 40 - 4$$
$$x_1 + x_2 + x_3 = 36$$

Therefore the remaining friends have 36 marbles.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GM6 2. 2GM7 3. 2GM8 4a. 2GM9 4b. 2GMB 4c. 2GMC 4d. 2GMD 5. 2GMF 6. 2GMG 7. 2GMH 8. 2GMJ 9. 2GMK 10. 2GMM 11. 2GMN

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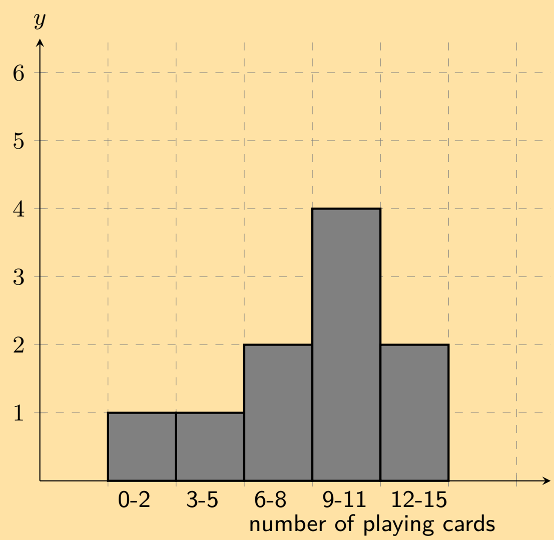
10.3 Grouping data

Exercise 10 – 3:

1. A group of 10 learners count the number of playing cards they each have. This is a histogram describing the data they collected:

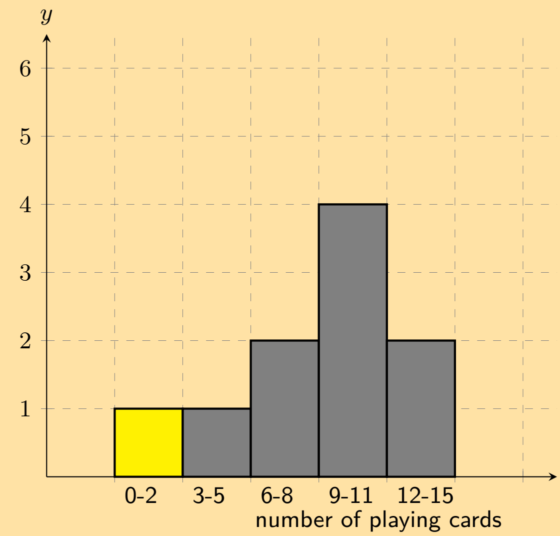


Count the number of playing cards in the following range: $0 \leq$ number of playing cards ≤ 2

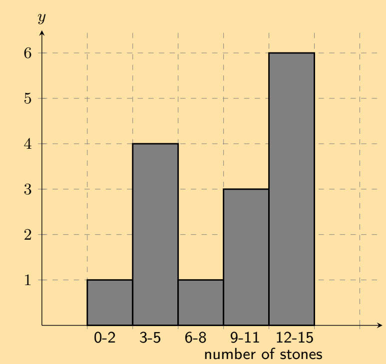
Solution:

From the graph the answer is: 1

From the histogram, we arrive at our answer by reading the height of the specified interval from the histogram.



2. A group of 15 learners count the number of stones they each have. This is a histogram describing the data they collected:

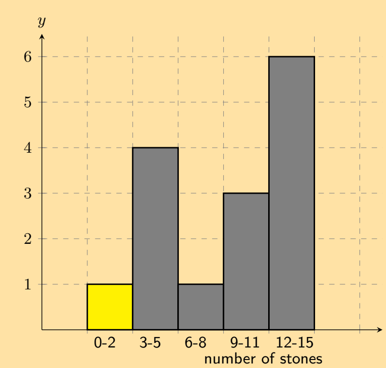


Count the number of stones in the following range: $0 \leq$ number of stones ≤ 2

Solution:

From the graph the answer is: 1

From the histogram, we arrive at our answer by reading the height of the specified interval from the histogram.



3. A group of **20** learners count the number of playing cards they each have. This is the data they collect:

14	9	11	8	13
2	3	4	16	17
9	19	10	14	4
16	16	11	2	17

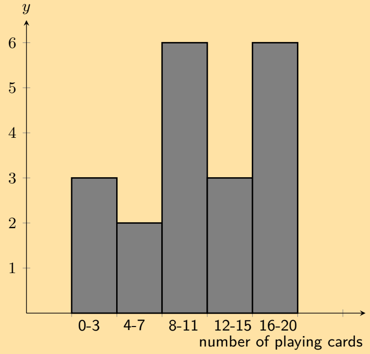
Count the number of learners who have from 12 up to 15 playing cards. In other words, how many learners have playing cards in the following range: 12 $\leq$ number of playing cards $\leq$ 15? It may be helpful for you to draw a histogram in order to answer the question.

Solution:

Firstly we sort the table into sequential order, starting with the smallest value.

2	2	3	4	4
8	9	9	10	11
11	13	14	14	16
16	16	17	17	19

Secondly, we draw a histogram of the data:



From the histogram you can see that the number of learners with playing cards in the range: 12 $\leq$ number of playing cards $\leq$ 15 is 3.

4. A group of **20** learners count the number of stones they each have. This is the data they collect:

16	6	11	19	20
17	13	1	5	12
5	2	16	11	16
6	10	13	6	17

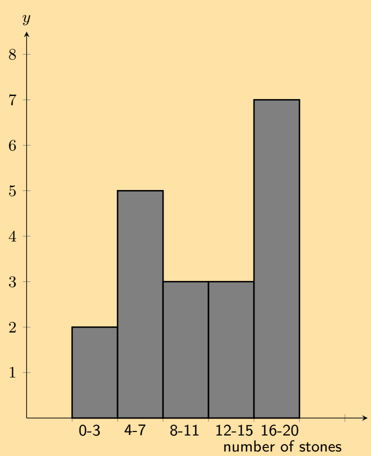
Count the number of learners who have from 4 up to 7 stones. In other words, how many learners have stones in the following range: 4 $\leq$ number of stones $\leq$ 7? It may be helpful for you to draw a histogram in order to answer the question.

Solution:

Firstly we sort the table into sequential order, starting with the smallest value.

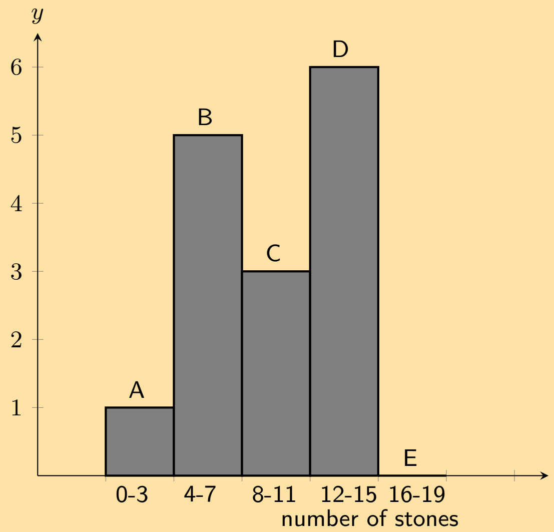
1	2	5	5	6
6	6	10	11	11
12	13	13	16	16
16	17	17	19	20

Secondly, we draw a histogram of the data:



From the histogram you can see that the number of learners with stones in the range: $4 \leq$ number of stones ≤ 7 is 5.

5. A group of 20 learners count the number of stones they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.



The data set below shows the correct information for the number of stones the learners have. Each value represents the number of stones for one learner.

$$\{4; 12; 15; 14; 18; 12; 17; 15; 1; 6; 6; 12; 6; 8; 6; 8; 17; 19; 16; 8\}$$

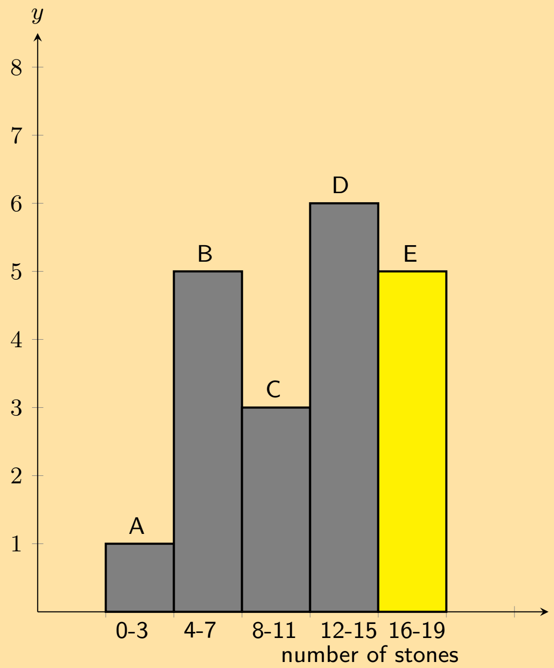
Help them figure out which **column** in the histogram is **incorrect**.

Solution:

We first need to order the data:

$$\{1; 4; 6; 6; 6; 6; 8; 8; 8; 12; 12; 12; 14; 15; 15; 16; 17; 17; 18; 19\}$$

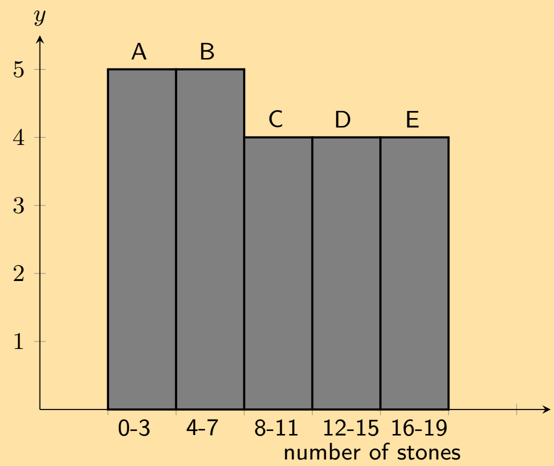
Using the ordered data set we can group the data and draw the correct histogram:



The column with the error in it was: E.

The learners used the incorrect value of **0**, when the correct value is 5.

6. A group of 20 learners count the number of stones they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.



The data set below shows the correct information for the number of stones the learners have. Each value represents the number of stones for one learner.

$$\{19; 11; 5; 2; 3; 4; 14; 2; 12; 19; 11; 14; 2; 19; 11; 5; 17; 10; 1; 12\}$$

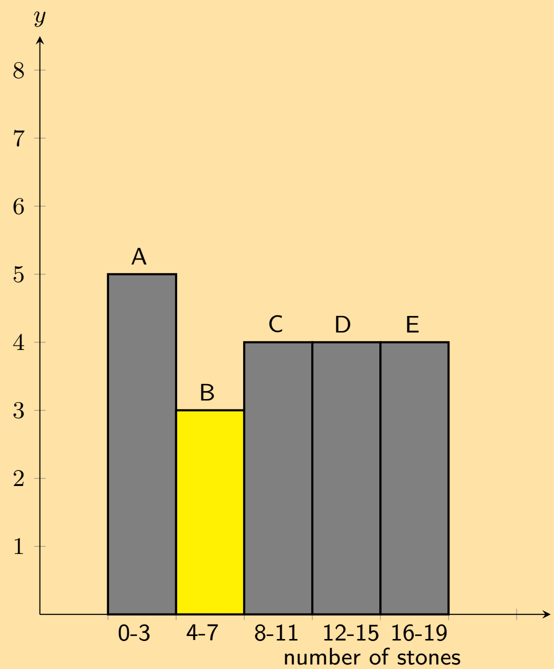
Help them figure out which **column** in the histogram is **incorrect**.

Solution:

We first need to order the data:

$$\{1; 2; 2; 2; 3; 4; 5; 5; 10; 11; 11; 11; 12; 12; 14; 14; 17; 19; 19; 19\}$$

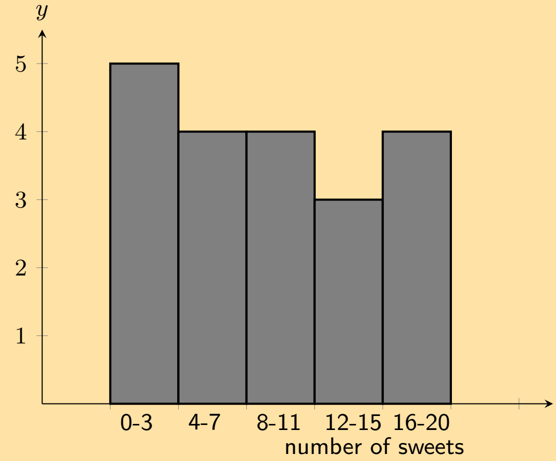
Using the ordered data set we can group the data and draw the correct histogram:



The column with the error in it was: B.

The learners used the incorrect value of **5**, when the correct value is 3.

7. A group of learners count the number of sweets they each have. This is a histogram describing the data they collected:



A cat jumps onto the table, and all their notes land on the floor, mixed up, by accident!
Help them find which of the following data sets match the above histogram:
Data Set A

2	1	20	10	5
3	10	2	6	1
2	2	17	3	18
3	7	10	8	18

Data Set B

2	9	12	10	5
9	9	10	13	6
5	11	10	7	7

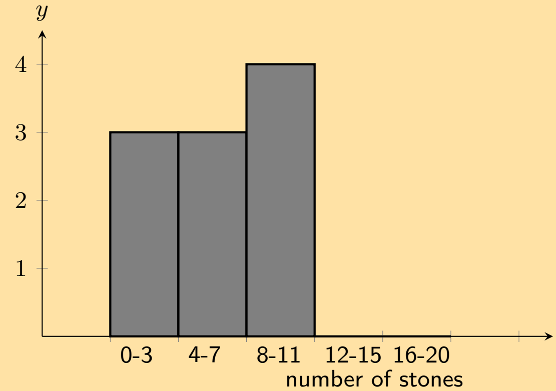
Data Set C

3	12	16	10	15
17	18	2	3	7
11	12	8	2	7
17	3	11	4	4

Solution:

The correct answer is: Data Set C

8. A group of learners count the number of stones they each have. This is a histogram describing the data they collected:



A cleaner knocks over their table, and all their notes land on the floor, mixed up, by accident!
Help them find which of the following data sets match the above histogram:
Data Set A

12	4	2	15	10
18	10	16	16	19
1	2	9	10	16
10	11	9	2	13

Data Set B

7	10	4	5	8
7	12	10	14	5
1	9	2	13	3

Data Set C

9	3	8	5	8
5	8	1	4	3

Solution:

The correct answer is: Data Set C

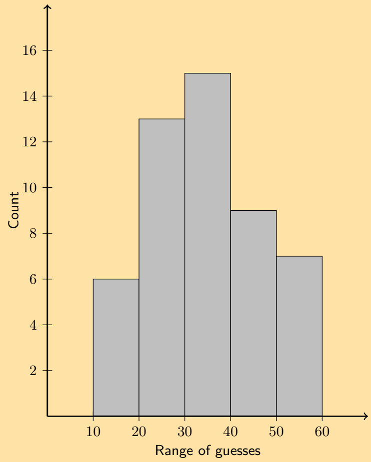
9. A class experiment was conducted and 50 learners were asked to guess the number of sweets in a jar. The following guesses were recorded:

56	49	40	11	33	33	37	29	30	59
21	16	38	44	38	52	22	24	30	34
42	15	48	33	51	44	33	17	19	44
47	23	27	47	13	25	53	57	28	23
36	35	40	23	45	39	32	58	22	40

a) Draw up a grouped frequency table using the intervals $10 < x \leq 20$, $20 < x \leq 30$, $30 < x \leq 40$, $40 < x \leq 50$ and $50 < x \leq 60$.
Solution:

Group	Frequency
$10 < x \leq 20$	6
$20 < x \leq 30$	13
$30 < x \leq 40$	15
$40 < x \leq 50$	9
$50 < x \leq 60$	7

b) Draw the histogram corresponding to the frequency table of the grouped data.
Solution:



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1. 2GMQ 2. 2GMR 3. 2GMS 4. 2GMT 5. 2GMV 6. 2GMW
7. 2GMX 8. 2GMY 9. 2GMZ

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Exercise 10 – 4:

1. Consider the following grouped data and calculate the mean, the modal group and the median group.

Mass (kg)	Count
$40 < m \leq 45$	7
$45 < m \leq 50$	10
$50 < m \leq 55$	15
$55 < m \leq 60$	12
$60 < m \leq 65$	6

Solution:
To find the mean we use the middle value for each group. The count then tells us how many times that value occurs in the data set. Therefore the mean is:

$$\begin{aligned} \text{mean} &= \frac{7(43) + 10(48) + 15(53) + 12(58) + 6(63)}{7 + 10 + 15 + 12 + 6} \\ &= \frac{2650}{50} \\ &= 53 \end{aligned}$$

The modal group is the group with the highest number of data values. This is $50 < m \leq 55$ with 15 data values.
The median group is the central group. There are 5 groups and so the central group is the third one: $50 < m \leq 55$.
Mean: 52; Modal group: $50 < m \leq 55$; Median group: $50 < m \leq 55$.

2. Find the mean, the modal group and the median group in this data set of how much time people needed to complete a game.

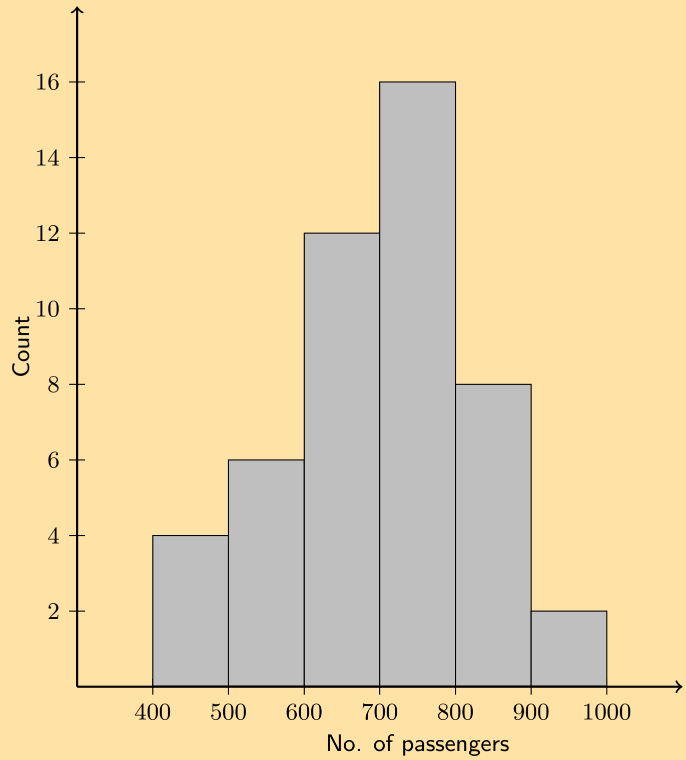
Time (s)	Count
$35 < t \leq 45$	5
$45 < t \leq 55$	11
$55 < t \leq 65$	15
$65 < t \leq 75$	26
$75 < t \leq 85$	19
$85 < t \leq 95$	13
$95 < t \leq 105$	6

Solution:
To find the mean we use the middle value for each group. The count then tells us how many times that value occurs in the data set. Therefore the mean is:

$$\begin{aligned} \text{mean} &= \frac{5(40{,}5) + 11(50{,}5) + 15(60{,}5) + 26(70{,}5) + 19(80{,}5) + 13(90{,}5) + 6(100{,}5)}{5 + 11 + 15 + 26 + 19 + 13 + 6} \\ &= \frac{6807{,}5}{95} \\ &= 71{,}66 \end{aligned}$$

The modal group is the group with the highest number of data values. This is $65 < m \leq 75$ with 26 data values.
The median group is the central group. There are 7 groups and so the central group is the fourth one: $65 < m \leq 75$.
Mean: 70,66; Modal group: $65 < t \leq 75$; Median group: $65 < t \leq 75$.

3. The histogram below shows the number of passengers that travel in Alfred's minibus taxi per week.



Calculate:

a) the modal interval

Solution:

The modal interval is the interval with the highest number of data values. For this data set it is: $700 < x \leq 800$ with 16 values.

b) the total number of passengers to travel in Alfred's taxi

Solution:

We add up the counts in each group and then multiply these counts with the central value for each group: $4(450) + 6(550) + 12(650) + 16(750) + 8(850) + 2(950) =$ 33 600.

c) an estimate of the mean

Solution:

There are 48 values in the data set. Therefore the mean is $\frac{33\ 600}{48} = 700$.

d) an estimate of the median

Solution:

We are looking for an estimate of the median rather than the median group here. In this case we note that there are 48 data values in the data set. Therefore the median will lie between the 24^{th} and 25^{th} values.

We note that 22 values in the first 3 groups and 38 values in the first four groups so the median must lie in the fourth group. Therefore we can estimate the median as the middle value of the fourth group: 750.

e) if it is estimated that every passenger travelled an average distance of 5 km, how much money would Alfred have made if he charged R 3,50 per km?

Solution:

$3{,}50 \times 5 \times 33\ 600 =$ R 588 000.

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10.4 Measures of dispersion

Range

Percentiles

Percentiles for grouped data

Ranges

Exercise 10 – 5:

1. A group of **15** learners count the number of sweets they each have. This is the data they collect:

$$\begin{array}{ccccc} 4 & 11 & 14 & 7 & 14 \\ 5 & 8 & 7 & 12 & 12 \\ 5 & 13 & 10 & 6 & 7 \end{array}$$

   Calculate the **range** of values in the data set.
   Solution:
   We first need to order the data set:

$$\{4; 5; 5; 6; 7; 7; 7; 8; 10; 11; 12; 12; 13; 14; 14\}$$

   Next we find the maximum value in the data set:

$$\text{maximum value} = 14$$

   Then we find the minimum value in the data set:

$$\text{minimum value} = 4$$

   Finally, we calculate the range of the data set:

$$\begin{aligned} \text{range} &= (\text{maximum value}) - (\text{minimum value}) \\ &= (14) - (4) \\ &= 10 \end{aligned}$$

2. A group of **10** learners count the number of playing cards they each have. This is the data they collect:

$$\begin{array}{ccccc} 5 & 1 & 3 & 1 & 4 \\ 10 & 1 & 3 & 3 & 4 \end{array}$$

   Calculate the **range** of values in the data set.
   Solution:
   We first need to order the data set:

$$\{1; 1; 1; 3; 3; 3; 4; 4; 5; 10\}$$

   Next we find the maximum value in the data set:

$$\text{maximum value} = 10$$

   Then we find the minimum value in the data set:

$$\text{minimum value } = 1$$

Finally, we calculate the range of the data set:

$$\begin{aligned}\text{range} &= (\text{maximum value}) - (\text{minimum value}) \\ &= 10 - 1 \\ &= 9\end{aligned}$$

3. Find the range of the data set

$$\{1; 2; 3; 4; 4; 4; 5; 6; 7; 8; 8; 9; 10; 10\}$$

Solution:
The data set is already ordered.
Firstly, we find the maximum value in the data set:

$$\text{maximum value } = 10$$

Secondly, we find the minimum value in the data set:

$$\text{minimum value } = 1$$

Finally, we calculate the range of the data set:

$$\begin{aligned}\text{range} &= (\text{maximum value}) - (\text{minimum value}) \\ &= 10 - 1 \\ &= 9\end{aligned}$$

4. What are the quartiles of this data set?

$$\{3; 5; 1; 8; 9; 12; 25; 28; 24; 30; 41; 50\}$$

Solution:
We first order the data set.

$$\{1; 3; 5; 8; 9; 12; 24; 25; 28; 30; 41; 50\}$$

Next we find the ranks of the quartiles. Using the percentile formula with $n = 12$, we can find the rank of the 25^{th}, 50^{th} and 75^{th} percentiles:

$$\begin{aligned}r_{25} &= \frac{25}{100}(12-1)+1 \\ &= 3{,}75 \\ r_{50} &= \frac{50}{100}(12-1)+1 \\ &= 6{,}5 \\ r_{75} &= \frac{75}{100}(12-1)+1 \\ &= 9{,}25\end{aligned}$$

Find the values of the quartiles. Note that each of these ranks is a fraction, meaning that the value for each percentile is somewhere in between two values from the data set.
For the 25^{th} percentile the rank is 3,75, which is between the third and fourth values. Therefore the 25^{th} percentile is $\frac{5+8}{2} = 6{,}5$.
For the 50^{th} percentile (the median) the rank is 6,5, meaning halfway between the sixth and seventh values. Therefore the median is $\frac{12+24}{2} = 18$. For the 75^{th} percentile the rank is 9,25, meaning between the ninth and tenth values. Therefore the 75^{th} percentile is $\frac{28+30}{2} = 29$.
Therefore we get the following values for the quartiles: $Q_1 = 6{,}5$; $Q_2 = 18$; $Q_3 = 29$.

5. A class of 12 learners writes a test and the results are as follows:

$$\{20; 39; 40; 43; 43; 46; 53; 58; 63; 70; 75; 91\}$$

Find the range, quartiles and the interquartile range.

Solution:

The data set is ordered.

The range is:

$$\begin{aligned} \text{range} &= (\text{maximum value}) - (\text{minimum value}) \\ &= (91) - (20) \\ &= 71 \end{aligned}$$

To find the quartiles we start by finding the ranks of the quartiles. Using the percentile formula with $n = 12$, we can find the rank of the 25^{th}, 50^{th} and 75^{th} percentiles:

$$\begin{aligned} r_{25} &= \frac{25}{100}(12-1)+1 \\ &= 3{,}75 \\ r_{50} &= \frac{50}{100}(12-1)+1 \\ &= 6{,}5 \\ r_{75} &= \frac{75}{100}(12-1)+1 \\ &= 9{,}25 \end{aligned}$$

Find the values of the quartiles. Note that each of these ranks is a fraction, meaning that the value for each percentile is somewhere in between two values from the data set.

For the 25^{th} percentile the rank is 3,75, which is between the third and fourth values. Therefore the 25^{th} percentile is $\frac{40+43}{2} = 41{,}5$.

For the 50^{th} percentile (the median) the rank is 6,5, meaning halfway between the sixth and seventh values. Therefore the median is $\frac{46+53}{2} = 49{,}5$. For the 75^{th} percentile the rank is 9,25, meaning between the ninth and tenth values. Therefore the 75^{th} percentile is $\frac{63+70}{2} = 66{,}5$.

Therefore we get the following values for the quartiles: $Q_1 = 41{,}5$; $Q_2 = 49{,}5$; $Q_3 = 66{,}5$.

Interquartile range:

$$\begin{aligned} \text{interquartile range} &= \text{quartile 3} - \text{quartile 1} \\ &= 66{,}5 - 41{,}5 \\ &= 25 \end{aligned}$$

6. Three sets of data are given:

Data set 1: $\{9; 12; 12; 14; 16; 22; 24\}$

Data set 2: $\{7; 7; 8; 11; 13; 15; 16\}$

Data set 3: $\{11; 15; 16; 17; 19; 22; 24\}$

For each data set find:

a) the range

Solution:

All three data sets are ordered. To find the range we subtract the minimum value from the maximum value. Doing so for each data set gives the following values for the range.

Data set 1: $24 - 9 = 15$

Data set 2: $16 - 7 = 9$

Data set 3: $24 - 11 = 13$

b) the lower quartile

Solution:

For each data set $n = 7$. Therefore the rank of the 25^{th} percentile is the same for each data set: $r_{25} = \frac{25}{100}(7-1)+1 = 2{,}5$. Therefore for each data set the lower quartile lies between the second and third values.

The lower quartile for each data set is:
Data set 1: 12
Data set 2: 7,5
Data set 3: 15,5

c) the median

Solution:

For each data set $n = 7$. Therefore the rank of the 50^{th} percentile is the same for each data set: $r_{50} = \frac{50}{100}(7-1)+1 = 4$. Therefore for each data set the median is the fourth value.
The median for each data set is:
Data set 1: 14
Data set 2: 11
Data set 3: 17

d) the upper quartile

Solution:

For each data set $n = 7$. Therefore the rank of the 75^{th} percentile is the same for each data set: $r_{75} = \frac{75}{100}(7-1)+1 = 5{,}5$. Therefore for each data set the lower quartile lies between the fifth and sixth values.
The upper quartile for each data set is:
Data set 1: 19
Data set 2: 14
Data set 3: 20,5

e) the interquartile range

Solution:

The interquartile range is calculated by subtracting the lower quartile from the upper quartile.
Data set 1: $19 - 12 = 7$
Data set 2: $14 - 7{,}5 = 6{,}5$
Data set 3: $20{,}5 - 15{,}5 = 5$

f) the semi-interquartile range

Solution:

The semi-interquartile range is half the interquartile range.
Data set 1: $\frac{7}{2} = 3{,}5$
Data set 2: $\frac{6{,}5}{2} = 3{,}25$
Data set 3: $\frac{5}{2} = 2{,}5$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GN5 2. 2GN6 3. 2GN7 4. 2GN8 5. 2GN9 6. 2GNB

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10.5 Five number summary

Wikihow shows a summary with short animations of how to make a box and whisker plot.

Exercise 10 – 6:

1. Lisa is working in a computer store. She sells the following number of computers each month:

$$\{27; 39; 3; 15; 43; 27; 19; 54; 65; 23; 45; 16\}$$

Give the five number summary and box-and-whisker plot of Lisa's sales.

Solution:

We first order the data set.

$$\{3; 15; 16; 19; 23; 27; 27; 39; 43; 45; 54; 65\}$$

Now we can read off the minimum as the first value (3) and the maximum as the last value (65).

Next we need to determine the quartiles.

There are 12 values in the data set. Using the percentile formula, we can determine that the median lies between the sixth and seventh values, making it:

$$\frac{27+27}{2} = 27$$

The first quartile lies between the third and fourth values, making it:

$$\frac{16+19}{2} = 17{,}5$$

The third quartile lies between the ninth and tenth values, making it:

$$\frac{43+45}{2} = 44$$

This provides the five number summary of the data set and allows us to draw the following box-and-whisker plot.

Five number summary:

Minimum: 3

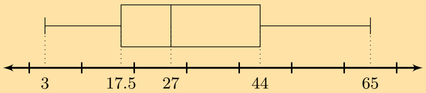
Q_1: 17,5

Median: 27

Q_3: 44

Maximum: 65

Box-and-whisker plot:



2. Zithulele works as a telesales person. He keeps a record of the number of sales he makes each month. The data below show how much he sells each month.

$$\{49; 12; 22; 35; 2; 45; 60; 48; 19; 1; 43; 12\}$$

Give the five number summary and box-and-whisker plot of Zithulele's sales.

Solution:

We first order the data set.

$$\{1; 2; 12; 12; 19; 22; 35; 43; 45; 48; 49; 60\}$$

Now we can read off the minimum as the first value (1) and the maximum as the last value (60).

Next we need to determine the quartiles.

There are 12 values in the data set. Using the percentile formula, we can determine that the median lies between the sixth and seventh values, making it:

$$\frac{22+35}{2} = 28{,}5$$

The first quartile lies between the third and fourth values, making it:

$$\frac{12+12}{2} = 12$$

The third quartile lies between the ninth and tenth values, making it:

$$\frac{45+48}{2} = 46{,}5$$

The five number summary is:

Minimum: 1

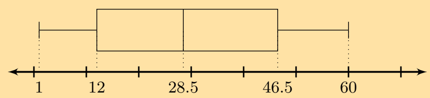
Q_1: 12

Median: 28,5

Q_3: 46,5

Maximum: 60

The box and whisker plot is:



3. Nombusa has worked as a florist for nine months. She sold the following number of wedding bouquets:

$$\{16; 14; 8; 12; 6; 5; 3; 5; 7\}$$

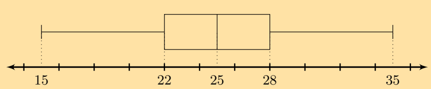
Give the five number summary of Nombusa's sales.
Solution:
We first order the data set.

$$\{3; 5; 5; 6; 7; 8; 12; 14; 16\}$$

Now we can read off the minimum as the first value (3) and the maximum as the last value (16).
Next we need to determine the quartiles.
There are 9 values in the data set. Using the percentile formula, we can determine that the median lies at the fifth value, making it 7.
The first quartile lies at the third value, making it 5.
The third quartile lies at the seventh values, making it 12.
The five number summary is:
Minimum: 3
Q_1: 5
Median: 7
Q_3: 12
Maximum: 16

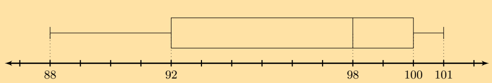
4. Determine the five number summary for each of the box-and-whisker plots below.

a)



Solution:
The box shows the interquartile range (the distance between $Q1$ and $Q3$). A line inside the box shows the median. The lines extending outside the box (the whiskers) show where the minimum and maximum values lie. Reading off the graph we obtain the following five number summary:
Minimum: 15
Q_1: 22
Median: 25
Q_3: 28
Maximum: 35

b)



Solution:
The box shows the interquartile range (the distance between $Q1$ and $Q3$). A line inside the box shows the median. The lines extending outside the box (the whiskers) show where the minimum and maximum values lie. Reading off the graph we obtain the following five number summary:
Minimum: 88
Q_1: 92
Median: 98
Q_3: 100
Maximum: 101

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1. 2GND 2. 2GNF 3. 2GNG 4a. 2GNH 4b. 2GNJ

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End of chapter Exercise 10 – 7:

1. The following data set of heights was collected from a class of learners.
{1,70 m; 1,41 m; 1,60 m; 1,32 m; 1,80 m; 1,40 m}
Categorise the data set.
Solution:
This data set is a set of numbers and so must be quantitative.
This data set is continuous since it cannot be represented by integers.
Therefore the data set is quantitative continuous.
2. The following data set of sandwich spreads was collected from learners at lunch.
{cheese; peanut butter; jam; cheese; honey}
Categorise the data set.
Solution:
This data set cannot be written as numbers and so must be qualitative.
This data set is categorical since it comes from a limited set of possibilities.
Therefore the data set is qualitative categorical. para>
3. Calculate the **mode** of the following data set:
$\{10; 10; 10; 18; 7; 10; 3; 10; 7; 10; 7\}$
Solution:
We first sort the data set: $\{3; 7; 7; 7; 10; 10; 10; 10; 10; 10; 18\}$. The mode is the value that occurs most often in the data set.
Therefore the mode is: 10.
4. Calculate the **median** of the following data set:
$\{5; 5; 10; 7; 10; 2; 16; 10; 10; 10; 7\}$
Solution:
We first need to order the data set:
$\{2; 5; 5; 7; 7; 10; 10; 10; 10; 10; 16\}$.
Since there are an odd number of values in this data set (11) the median lies at the sixth place.
The median is: 10.
5. In a park, the tallest 7 trees have heights (in metres):
$$\{41; 60; 47; 42; 44; 42; 47\}$$
Find the median of their heights.
Solution:
We first need to order the data set:
$\{41; 42; 42; 44; 47; 47; 60\}$.
Since there are an odd number of values in this data set (7) the median lies at the fourth place.
The median is: 44.
6. The learners in Ndeme's class have the following ages:
$$\{5; 6; 7; 5; 4; 6; 6; 6; 7; 4\}$$
Find the mode of their ages.
Solution:
We first sort the data set: $\{4; 4; 5; 5; 6; 6; 6; 6; 7; 7\}$. The mode is the value that occurs most often in the data set.
Therefore the mode is: 6.
7. A group of 7 friends each have some sweets. They work out that the **mean** number of sweets they have is 6. Then 4 friends leave with an unknown number (x) of sweets. The remaining 3 friends work out that the **mean** number of sweets they have left is 10,67.
When the 4 friends left, how many sweets did they take with them?
Solution:
If the **mean** number of sweets the group originally had was **6** then the total number of sweets must have been:

$$\text{mean} = \frac{\text{number of sweets (before)}}{\text{group size}}$$
$$\text{number of sweets (before)} = \text{mean} \times \text{group size}$$
$$\text{number of sweets (before)} = (6) \times (7)$$
$$\text{number of sweets (before)} = 42$$

We are then told that **4** friends leave and thereafter the **mean** number of sweets left is **10,67**. Let us work out the remaining number of sweets.

$$\text{mean} = \frac{\text{number of sweets (after)}}{\text{group size}}$$
$$\text{number of sweets (after)} = \text{mean} \times \text{group size}$$
$$\text{number of sweets (after)} = (10{,}67) \times (3)$$
$$\text{number of sweets (after)} = 32$$

Now we can calculate how many sweets were taken by the 4 friends who left the group.

$$\text{number of sweets removed } (x) = \text{items before} - \text{items after}$$
$$\text{number of sweets removed } (x) = (42) - (32)$$
$$\text{number of sweets removed } (x) = 10$$

8. A group of 10 friends each have some sweets. They work out that the **mean** number of sweets they have is 3. Then 5 friends leave with an unknown number (x) of sweets. The remaining 5 friends work out that the **mean** number of sweets they have left is 3.
When the 5 friends left, how many sweets did they take with them?
Solution:
If the **mean** number of sweets the group originally had was **3** then the total number of sweets must have been:

$$\text{mean} = \frac{\text{number of sweets (before)}}{\text{group size}}$$
$$\text{number of sweets (before)} = \text{mean} \times \text{group size}$$
$$\text{number of sweets (before)} = (3) \times (10)$$
$$\text{number of sweets (before)} = 30$$

We are then told that **5** friends leave and thereafter the **mean** number of sweets left is **3**. Let us work out the remaining number of sweets.

$$\text{mean} = \frac{\text{number of sweets (after)}}{\text{group size}}$$
$$\text{number of sweets (after)} = \text{mean} \times \text{group size}$$
$$\text{number of sweets (after)} = (3) \times (5)$$
$$\text{number of sweets (after)} = 15$$

Now we can calculate how many sweets were taken by the 5 friends who left the group.

$$\text{number of sweets removed } (x) = \text{items before} - \text{items after}$$
$$\text{number of sweets removed } (x) = (30) - (15)$$
$$\text{number of sweets removed } (x) = 15$$

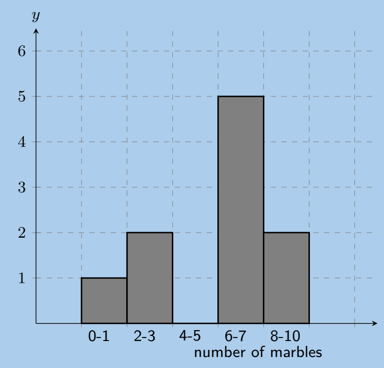
9. Five data values are represented as follows: $3x; x + 2; x{-}3; x + 4; 2x{-}5$, with a mean of 30. Solve for x.
Solution:

$$\bar{x} = \frac{x_1 + x_2 + ...x_n}{N}$$
$$30 = \frac{3x + x + 2 + x - 3 + x + 4 + 2x - 5}{5}$$
$$150 = 8x - 2$$
$$152 = 8x$$
$$x = \frac{152}{8}$$
$$\therefore x = 19$$

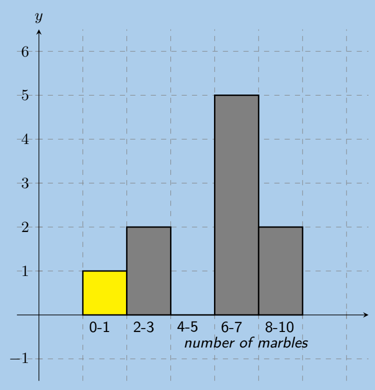
10. Five data values are represented as follows: $p + 1; p + 2; p + 9$. Find the mean in terms of p.
Solution:

$$\bar{x} = \frac{x_1 + x_2 + ...x_n}{N}$$
$$= \frac{p + 1 + p + 2 + p + 9}{3}$$
$$= \frac{3p + 12}{3}$$
$$\therefore \bar{x} = p + 4$$

11. A group of 10 learners count the number of marbles they each have. This is a histogram describing the data they collected:



Count the number of marbles in the following range: $0 \leq$ number of marbles ≤ 1
Solution:
From the graph the answer is: 1
From the histogram, we arrive at our answer by reading the height of the specified interval from the histogram.



12. A group of **20** learners count the number of playing cards they each have. This is the data they collect:

12	1	5	4	17	14	7	5	1	3
9	4	12	17	5	19	1	19	7	15

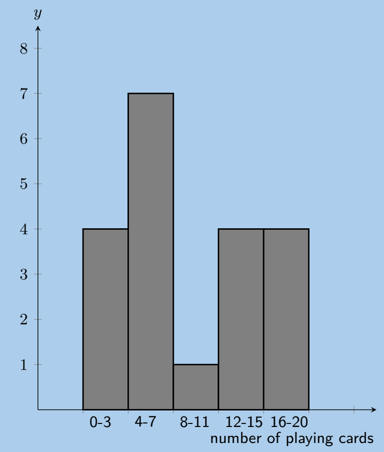
Count the number of learners who have from 0 up to 3 playing cards. In other words, how many learners have playing cards in the following range: $0 \leq$ number of playing cards ≤ 3? It may be helpful for you to draw a histogram in order to answer the question.

Solution:

Firstly we sort the table into sequential order, starting with the smallest value.

1	1	1	3	4	4	5	5	5	7
7	9	12	12	14	15	17	17	19	19

Secondly, we draw a histogram of the data:



From the histogram you can see that the number of learners with playing cards in the range: $0 \leq$ number of playing cards ≤ 3 is 4.

13. A group of **20** learners count the number of coins they each have. This is the data they collect:

17	11	1	15	14	3	4	18	5	14
18	19	4	18	15	16	13	20	8	18

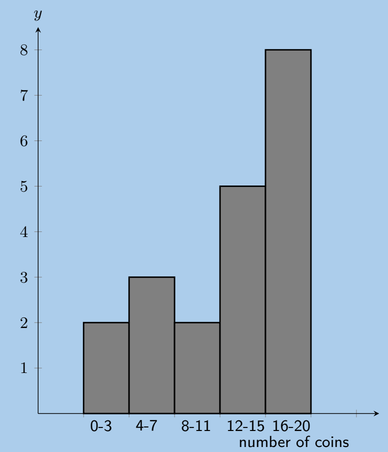
Count the number of learners who have from 4 up to 7 coins. In other words, how many learners have coins in the following range: $4 \leq$ number of coins ≤ 7? It may be helpful for you to draw a histogram in order to answer the question.

Solution:

Firstly we sort the table into sequential order, starting with the smallest value.

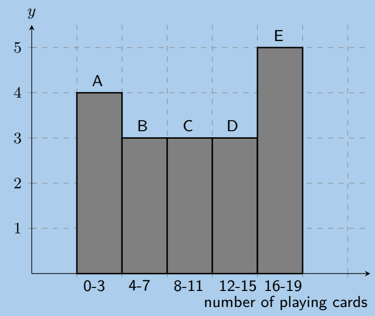
1	3	4	4	5	8	11	13	14	14
15	15	16	17	18	18	18	18	19	20

Secondly, we draw a histogram of the data:



From the histogram you can see that the number of learners with coins in the range: $4 \leq$ number of coins ≤ 7 is 3.

14. A group of 20 learners count the number of playing cards they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.



The data set below shows the correct information for the number of playing cards the learners have. Each value represents the number of playing cards for one learner.

$$\{18; 10; 3; 2; 19; 15; 2; 13; 11; 14; 10; 3; 5; 9; 4; 18; 11; 18; 16; 5\}$$

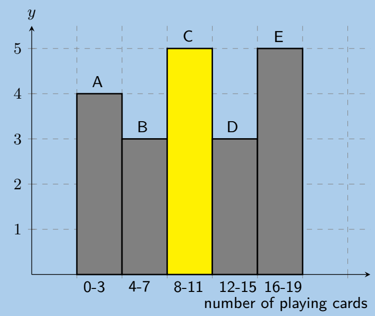
Help them figure out which **column** in the histogram is **incorrect**.

Solution:

We first need to order the data:

$$\{2; 2; 3; 3; 4; 5; 5; 9; 10; 10; 11; 11; 13; 14; 15; 16; 18; 18; 18; 19\}$$

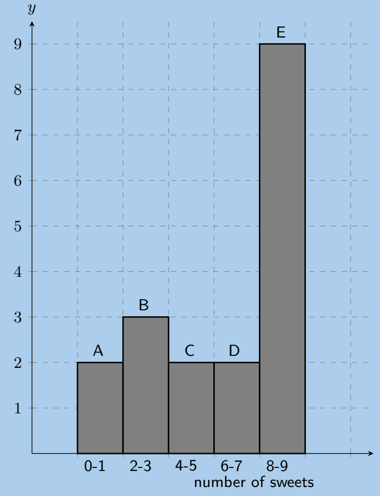
Using the ordered data set we can group the data and draw the correct histogram:



The column with the error in it was: C.

The learners used the incorrect value of **3**, when the correct value is 5.

15. A group of 10 learners count the number of sweets they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.



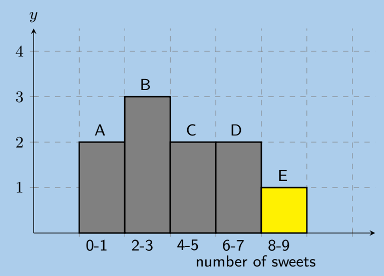
The data set below shows the correct information for the number of sweets the learners have. Each value represents the number of sweets for one learner.

$$\{1; 3; 7; 4; 5; 8; 2; 2; 1; 7\}$$

Help them figure out which **column** in the histogram is **incorrect**.
Solution:
We first need to order the data:

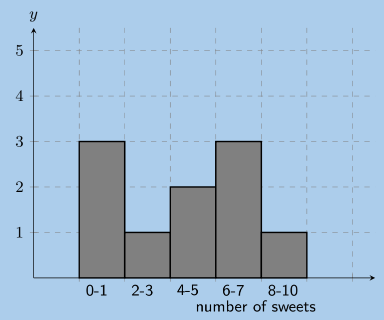
$$\{1; 1; 2; 2; 3; 4; 5; 7; 7; 8\}$$

Using the ordered data set we can group the data and draw the correct histogram:



The column with the error in it was: E.
The learners used the incorrect value of **9**, when the correct value is 1.

16. A group of learners count the number of sweets they each have. This is a histogram describing the data they collected:



A cleaner knocks over their table, and all their notes land on the floor, mixed up, by accident!
Help them find which of the following data sets match the above histogram:
Data set A

1	8	4	8	8
6	1	5	7	5

Data set B

5	6	9	2	1
6	6	4	4	6

Data set C

7	2	4	1	5
1	1	7	8	6

Solution:
In order to determine which data set is correct we need to order each data set:
Data set A

1	1	4	5	5
6	7	8	8	8

Data set B

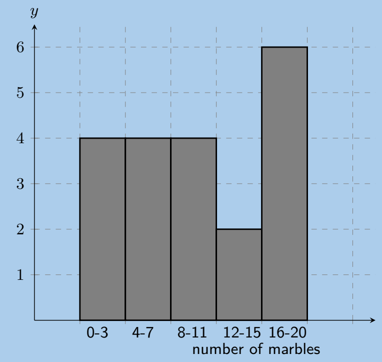
1	2	4	4	5
6	6	6	6	9

Data set C

1	1	1	2	4
5	6	7	7	8

Now we can group the data for each data set and compare the grouped data to the histogram. Doing so we find that data set C is the correct data set.

17. A group of learners count the number of marbles they each have. This is a histogram describing the data they collected:



A cat jumps onto the table, and all their notes land on the floor, mixed up, by accident!
Help them find which of the following data sets match the above histogram:
Data set A

7	13	15	13	12
13	8	14	3	15
1	7	4	11	1

Data set B

17	1	5	4	11
13	6	19	6	20
19	1	14	9	17
3	16	3	10	10

Data set C

10	3	5	5	6
5	2	1	4	3

Solution:
In order to determine which data set is correct we need to order each data set:
Data set A

1	1	3	4	7
7	8	11	12	13
13	13	14	15	15

Data set B

1	1	3	3	4
5	6	6	9	10
10	11	13	14	16
17	17	19	19	20

Data set C

1	2	3	3	4
5	5	5	6	10

Now we can group the data for each data set and compare the grouped data to the histogram. Doing so we find that data set B is the correct data set.

18. A group of **20** learners count the number of marbles they each have. This is the data they collect:

11	8	17	13	9
12	2	6	15	7
14	15	1	6	6
13	19	9	6	19

Calculate the **range** of values in the data set.

Solution:

We need to order the data set:

1	2	6	6	6
6	7	8	9	9
11	12	13	13	14
15	15	17	19	19

Now we find the maximum value in the data set:

$$\text{maximum value} = 19$$

Next we find the minimum value in the data set:

$$\text{minimum value} = 1$$

Finally, we calculate the range of the data set.

$$\begin{aligned} \text{range} &= (\text{maximum value}) - (\text{minimum value}) \\ &= 19 - 1 \\ &= 18 \end{aligned}$$

19. A group of **15** learners count the number of sweets they each have. This is the data they collect:

5	13	4	15	5
6	1	3	13	13
15	14	7	2	4

Calculate the **range** of values in the data set.

Solution:

We first need to order the data set:

1	2	3	4	4
5	5	6	7	13
13	13	14	15	15

Next we find the maximum value in the data set.

$$\text{maximum value} = 15$$

Then we find the minimum value in the data set.

$$\text{minimum value} = 1$$

Finally, we calculate the range of the data set.

$$\begin{aligned} \text{range} &= (\text{maximum value}) - (\text{minimum value}) \\ &= 15 - 1 \\ &= 14 \end{aligned}$$

20. An engineering company has designed two different types of engines for motorbikes. The two different motorbikes are tested for the time (in seconds) it takes for them to accelerate from 0 km·h^{-1} to 60 km·h^{-1}.

Test	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Bike 1	1,55	1,00	0,92	0,80	1,49	0,71	1,06	0,68	0,87	1,09
Bike 2	0,9	1,0	1,1	1,0	1,0	0,9	0,9	1,0	0,9	1,1

a) Which measure of central tendency should be used for this information?

Solution:

Mean and mode. The mean will give us the average acceleration time, while the mode will give us the time that is most often obtained.

If we used the median we would not get any useful information as all the median tells us is what the central value is. The mean and mode provide more information about the data set as a whole.

b) Calculate the measure of central tendency that you chose in the previous question, for each motorbike.

Solution:

We first sort the data.

Bike 1: $\{0{,}68; 0{,}71; 0{,}80; 0{,}87; 0{,}92; 1{,}00; 1{,}06; 1{,}09; 1{,}49; 1{,}55\}$.

Bike 2: $\{0{,}9; 0{,}9; 0{,}9; 0{,}9; 1{,}0; 1{,}0; 1{,}0; 1{,}0; 1{,}1; 1{,}1\}$.

Next we can calculate the mean for each bike:

$$\begin{aligned}\text{mean bike 1} &= \frac{0{,}68 + 0{,}71 + 0{,}80 + 0{,}87 + 0{,}92 + 1{,}00 + 1{,}06 + 1{,}09 + 1{,}49 + 1{,}55}{10} \\ &= 1{,}02\end{aligned}$$

$$\begin{aligned}\text{mean bike 2} &= \frac{0{,}9 + 0{,}9 + 0{,}9 + 0{,}9 + 1{,}0 + 1{,}0 + 1{,}0 + 1{,}0 + 1{,}1 + 1{,}1}{10} \\ &= 1{,}0\end{aligned}$$

For bike 1 the mean is 1,02 s and there is no mode, because there is no value that occurs more than once.

For bike 2 the mean is 1,0 s and there are two modes, 1,0 and 0,9.

c) Which motorbike would you choose based on this information? Take note of the accuracy of the numbers from each set of tests.

Solution:

It would be difficult to choose. Although bike 1 appears to do better than bike 2 from the mean, the data for bike 2 is less accurate than that for bike 1 (it only has 1 decimal place). If we were to calculate the mean for bike 1 using only 1 decimal place we would get 0,9 s. This would make bike 2 better. Also bike 2 produces more consistent numbers. So bike 2 would likely be a good choice, but more information or more accurate information should be obtained.

21. In a traffic survey, a random sample of 50 motorists were asked the distance they drove to work daily. This information is shown in the table below.

Distance (km)	Count
$0 < d \leq 5$	4
$5 < d \leq 10$	5
$10 < d \leq 15$	9
$15 < d \leq 20$	10
$20 < d \leq 25$	7
$25 < d \leq 30$	8
$30 < d \leq 35$	3
$35 < d \leq 40$	2
$40 < d \leq 45$	2

a) Find the approximate mean of the data.

Solution:

To find the approximate mean we need to use the central value from each group. We are told that 50 motorists were surveyed and so the total number of data values is 50.

$$\begin{aligned}\text{mean} &= \frac{4(3) + 5(8) + 9(13) + 10(18) + 7(23) + 8(28) + 3(33) + 2(38) + 2(43)}{50} \\ &= 19{,}9\end{aligned}$$

b) What percentage of drivers had a distance of

i. less than or equal to 15 km?

ii. more than 30 km?

iii. between 16 km and 30 km?

Solution:

i. The first three groups all drive less than or equal to 15 km. We can add up the number of drivers in these three groups and then divide this by the total number of drivers to find the percentage of drivers: $\frac{18}{50} \times 100 = 38\%$.

ii. The last three groups all drive more than 30 km. We can add up the number of drivers in these three groups and then divide this by the total number of drivers to find the percentage of drivers: $\frac{6}{50} \times 100 = 12\%$.

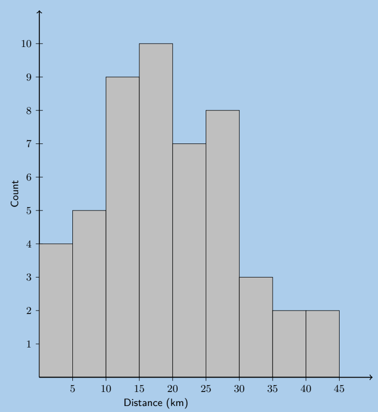
iii. The middle three groups fall into this range. We can add up the number of drivers in these three groups and then divide this by the total number of drivers to find the percentage of drivers: $\frac{25}{50} \times 100 = 50\%$.

Note that the three percentages we have just calculated all add up to 100%.

c) Draw a histogram to represent the data.

Solution:

We are given the groupings and the counts for each group. So we can draw the following histogram to represent the data:



22. A company wanted to evaluate the training programme in its factory. They gave the same task to trained and untrained employees and timed each one in seconds.

Trained	121	137	131	135	130
	128	130	126	132	127
	129	120	118	125	134
Untrained	135	142	126	148	145
	156	152	153	149	145
	144	134	139	140	142

a) Find the medians and quartiles for both sets of data.

Solution:

First order the data sets for both trained and untrained employees.

Trained: $118, 120, 121, 125, 126, 127, 128, 129, 130, 130, 131, 132, 134, 135, 137$.

Untrained: $126, 134, 135, 139, 140, 142, 142, 144, 145, 145, 148, 149, 152, 153, 156$.

There are 15 values in each data set.

Using the percentile formula with $n = 15$, we can find the rank of the 25^{th}, 50^{th} and 75^{th} percentiles:

$$\begin{aligned} r_{25} &= \frac{25}{100}(15-1)+1 \\ &= 4{,}5 \\ r_{50} &= \frac{50}{100}(15-1)+1 \\ &= 8 \\ r_{75} &= \frac{75}{100}(15-1)+1 \\ &= 11{,}5 \end{aligned}$$

For the 25^{th} percentile the rank is 4,5, which is between the fourth and fifth values. For the 50^{th} percentile (the median) the rank is 8. Therefore the median lies at the eighth value. For the 75^{th} percentile the rank is 11,5, meaning between the eleventh and 12th values.

For the trained employees we get:

25^{th} percentile: 125,5; median: 129; 75^{th} percentile: 131,5.

For the untrained employees we get:

25^{th} percentile: 139,5; median: 144; 75^{th} percentile: 148,5.

b) Find the interquartile range for both sets of data.

Solution:

Interquartile range for the trained employees: $Q_3 - Q_1 = 6$.

Interquartile range for the untrained employees: $Q_3 - Q_1 = 9$.

c) Comment on the results.

Solution:

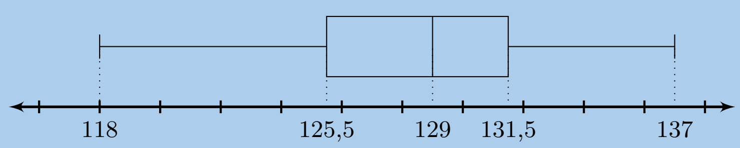
The median of the untrained employees is higher than that of the trained employees. Also the untrained employees have a larger interquartile range than the trained employees. There is some evidence to suggest that the training programme may be working.

d) Draw a box-and-whisker diagram for each data set to illustrate the five number summary.

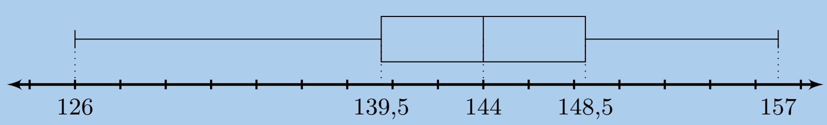
Solution:

A box-and-whisker plot shows the five number summary. The box shows the interquartile range (the distance between $Q1$ and $Q3$). A line inside the box shows the median. The lines extending outside the box (the whiskers) show where the minimum and maximum values lie.

Trained employees:



Untrained employees:



23. A small firm employs 9 people. The annual salaries of the employees are:

R 600 000	R 250 000	R 200 000
R 120 000	R 100 000	R 100 000
R 100 000	R 90 000	R 80 000

a) Find the mean of these salaries.

Solution:

$$\begin{aligned} \text{mean} &= \frac{600\ 000 + 250\ 000 + 200\ 000 + 120\ 000 + 3(100\ 000) + 90\ 000 + 80\ 000}{9} \\ &= \frac{1\ 640\ 000}{9} \\ &= \text{R } 182\ 222{,}22 \end{aligned}$$

b) Find the mode.

Solution:

The mode is R 100 000 (this value occurs 3 times in the data set).

c) Find the median.

Solution:

First order the data. To make the numbers easier to work with we will divide each one by 100 000.

The ordered set is $\{80; 90; 100; 100; 100; 120; 200; 250; 600\}$.

The median is at position 5 and is R 100 000.

d) Of these three figures, which would you use for negotiating salary increases if you were a trade union official? Why?

Solution:

Either the mode or the median.The mean is skewed (shifted) by the one salary of R 600 000. The mode gives us a better estimate of what the employees are actually earning. The median also gives us a fairly accurate representation of what the employees are earning.

24. The stem-and-leaf diagram below indicates the pulse rate per minute of ten Grade 10 learners.

7	8
8	1 3 5 5
9	0 1 1
10	3 5

Key: $7|8 = 78$

a) Determine the mean and the range of the data.
Solution:
The data set is $\{78; 81; 83; 85; 85; 90; 91; 91; 103; 105\}$.

$$\begin{aligned}\bar{x} &= \frac{x_1 + x_2 + ...x_n}{N}\\ &= \frac{892}{10}\\ \bar{x} &= 89{,}2\end{aligned}$$

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ &= 105 - 78\\ \text{range} &= 27\end{aligned}$$

The mean and range are 89,2 and 27 respectively.

b) Give the five-number summary and create a box-and-whisker plot for the data.
Solution:

$$\begin{aligned}r &= \frac{p}{100}(n-1) + 1\\ r_{25} &= \frac{25}{100}(10-1) + 1\\ &= 3{,}25\end{aligned}$$

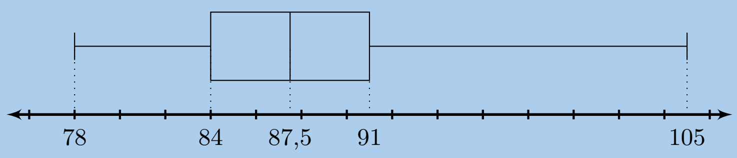
$$\begin{aligned}\therefore Q_1 &= \frac{83 + 85}{2}\\ &= 84\end{aligned}$$

$$\begin{aligned}r_{50} &= \frac{50}{100}(10-1) + 1\\ &= 5{,}5\end{aligned}$$

$$\begin{aligned}\therefore Q_2 &= \frac{85 + 90}{2}\\ &= 87{,}5\end{aligned}$$

$$\begin{aligned}r_{75} &= \frac{75}{100}(10-1) + 1\\ &= 7{,}75\\ \therefore Q_3 &= \frac{91 + 91}{2}\\ &= 91\end{aligned}$$

The five-number summary is: 78; 84; 87,5; 91; 105.
Using this we can draw the box-and-whisker plot.



25. The following is a list of data: $3; 8; 8; 5; 9; 1; 4; x$

In each separate case, determine the value of x if the:

a) range = 16

Solution:

The set is: $\{1; 3; 4; 5; 8; 8; 9; x\}$. We have ordered all the numbers and then added x on the end as we do not know the value of x.

We are told that the range must be equal to 16. If $x < 9$ the range would be $9 - 1 = 8$. Therefore $x > 9$ and so x must be the maximum value.

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ 16 &= x - 1\\ \therefore x &= 17\end{aligned}$$

b) mode = 8

Solution:

8 is already the mode if we exclude x, to maintain this x is any integer with $x \neq \{1; 3; 4; 5; 9\}$.

c) median = 6

Solution:

First consider the set without x: $\{1; 3; 4; 5; 8; 8; 9\}$. The median in this set is 5 (there is an odd number of values in the set and the median lies at position 4).

Next we consider the full set: $\{1; 3; 4; 5; 8; 8; 9; x\}$ There is an even number of values (8) in the full set. Therefore the median must lie between the fourth and fifth values.

Now we need to think about where x could fit into the set. x could be the fourth value, the fifth value or somewhere else in the set. If x is either the fourth or fifth value we will get the same median.

First try the case where x is the fourth or fifth value:

$$\begin{aligned}6 &= \frac{5 + x}{2}\\ x + 5 &= 12\\ \therefore x &= 7\end{aligned}$$

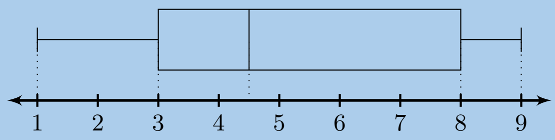
Next we check the case where x is not the fourth or fifth value. In this case the median is $\frac{5+8}{2}$ = 6,5. Therefore we can say that $x = 7$.

d) mean = 6

Solution:

$$\begin{aligned}\bar{x} &= \frac{x_1 + x_2 + ...x_n}{N}\\ 6 &= \frac{x + 38}{8}\\ x + 38 &= 48\\ \therefore x &= 10\end{aligned}$$

e) box-and whisker plot



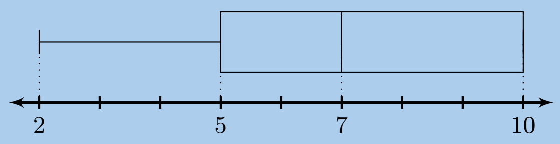
Solution:

In part a we found that if the range is 8, then $x < 9$. The range here is 8, therefore $x < 9$.

Therefore we will use the median to help us find x. The median on the box-and-whiskers plot is 4,5. From our reasoning in part c we know that this means x is either the fourth or fifth value. So we can calculate x as follows:

$$\begin{aligned}4{,}5 &= \frac{5 + x}{2}\\ x + 5 &= 9\\ \therefore x &= 4\end{aligned}$$

26. Write down one list of numbers that satisfies the box-and-whisker plot below:



Solution:

From the box-and-whisker plot we get the five number summary: $2; 5; 7; 10; 10$. Note that the third quartile is also the maximum value in this case.

From this we can state that the data set must have a minimum value of 2 and a maximum value of 10.

The data set can contain any numbers in the range $2 \leq x \leq 10$ such that the first quartile is 5, the median is 7 and the third quartile is 10. There is also no restriction on how many values are in the data set.

One possible set that satisfies this set of numbers is $\{2; 5; 7; 10; 10\}$. You can check that this set works by calculating the quartiles.

27. Given ϕ (which represents the golden ratio) to 20 decimal places: 1,61803398874989484820

a) For the first 20 decimal digits of (Φ), determine the:

i. median

ii. mode

iii. mean

Solution:

i. The ordered set is: $\{0; 0; 1; 2; 3; 3; 4; 4; 4; 6; 7; 8; 8; 8; 8; 8; 8; 9; 9; 9\}$

$$\begin{aligned} r_{50} &= \frac{50}{100}(20-1)+1 \\ &= 10{,}5 \end{aligned}$$

$$\begin{aligned} \text{median} &= \frac{6+7}{2} \\ &= 6{,}5 \end{aligned}$$

ii. The mode is 8.

iii.

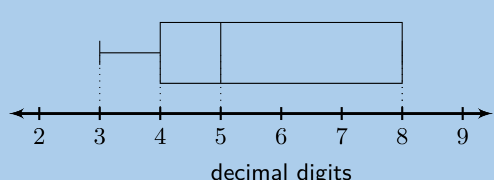
$$\begin{aligned} \bar{x} &= \frac{x_1 + x_2 + \ldots x_n}{N} \\ &= \frac{109}{20} \\ \bar{x} &= 5{,}45 \end{aligned}$$

b) If the mean of the first 21 decimal digits of (Φ) is 5,38095 determine the 21$^{\text{st}}$ decimal digit.

Solution:

$$\begin{aligned} \bar{x} &= \frac{x_1 + x_2 + \ldots x_n}{N} \\ 5{,}38095 &= \frac{109 + \phi_{21}}{21} \\ \phi_{21} &= 21(5{,}38095) - 109 \\ \phi_{21} &= 4 \end{aligned}$$

c) Below is a box-and-whisker plot of the 21$^{\text{st}}$ - 27$^{\text{th}}$ decimal digits. Write down one list of numbers that satisfies this box-and-whisker plot.



Solution:

From the box-and-whisker plot we get the five number summary: $3; 4; 5; 8; 8$. Note that the third quartile is also the maximum value in this case.

From this we can state that the data set must have a minimum value of 3 and a maximum value of 8.

The data set can contain any numbers in the range $3 \leq x \leq 8$ such that the first quartile is 4, the median is 5 and the third quartile is 8. However, we know that the data set consists of the 21st - 27th decimal digits of Φ and so the data set must contain 7 values.

The median will be at the fourth position and so the fourth number in the set is 5. The first quartile will lie between the second and third values while the third quartile will lie between the fifth and sixth values.

Let the data set be: $\{3; x; y; 5; a; b; 8\}$

The first quartile is:

$$\begin{aligned} 4 &= \frac{x+y}{2} \\ 8 &= x+y \end{aligned}$$

x and y can be any integers that add up to 8. However x and y must be greater than or equal to 3 and less than or equal to 5. Therefore the possible values are: 3 and 5 or 4 and 4.

The third quartile is:

$$\begin{aligned} 8 &= \frac{a+b}{2} \\ 16 &= a+b \end{aligned}$$

a and b can be any integers that add up to 16. However a and b must be greater than or equal to 5 and less than or equal to 8. Therefore the only possible values are: 8 and 8.

There are two possible sets: $\{3; 3; 5; 5; 8; 8; 8\}$ or $\{3; 4; 4; 5; 8; 8; 8\}$.

28. There are 14 men working in a factory. Their ages are : $22; 25; 33; 35; 38; 48; 53; 55; 55; 55; 55; 56; 59; 64$

 a) Write down the five number summary.

 Solution:

$$\begin{aligned} r &= \frac{p}{100}(n-1)+1 \\ r_{25} &= \frac{25}{100}(14-1)+1 \\ &= 4{,}25 \end{aligned}$$

$$\begin{aligned} \therefore Q_1 &= \frac{35+38}{2} \\ &= 36{,}5 \end{aligned}$$

$$\begin{aligned} r_{50} &= \frac{50}{100}(14-1)+1 \\ &= 7{,}5 \end{aligned}$$

$$\begin{aligned} \therefore Q_2 &= \frac{53+55}{2} \\ &= 54 \end{aligned}$$

$$\begin{aligned} r_{75} &= \frac{75}{100}(14-1)+1 \\ &= 10{,}75 \end{aligned}$$

$$\begin{aligned} \therefore Q_3 &= \frac{55+55}{2} \\ &= 55 \end{aligned}$$

The five number summary is: 22; 36,5; 50; 55; 64

b) If 3 men had to be retrenched, but the median had to stay the same, show the ages of the 3 men you would retrench.

Solution:

Retrenching 3 men will leave 11 men. For an odd numbered set the median must be the same as one of the ages. None of the men are 54 years.

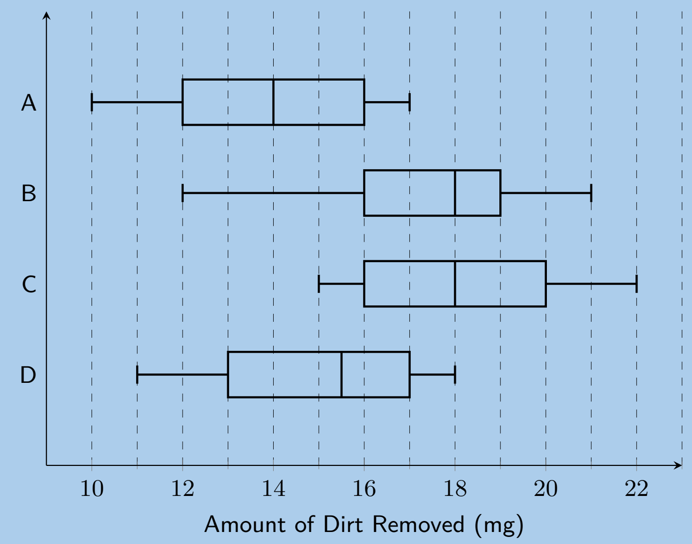
Therefore no men can be retrenched to keep the median the same.

c) Find the mean age of the men in the factory using the original data.

Solution:

$$\begin{aligned} \bar{x} &= \frac{x_1 + x_2 + ...x_n}{N} \\ &= \frac{597}{14} \\ \therefore \bar{x} &= 42{,}643 \end{aligned}$$

29. The example below shows a comparison of the amount of dirt removed by four brands of detergents (brands A to D).



a) Which brand has the biggest range, and what is this range?

Solution:

A: range $= 17 - 10 = 7$

B: range $= 21 - 12 = 9$

C: range $= 22 - 15 = 7$

D: range $= 18 - 11 = 7$

B has the biggest range. The range is 9.

b) For brand C, what does the number 18 mg represent?

Solution:

18 mg represents the median.

c) Give the interquartile range for brand B.

Solution:

$$\begin{aligned} \text{interquartile range} &= Q_3 - Q_1 \\ &= 19 - 16 \\ &= 3 \end{aligned}$$

d) Which brand of detergent would you buy? Explain your answer.

Solution:

We need to compare several values to help us decide. These values are shown in the table below.

Brand	Minimum value	Maximum value	Range	Interquartile range
A	10	17	7	4
B	12	21	9	3
C	15	22	7	4
D	11	18	7	4

From this we see that brand C has the highest minimum value. Brand B has the smallest interquartile range but the largest range. It is possible that the minimum value for brand B is an outlier which would make brand B a better choice than brand C.

Considering all the data available it would be hard to choose between brand C and brand B. We can however say that brands A and D are not very good choices as they both have low minimum and maximum values.

Since brand C does not have a potential outlier this might be the best brand to choose.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GNM 2. 2GNN 3. 2GNP 4. 2GNQ 5. 2GNR 6. 2GNS
7. 2GNT 8. 2GNV 9. 2GNW 10. 2GNX 11. 2GNY 12. 2GNZ
13. 2GP2 14. 2GP3 15. 2GP4 16. 2GP5 17. 2GP6 18. 2GP7
19. 2GP8 20. 2GP9 21. 2GPB 22. 2GPC 23. 2GPD 24. 2GPF
25. 2GPG 26. 2GPH 27. 2GPJ 28. 2GPK 29. 2GPM

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CHAPTER 11

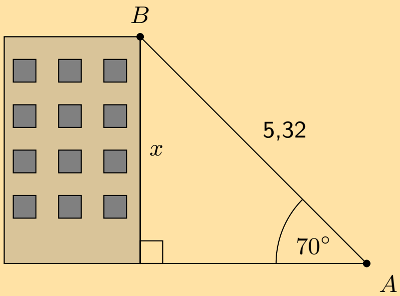
Trigonometry

- This chapter covers solving problems in two-dimensions using trigonometry.
- Emphasise the value and importance of making sketches, where appropriate.
- Prior to starting this chapter it may be appropriate to quickly revise the earlier content on trigonometry.

11.1 Two-dimensional problems

Exercise 11 – 1:

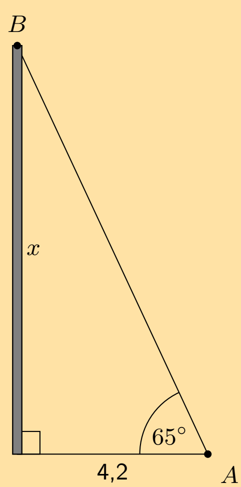
1. A person stands at point A, looking up at a bird sitting on the top of a building, point B.
   The height of the building is x meters, the line of sight distance from point A to the top of the building (point B) is 5,32 meters, and the angle of elevation to the top of the building is $70°$.
   Calculate the height of the building (x) as shown in the diagram below:

   Solution:

   $$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \sin 70° &= \frac{x}{5{,}32} \\ x &= 4{,}99916... \\ &\approx 5 \end{aligned}$$

   The height of the building is 5 m.
2. A person stands at point A, looking up at a bird sitting on the top of a pole (point B).
   The height of the pole is x meters, point A is 4,2 meters away from the foot of the pole, and the angle of elevation to the top of the pole is $65°$.
   Calculate the height of the pole (x), to the nearest metre.

Solution:

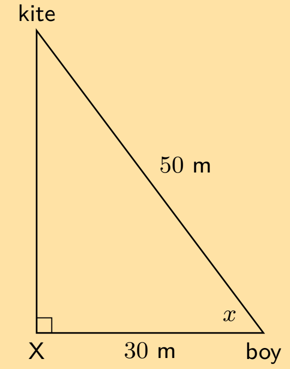
$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}}\\ \tan 65° &= \frac{x}{4{,}2}\\ x &= 9{,}0069...\\ &\approx 9\end{aligned}$$

The height of the pole is 9 m.

3. A boy flying a kite is standing 30 m from a point directly under the kite. If the kite's string is 50 m long, find the angle of elevation of the kite.

Solution:

First draw a sketch:



We can use the cosine ratio to find the angle of elevation (x):

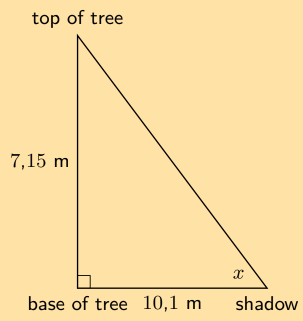
$$\begin{aligned}\cos x &= \frac{30}{50}\\ x &= 53{,}1301...\\ &\approx 53{,}13°\end{aligned}$$

The angle of elevation of the kite is 53,13°.

4. What is the angle of elevation of the sun when a tree 7,15 m tall casts a shadow 10,1 m long?

Solution:

First draw a sketch:

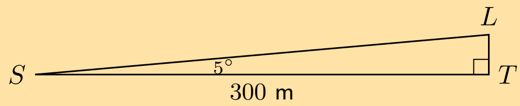


We can use the tangent ratio to find the angle of elevation (x):

$$\begin{aligned}\tan x &= \frac{7{,}15}{10{,}1}\\ x &= 35{,}2954...\\ &\approx 35{,}30°\end{aligned}$$

The angle of elevation of the sun is 35,30°.

5. From a distance of 300 m, Susan looks up at the top of a lighthouse. The angle of elevation is 5°. Determine the height of the lighthouse to the nearest metre.
Solution:
First draw a sketch:

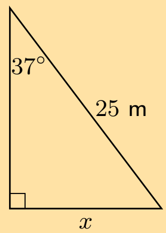


We need to find LT. We can use the tangent ratio:

$$\begin{aligned} \tan \hat{S} &= \frac{LT}{ST} \\ LT &= 300 \tan 5° \\ &= 26{,}2465... \\ &\approx 26 \text{ m} \end{aligned}$$

The height of the lighthouse is 26 m.

6. A ladder of length 25 m is resting against a wall, the ladder makes an angle 37° to the wall. Find the distance between the wall and the base of the ladder to the nearest metre.
Solution:
First draw a sketch:



Notice that we are given the angle that the ladder makes with the wall, not the angle that the ladder makes with the ground.
Now we can use the sine function to find x:

$$\begin{aligned} \sin 37° &= \frac{x}{25} \\ x &= 25 \sin 37° \\ &= 15{,}04537... \\ &\approx 15 \text{ m} \end{aligned}$$

The base of the ladder is 15 m away from the wall.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GPN 2. 2GPP 3. 2GPQ 4. 2GPR 5. 2GPS 6. 2GPT

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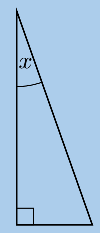
11.2 Chapter summary

End of chapter Exercise 11 – 2:

1. A ladder of length 15 m is resting against a wall, the base of the ladder is 5 m from the wall. Find the angle between the wall and the ladder.

Solution:

First draw a sketch:



Notice that we want to find the angle that the ladder makes with the wall, not the angle that the ladder makes with the ground.

Now we use $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$:

$$\begin{aligned}\sin x &= \frac{5}{15}\\ &= 0{,}3333...\\ x &= 19{,}4712...\\ &\approx 19{,}47°\end{aligned}$$

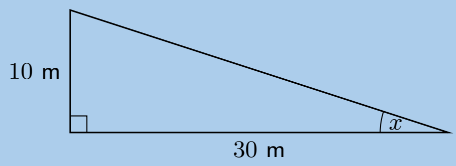
The angle between the ladder and the wall is 19,47°.

2. Captain Jack is sailing towards a cliff with a height of 10 m.

a) The distance from the boat to the bottom of the cliff is 30 m. Calculate the angle of elevation from the boat to the top of the cliff (correct to the nearest degree).

Solution:

First draw a sketch:



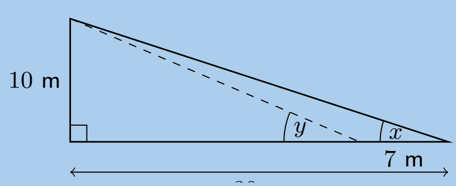
$$\begin{aligned}\tan x &= \frac{\text{opposite}}{\text{hypotenuse}}\\ &= \frac{10}{30}\\ x &= 18{,}4349...°\\ &= 18°\end{aligned}$$

The angle of elevation is 18°.

b) If the boat sails 7 m closer to the cliff, what is the new angle of elevation from the boat to the top of the cliff?

Solution:

We redraw the sketch with the new information:

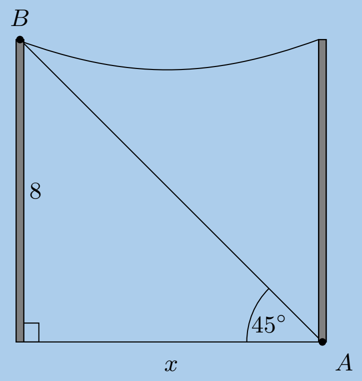


The new distance from the boat to the cliffs is 30 m − 7 m = 23 m. The height of the cliffs has not changed.

$$\tan y = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{10}{23}$$
$$y = 23{,}49856...^\circ$$
$$= 23^\circ$$

The new angle of elevation is $23°$.

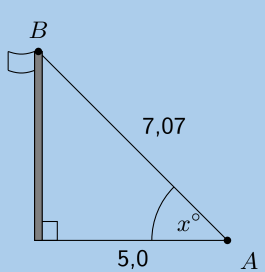
3. Jim stands at point A at the base of a telephone pole, looking up at a bird sitting on the top of another telephone pole (point B).
   The height of each of the telephone poles is 8 meters, and the angle of elevation from A to the top of B is $45°$.
   Calculate the distance between the telephone poles (x) as shown in the diagram below:

   Solution:

   $$\tan x = \frac{\text{opposite}}{\text{adjacent}}$$
   $$\tan 45^\circ = \frac{8}{x}$$
   $$x = \frac{8}{\tan 45^\circ}$$
   $$= 8$$

   The distance between the telephone poles is 8 m.
4. Alfred stands at point A, looking up at a flag on a pole (point B).
   Point A is 5,0 meters away from the bottom of the flag pole, the line of sight distance from point A to the top of the flag pole (point B) is 7,07 meters, and the angle of elevation to the top of the flag pole is $x°$.
   Calculate the angle of elevation to the top of the flag pole (x) as shown in the diagram below:

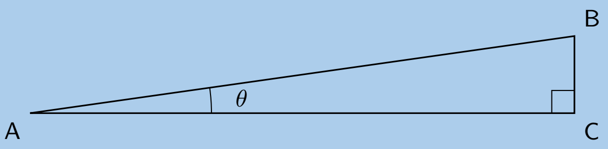
   Solution:

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos x^\circ = \frac{5{,}0}{7{,}07}$$
$$x = 44{,}9913...$$
$$\approx 44{,}99^\circ$$

The angle of elevation is $44{,}99^\circ$.

5. A rugby player is trying to kick a ball through the poles. The rugby crossbar is 3,4 m high. The ball is placed 24 m from the poles. What is the minimum angle he needs to launch the ball to get it over the bar?

Solution:

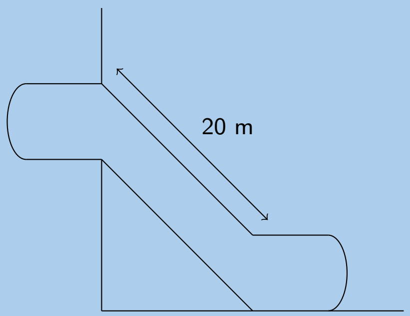


CA is the distance from the poles, 24 m; BC is the crossbar height 3,4 m. The minimum angle is the angle of elevation.

$$\tan\theta = \frac{BC}{CA}$$
$$= \frac{3{,}4}{24}$$
$$\theta = 8{,}0632...$$
$$\approx 8^\circ$$

Therefore he needs to kick the ball with a minimum angle of 8°.

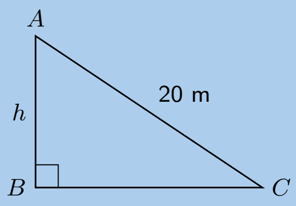
6. The escalator at a mall slopes at an angle of 30° and is 20 m long.



Through what height would a person be lifted by travelling on the escalator?

Solution:

We note that we have the following right-angled triangle:



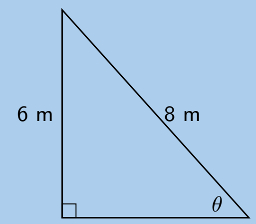
We can use the sine ratio to find the height:

$$\sin 30^\circ = \frac{h}{20}$$
$$h = 20\sin 30^\circ$$
$$= 10 \text{ m}$$

A person travelling on the escalator would be lifted through a height of 10 m.

7. A ladder is 8 metres long. It is leaning against the wall of a house and reaches 6 metres up the wall.

   a) Draw a sketch of the situation.
   Solution:



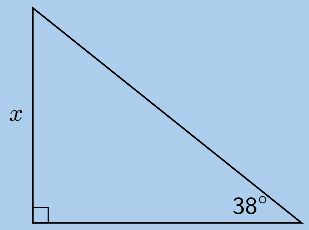
   b) Calculate the angle which the ladder makes with the flat (level) ground.
   Solution:

$$\begin{aligned}\sin\theta &= \frac{6}{8}\\ \theta &= 48{,}5903\ldots\\ \theta &\approx 48{,}59°\end{aligned}$$

   The ladder makes an angle of 48,59° with the ground.

8. Nandi is standing on level ground 70 metres away from a tall tower. From her position, the angle of elevation of the top of the tower is 38°.

   a) Draw a sketch of the situation.
   Solution:

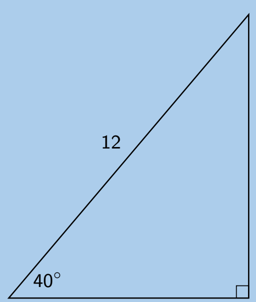


   b) What is the height of the tower?
   Solution:
   We can use the tangent ratio to find the height of the tower.

$$\begin{aligned}\tan 38 &= \frac{x}{70}\\ x &= 70\tan 38\\ &= 54{,}6899\ldots\\ &\approx 54{,}69\text{ m}\end{aligned}$$

   The height of the tower is 54,69 m.

9. The top of a pole is anchored by a 12 m cable which makes an angle of 40 degrees with the horizontal. What is the height of the pole?

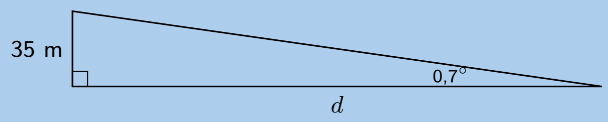


Solution:

$$\begin{aligned}\sin 40 &= \frac{h}{12}\\ h &= 12\sin 40\\ &= 7{,}713...\\ &\approx 7{,}71 \text{ m}\end{aligned}$$

The height of the pole is 7,71 m.

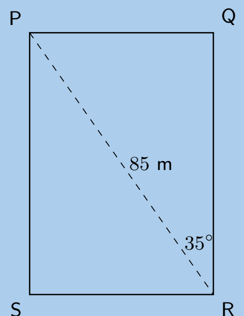
10. A ship's navigator observes a lighthouse on a cliff. According to the navigational charts the top of the lighthouse is 35 metres above sea level. She measures the angle of elevation of the top of the lighthouse to be 0,7°.
    Ships have been advised to stay at least 4 km away from the shore. Is the ship safe?
    Solution:
    First draw a diagram:



$$\begin{aligned}\tan 0{,}7 &= \frac{35}{d}\\ d &= \frac{35}{\tan 0{,}7}\\ &= 2864{,}6464...\\ &\approx 2864{,}65 \text{ m}\\ &\approx 2{,}86 \text{ km}\end{aligned}$$

Therefore the ship is not safe.

11. Determine the perimeter of rectangle $PQRS$:



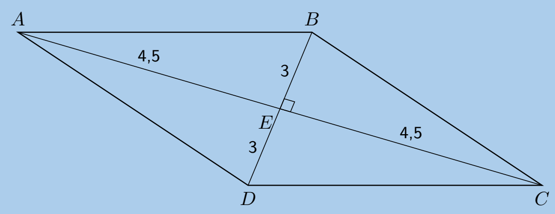
Solution:
Using trigonometric ratios we can calculate QR and PQ.

$$\begin{aligned}QR &= 85\cos 35\\ PQ &= 85\sin 35\end{aligned}$$

$$\begin{aligned}P &= 2\times(h+b)\\ &= 2(85\cos 35 + 85\sin 35)\\ &= 2(85(\cos 35+\sin 35))\\ &= 2(236{,}76 \text{ m})\\ &= 473{,}52 \text{ m}\end{aligned}$$

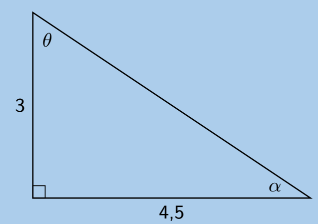
Therefore the perimeter is 473,52 m.

12. A rhombus has diagonals of lengths 6 cm and 9 cm. Calculate the sizes of its interior vertex angles.



Solution:

There are four small right-angled triangles in the rhombus: $\triangle ABE$, $\triangle BEC$, $\triangle CED$ and $\triangle DEA$. Since the diagonals bisect each other and we are given the lengths of the diagonals, we know the lengths of the sides of the triangles:



We can calculate the two angles:

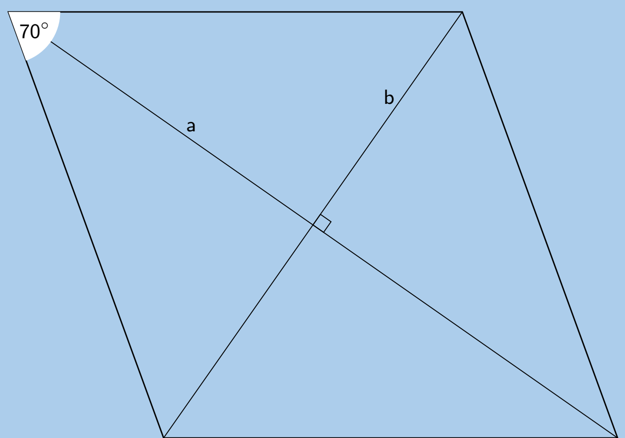
$$\begin{aligned}\tan\theta &= \frac{4{,}5}{3}\\ &= 41{,}9872...\\ \theta &\approx 53{,}31^\circ\end{aligned}$$

$$\begin{aligned}\tan\alpha &= \frac{3}{4{,}5}\\ &= 33{,}6900...\\ \alpha &\approx 33{,}69^\circ\end{aligned}$$

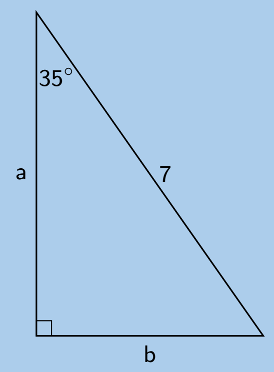
Now we note that there are two different interior angles. One of these angles is 2α and the other is 2θ.

Therefore the two angles are 106,62° and 67,38°

13. A rhombus has edge lengths of 7 cm. Its acute interior vertex angles are both 70°. Calculate the lengths of both of its diagonals.



Solution:

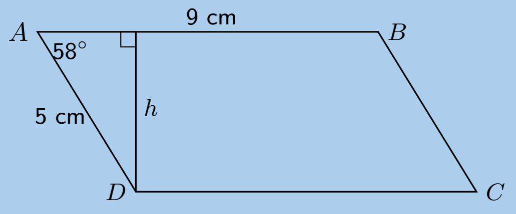


$$\begin{aligned} \cos 35° &= \frac{a}{7} \\ a &= 7\cos 35° \\ &= 5{,}734... \\ \text{diagonal 1} &= 11{,}47 \text{ cm} \end{aligned}$$

$$\begin{aligned} \sin 35° &= \frac{b}{7} \\ b &= 7\sin 35° \\ &= 4{,}015... \\ \text{diagonal 2} &= 8{,}03 \text{ cm} \end{aligned}$$

Therefore the two diagonals are 11,47 cm and 8,03 cm

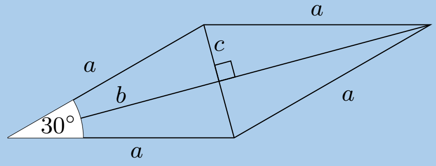
14. A parallelogram has edge-lengths of 5 cm and 9 cm respectively, and an angle of 58° between them. Calculate the perpendicular distance between the two longer edges.



Solution:

$$\begin{aligned} \sin 58° &= \frac{h}{5} \\ h &= 5\sin 58° \\ &= 4{,}24 \text{ cm} \end{aligned}$$

15. One of the angles of a rhombus with perimeter 20 cm is 30°.
    a) Find the lengths of the sides of the rhombus.
    Solution:
    First draw a sketch:



The perimeter is found by adding each side together. All the sides are equal in length, therefore the perimeter $= 4a$.

$$\begin{aligned} 20 &= 4a \\ a &= 5 \end{aligned}$$

Therefore the sides are all 5 cm in length.

b) Find the length of both diagonals.

Solution:

The diagonals of a rhombus bisect the angle, so working in one of the small triangles we can use trigonometric ratios to find b:

$$\begin{aligned} \cos 15° &= \frac{b}{5} \\ b &= 5 \cos 15° \\ &= 4{,}83 \end{aligned}$$

By Pythagoras $c^2 = a^2 - b^2$:

$$\begin{aligned} c^2 &= (5)^2 - (4{,}83)^2 \\ &= 25 - 23{,}33 \\ &= 1{,}67 \\ c &= 1{,}29 \end{aligned}$$

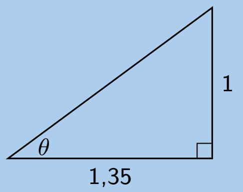
Since the diagonals bisect each other we know that the total length of each diagonal is either $2b$ or $2c$, depending on which diagonal we examine.

The one diagonal is $2(4{,}83) = 9{,}66$ cm and the other diagonal is $2(1{,}29) = 2{,}58$ cm.

16. Upright sticks and the shadows they cast can be used to judge the sun's altitude in the sky (the angle the sun makes with the horizontal) and the heights of objects.

a) An upright stick, 1 metre tall, casts a shadow which is 1,35 metres long. What is the altitude of the sun?

Solution:

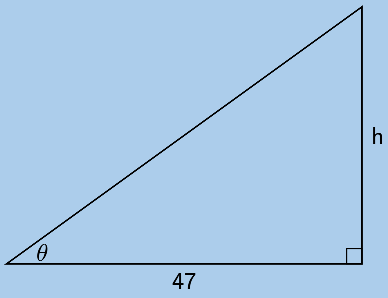


$$\begin{aligned} \tan\theta &= \frac{1}{1{,}35} \\ \theta &= 36{,}53° \end{aligned}$$

b) At the same time, the shadow of a building is found to be 47 metres long. What is the height of the building?

Solution:

We know the angle that the sun makes with the horizontal and now we can use that to find the height of the building.



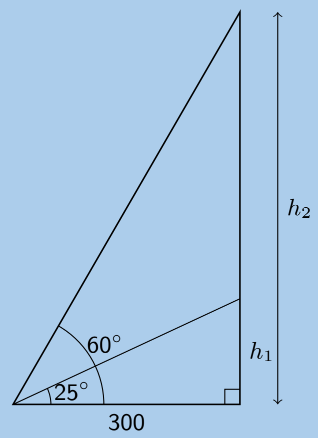
In the figure above θ is the angle that the sun makes with the horizontal.

$$\begin{aligned}\tan 36{,}53° &= \frac{h}{47}\\ h &= 47 \tan 36{,}53°\\ &= 34{,}82 \text{ m}\end{aligned}$$

17. The angle of elevation of a hot air balloon, climbing vertically, changes from 25 degrees at 11:00 am to 60 degrees at 11:02 am. The point of observation of the angle of elevation is situated 300 metres away from the take off point.

a) Draw a sketch of the situation.

Solution:



b) Calculate the increase in height between 11:00 am and 11:02 am.

Solution:

$$\begin{aligned}\tan 25° &= \frac{h_1}{300}\\ h_1 &= 300 \tan 25°\\ &= 139{,}89 \text{ m}\end{aligned}$$

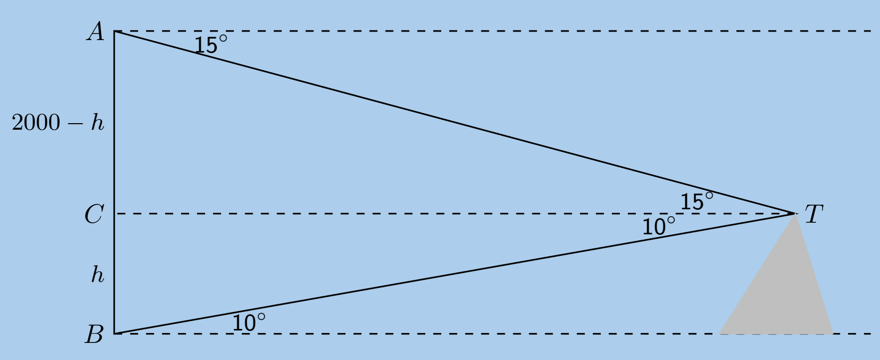
$$\begin{aligned}\tan 60° &= \frac{h_2}{300}\\ h_2 &= 300 \tan 60°\\ &= 519{,}62 \text{ m}\end{aligned}$$

The difference is:

$$519{,}62 \text{ m} - 129{,}89 \text{ m} = 379{,}73 \text{ m}$$

18. When the top, T, of a mountain is viewed from point A, 2000 m from the ground, the angle of depression (a) is equal to 15°. When it is viewed from point B on the ground, the angle of elevation (b) is equal to 10°. If the points A and B are on the same vertical line, find the height, h, of the mountain. Round your answer to one decimal place.

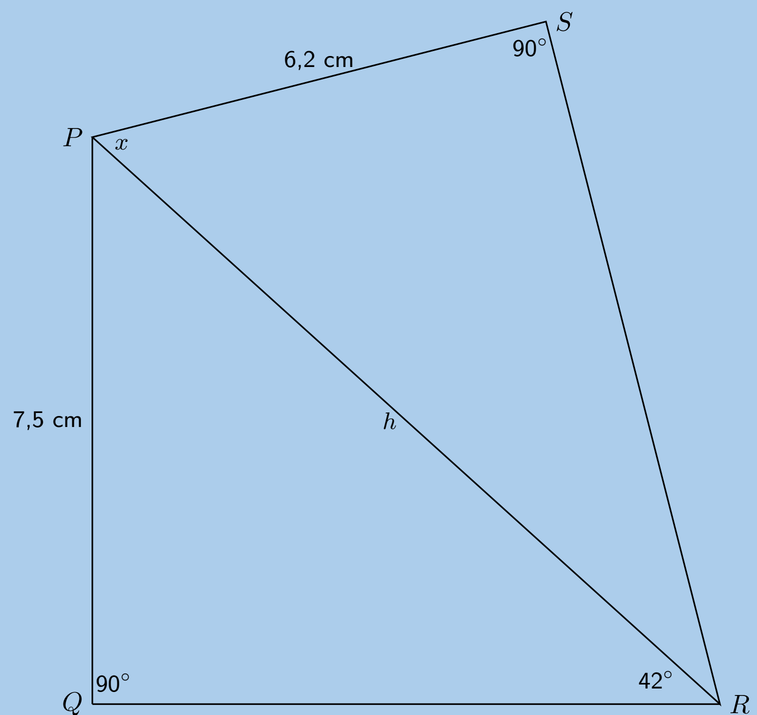
Solution:



$$\tan 10° = \frac{h}{CT}$$
$$CT = \frac{h}{\tan 10°}$$
$$\tan 15° = \frac{2000 - h}{CT}$$
$$CT = \frac{2000 - h}{\tan 15°}$$

$$\therefore \frac{h}{\tan 10°} = \frac{2000 - h}{\tan 15°}$$
$$h = 793{,}77 \text{ m}$$

19. The diagram below shows quadrilateral $PQRS$, with $PQ = 7{,}5$ cm, $PS = 6{,}2$ cm, angle $R = 42°$ and angles S and Q are right angles.



a) Find PR, correct to 2 decimal places.
Solution:

$$\frac{7{,}5}{PR} = \sin 42°$$
$$\frac{7{,}5}{\sin 42°} = PR$$
$$\therefore PR = 11{,}21 \text{ cm}$$

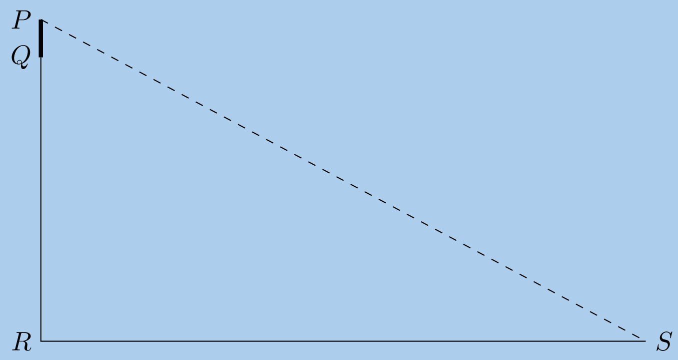
b) Find the size of the angle marked x, correct to one decimal place.
Solution:

$$\cos x = \frac{6{,}2}{11{,}21}$$
$$\therefore x = 56{,}4°$$

20. From a boat at sea (S), the angle of elevation of the top of a lighthouse PQ, on a cliff QR, is 27°. The lighthouse is 10 m high and the cliff top is 75 m above sea level. How far is the boat from the base of the cliff, to the nearest metre?

Solution:

First draw a sketch:



The distance PR is equal to the height of the lighthouse, PQ, plus the height of the cliffs, QR.

$$\frac{85}{RS} = \tan 27°$$
$$\frac{85}{\tan 27°} = RS$$
$$\therefore RS = 167 \text{ m}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GPW 2. 2GPX 3. 2GPY 4. 2GPZ 5. 2GQ2 6. 2GQ3
7. 2GQ4 8. 2GQ5 9. 2GQ6 10. 2GQ7 11. 2GQ8 12. 2GQ9
13. 2GQB 14. 2GQC 15. 2GQD 16. 2GQF 17. 2GQG 18. 2GQH
19. 2GQJ 20. 2GQK

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CHAPTER 12

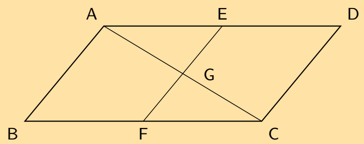
Euclidean geometry

- This chapter focuses on solving problems in Euclidean geometry and proving riders.
- It must be explained that a single counter example can disprove a conjecture but numerous specific examples supporting a conjecture do not constitute a general proof.
- To prove that a quadrilateral is one of the special quadrilaterals learners need to show that a unique property of that quadrilateral is true. For example to prove a quadrilateral is a parallelogram it is not enough to show that both pairs of sides are parallel, learners will also need to show that either the opposite angles are equal or both pairs of opposite sides are equal in length.

12.1 Proofs and conjectures

Exercise 12 – 1:

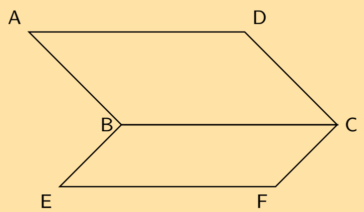
1. In the diagram below, AC and EF bisect each other at G. E is the midpoint of AD, and F is the midpoint of BC.
   a) Prove $AECF$ is a parallelogram.



   Solution:
   AC and EF bisect each other (given).
   $AECF$ is a parallelogram (diagonals bisect each other).

   b) Prove $ABCD$ is a parallelogram.
   Solution:
   $AD \parallel BC (AE \parallel CF,\ AECF$ is a parallelogram)
   $AD = 2AE$ (mid-point, given)
   $CF = AE$ ($AECF$ is a parallelogram)
   $\therefore AD = 2AE = 2CF = BC$
   $ABCD$ is a parallelogram (two sides are parallel and equal)

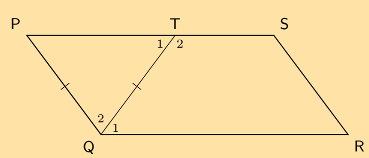
2. Parallelogram $ABCD$ and $BEFC$ are shown below. Prove $AD = EF$.



   Solution:

$$\begin{aligned} AD &= BC \text{ (opp sides of } \parallel \text{m)} \\ BC &= EF \text{ (opp sides of } \parallel \text{m)} \\ \therefore AD &= EF \end{aligned}$$

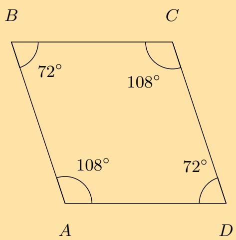
3. $PQRS$ is a parallelogram. $PQ = TQ$. Prove $\hat{Q}_1 = \hat{R}$



Solution:

$$\hat{P} = \hat{T}_1 \quad (\angle\text{s opp equal sides})$$
$$\hat{T}_1 = \hat{Q}_1 \quad (\text{alt } \angle\text{s}; (PS \parallel QR))$$
$$\hat{P} = \hat{Q}_1$$
$$\hat{P} = \hat{R} \text{ (opp}\angle\text{s of } \parallel \text{m)}$$
$$\therefore \hat{Q}_1 = \hat{R}$$

4. Study the quadrilateral $ABCD$ with opposite angles $\hat{A} = \hat{C} = 108°$ and angles $\hat{B} = \hat{D} = 72°$ carefully. Fill in the missing reasons and steps to prove that the quadrilateral $ABCD$ is a parallelogram.



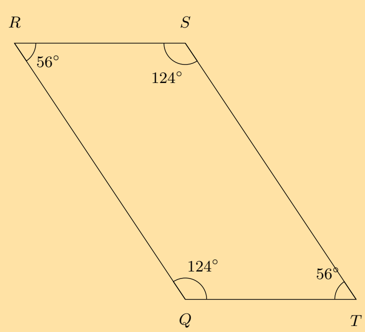
Steps	Reasons
?	given both $\angle$'s $= 108°$
$A\hat{B}C = A\hat{D}C$	given both$\angle$'s $= 72°$
$\hat{A} + \hat{B} + \hat{C} + \hat{D} = 360°$	sum of $\angle$s in quad
$B\hat{A}D + A\hat{D}C = 180°$	given $108° + 72° = 180°$
$\therefore AB \parallel DC$	?
$\therefore BC \parallel AD$	?
$\therefore ABCD$ is a parallelogram	opp sides of quad $\parallel$

Solution:

Here is the completed proof with the correct steps and reasons.

Steps	Reasons
$B\hat{A}D = B\hat{C}D$	given both $\angle$'s $= 108°$
$A\hat{B}C = A\hat{D}C$	given both $\angle$'s $= 72°$
$\hat{A} + \hat{B} + \hat{C} + \hat{D} = 360°$	sum of $\angle$s in quad
$B\hat{A}D + A\hat{D}C = 180°$	given $108° + 72° = 180°$
$\therefore AB \parallel DC$	co-int $\angle$s; $AB \parallel DC$
$\therefore BC \parallel AD$	co-int $\angle$s; $BC \parallel AD$
$\therefore ABCD$ is a parallelogram	opp sides of quad $\parallel$

5. Study the quadrilateral $QRST$ with opposite angles $Q = S = 124°$ and angles $R = T = 56°$ carefully. Fill in the missing reasons and steps to prove that the quadrilateral $QRST$ is a parallelogram.



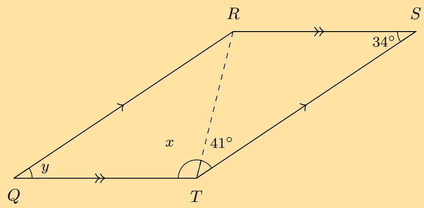
Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle$s $= 124°$
?	given both $\angle$s $= 56°$
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360°$	?
$R\hat{Q}T + Q\hat{T}S = 180°$	?
$\therefore QR \parallel TS$	co-int $\angle$s; $QR \parallel TS$
$\therefore RS \parallel QT$	co-int $\angle$s; $RS \parallel QT$
?	opp sides of quad $\parallel$

Solution:

Here is the completed proof with the correct steps and reasons.

Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle$s $= 124°$
$Q\hat{R}S = Q\hat{T}S$	given both $\angle$s $= 56°$
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360°$	sum of $\angle$s in quad
$R\hat{Q}T + Q\hat{T}S = 180°$	given $124° + 56° = 180°$
$\therefore QR \parallel TS$	co-int $\angle$s; $QR \parallel TS$
$\therefore RS \parallel QT$	co-int $\angle$s; $RS \parallel QT$
$\therefore QRST$ is a parallelogram	opp sides of quad $\parallel$

6. a) Quadrilateral $QRST$ with sides $QR \parallel TS$ and $QT \parallel RS$ is given. You are also given that: $\hat{Q} = y$ and $\hat{S} = 34°$; $Q\hat{T}R = x$ and $R\hat{T}S = 41°$. Prove that $QRST$ is a parallelogram.



Solution:

Steps	Reasons
$Q\hat{T}R = T\hat{R}S$	alt $\angle$s $QT \parallel RS$
$S\hat{T}R = Q\hat{R}T$	alt $\angle$s $QR \parallel TS$
In$\triangle QRT$ and $\triangle RST$ side $RT = RT$	common side
$\therefore \triangle QRT \equiv \triangle STR$	congruent (AAS)
$\hat{Q} = \hat{S}$	congruent triangles (AAS)
$QR = TS$ and $RS = QT$	congruent triangles (AAS)
$\therefore QRST$ is a parallelogram	opp. sides of quad are $=$

b) Find the value of y.

Solution:

$QRST$ is a parallelogram (proved above).

$\hat{Q} = \hat{S}$ and $\hat{R} = \hat{T}$ (opp $\angle$s of $\parallel$m).

Therefore, $y = 34°$.

c) Find the value of x.

Solution:

We can solve this problem in two ways: using the sum of angles in a triangle or using the sum of the interior angles in a quadrilateral.

Option 1: sum of angles in a triangle.

$\hat{Q} + Q\hat{R}T + Q\hat{T}R = 180°$ (sum of$\angle$s in $\triangle = 180°$).

We know that $\hat{Q} = \hat{S} = 34°$ and that $R\hat{T}S = 41°$.

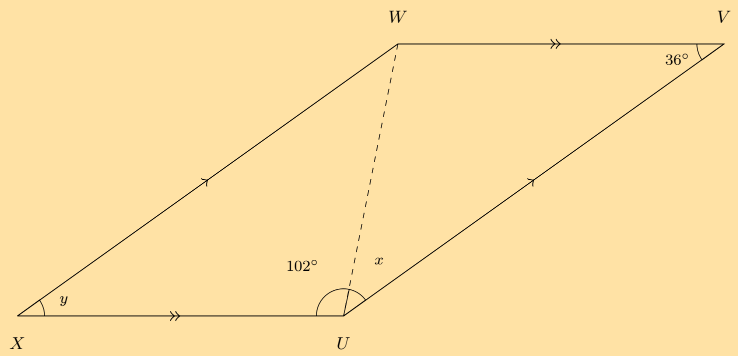
$\therefore x = 180° - 34° - 41° = 105°$.

Option 2: sum of angles in a quadrilateral.

The sum of the interior $\angle$'s in a quadrilateral is $360°$.

$$\begin{aligned}
\therefore \hat{Q} + \hat{R} + \hat{S} + \hat{T} &= 360° \quad \text{(sum of } \angle\text{s in quad} \\
34° + 34° + \hat{R} + \hat{T} &= 360° \\
\hat{R} &= \hat{T} \quad \text{opp } \angle\text{s of } \parallel \text{m)} \\
68° + 2\hat{R} &= 360° \\
\hat{R} &= \frac{292}{2} \\
&= 146° \\
x &= 146° - 41° = 105°
\end{aligned}$$

7. a) Quadrilateral $XWVU$ with sides $XW \parallel UV$ and $XU \parallel WV$ is given. Also given is $\hat{X} = y$ and $\hat{V} = 36°$; $X\hat{U}W = 102°$ and $W\hat{U}V = x$. Prove that $XWVU$ is a parallelogram.



Solution:

Steps	Reasons
$X\hat{U}W = U\hat{W}V$	alt $\angle$s; $XU \parallel WV$
$V\hat{U}W = X\hat{W}U$	alt $\angle$s; $XW \parallel UV$
In$\triangle XWU$ and $\triangle WVU$ side $WU = WU$	common side
$\therefore \triangle XWU \equiv \triangle VUW$	congruent (AAS)
$\therefore XW = UV and XU = WV$	congruent triangles (AAS)
$\hat{X} = \hat{V}$	congruent triangles (AAS)
$\therefore XWVU$is a parallelogram	opp sides of quad are $=$

b) Determine the value of y.

Solution:

$XWVU$ is a parallelogram, $\therefore \hat{X} = \hat{V}$.

Opposite $\angle$'s of a parallelogram are equal: $\hat{X} = \hat{V}$ and $\hat{W} = \hat{U}$.

Therefore, $y = 36°$.

c) Determine the value of x.

Solution:

We can solve this problem in two ways: using the sum of angles in a triangle or using the sum of interior angles in a quadrilateral.

Option 1: sum of angles in a triangle.

$\angle$'s in a $\triangle = 180°$

$\therefore \hat{X} + X\hat{W}U + X\hat{U}W = 180°$

Now we know that $\hat{X} = \hat{V} = 36°$ and that $X\hat{U}W = 42°$.

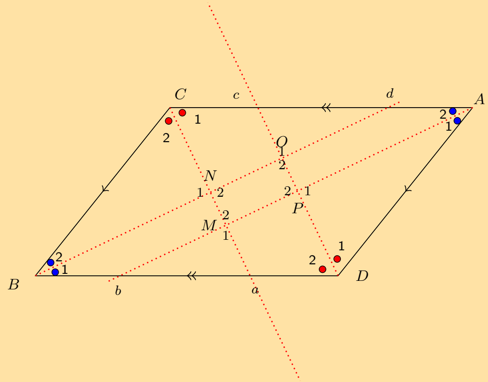
$\therefore \hat{x} = 180° - 36° - 102° = 42°$.

Option 2: sum of interior angles in a quadrilateral.

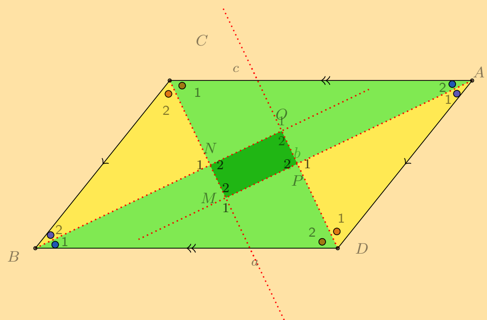
The sum of the interior $\angle$'s in a quadrilateral is $360°$.

$$\begin{aligned}
\therefore \hat{X} + \hat{W} + \hat{V} + \hat{U} &= 360° \quad \text{(sum of } \angle\text{s in quad} \\
36° + 36° + \hat{U} + \hat{W} &= 360° \\
\hat{U} = \hat{W} & \quad \text{opp } \angle\text{s of } \parallel \text{m)} \\
72° + 2\hat{U} &= 360° \\
\hat{U} &= \frac{298}{2} \\
&= 149° \\
x = 149° - 102° &= 42°
\end{aligned}$$

8. In parallelogram $ADBC$, the bisectors of the angles (A, D, B, C) have been constructed, indicated with the red lines below. You are also given $AD = CB$, $DB = AC$, $AD \parallel CB$, $DB \parallel AC$, $\hat{A} = \hat{B}$ and $\hat{D} = \hat{C}$.
Prove that the quadrilateral $MNOP$ is a parallelogram.
Note the diagram is drawn to scale.

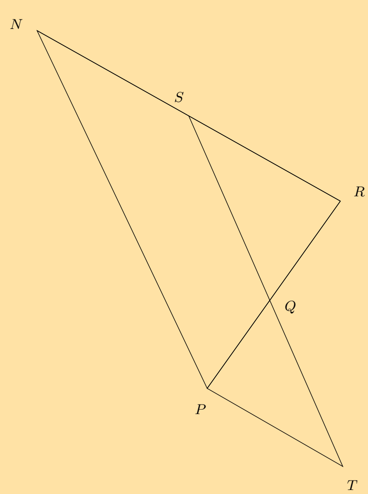


Solution:



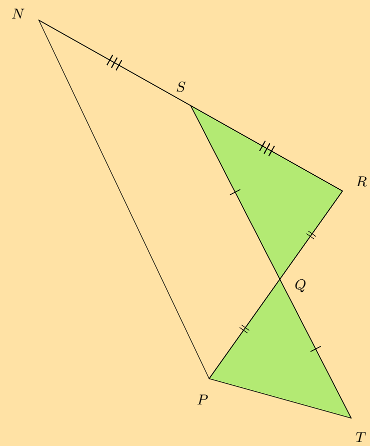
Steps	Reasons
In $\triangle AMC$ and $\triangle BOD \hat{D}_2 = \hat{C}_1$	(given)
In $\triangle AMC$ and $\triangle DOB \hat{A}_2 = \hat{B}_1$	(given)
In $\triangle AMC$ and $\triangle DOB side AC = DB$	(given)
$\therefore \triangle AMC \equiv \triangle DOB$	(AAS)
$\therefore \hat{M}_2 = \hat{O}_2$	corresp $\angle$s proved with $\triangle AMC \equiv \triangle DOB$
In $\triangle ADP$ and $\triangle CBN \hat{A}_1 = \hat{B}_2$	(given)
In $\triangle ADP$ and $\triangle CBN \hat{D}_1 = \hat{C}_2$	(given)
In $\triangle ADP$ and $\triangle CBN sides AD = CB$	(given)
$\therefore \triangle ADP \equiv \triangle CBN$	(AAS)
$\therefore \hat{P}_1 = \hat{N}_1$	corresp $\angle$s proved with $\triangle ADP \equiv \triangle CBN$
but $\hat{P}_1 = \hat{P}_2$ and $\hat{N}_1 = \hat{N}_2$	vert. opp. $\angle$'s
$\therefore MNOP$ is a parallelogram	(opp $\angle$s of quad are equal)

9. Study the diagram below; it is not necessarily drawn to scale. Two triangles in the figure are congruent: $\triangle QRS \equiv \triangle QPT$. Additionally, $SN = SR$. You need to prove that $NPTS$ is a parallelogram.



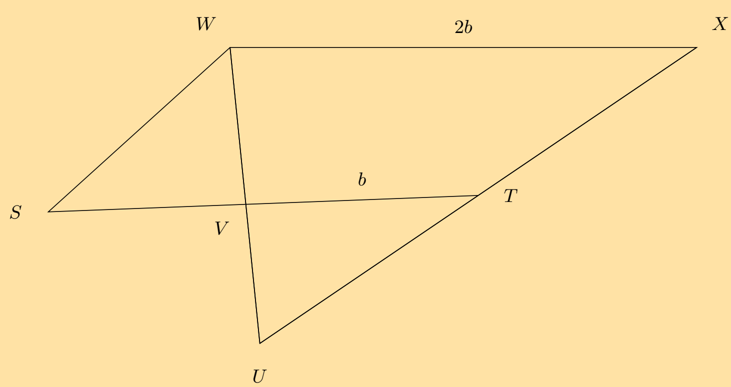
Solution:

Redraw the diagram and mark all given and known information:



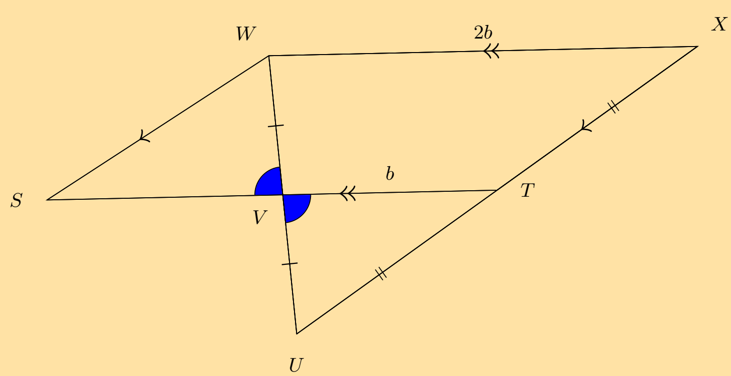
Steps	Reasons
$QR = PQ$	corresp. sides of congruent triangles
Q is the mid-point of PR	Midpt Theorem
$S$is a mid-point	given $SN = SR$
$ST \parallel NP$	Midpt Theorem
$R\hat{S}Q = P\hat{T}Q$	corresp $\angle$s in congruent triangles
$NR \parallel PT$	alt $\angle$s $=$
$NPTS$ is a parallelogram	opp sides of quad are $\parallel$

10. Study the diagram below; it is not necessarily drawn to scale. Quadrilateral $XWST$ is a parallelogram and TV and XW have lengths b and $2b$, respectively, as shown. You need to prove that $\triangle TVU \equiv \triangle SVW$.



Solution:

Redraw the diagram and fill in all given and known information.



T and V are mid-points

Steps	Reasons
$WV = VU$	definition of mid-point
$T\hat{V}U = S\hat{V}W$	vert opp $\angle$s $=$
$TV + VS = XW$	opp sides parm are equal
$b + VS = 2b$	substitute given values: $TV = b$ and $XW = 2b$
$VS = b = VT$	solve for VS; note that it is equal to VT
$\triangle TVU \equiv \triangle SVW$	SAS

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GQP 2. 2GQQ 3. 2GQR 4. 2GQS 5. 2GQT 6. 2GQV 7. 2GQW 8. 2GQX 9. 2GQY 10. 2GQZ

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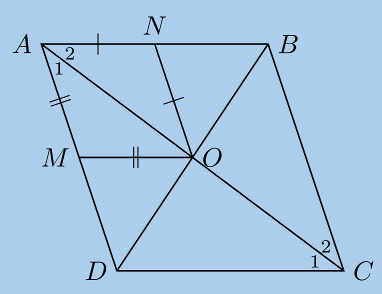
12.2 Chapter summary

End of chapter Exercise 12 – 2:

1. $ABCD$ is a rhombus with $AM = MO$ and $AN = NO$. Prove $ANOM$ is also a rhombus.



Solution:

In $\triangle AMO$ and $\triangle ANO$

$\hat{A}_1 = \hat{A}_2$ (given rhombus $ABCD$, diagonal AC bisects $\hat{A}$)

$\therefore \hat{A}_1 = A\hat{O}M$ ($\angle$s opp equal sides)

similarly $\hat{A}_2 = A\hat{O}N$

$\therefore \hat{A}_2 = A\hat{O}M$ and $\hat{A}_1 = A\hat{O}N$

but these are alternate interior $\angle$s

$\therefore AN \parallel MO$ and $AM \parallel NO$

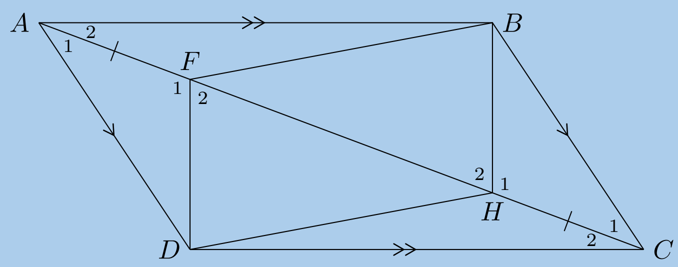
$\therefore ANOM$ is a parallelogram

$\therefore AM = NO$ (opp sides of $\|$m)

$\therefore AM = MO = ON = NO$

$\therefore ANOM$ is a rhombus (all sides equal and two pairs of sides parallel)

2. $ABCD$ is a parallelogram with diagonal AC. Given that $AF = HC$, show that:



a) $\triangle AFD \equiv \triangle CHB$
Solution:

$$\begin{aligned} \hat{A}_1 &= \hat{C}_1 && \text{(alt } \angle\text{s; } AD \parallel BC\text{)} \\ AD &= BC && \text{(opp sides of } \parallel \text{m)} \\ AF &= HC && \text{(given)} \\ \therefore \triangle AFD &\equiv \triangle CHB && \text{(SAS)} \end{aligned}$$

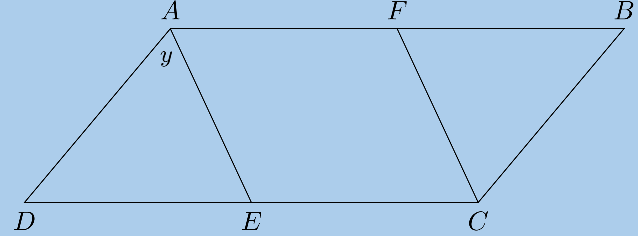
b) $DF \parallel HB$
Solution:

$$\begin{aligned} \hat{F}_1 &= \hat{H}_1 && (\triangle AFD \equiv \triangle CHB) \\ \therefore \hat{F}_1 + \hat{F}_2 &= 180° && (\angle\text{s on str line}) \\ \text{and } \hat{H}_1 + \hat{H}_2 &= 180° && (\angle\text{s on str line}) \\ \therefore \hat{F}_1 &= 180° - \hat{F}_2 \\ \text{and } \hat{H}_1 &= 180° - \hat{H}_2 \\ \therefore 180° - \hat{F}_2 &= 180° - \hat{H}_2 \\ \therefore \hat{F}_2 &= \hat{H}_2 \\ \therefore DF &\parallel HB && \text{(corresp } \angle\text{s equal)} \end{aligned}$$

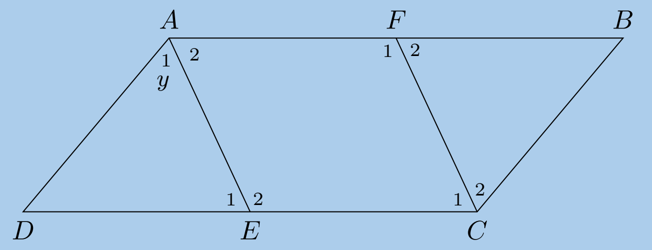
c) $DFBH$ is a parallelogram
Solution:

$$\begin{aligned} FD &= HB && (\triangle AFD \equiv \triangle CHB) \\ \text{and } DF &\parallel HB && \text{(proved above)} \\ \therefore DFBH & \text{ is a parallelogram} && \text{(one pair opp sides equal and parallel)} \end{aligned}$$

3. Given parallelogram $ABCD$ with AE bisecting $\hat{A}$ and FC bisecting $\hat{C}$.



a) Write all interior angles in terms of y.
Solution:
First number the angles:



$$\hat{A}_2 = y \quad \text{(given } AE \text{ bisects } \hat{A}\text{)}$$
$$\hat{E}_1 = y \quad \text{(alt } \angle\text{s; } AB \parallel DC\text{)}$$
$$\therefore \hat{E}_2 = 180° - y \quad (\angle\text{s on str line})$$

$$\hat{C}_1 = \hat{C}_2 \quad \text{(given } FC \text{ bisects } \hat{C}\text{)}$$
$$\text{and } \hat{A} = \hat{C} \quad \text{(opp } \angle\text{s of } \parallel \text{m)}$$
$$\therefore \hat{C}_1 = \hat{C}_2 = y$$

$$\therefore \hat{F}_2 = \hat{C}_1 = y \quad \text{(alt } \angle\text{s; } AB \parallel DC\text{)}$$
$$\therefore \hat{F}_1 = 180° - y$$

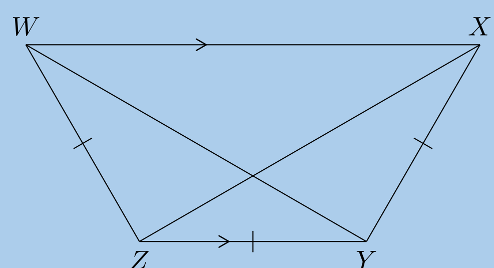
In $\triangle ADE$

$$\hat{D} + \hat{A}_1 + \hat{E}_1 = 180° \quad \text{(sum of } \angle\text{s in } \triangle\text{)}$$
$$\therefore \hat{D} + y + y = 180°$$
$$\hat{D} = 180° - 2y$$
$$= 90° - y$$
$$\therefore \hat{B} = 90° - y \quad \text{(opp } \angle\text{s of } \parallel \text{m)}$$

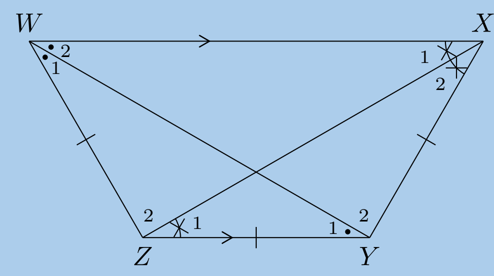
b) Prove that $AFCE$ is a parallelogram.
Solution:

$$AF \parallel EC \quad \text{(opp sides of } \parallel \text{m)}$$
$$\text{and } \hat{C}_1 + \hat{E}_2 = y + (180° - y)$$
$$\therefore \text{the sum of the co-interior angles is } 180°$$
$$\therefore AE \parallel FC$$
$$\therefore AFCE \text{ is a parallelogram (both pairs opp. sides parallel)}$$

4. Given that $WZ = ZY = YX$, $\hat{W} = \hat{X}$ and $WX \parallel ZY$, prove that:



a) XZ bisects $\hat{X}$
Solution:
First label the angles:



In $\triangle XYZ$

$\hat{X}_2 = \hat{Z}_2$ ($\angle$s opp equal sides)

and $\hat{X}_1 = \hat{Z}_2$ (alt $\angle$s; $WX \parallel ZY$)

$\therefore \hat{X}_1 = \hat{X}_2$

$\therefore XZ$ bisects $\hat{X}$

b) $WY = XZ$

Solution:

Similarly, WY bisects $\hat{W}$

$\therefore \hat{W}_1 = \hat{W}_2$

and $\hat{W} = \hat{X}$ (given)

$\therefore \hat{W}_1 = \hat{W}_2 = \hat{X}_1 = \hat{X}_2$

and $\hat{W}_1 = \hat{Y}_1$ ($\angle$s opp equal sides)

In $\triangle WZY$ and $\triangle XYZ$

$WZ = XY$ (given)

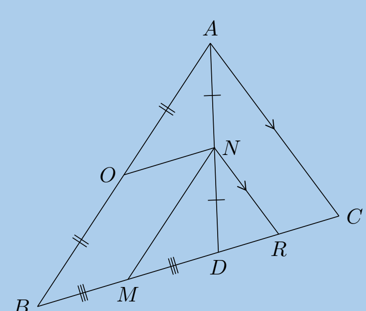
ZY is a common side

$\hat{Z} = \hat{Y}$(third $\angle$ in $\triangle$)

$\therefore \triangle WZY \equiv \triangle XYZ$ (SAS)

$\therefore WY = XZ$

5. D is a point on BC, in $\triangle ABC$. N is the mid-point of AD. O is the mid-point of AB and M is the mid-point of BD. $NR \parallel AC$.



a) Prove that $OBMN$ is a parallelogram.

Solution:

$AO = OB$ (given)

$AN = ND$ (given)

$\therefore ON \parallel BD$ (Midpt Theorem)

$BM = MD$ (given)

$AN = ND$ (given)

$\therefore MN \parallel AB$ (Midpt Theorem)

$\therefore OBMN$ is a parallelogram (both pairs opp. sides parallel)

b) Prove that $BC = 2MR$.

Solution:

$AN = NC$ (given)

$NR \parallel AC$ (given)

$\therefore DR = RC$ (Midpt Theorem)

$\therefore DR = \frac{1}{2}DC$

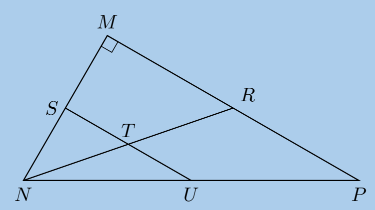
$MD = \frac{1}{2}BD$ (given)

$\therefore MD + DR = \frac{1}{2}(BD + DC)$

$MR = \frac{1}{2}BC$

$\therefore BC = 2MR$

6. In $\triangle MNP$, $\hat{M} = 90°$, S is the mid-point of MN and T is the mid-point of NR.



a) Prove U is the mid-point of NP.

Solution:

$NS = SM$ (given)

$NT = TR$ (given)

$\therefore ST \parallel MR$ (Midpt Theorem)

$\therefore U$ is the mid-point of NP (converse of Midpt Theorem)

b) If $ST = 4$ cm and the area of $\triangle SNT$ is 6 cm^2, calculate the area of $\triangle MNR$.

Solution:

$$N\hat{S}T = 90° \text{ (corresp } \angle\text{s; } ST \parallel MR)$$

$$\therefore \text{ area } \triangle SNT = \frac{1}{2}ST \times SN$$

$$6 = \frac{1}{2}(4)SN$$

$$\therefore SN = 3 \text{ cm}$$

$$\therefore MN = 6 \text{ cm}$$

$$MR = 2ST = 8 \text{ cm}$$

$$\text{area } \triangle MNR = \frac{1}{2}MR \times MN$$

$$= \frac{1}{2}(8)(6)$$

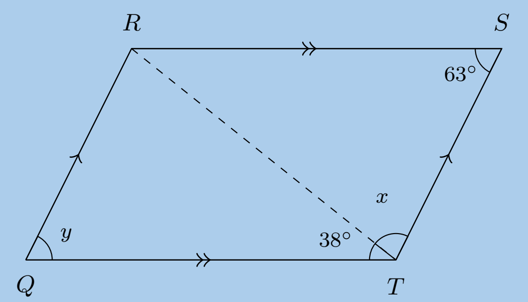
$$= 24 \text{ cm}^2$$

c) Prove that the area of $\triangle MNR$ will always be four times the area of $\triangle SNT$, let $ST = x$ units and $SN = y$ units.

Solution:

$$\begin{aligned}
\text{Let } ST \text{ be } x \text{ units} \\
\therefore MR \text{ will be } 2x \\
\text{Let } SN \text{ be } y \text{ units} \\
\therefore MN \text{ will be } 2y \\
\text{area } \triangle SNT &= \frac{1}{2}xy \\
\text{area } \triangle MNR &= \frac{1}{2}(2x)(2y) \\
&= 2xy \\
\therefore \text{ area } \triangle MNR &= 4\left(\frac{1}{2}xy\right) \\
&= 4(\text{area } \triangle SNT)
\end{aligned}$$

7. a) Given quadrilateral $QRST$ with sides $QR \parallel TS$ and $QT \parallel RS$. Also given: $\hat{Q} = y$ and $\hat{S} = 63°$; $Q\hat{T}R = 38°$ and $R\hat{T}S = x$. Complete the proof below to prove that $QRST$ is a parallelogram.



Steps	Reasons
$Q\hat{T}R = T\hat{R}S$	alt $\angle$s; $QT \parallel RS$
$S\hat{T}R = Q\hat{R}T$	alt $\angle$s; $QR \parallel TS$
?	?
$\therefore \triangle QRT \equiv \triangle STR$	(AAS)
?	congruent triangles
$\hat{Q} = \hat{S}$	congruent triangles
$\therefore QRST$ is a parallelogram	?

Solution:

The completed proof looks like this:

Steps	Reasons
$Q\hat{T}R = T\hat{R}S$	alt $\angle$s; $QT \parallel RS$
$S\hat{T}R = Q\hat{R}T$	alt $\angle$s; $QR \parallel TS$
In $\triangle QRT$ and $\triangle RST$ side $RT = RT$	common side
$\therefore \triangle QRT \equiv \triangle STR$	congruent (AAS)
$\therefore QR = TS$ and $QT = RS$	congruent triangles
$\hat{Q} = \hat{S}$	congruent triangles
$\therefore QRST$ is a parallelogram	opp sides of quad are $=$

b) Calculate the value of y.

Solution:

$QRST$ is a parallelogram, $\therefore \hat{Q} = \hat{S}$.

Opposite $\angle$'s of parallelogram are equal. $\hat{Q} = \hat{S}$ and $\hat{R} = \hat{T}$.

Therefore, $y = 63°$.

c) Calculate the value of x.

Solution:

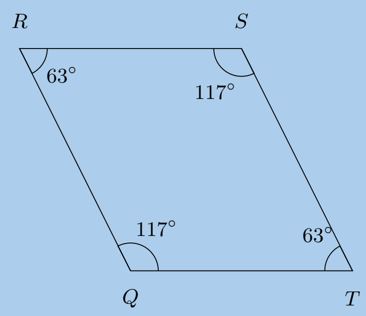
$\angle$'s in a $\triangle = 180°$

$\therefore \hat{Q} + Q\hat{R}T + Q\hat{T}R = 180°$

Now we know that $\hat{Q} = \hat{S} = 63°$ and that $R\hat{T}S = 79°$.

$\therefore \hat{x} = 180° - 63° - 79° = 79°$.

8. Study the quadrilateral $QRST$ with opposite angles $\hat{Q} = \hat{S} = 117°$ and angles $\hat{R} = \hat{T} = 63°$ carefully. Fill in the correct reasons or steps to prove that the quadrilateral $QRST$ is a parallelogram.

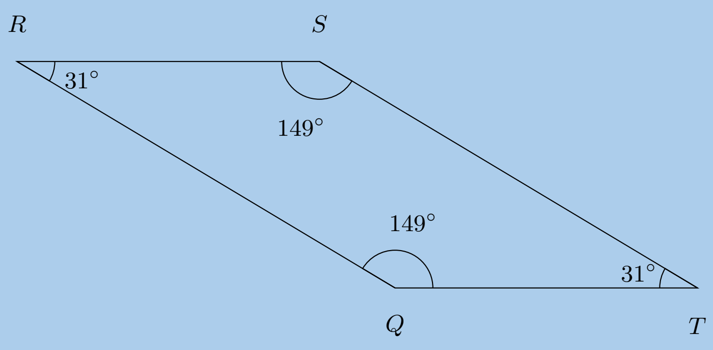


Steps	Reasons
?	given both $\angle$s $= 117°$
$Q\hat{R}S = Q\hat{T}S$	given both $\angle$s $= 63°$
?	sum of $\angle$s in quad
$R\hat{Q}T + Q\hat{T}S = 180°$	$117° + 63° = 180°$
$\therefore QR \parallel TS$	co-int $\angle$s; $QR \parallel TS$
$\therefore RS \parallel QT$	?
$\therefore QRST$ is a parallelogram	?

Solution:

Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle$s $= 117°$
$Q\hat{R}S = Q\hat{T}S$	given both $\angle$s $= 63°$
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360°$	sum of $\angle$s in quad
$R\hat{Q}T + Q\hat{T}S = 180°$	$117° + 63° = 180°$
$\therefore QR \parallel TS$	co-int $\angle$s; $QRTS$
$\therefore RS \parallel QT$	co-int $\angle$s; $RS \parallel QT$
$\therefore QRST$ is a parallelogram	opp. sides parallel

9. Study the quadrilateral $QRST$ with $\hat{Q} = \hat{S} = 149°$ and $\hat{R} = \hat{T} = 31°$ carefully. Fill in the correct reasons or steps to prove that the quadrilateral $QRST$ is a parallelogram.

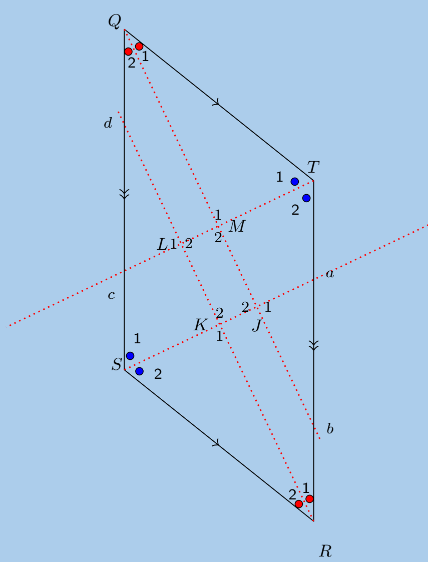


Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle$s $= 149°$
$Q\hat{R}S = Q\hat{T}S$	?
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360°$	sum of $\angle$s in quad
$R\hat{Q}T + Q\hat{T}S = 180°$	?
?	co-int $\angle$s; $QR \parallel TS$
?	co-int $\angle$s; $RS \parallel QT$
$\therefore QRST$ is a parallelogram	opp. sides are parallel

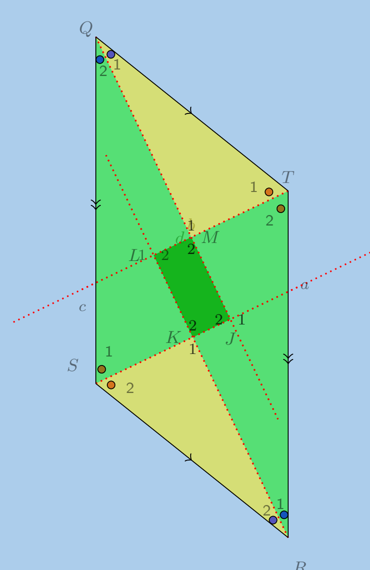
Solution:

Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle$s $= 149°$
$Q\hat{R}S = Q\hat{T}S$	given both $\angle$s $= 31°$
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360°$	sum of $\angle$s in quad
$R\hat{Q}T + Q\hat{T}S = 180°$	$149° + 31° = 180°$
$\therefore QR \parallel TS$	co-int $\angle$s; $QR \parallel TS$
$\therefore RS \parallel QT$	co-int $\angle$s; $RS \parallel QT$
$\therefore QRST$ is a parallelogram	opp. sides parallel

10. In parallelogram $QTRS$, the bisectors of the angles have been constructed, indicated with the red lines below. You are also given $QT = SR$, $TR = QS$, $QT \parallel SR$, $TR \parallel QS$, $\hat{Q} = \hat{R}$ and $\hat{T} = \hat{S}$.
Prove that the quadrilateral $JKLM$ is a parallelogram.
Note the diagram is drawn to scale.

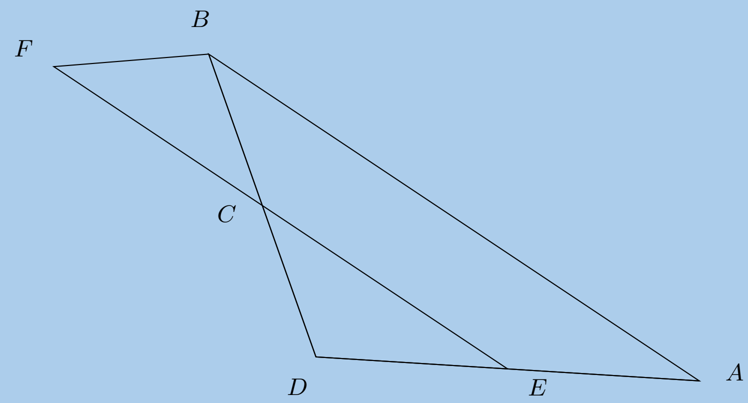


Solution:
Redraw the diagram and mark all the known information:



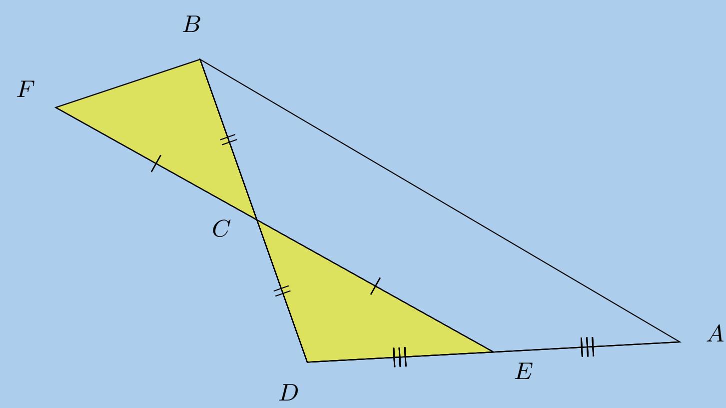
Steps	Reasons
In $\triangle QJS$ and $\triangle RLT$	
$\hat{T}_2 = \hat{S}_1$	(given)
$\hat{Q}_2 = \hat{R}_1$	(given)
$QS = TR$	(given)
$\therefore \triangle QJS \equiv \triangle TLR$	(AAS)
$\therefore \hat{J}_2 = \hat{L}_2$	corresp. $\angle$'s proved with $\triangle QJS \equiv \triangle TLR$
In $\triangle QTM$ and $\triangle SRK$	
$\hat{Q}_1 = \hat{R}_2$	(given)
$\hat{T}_1 = \hat{S}_2$	(given)
$QT = SR$	(given)
$\therefore \triangle QTM \equiv \triangle SRK$	(AAS)
$\therefore \hat{M}_1 = \hat{K}_1$	corresp $\angle$s proved with $\triangle QTM \equiv \triangle SRK$
but $\hat{M}_1 = \hat{M}_2$ and $\hat{K}_1 = \hat{K}_2$	vert opp $\angle$s $=$
$\therefore JKLM$ is a parallelogram	(opp. $\angle$s are equal)

11. Study the diagram below; it is not necessarily drawn to scale. Two triangles in the figure are congruent: $\triangle CDE \equiv \triangle CBF$. Additionally, $EA = ED$. You need to prove that $ABFE$ is a parallelogram.



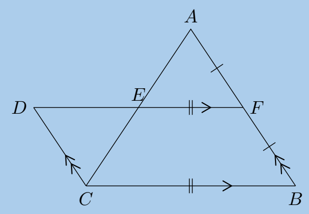
Solution:

Redraw the diagram and mark all known and given information:



Steps	Reasons
$CD = BC$	corresp sides of congruent triangles
C is the mid-point of BD	definition of mid-point
$E$is a mid-point	given: $EA = ED$
$EF \parallel AB$	Midpt Theorem
$D\hat{E}C = B\hat{F}C$	corresp$\angle$s in congruent triangles
$AD \parallel BF$	alt $\angle$s equal
$ABFE$ is a parallelogram	both pairs opp. sides parallel

12. Given the following diagram:



a) Show that $BCDF$ is a parallelogram.
Solution:

$$\begin{aligned} DF &\parallel CB \text{ (given)} \\ DC &\parallel FB \text{ (given)} \\ \therefore BCDF &\text{ is a parallelogram (both pairs opp. sides } \parallel \text{)} \end{aligned}$$

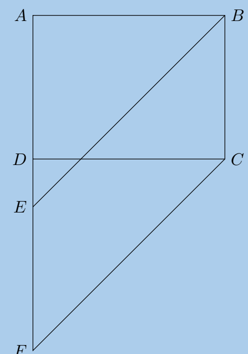
b) Show that $ADCF$ is a parallelogram.
Solution:

$$\begin{aligned} &\text{In } \triangle DEC \text{ and } \triangle FEA \\ &C\hat{A}F = A\hat{C}D \text{ (alt } \angle\text{s; } AB \parallel DC\text{)} \\ &A\hat{F}D = C\hat{D}F \text{ (alt } \angle\text{s; } AB \parallel DC\text{)} \\ &DC = FB \text{ (opp sides parm eq)} \\ &\therefore DC = FA = FB \\ &\therefore \triangle DEC \equiv \triangle FEA \text{ (ASA)} \\ &\therefore DE = EF \text{ and } CE = EA \end{aligned}$$

But AE and DF are diagonals of $ADCF$, $\therefore$ $ADCF$ is a parallelogram (diagonals bisect each other).

c) Prove that $AE = EC$.
Solution:
$AE = EC$ (proved above).

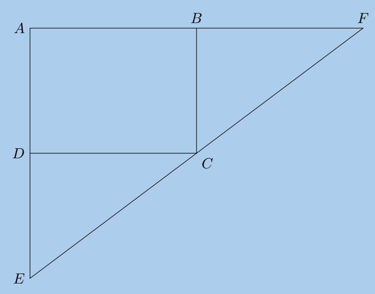
13. $ABCD$ is a parallelogram. $BEFC$ is a parallelogram. $ADEF$ is a straight line. Prove that $AE = DF$.



Solution:

$$\begin{aligned} BC &= EF \text{ (opp sides of } \parallel \text{m)} \\ BC &= AD \text{ (opp sides of } \parallel \text{m)} \\ \therefore EF &= ED \\ AD + DE &= AE \\ EF + DE &= DF \\ &\text{but } DE \text{ is common} \\ \therefore AE &= DF \end{aligned}$$

14. In the figure below $AB = BF$, $AD = DE$. $ABCD$ is a parallelogram. Prove EF is a straight line.



Solution:

We note that:

$$
\begin{aligned}
B\hat{A}D &= B\hat{C}D \text{ (opp } \angle\text{s } \parallel \text{m)}\\
C\hat{D}E &= B\hat{C}D \text{ (alt } \angle\text{s; } AE \parallel BC)\\
F\hat{B}C &= B\hat{C}D \text{ (alt } \angle\text{s; } AF \parallel DC)\\
\therefore C\hat{D}E &= F\hat{B}C
\end{aligned}
$$

We also note that:

$$
\begin{aligned}
AD &= BC \text{ (opp sides parm eq)}\\
AB &= DC \text{ (opp sides parm eq)}
\end{aligned}
$$

Now we can show that $\triangle DEC$ is congruent to $\triangle BCF$:

$$
\begin{aligned}
&\text{in } \triangle DEC \text{ and } \triangle BCF\\
&\quad C\hat{D}E = F\hat{B}C \qquad \text{(proven above)}\\
&DC = AB = BF \qquad \text{(given)}\\
&DE = AD = BC \qquad \text{(given)}\\
&\therefore \triangle DEC \equiv \triangle BCF \text{ (SAS)}
\end{aligned}
$$

Finally we can show that ECF is a straight line:

$$
\begin{aligned}
\therefore B\hat{F}C &= D\hat{C}E \;(\triangle DEC \equiv \triangle BCF)\\
B\hat{C}F &= D\hat{E}C \;(\triangle DEC \equiv \triangle BCF)\\
\text{but } F\hat{B}C + B\hat{F}C + B\hat{C}F &= 180°\text{(sum of } \angle\text{s in } \triangle)\\
\therefore D\hat{C}E + B\hat{C}F + B\hat{C}D &= 180°\\
\therefore ECF &\text{ is a str line}
\end{aligned}
$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GR4 2. 2GR5 3. 2GR6 4. 2GR7 5. 2GR8 6. 2GR9
7. 2GRB 8. 2GRC 9. 2GRD 10. 2GRF 11. 2GRG 12. 2GRH
13. 2GRJ 14. 2GRK

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CHAPTER 13

Measurements

- Content covered in this chapter includes revision of volume and surface area for right-prisms and cylinders. This work is then extended to spheres, right pyramids and cones. Finally learners investigate the effects of multiplying any dimension by a constant factor k.
- Restrict pyramids to those with square or equilateral triangles as bases.
- Composite figures must be included e.g. a square pyramids on top of a cube or two cones stuck together.

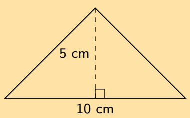
For revision of earlier grades content on surface area and perimeter you can have learners look up the regulations for the size of different sports fields and calculate the perimeter and surface area for different parts of the field.

13.1 Area of a polygon

Exercise 13 – 1:

1. Find the area of each of the polygons below:

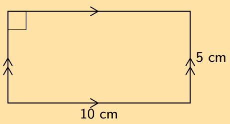
a)



Solution:

$$\begin{aligned} A &= \frac{1}{2}b \times h \\ &= \frac{1}{2}(10)(5) \\ &= 25 \text{ cm}^2 \end{aligned}$$

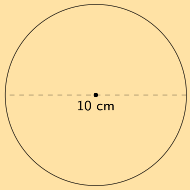
b)



Solution:

$$\begin{aligned} A &= b \times h \\ &= (10)(5) \\ &= 50 \text{ cm}^2 \end{aligned}$$

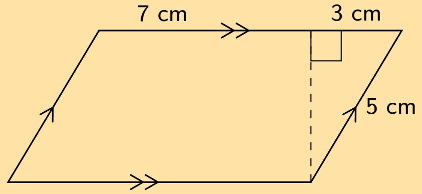
c)



Solution:
The radius is half the diameter, therefore the radius is 5 cm.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(5)^2 \\ &= 78{,}5398... \\ &\approx 78{,}54 \text{ cm}^2 \end{aligned}$$

d)



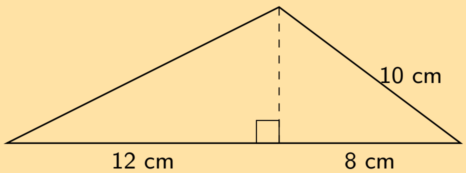
Solution:
We first need to work out the height using the theorem of Pythagoras:

$$\begin{aligned} h^2 &= 5^2 - 3^2 \\ &= 16 \\ \therefore h &= 4 \text{ cm} \end{aligned}$$

Now we can calculate the area:

$$\begin{aligned} A &= b \times h \\ &= (10)(4) \\ &= 40 \text{ cm}^2 \end{aligned}$$

e)



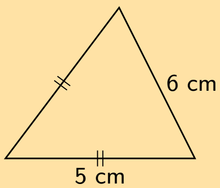
Solution:
We first need to work out the height using the theorem of Pythagoras:

$$\begin{aligned} h^2 &= 10^2 - 8^2 \\ &= 36 \\ \therefore h &= 6 \text{ cm} \end{aligned}$$

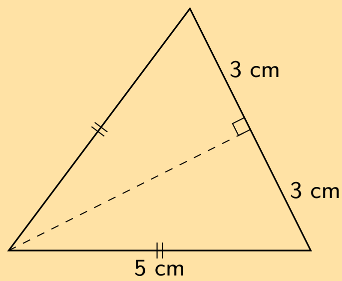
Now we can calculate the area:

$$\begin{aligned} A &= \frac{1}{2} b \times h \\ &= \frac{1}{2}(6)(20) \\ &= 60 \text{ cm}^2 \end{aligned}$$

f)



Solution:
We first need to construct the vertical (or perpendicular) height. For an isosceles triangle if we construct the perpendicular height between the two equal sides then this line will bisect the third side.



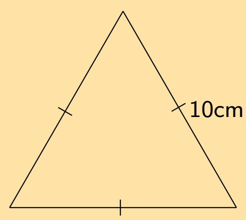
Now we can calculate the height using the theorem of Pythagoras:

$$\begin{aligned} h^2 &= 5^2 - 3^2 \\ &= 16 \\ \therefore h &= 4 \text{ cm} \end{aligned}$$

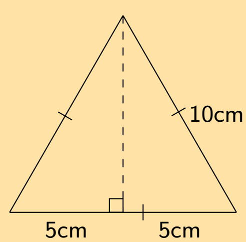
Now we can calculate the area:

$$\begin{aligned} A &= \frac{1}{2} b \times h \\ &= \frac{1}{2}(6)(4) \\ &= 12 \text{ cm}^2 \end{aligned}$$

g)



Solution:
We first construct the vertical (perpendicular) height. For an equilateral triangle the perpendicular height will bisect the third side.



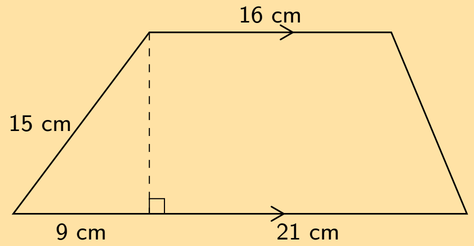
Now we can calculate the height using the theorem of Pythagoras:

$$\begin{aligned} h^2 &= 10^2 - 5^2 \\ &= 75 \\ \therefore h &= \sqrt{75} \text{ cm} \end{aligned}$$

Now we can calculate the area:

$$A = \frac{1}{2}\text{base} \times \text{height}$$
$$A = \frac{1}{2}(10)(\sqrt{75})$$
$$A = 43{,}30 \text{ cm}^2$$

h)



Solution:

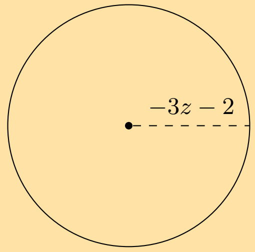
We first find the height using the theorem of Pythagoras:

$$\begin{aligned} h^2 &= 15^2 - 9^2 \\ &= 144 \\ h &= 12 \end{aligned}$$

Now we can calculate the area:

$$\begin{aligned} A &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2}(16 + (21+9))(12) \\ &= \frac{1}{2}(46)(12) \\ A &= 276 \text{ cm}^2 \end{aligned}$$

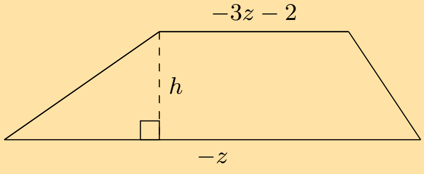
2. a) Find an expression for the area of this figure in terms of z and π. The circle has a radius of $-3z-2$. Write your answer in expanded form (not factorised).



Solution:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(-3z-2)^2 \\ &= 9\pi z^2 + 12\pi z + 4\pi \end{aligned}$$

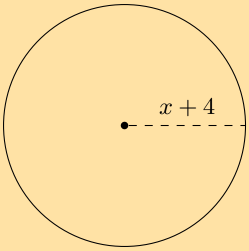
b) Find an expression for the area of this figure in terms of z and h. The height of the figure is h, and two sides are labelled as $-3z-2$ and $-z$. Write your answer in expanded form (not factorised).



Solution:

$$\begin{aligned} A &= \frac{h}{2}(a+b) \\ &= \frac{h}{2}((-3z-2)+(-z)) \\ &= -2hz-h \end{aligned}$$

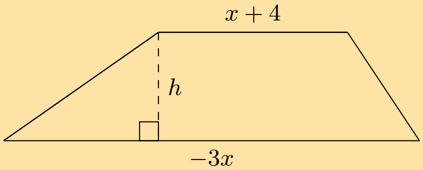
3. a) Find an expression for the area of this figure in terms of x and π. The circle has a radius of $x+4$. Write your answer in expanded form (not factorised).



Solution:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(x+4)^2 \\ &= \pi x^2 + 8\pi x + 16\pi \end{aligned}$$

b) Find an expression for the area of this figure in terms of x and h. The height of the figure is h, and two sides are labelled as $x+4$ and $-3x$. Write your answer in expanded form (not factorised).



Solution:

$$\begin{aligned} A &= \frac{h}{2}(a+b) \\ &= \frac{h}{2}((x+4)+(-3x)) \\ &= -hx+2h \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2GRP 1b. 2GRQ 1c. 2GRR 1d. 2GRS 1e. 2GRT 1f. 2GRV
1g. 2GRW 1h. 2GRX 2. 2GRY 3. 2GRZ

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13.2 Right prisms and cylinders

Surface area of prisms and cylinders

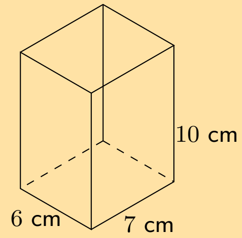
It may be useful to have some nets of the different polyhedra available for learners to see how they fold up to form the polyhedra.

You can download and print out nets of various polyhedra from senteacher to use in your classroom.

Exercise 13 – 2:

1. Calculate the surface area of the following prisms:

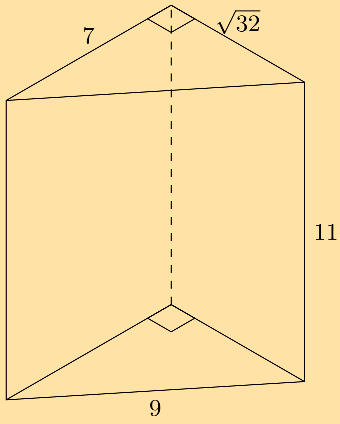
 a)



Solution:

$$\begin{aligned}
\text{Area of large rectangle} &= \text{perimeter of small rectangle} \times \text{length} \\
&= (10 + 7 + 10 + 7) \times 6 \\
&= 34 \times 6 \\
&= 204 \text{ cm}^2 \\
\text{Area of } 2 \times \text{ small rectangle} &= 2(7 \times 10) \\
&= 2(70) \\
&= 140 \text{ cm}^2 \\
\text{Area of large rectangle} + 2 \times \text{(small rectangle)} &= 204 + 140 = 344 \text{ cm}^2
\end{aligned}$$

 b)



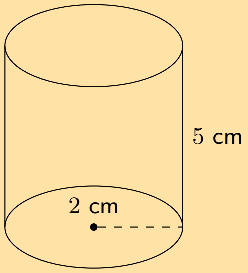
Solution:

There are three different sized rectangles that make up the sides of this triangular prism. We need to find the area of each one of them. All of the rectangles have a height of 11 but each rectangle has a different base.

$$\begin{aligned}
\text{area of } 2 \times \text{ triangle} &= 2\left(\frac{1}{2}b \times h\right) \\
&= (\sqrt{32})(7) \\
&= 39{,}5979...
\end{aligned}$$

$$
\begin{aligned}
\text{area of rectangle 1} &= b \times h \\
&= (7)(11) \\
&= 77 \\
\text{area of rectangle 2} &= b \times h \\
&= (9)(11) \\
&= 99 \\
\text{area of rectangle 3} &= b \times h \\
&= (\sqrt{32})(11) \\
&= 62{,}2253... \\
A_{\text{triangular prism}} &= 39{,}5979... + 77 + 99 + 62{,}2253... \\
&= 277{,}82 \text{ cm}^2
\end{aligned}
$$

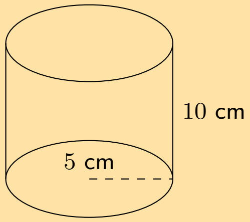
c)



Solution:

$$
\begin{aligned}
\text{Area of large rectangle} &= \text{circumference of circle} \times \text{length} \\
&= 2\pi \times r \times l \\
&= 2\pi \times (2) \times 5 \\
&= 20\pi \\
\text{Area of circle} &= \pi r^2 \\
&= \pi(2)^2 \\
&= 4\pi \\
\text{Surface area} &= \text{area large rectangle} + 2(\text{area of circle}) \\
&= 20\pi + 2(4\pi) \\
&= 28\pi \\
&\approx 87{,}96 \text{ cm}^2
\end{aligned}
$$

d)

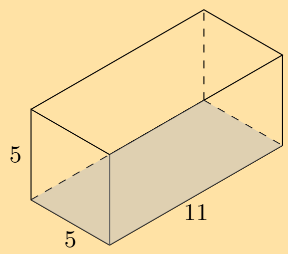


Solution:

$$
\begin{aligned}
\text{Area of large rectangle} &= \text{circumference of circle} \times \text{length} \\
&= 2\pi \times r \times l \\
&= 2\pi \times (5) \times 10 \\
&= 100\pi
\end{aligned}
$$

$$\begin{aligned}\text{Area of circle} &= \pi r^2 \\ &= \pi(5)^2 \\ &= 25\pi \\ \text{Surface area} &= \text{area large rectangle} + 2(\text{area of circle}) \\ &= 100\pi + 2(25\pi) \\ &= 150\pi \\ &\approx 471{,}24 \text{ cm}^2\end{aligned}$$

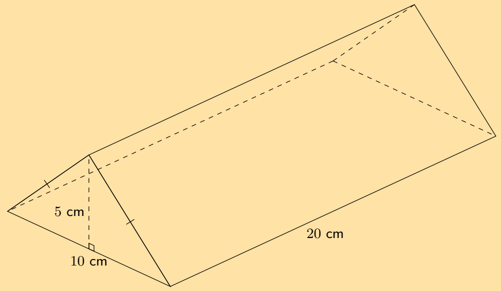
e)



Solution:
There are 4 rectangles and 2 squares that make up this rectangular prism. The square has a side length of 5. The rectangles have a base of 5 and a height of 11.

$$\begin{aligned}A_{\text{rectangular prism}} &= 4 \times \text{area rectangle} + 2 \times \text{area square} \\ &= 4(b \times h) + 2(s^2) \\ &= 4(11 \times 5) + 2(5^2) \\ &= 4(55) + 2(25) \\ &= 270\end{aligned}$$

f)



Solution:
We first need to find the missing side of the triangle. We can do this using the theorem of Pythagoras.

$$\begin{aligned}x^2 &= 5^2 + \left(\frac{10}{2}\right)^2 \\ x^2 &= 5^2 + 5^2 \\ &= 25 + 25 \\ x &= \sqrt{50}\end{aligned}$$

Now we can find the area of the triangular prism:

$$\begin{aligned}\text{perimeter of triangle} &= 10 + \sqrt{50} + \sqrt{50} \\ &= 24{,}1421\ldots\end{aligned}$$

$$\begin{aligned}
\text{area of large rectangle} &= \text{perimeter of triangle} \times \text{length} \\
&= 24{,}1421... \times 20 \\
&= 482{,}8427... \\
\text{area of triangle} &= \frac{1}{2}b \times h \\
&= \frac{1}{2} \times 5 \times 10 \\
&= 25 \\
\text{surface area} &= \text{area large rectangle} + 2(\text{area of triangle}) \\
&= 482{,}8427... + 2(25) \\
&= 532{,}84 \text{ cm}^2
\end{aligned}$$

2. If a litre of paint covers an area of 2 m^2, how much paint does a painter need to cover:

   a) a rectangular swimming pool with dimensions 4 m $\times$ 3 m $\times$ 2,5 m (the inside walls and floor only);

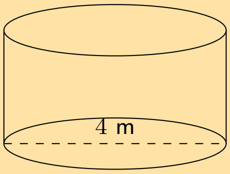
   Solution:

   We need to find the surface area of the pool. In this case we have a rectangular prism but with one rectangle missing (which would be the top of the pool).

$$\begin{aligned}
\text{surface area} &= \text{area of bottom of pool} + 2(\text{area of long sides}) \\
&\quad + 2(\text{area of short sides}) \\
&= (4 \times 3) + 2(4 \times 2{,}5) + 2(3 \times 2{,}5) \\
&= 12 + 20 + 15 \\
&= 47 \text{ m}^2
\end{aligned}$$

   The painter needs one litre of paint for every 2 m^2 of area. So we must divide the surface area by 2 to find the total amount of paint needed. Therefore, the painter will need $\frac{47}{2} = 24$ l of paint (rounded up to the nearest litre).

   b) the inside walls and floor of a circular reservoir with diameter 4 m and height 2,5 m.



   Solution:

   We need to find the surface area of the reservoir. In this case we have a cylinder but with one circle missing (which would be the top of the reservoir).

   We are given the diameter of the reservoir. The radius is half the diameter and so $r = 2$ m.

$$\begin{aligned}
\text{surface area} &= \text{area of bottom of reservoir} + \text{area of inside of reservoir} \\
&= (\pi r^2) + (\text{circumference of base} \times \text{height of reservoir}) \\
&= (\pi(2)^2) + (2(\pi)(2) \times 2{,}5) \\
&= 14\pi \\
&\approx 44 \text{ m}^2
\end{aligned}$$

   The painter needs one litre of paint for every 2 m^2 of area. So we must divide the surface area by 2 to find the total amount of paint needed. Therefore, the painter will need $\frac{44}{2} = 22$ l of paint (rounded up to the nearest litre).

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2GS3 1b. 2GS4 1c. 2GS5 1d. 2GS6 1e. 2GS7 1f. 2GS8 2. 2GS9

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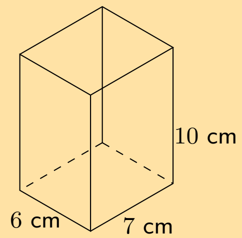
Exercise 13 – 3:

Calculate the volumes of the following prisms (correct to 1 decimal place):

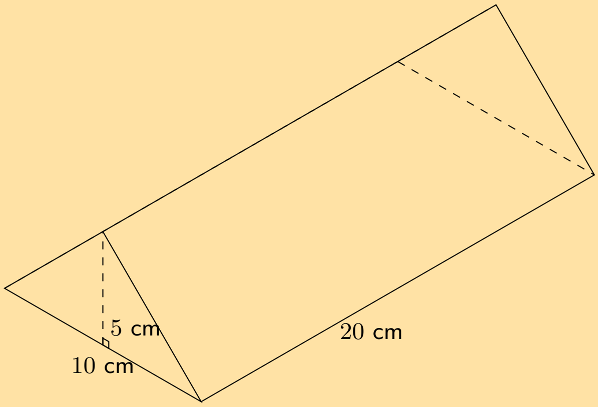
1.



Solution:

$$\begin{aligned} V &= l \times b \times h \\ &= 6 \times 7 \times 10 \\ &= 420 \text{ cm}^3 \end{aligned}$$

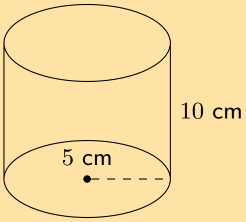
2.



Solution:

$$\begin{aligned} V &= \frac{1}{2} \times b \times h \times H \\ &= \frac{1}{2} \times 10 \times 5 \times 20 \\ &= 500 \text{ cm}^3 \end{aligned}$$

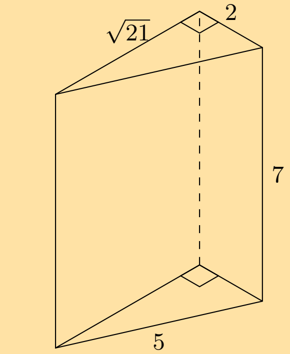
3.



Solution:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(5)^2(10) \\ &= 250\pi \\ &\approx 785{,}4 \text{ cm}^3 \end{aligned}$$

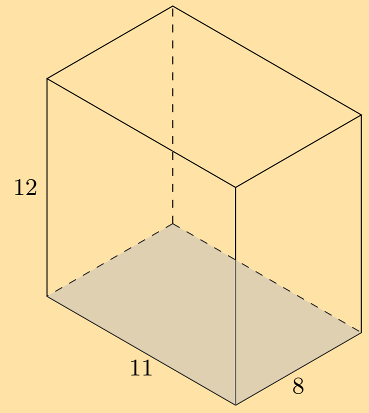
4. The figure here is a triangular prism. The height of the prism is 7 units; the triangles, which both contain right angles, have sides which are 2, $\sqrt{21}$ and 5 units long. Calculate the volume of the figure. Round to two decimal places if necessary.



Solution:

$$\begin{aligned} V &= \text{area of base} \times \text{height} \\ &= \left[\frac{1}{2} b_{\triangle} h_{\triangle}\right](H) \\ &= \left[\frac{1}{2}(2)(\sqrt{21})\right](7) \\ &= (\sqrt{21})(7) \\ &\approx 32{,}06 \end{aligned}$$

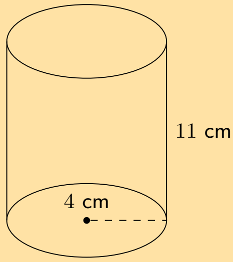
5. The figure here is a rectangular prism. The height of the prism is 12 units; the other dimensions of the prism are 11 and 8 units. Find the volume of the figure.



Solution:

$$\begin{aligned} V_{\text{rectangular prism}} &= \text{area of base} \times \text{height} \\ &= (bh)(H) \\ &= (8 \times 11)(12) \\ &= 1056 \end{aligned}$$

6. The picture below shows a cylinder. The height of the cylinder is 11 units; the radius of the cylinder is $r = 4$ units. Determine the volume of the figure. Round your answer to two decimal places.



Solution:

$$\begin{aligned} V_{\text{cylinder}} &= (\text{area of circle})(H) \\ &= \left[\pi r^2\right](H) \\ &= \left[\pi(4)^2\right](11) \\ &= [16\pi](11) \\ &= 176\pi \\ &\approx 552{,}92 \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GSC 2. 2GSD 3. 2GSF 4. 2GSG 5. 2GSH 6. 2GSJ

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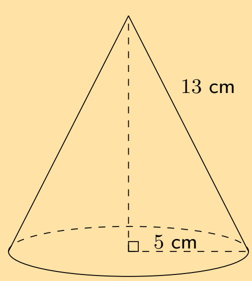
13.3 Right pyramids, right cones and spheres

Surface area of pyramids, cones and spheres

Exercise 13 – 4:

1. Find the total surface area of the following objects (correct to 1 decimal place if necessary):

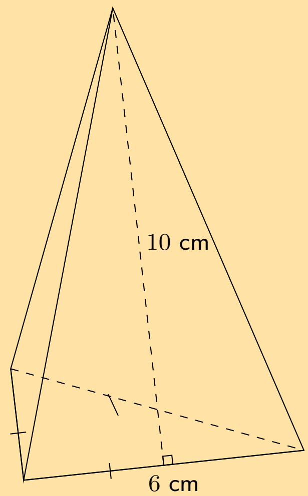
a)



Solution:

$$\begin{aligned} \text{Surface area} &= \text{area of base} + \text{area of walls} \\ &= \pi r(r + h_s) \\ &= \pi(5)(5 + 13) \\ &\approx 282{,}7\text{cm}^2 \end{aligned}$$

b)



Solution:

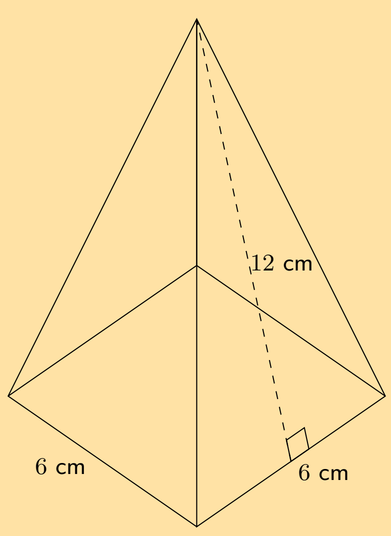
We first need to find h_b by constructing the vertical (perpendicular) height and using the theorem of Pythagoras:

$$\begin{aligned}(h_b)^2 &= (b)^2 - \left(\frac{b}{2}\right)^2 \\ &= 6^2 - \left(\frac{6}{2}\right)^2 \\ &= 36 - 9 \\ &= 27 \\ h_b &= \sqrt{27}\text{ cm}\end{aligned}$$

Now we can find the surface area:

$$\begin{aligned}\text{surface area} &= \text{area of base} + \text{area of triangular sides} \\ &= \frac{1}{2}b(h_b + 3h_s) \\ &= \frac{1}{2}(6)(\sqrt{27} + 10) \\ &\approx 45{,}6\text{ cm}^2\end{aligned}$$

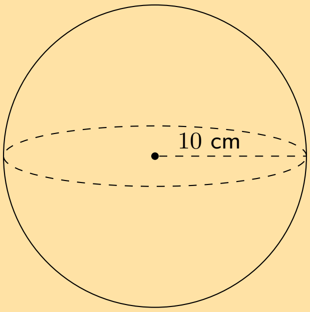
c)



Solution:

$$\begin{aligned}\text{surface area} &= \text{area of base} + \text{area of triangular sides}\\ &= b(b + 2h_s)\\ &= 6(6 + 2(12))\\ &= 180 \text{ cm}^2\end{aligned}$$

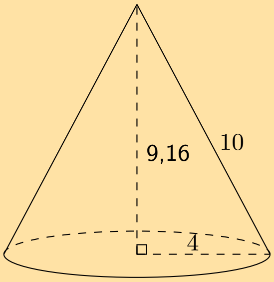
d)



Solution:

$$\begin{aligned}\text{surface area} &= 4\pi r^2\\ &= 4\pi(10)^2\\ &\approx 1256{,}6 \text{ cm}^2\end{aligned}$$

2. The figure here is a cone. The vertical height of the cone is $H = 9{,}16$ units and the slant height of the cone is $h = 10$ units; the radius of the cone is shown, $r = 4$ units. Calculate the surface area of the figure. Round your answer to two decimal places.

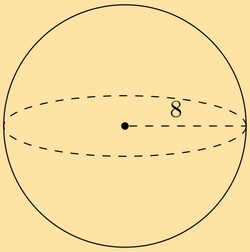


Solution:

$$\begin{aligned}A_{\text{cone}} &= \pi r(r + h)\\ &= \pi(4)(4 + 10)\\ &= 56\pi\\ &= 175{,}9291...\end{aligned}$$

Therefore the surface area for the cone is 175,93 square units.

3. The figure here is a sphere. The radius of the sphere is $r = 8$ units. Calculate the surface area of the figure. Round your answer to two decimal places.

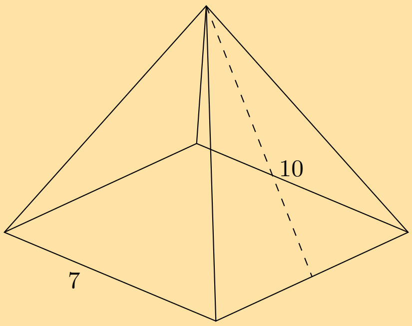


Solution:

$$\begin{aligned} A_{\text{sphere}} &= 4\pi r^2 \\ &= 4\pi(8)^2 \\ &= 256\pi \\ &= 804{,}2477... \end{aligned}$$

Therefore the surface area is 804,25 square units.

4. The figure here shows a pyramid with a square base. The sides of the base are each 7 units long. The vertical height of the pyramid is 9,36 units, and the slant height of the pyramid is 10 units. Determine the surface area of the pyramid.



Solution:

$$\begin{aligned} A_{\text{square pyramid}} &= b(b + 2h_s) \\ &= (7)(7 + 2(10)) \\ &= 189 \end{aligned}$$

The surface area for the pyramid is 189 square units.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2GSK 1b. 2GSM 1c. 2GSN 1d. 2GSP 2. 2GSQ 3. 2GSR 4. 2GSS

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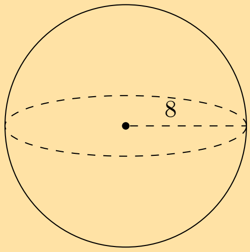
Volume of pyramids, cones and spheres

Exercise 13 – 5:

1. The figure below shows a sphere. The radius of the sphere is $r = 8$ units. Determine the volume of the figure. Round your answer to two decimal places.

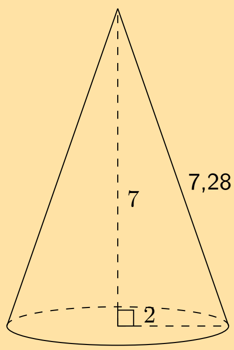


Solution:

$$\begin{aligned} V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(8)^3 \\ &= \frac{4}{3}\pi(512) \\ &= \frac{2048}{3}\pi \\ &= 2144{,}6605... \end{aligned}$$

Therefore the volume for the sphere is 2144,66 units^3.

2. The figure here is a cone. The vertical height of the cone is $H = 7$ units and the slant height is $h = 7{,}28$ units; the radius of the cone is shown, $r = 2$ units. Calculate the volume of the figure. Round your answer to two decimal places.

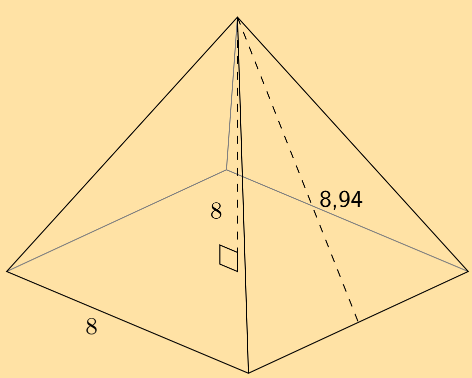


Solution:

$$\begin{aligned} V_{\text{cone}} &= \frac{1}{3} \times \pi r^2 H \\ &= \frac{1}{3} \times \pi(2)^2(7) \\ &= \frac{1}{3}\pi(4)(7) \\ &= \frac{28}{3}\pi \\ &= 29{,}3215... \end{aligned}$$

Therefore the volume of the cone is 29,32 units^3.

3. The figure here is a pyramid with a square base. The vertical height of the pyramid is $H = 8$ units and the slant height is $h = 8{,}94$ units; each side of the base of the pyramid is $b = 8$ units. Round your answer to two decimal places.



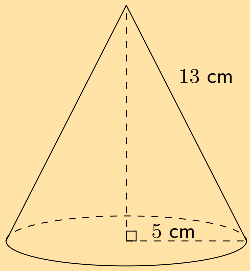
Solution:

$$
\begin{aligned}
V_{\text{square pyramid}} &= \frac{1}{3}b^2H \\
&= \frac{1}{3}(8)^2(8) \\
&= \frac{1}{3}(64)(8) \\
&= \frac{512}{3} \\
&\approx 170{,}6666\ldots
\end{aligned}
$$

Therefore the volume of the square pyramid is: 170,67 units^3.

4. Find the volume of the following objects (round off to 1 decimal place if needed):

a)



Solution:

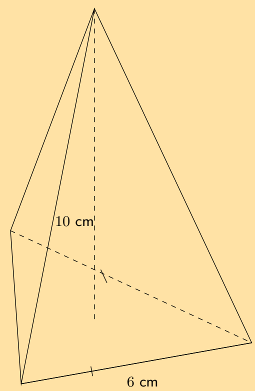
We are given the radius of the cone and the slant height. We can use this to find the vertical height (H) of the cone:

$$
\begin{aligned}
H^2 &= 13^2 - 5^2 \\
&= 144 \\
H &= 12
\end{aligned}
$$

Now we can calculate the volume of the cone:

$$
\begin{aligned}
V &= \frac{1}{3} \times \pi(r)^2 \times H \\
&= \frac{1}{3}\pi(5)^2(12) \\
&= 100\pi \\
&\approx 314{,}2\text{ cm}^3
\end{aligned}
$$

b)



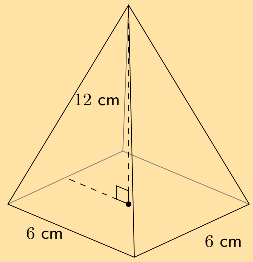
Solution:
We first need to find the vertical (perpendicular) height of the triangle (h) using the theorem of Pythagoras:

$$\begin{aligned} h^2 &= b^2 - \left(\frac{b}{2}\right)^2 \\ &= 36 - 9 \\ &= 27 \\ h &= \sqrt{27} \text{ cm} \end{aligned}$$

Now we can find the volume:

$$\begin{aligned} V &= \frac{1}{3} \times \frac{1}{2} bh \times H \\ &= \frac{1}{3} \times \frac{1}{2} (\sqrt{27})(6) \times (10) \\ &= 10\sqrt{27} \\ &\approx 52{,}0 \text{ cm}^3 \end{aligned}$$

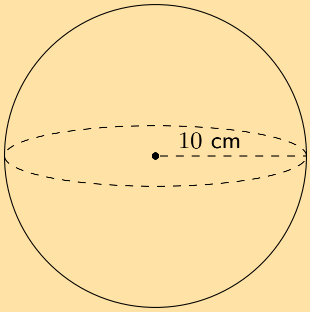
c)



Solution:

$$\begin{aligned} V &= \frac{1}{3} \times b^2 \times H \\ &= \frac{1}{3}(6)^2(12) \\ &= 144 \text{ cm}^3 \end{aligned}$$

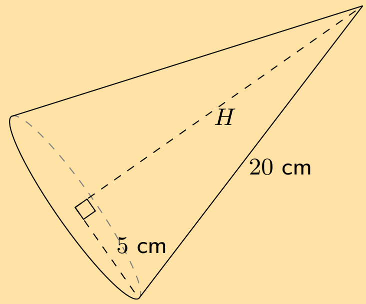
d)



Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(10)^3 \\ &\approx 4188{,}8 \text{ cm}^3 \end{aligned}$$

5. Find the surface area and volume of the cone shown here. Round your answers to the nearest integer.



Solution:
The surface area of the cone is:

$$
\begin{aligned}
A_{\text{cone}} &= \pi r(r+h) \\
&= \pi(5)(5+20) \\
&= 392{,}69908...\text{ cm}^2 \\
&\approx 393\text{ cm}^2
\end{aligned}
$$

For the volume we first need to find the perpendicular (or vertical) height using the theorem of Pythagoras:

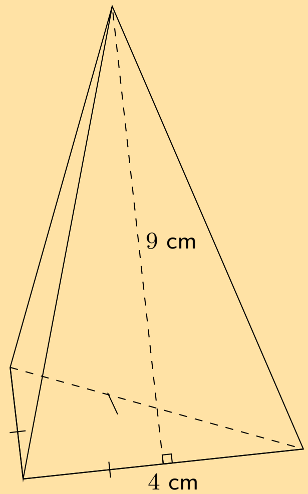
$$
\begin{aligned}
H^2 &= 20^2 - 5^2 \\
H &= \sqrt{400-25} \\
&= \sqrt{375}
\end{aligned}
$$

Now we can calculate the volume of the cone:

$$
\begin{aligned}
V_{\text{cone}} &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi(5)^2(\sqrt{375}) \\
&= 506{,}97233...\text{ cm}^3 \\
&\approx 507\text{ cm}^3
\end{aligned}
$$

Therefore the surface area is 393 cm^2 and the volume is 507 cm^3.

6. Calculate the following properties for the pyramid shown below. Round your answers to two decimal places.



a) Surface area
Solution:

We first calculate the vertical (perpendicular) height of the base triangle:

$$\begin{aligned} h_b^2 &= 4^2 - 2^2 \\ &= 16 - 4 \\ h_b &= \sqrt{12} \end{aligned}$$

Now we can calculate the surface area of the pyramid:

$$\begin{aligned} A_{\text{pyramid}} &= \frac{1}{2}b(h_b + 3h_s) \\ &= \frac{1}{2}(6)(\sqrt{12} + 3(9)) \\ &= \text{91,39230... cm}^2 \end{aligned}$$

Therefore the surface area of the triangular pyramid is: 91,39 cm^2.

b) Volume
Solution:
We first need to find the vertical height (H):

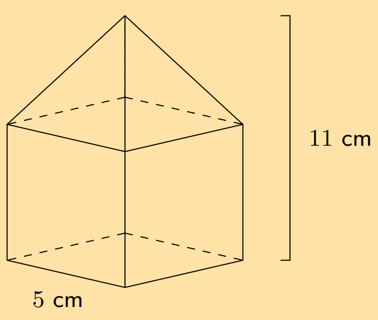
$$\begin{aligned} H^2 &= 9^2 - 3^2 \\ &= 81 - 9 \\ H &= \sqrt{72} \end{aligned}$$

Now we can find the volume:

$$\begin{aligned} V_{\text{pyramid}} &= \frac{1}{3} \times \frac{1}{2}(b)(h_b) \times H \\ &= \frac{1}{6}(6)(\sqrt{12}) \times (\sqrt{72}) \\ &= \text{29.39387... cm}^3 \end{aligned}$$

Therefore the volume of the pyramid is: 29.39 cm^3.

7. The solid below is made up of a cube and a square pyramid. Find its volume and surface area (correct to 1 decimal place):



Solution:
The height of the cube is 5 cm. Since the total height of the object is 11 cm, the height of the pyramid must be 6 cm. We will work out the volume first:

$$\begin{aligned} \text{Volume} &= \text{volume of cube} + \text{volume of square pyramid} \\ &= s^3 + \frac{1}{3}bH \\ &= (5)^3 + \frac{1}{3}(5)^2(6) \\ &= 175 \text{ cm}^3 \end{aligned}$$

For the surface area we note that one face of the cube is covered by the pyramid. We also note that the base of the pyramid is covered by the cube. So we only need to find the area of 5 sides of the cube and the four triangular faces of the pyramid.

For the triangular faces we need the slant height. We can calculate this using the theorem of Pythagoras:

$$\begin{aligned} h_s^2 &= H^2 + \left(\frac{b}{2}\right)^2 \\ &= (6)^2 + (2{,}5)^2 \\ h_s &= \sqrt{42{,}25} \end{aligned}$$

The surface area is:

$$\begin{aligned} \text{Surface area } &= 5(\text{sides of cube}) + 4(\text{triangle faces of pyramid}) \\ &= 5(s^2) + 4\left(\frac{1}{2}bh_s\right) \\ &= 5(5^2) + 4\left(\frac{1}{2}(5)(\sqrt{42{,}25})\right) \\ &= 125 + 10\sqrt{42{,}25} \\ &= 190 \text{ cm}^2 \end{aligned}$$

The surface area is 190 cm^2 and the volume is 175 cm^3.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GSW 2. 2GSX 3. 2GSY 4a. 2GSZ 4b. 2GT2 4c. 2GT3
4d. 2GT4 5. 2GT5 6. 2GT6 7. 2GT7

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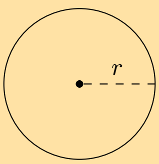
13.4 The effect of multiplying a dimension by a factor of k

Exercise 13 – 6:

1. If the length of the radius of a circle is a third of its original size, what will the area of the new circle be?



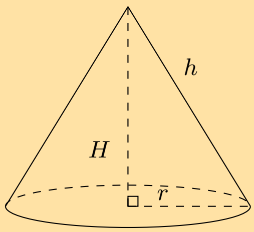
Solution:

The area of the original circle is: πr^2. Now we reduce the radius by a third. In other words we multiply r by one third. The new area is:

$$\begin{aligned} A_{\text{new}} &= \pi\left(\frac{1}{3}r\right)^2 \\ &= \frac{1}{9}\pi r^2 \\ &= \frac{1}{9}A \end{aligned}$$

Therefore, if the radius of a circle is a third of its original size, the area of the new circle will be $\frac{1}{9}$ the original area.

2. If the length of the base's radius and height of a cone is doubled, what will the surface area of the new cone be?



Solution:

We can find the new area by noting that the area will change by a factor of k when we change the dimensions of the cone. In this case we are changing two dimensions of the cone and so the new area will be: $A_{\text{new}} = k^2 A$

The value of k comes from the word "doubled" in the question: the value of k is 2.

So the new area of the cone will be $A_{\text{new}} = 4 \times A$ if we double the height and the base's radius of the cone.

Therefore the surface area of the new cone will be 4 times the original surface area.

3. If the height of a prism is doubled, how much will its volume increase by?

Solution:

We do not know if we have a rectangular prism or a triangular prism. However we do know that the volume of a prism is given by:

$$V = \text{area of base} \times \text{height of prism}$$

Now we are changing just one dimension of the prism: the height. Therefore the new volume is given by:

$$\begin{aligned} V_{\text{new}} &= \text{area of base} \times 2(\text{height of prism}) \\ &= 2V \end{aligned}$$

Therefore the volume of the prism doubles if the height is doubled.

4. Describe the change in the volume of a rectangular prism if the:

a) length and breadth increase by a constant factor of 3.

Solution:

The volume of a rectangular prism is given by $V = l \times b \times h$. If we increase the length and breadth by a constant factor of 3 the volume is:

$$\begin{aligned} V_{\text{new}} &= 3(l) \times 3(b) \times h \\ &= 9V \end{aligned}$$

Therefore the volume of the prism increaases by a factor of 9 when the length and breadth are increased by a constant factor of 3.

b) length, breadth and height are multiplied by a constant factor of 3.

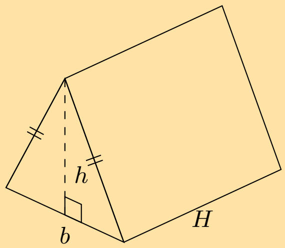
Solution:

The volume of a rectangular prism is given by $V = l \times b \times h$. If we increase the length, breadth and height by a constant factor of 3 the volume is:

$$\begin{aligned} V_{\text{new}} &= 3(l) \times 3(b) \times 3(h) \\ &= 27V \end{aligned}$$

Therefore the volume of the prism increaases by a factor of 27 when the length, breadth and height are increased by a constant factor of 3.

5. If the length of each side of a triangular prism is quadrupled, what will the volume of the new triangular prism be?



Solution:

When multiplied by a factor of k the volume of a shape will increase by k^3. We are told that the dimensions are quadrupled. This means that each dimension is multiplied by 4. Therefore $k = 4$.

Now we can calculate k^3.

$$k^3 = (4)^3 = 64$$

Therefore, if each side of a triangular prism is quadrupled, the volume of the new triangular prism will be 64 times the original shape's volume.

6. Given a prism with a volume of 493 cm^3 and a surface area of 6007 cm^2, find the new surface area and volume for a prism if all dimensions are increased by a constant factor of 4.

   Solution:

   We are increasing all the dimensions by 4 and so the volume will increase by 4^3. The surface area will increase by 4^2.

$$\begin{aligned} V &= 493 \times 4^3 \\ &= 31\ 552 \text{ cm}^3 \\ \text{Surface area} &= 6007 \times 4^2 \\ &= 96\ 112 \text{ cm}^2 \end{aligned}$$

   Therefore the volume is 31 552 cm^3 and the surface area is 96 112 cm^2.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GT8 2. 2GT9 3. 2GTB 4a. 2GTC 4b. 2GTD 5. 2GTF
6. 2GTG

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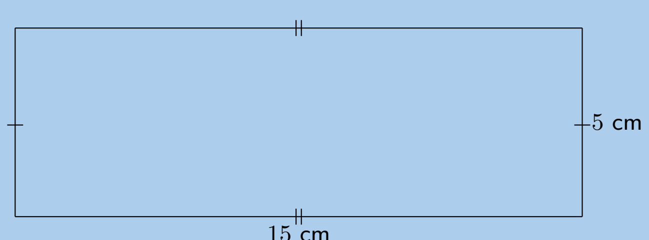
13.5 Chapter summary

End of chapter Exercise 13 – 7:

1. Find the area of each of the shapes shown. Round your answer to two decimal places if necessary.

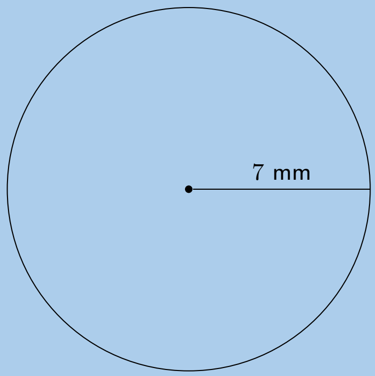
   a)



Solution:

$$\begin{aligned} A_{\text{rectangle}} &= l \times b \\ &= (15)(5) \\ &= 75 \text{ cm}^2 \end{aligned}$$

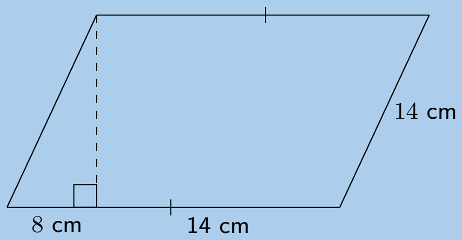
b)



Solution:

$$\begin{aligned} A_{\text{circle}} &= \pi r^2 \\ &= (3,1415...)(7)^2 \\ &= 153{,}93804... \\ &\approx 153{,}94 \text{ mm}^2 \end{aligned}$$

c)



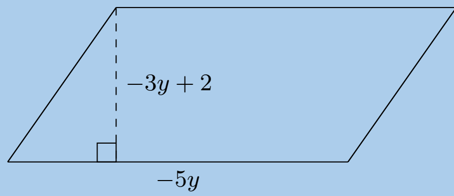
Solution:
We first need to find the height:

$$\begin{aligned} h^2 &= (14)^2 - (8)^2 \\ h &= \sqrt{132} \end{aligned}$$

Now we can find the area of the parallelogram. Note that the length of the base is $b = 8 + 14 = 22$.

$$\begin{aligned} A_{\text{parallelogram}} &= b \times h \\ &= (22)(\sqrt{132}) \\ &\approx 252{,}76 \text{ cm}^2 \end{aligned}$$

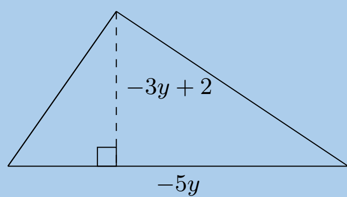
2. a) Find an expression for the area of this figure in terms of y. The dimensions of the figure are labelled $-5y$ and $-3y + 2$. Write your answer in expanded form (not factorised).



Solution:

$$
\begin{aligned}
A &= bh \\
A &= (-5y)(-3y+2) \\
A &= 15y^2 - 10y
\end{aligned}
$$

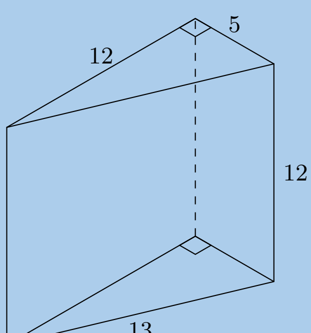
b) Find an expression for the area of this figure in terms of y. The figure has dimensions of $-5y$ and $-3y+2$, as labelled. Write your answer in expanded form (not factorised).



Solution:

$$
\begin{aligned}
A &= \frac{1}{2}bh \\
A &= \frac{1}{2}(-5y)(-3y+2) \\
A &= \frac{15y^2}{2} - 5y
\end{aligned}
$$

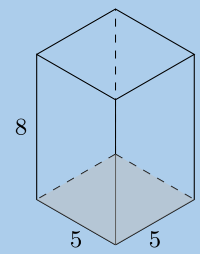
3. The figure here is a triangular prism. The height of the prism is 12 units; the triangles, which are both right triangles, have sides which are 5, 12 and 13 units long. Find the surface area of the figure.



Solution:
A triangular prism is made up of 2 triangles and 3 rectangles. In this case the triangles are right-angled triangles and so we have the height of the triangle. We also note that each rectangle has a different length and breadth.

$$
\begin{aligned}
A_{\triangle\text{ prism}} &= 2A_{\text{triangles}} + 3A_{\text{rectangles}} \\
&= 2\left[\frac{1}{2}b_{\triangle}h_{\triangle}\right] + b_1h_1 + b_2h_2 + b_3h_3 \\
&= 2\left[\frac{1}{2}(5)(12)\right] + (12)(12) + (5)(12) + (13)(12) \\
&= 60 + 144 + 60 + 156 \\
&= 420
\end{aligned}
$$

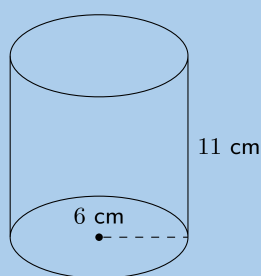
4. The figure here is a rectangular prism. The height of the prism is 5 units; the other dimensions of the prism are 8 and 5 units. Find the surface area of the figure.



Solution:
A rectangular prism is made up of 6 rectangles. In this case there are 4 rectangles with a breadth of 5 units and a height of 8 units and two rectangles with a breadth of 5 units and a height of 5 units.

$$\begin{aligned} A_{\text{rectangular prism}} &= A_{\text{6 rectangles}} \\ &= 4(b_1 h_1) + 2(b_2 h_2) \\ &= 4(8)(5) + 2(5)(5) \\ &= 4(40) + 2(25) \\ &= 210 \end{aligned}$$

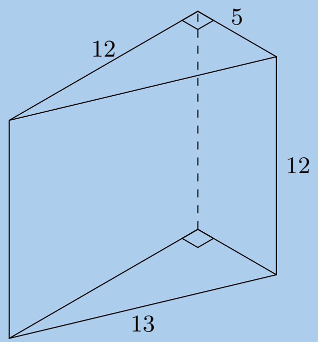
5. A cylinder is shown below. The height of the cylinder is 11 cm; the radius of the cylinder is $r = 6$ cm, as shown. Find the surface area of the figure. Round your answer to two decimal places.



Solution:
A cylinder is made up of two circles and a rectangle. We can find the area of each of these and add them up to find the surface area of the cylinder. For the rectangle we note that the length is the circumference of the circle.

$$\begin{aligned} A_{\text{cylinder}} &= A_{\text{2 circles}} + A_{\text{1 rectangle}} \\ &= 2(\pi r^2) + b(2\pi r) \\ &= 2\left(\pi(6)^2\right) + (11)(2\pi(6)) \\ &= 2\left(36\pi\right) + 132\pi \\ &= 204\pi \\ &\approx 640{,}88 \end{aligned}$$

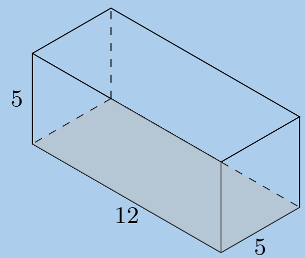
6. The figure here is a triangular prism. The height of the prism is 12 units; the triangles, which both contain right angles, have sides which are 5, 12 and 13 units long. Determine the volume of the figure.



Solution:

$$
\begin{aligned}
V_{\triangle\text{prism}} &= \text{area of base} \times \text{height} \\
&= \left[\frac{1}{2} b_{\triangle} h_{\triangle}\right](H) \\
&= \left[\frac{1}{2}(5)(12)\right](12) \\
&= (30)(12) \\
&= 360
\end{aligned}
$$

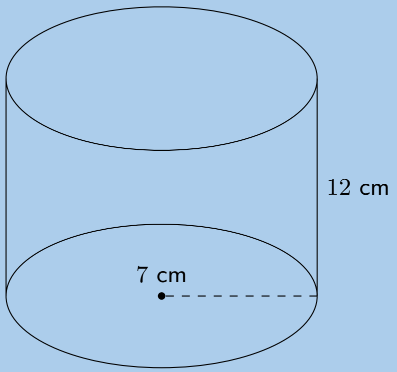
7. The figure here is a rectangular prism. The height of the prism is 5 units; the other dimensions of the prism are 12 and 5 units. Calculate the volume of the figure.



Solution:

$$
\begin{aligned}
V_{\text{rectangular prism}} &= \text{area of base} \times \text{height} \\
&= (bh)(H) \\
&= (5 \times 12)(5) \\
&= 300
\end{aligned}
$$

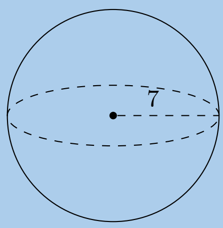
8. The picture below shows a cylinder. The height of the cylinder is 12 cm; the radius of the cylinder is $r = 7$ cm. Calculate the volume of the figure. Round your answer two decimal places.



Solution:

$$\begin{aligned} V_{\text{cylinder}} &= (\text{area of circle})(H) \\ &= \left[\pi r^2\right](H) \\ &= \left[\pi(7)^2\right](12) \\ &= [\pi 49](12) \\ &\approx 1847{,}26 \end{aligned}$$

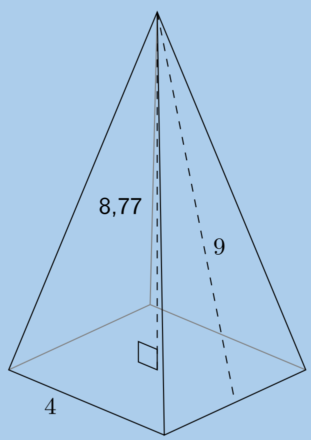
9. The figure here is a sphere. The radius of the sphere is $r = 7$ units. Find the surface area of the figure. Round your answer two decimal places.



Solution:

$$\begin{aligned} A_{\text{sphere}} &= 4\pi r^2 \\ &= 4\pi(7)^2 \\ &= 4\pi(49) \\ &= 196\pi \\ &\approx 615{,}75 \end{aligned}$$

10. The figure here shows a pyramid with a square base. The sides of the base are each 4 units long. The vertical height of the pyramid is 8,77 units, and the slant height of the pyramid is 9 units. Determine the surface area of the pyramid.

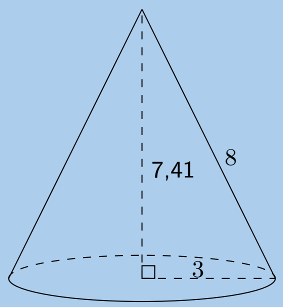


Solution:

$$\begin{aligned} A_{\text{square pyramid}} &= A_{\text{1 square}} + A_{\text{4 triangles}} \\ &= (b)^2 + 4\left(\frac{1}{2}bh_s\right) \\ &= (4)^2 + 4\left(\frac{1}{2}(4)(9)\right) \\ &= 16 + 2(36) \\ &= 88 \end{aligned}$$

The total surface area for the pyramid is: 88 square units.

11. The figure here is a cone. The vertical height of the cone is $H = 7{,}41$ units and the slant height of the cone is $h = 8$ units; the radius of the cone is shown, $r = 3$ units. Find the surface area of the figure. Round your answer two decimal places.

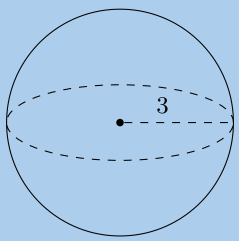


Solution:

$$\begin{aligned} A_{\text{cone}} &= \pi r(r + h) \\ &= \pi(3)(3 + 8) \\ &= 33\pi \\ &= 103{,}6725... \end{aligned}$$

Therefore the total surface area for the cone is 103,67 square units.

12. The figure below shows a sphere. The radius of the sphere is $r = 3$ units. Determine the volume of the figure. Round your answer to two decimal places.

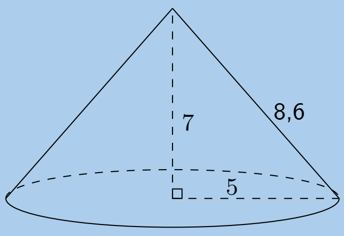


Solution:

$$\begin{aligned} V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(3)^3 \\ &= \frac{4}{3}\pi(27) \\ &= 36\pi \\ &= 113{,}0973... \end{aligned}$$

Therefore the volume for the sphere is 113,1 units^3.

13. The figure here is a cone. The vertical height of the cone is $H = 7$ units and the slant height is $h = 8{,}60$ units; the radius of the cone is shown, $r = 5$ units. Find the volume of the figure. Round your answer to two decimal places.

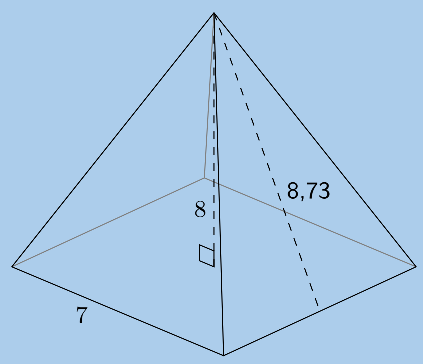


Solution:

$$\begin{aligned} V_{\text{cone}} &= \frac{1}{3} \times \pi r^2 H \\ &= \frac{1}{3} \times \pi(5)^2(7) \\ &= \frac{1}{3}\pi(25)(7) \\ &= \frac{175}{3}\pi \\ &= 183{,}2595... \end{aligned}$$

Therefore the volume is 183,26 units^3.

14. The figure here is a pyramid with a square base. The vertical height of the pyramid is $H = 8$ units and the slant height is $h = 8{,}73$ units; each side of the base of the pyramid is $b = 7$ units. Find the volume of the figure. Round your answer to two decimal places.

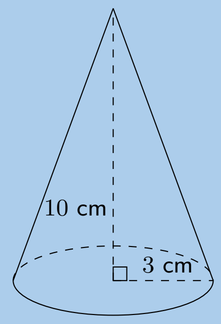


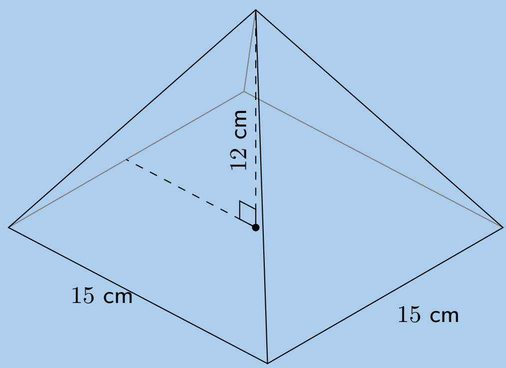
Solution:

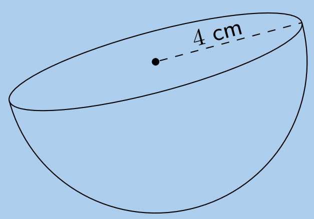
$$\begin{aligned} V_{\text{square pyramid}} &= \frac{1}{3}b^2 H \\ &= \frac{1}{3}(7)^2(8) \\ &= \frac{1}{3}(49)(8) \\ &= \frac{392}{3} \end{aligned}$$

Therefore the volume is: 130,67 units cubed

15. Consider the solids below:







a) Calculate the surface area of each solid.

Solution:

Cone

We first need to calculate the slant height:

$$\begin{aligned} h_s^2 &= r^2 + h^2 \\ &= 3^2 + 10^2 \\ h_s &= \sqrt{109} \end{aligned}$$

Now we can calculate the surface area:

$$\begin{aligned} \text{Surface area} &= \pi r(r + h_s) \\ &= \pi(3)(3 + \sqrt{109}) \\ &\approx 126{,}67 \text{ cm}^2 \end{aligned}$$

Square pyramid

We first need to calculate the slant height:

$$\begin{aligned} h_s^2 &= b^2 + h^2 \\ &= (7{,}5)^2 + 12^2 \\ h_s &= \sqrt{200{,}25} \end{aligned}$$

Now we can calculate the surface area:

$$\begin{aligned} \text{Surface area} &= b(b + 2h_s) \\ &= (15)(15 + 2\sqrt{200{,}25}) \\ &\approx 437{,}26 \text{ cm}^2 \end{aligned}$$

Half sphere

For a half sphere we need to divide the surface area of a sphere by 2. We also need to include the area of a circle.

$$\text{Surface area} = \frac{4\pi r^2}{2} + \pi r^2$$
$$= 2\pi(4)^2 + \pi(4)^2$$
$$= 48\pi$$
$$\approx 150{,}80 \text{ cm}^2$$

The surface area of each of the objects is: $A_{\text{cone}} = 126{,}67 \text{ cm}^2$ $A_{\text{square pyramid}} = 437{,}26 \text{ cm}^2$ $A_{\text{half sphere}} = 150{,}80 \text{ cm}^2$.

b) Calculate the volume of each solid.

Solution:

Cone

$$V = \frac{1}{3}\pi r^2 H$$
$$= \frac{1}{3}\pi(3)^2 \times 10$$
$$= 30\pi$$
$$\approx 94{,}25 \text{ cm}^3$$

Square pyramid:

$$V = \frac{1}{3}b^2 \times H$$
$$= \frac{1}{3}(15)^2 \times 12$$
$$= 900 \text{ cm}^3$$

Half sphere

The volume of a half sphere is half the volume of a sphere.

$$V = \frac{4}{3}\pi r^3 \times \frac{1}{2}$$
$$= \frac{4}{3}\pi(4)^3 \times \frac{1}{2}$$
$$= 18\pi$$
$$\approx 134{,}04 \text{ cm}^3$$

The volume of each of the objects is: $V_{\text{cone}} = 94{,}25 \text{ cm}^3$ $V_{\text{square pyramid}} = 900 \text{ cm}^3$ $V_{\text{half sphere}} = 134{,}04 \text{ cm}^3$

16. If the length of each side of a square is a quarter of its original size, what will the area of the new square be?

Solution:

When we multiply the sides of a square by a factor of k the area of the square will increase by k^2.

In this case we are making each side of the square a quarter of the original size so we get:

$$A_{\text{new}} = \left(\frac{1}{4}s\right)^2$$
$$= \frac{1}{16}s^2$$
$$= \frac{1}{16}A$$

Therefore, if each side of a square is a quarter of its original size, the area of the new square will be $\frac{1}{16}$ times the original square's area.

17. If the length of each side of a square pyramid is a third of its original size, what will the surface area of the new square pyramid be?

Solution:

When we multiply two dimensions of a square pyramid by a factor of k the area of the square pyramid will change by k^2.

In this case the length each side of the square pyramid is a third of the original size so we get:

$$\begin{aligned} A_{\text{new}} &= \frac{1}{3}\left(\frac{1}{3}b\right)^2 H \\ &= \frac{1}{9}\left(\frac{1}{3}b^2 H\right) \\ &= \frac{1}{9}A \end{aligned}$$

Therefore, if each side of a square pyramid is a third of its original size, the surface area of the new square pyramid will be $\frac{1}{9}$ times the original shape's surface area.

18. If the length of the base's radius and the height of a cylinder is halved, what will the volume of the new cylinder be?

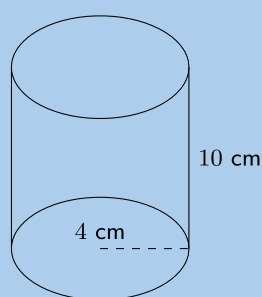
Solution:

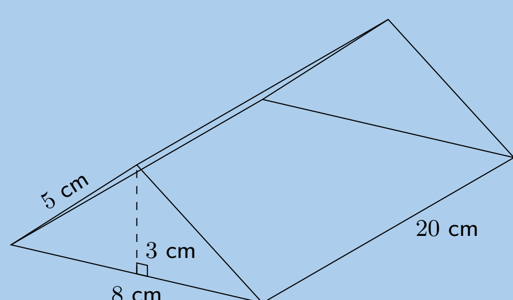
In this case the base's radius and the height of a cylinder is half of the original size so we get:

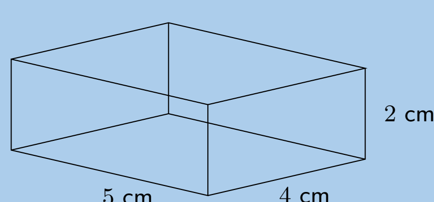
$$\begin{aligned} A_{\text{new}} &= \frac{1}{3}\pi\left(\frac{1}{2}r\right)^2\left(\frac{1}{2}H\right) \\ &= \frac{1}{8}\left(\frac{1}{3}\pi r^2 H\right) \\ &= \frac{1}{8}A \end{aligned}$$

Therefore, if the base's radius and the height of a cylinder is halved, the volume of the new cylinder will be $\frac{1}{8}$ times the original shape's volume.

19. Consider the solids below and answer the questions that follow (correct to 1 decimal place, if necessary):







a) Calculate the surface area of each solid.

Solution:

Cylinder

A cylinder is composed of two circles and a rectangle. The breadth of the rectangle is the circumference of the circle.

$$\begin{aligned}\text{Surface area} &= 2\pi r^2 + 2\pi rh\\ &= 2\pi(4)^2 + 2\pi(4)(10)\\ &= 112\pi\\ &\approx 351{,}9 \text{ cm}^2\end{aligned}$$

Triangular prism

A triangular prism is composed of three rectangles and two triangles. We are given the vertical height of the triangles as well as the slant height.

$$\begin{aligned}\text{Surface area} &= 2(\frac{1}{2}b \times h) + 2(H \times h_s) + (H \times b)\\ &= 2(\frac{1}{2}(8)(3)) + 2(20 \times 5) + (20 \times 8)\\ &= 384 \text{ cm}^2\end{aligned}$$

Rectangular prism

A rectangular prism is composed of 6 rectangles. We have the dimensions of all the rectangles.

$$\begin{aligned}\text{Surface area} &= 2[(L \times b) + (b \times h) + (L \times h)]\\ &= 2[(5 \times 4) + (4 \times 2) + (5 \times 2)]\\ &= 76 \text{ cm}^2\end{aligned}$$

The surface area of each shape is: $A_{\text{cylinder}} = 351{,}9 \text{ cm}^2$ $A_{\text{triangular prism}} = 384 \text{ cm}^2$ $A_{\text{rectangular prism}} = 76 \text{ cm}^2$.

b) Calculate volume of each solid.

Solution:

Cylinder

$$\begin{aligned}V &= \pi r^2 h\\ &= \pi(4)^2(10)\\ &= 160\pi\\ &= 502{,}7 \text{ cm}^3\end{aligned}$$

Triangular prism

$$\begin{aligned}V &= \frac{1}{2} \times h \times b \times H\\ &= \frac{1}{2}(3)(8)(20)\\ &= 240 \text{ cm}^3\end{aligned}$$

Rectangular prism

$$\begin{aligned}\text{Volume} &= l \times b \times h\\ &= 5 \times 4 \times 2\\ &= 40 \text{ cm}^3\end{aligned}$$

The volume of each shape is: $V_{\text{cylinder}} = 502{,}7 \text{ cm}^3$ $V_{\text{triangular prism}} = 240 \text{ cm}^3$ $V_{\text{rectangular prism}} = 40 \text{ cm}^3$.

c) If each dimension of the solids is increased by a factor of 3, calculate the new surface area of each solid.

Solution:

Cylinder

$$\begin{aligned}\text{Surface area} &= 2\pi(3r)^2 + 2\pi(3r)(3h)\\ &= 2\pi 9(4)^2 + 2\pi(9)(4)(10)\\ &= 1008\pi\\ &\approx 3166{,}7 \text{ cm}^2\end{aligned}$$

Triangular prism

$$\begin{aligned}\text{Surface area} &= 2(\frac{1}{2}b \times h) + 2(H \times S) + (H \times b)\\ &= 2(\frac{9}{2}(8)(3)) + 18(20 \times 5) + 9(20 \times 8)\\ &= 3456 \text{ cm}^2 3456\end{aligned}$$

Rectangular prism

$$\begin{aligned}\text{Surface area} &= 2[9(L \times b) + 9(b \times h) + 9(L \times h)]\\ &= 2[9(5 \times 4) + 9(4 \times 2) + 9(5 \times 2)]\\ &= 684 \text{ cm}^2\end{aligned}$$

The new surface area of each shape is: $A_{\text{cylinder}} = 3166{,}7 \text{ cm}^2$ $A_{\text{triangular prism}} = 3456 \text{ cm}^2$ $A_{\text{rectangular prism}} = 684 \text{ cm}^2$.

d) If each dimension of the solids is increased by a factor of 3, calculate the new volume of each solid.
Solution:
Cylinder

$$\begin{aligned}V &= \pi(3r)^2 3h\\ &= \pi(3(4))^2(3(10))\\ &= 4320\pi\\ &\approx 13\,571{,}9 \text{ cm}^3\end{aligned}$$

Triangular prism

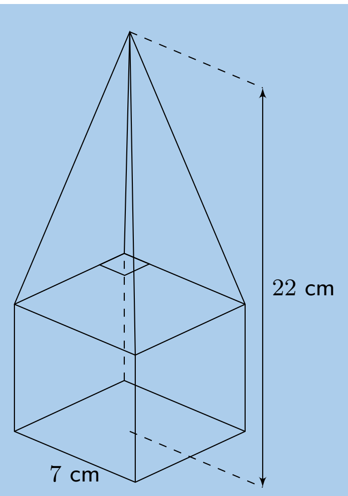
$$\begin{aligned}V &= \frac{1}{2} \times h \times b \times H\\ &= \frac{27}{2}(3)(8)(20)\\ &= 6480 \text{ cm}^3\end{aligned}$$

Rectangular prism

$$\begin{aligned}V &= 27(L \times b \times h)\\ &= 27(5 \times 4 \times 2)\\ &= 1080 \text{ cm}^3\end{aligned}$$

The new volume of each shape is: $V_{\text{cylinder}} = 13\,571{,}9 \text{ cm}^3$ $V_{\text{triangular prism}} = 6480 \text{ cm}^3$ $V_{\text{rectangular prism}} = 1080 \text{ cm}^3$

20. The solid below is made of a cube and a square pyramid. Answer the following:



a) Find the surface area of the solid shown. Give your answers to two decimal places.
Solution:
Start with the faces of the cube, which are all squares:

$$\begin{aligned} A_{\text{five squares}} &= 5 \times s^2 \\ &= 5 \times (7)^2 \\ &= 245 \text{ cm}^2 \end{aligned}$$

Next we note that the height of the pyramid is:

$$h_{\text{pyramid}} = 22 - 7 = 15$$

And we need to calculate the slant height using the theorem of Pythagoras:

$$\begin{aligned} h_s &= \sqrt{(15)^2 + \left(\frac{1}{2}(7)\right)^2} \\ &= \sqrt{(225 + 12{,}25)} \\ &= \sqrt{237{,}25} \end{aligned}$$

Now we can calculate the area of each of the four triangles:

$$\begin{aligned} A_{\text{four triangles}} &= 4 \times \frac{1}{2} b h_s \\ &= 4 \times \frac{1}{2}(7)(\sqrt{237{,}25}) \\ &= 14\sqrt{237{,}25} \end{aligned}$$

Finally we can calculate the total surface area:

$$\begin{aligned} A_{\text{total}} &= A_{\text{triangles}} + A_{\text{squares}} \\ &= 14\sqrt{237{,}25} + 245 \\ &= 460{,}6409\ldots \text{ cm}^2 \\ &\approx 460{,}64 \text{ cm}^2 \end{aligned}$$

Therefore the surface area is: $460{,}64 \text{ cm}^2$.

b) Now determine the volume of the shape. Give your answer to the nearest integer value.
Solution:
Volume of the pyramid:

$$\begin{aligned} V_{\text{pyramid}} &= \frac{1}{3}b^2H \\ &= \frac{1}{3}(7)^2(15) \\ &= 245 \text{ cm}^3 \end{aligned}$$

Volume of the cube:

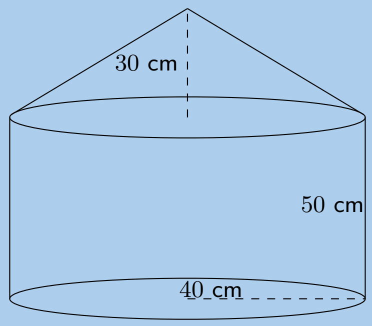
$$\begin{aligned} V_{\text{cube}} &= l^3 \\ &= (7)^3 \\ &= 343 \text{ cm}^3 \end{aligned}$$

Total volume:

$$\begin{aligned} V_{\text{total}} &= V_{\text{cube}} + V_{\text{pyramid}} \\ &= 343 + 245 \\ &= 588 \text{ cm}^3 \end{aligned}$$

Therefore the total volume is: 588 cm^3.

21. Calculate the volume and surface area of the solid below (correct to 1 decimal place):



Solution:
Surface area
Cylinder:

$$\begin{aligned} \text{Surface area} &= \pi r^2 + 2\pi r h \\ &= \pi(40)^2 + 2\pi(40)(50) \\ &= 17\ 592{,}91 \text{ cm}^2 \end{aligned}$$

Cone:

$$\begin{aligned} \text{Surface area} &= 2\pi r\sqrt{r^2+h^2} \\ &= 2\pi(40)\sqrt{40^2+30^2} \\ &= 12\ 566{,}4 \text{ cm}^2 \end{aligned}$$

Total surface area: 17 592,92 + 12 566,4 = 30 159,52 cm^2.
Volume
Cylinder:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(40)^2(50) \\ &= 251\ 327{,}4 \text{ cm}^3 \end{aligned}$$

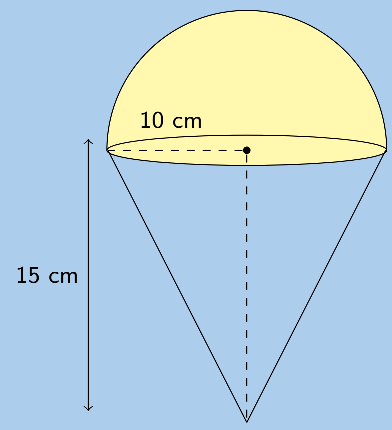
Cone:

$$
\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi(40)^2(30) \\
&= 50\ 265{,}48 \text{ cm}^3
\end{aligned}
$$

Total volume $= 251\ 327{,}4 + 50\ 265{,}48 = 301\ 592{,}88 \text{ cm}^3$.
The total surface area and volume is 30 159,52 cm^2 and 301 592,88 cm^3 respectively.

22. Find the volume and surface areas of the following composite shapes.

a)



Solution:
The shape is a half sphere on top of a right cone. We can calculate the volume of a cone and add this to half the volume of a sphere. The volume is:

$$
\begin{aligned}
V &= \frac{1}{3}\pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) \\
&= \frac{1}{3}\pi(10)^2(15) + \frac{2}{3}\pi(10)^3 \\
&= \frac{3500}{3}\pi \\
&= 3665{,}19 \text{ cm}^3
\end{aligned}
$$

For the surface area we first need to find the slant height:

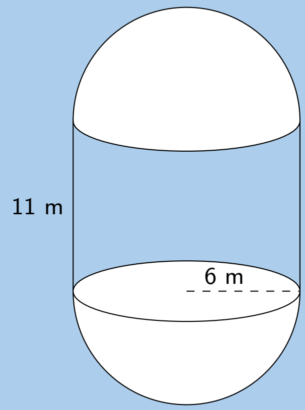
$$
\begin{aligned}
h_s^2 &= 10^2 + 15^2 \\
&= 325 \\
h_s &= 5\sqrt{13}
\end{aligned}
$$

We have a half sphere on top of a cone. The half sphere covers the circle on top of the cone and so we need to exclude this part from our calculation. For the half sphere we can use half the surface area of a sphere as this does not include the circle at the base of the half sphere.
The surface area is:

$$
\begin{aligned}
\text{surface area } &= \pi r h_s + 2\pi r^2 \\
&= \pi\,(10)\left(5\sqrt{13}\right) + 2\pi\,(10)^2 \\
&\approx 1194{,}68 \text{ cm}^2
\end{aligned}
$$

Therefore the volume and surface area are 3665,19 cm^3 and 1194,67 cm^2 respectively.

b)



Solution:

We have a cylinder with two half spheres. We can calculate the volume of a cylinder and add the volume of a sphere to this. The volume is:

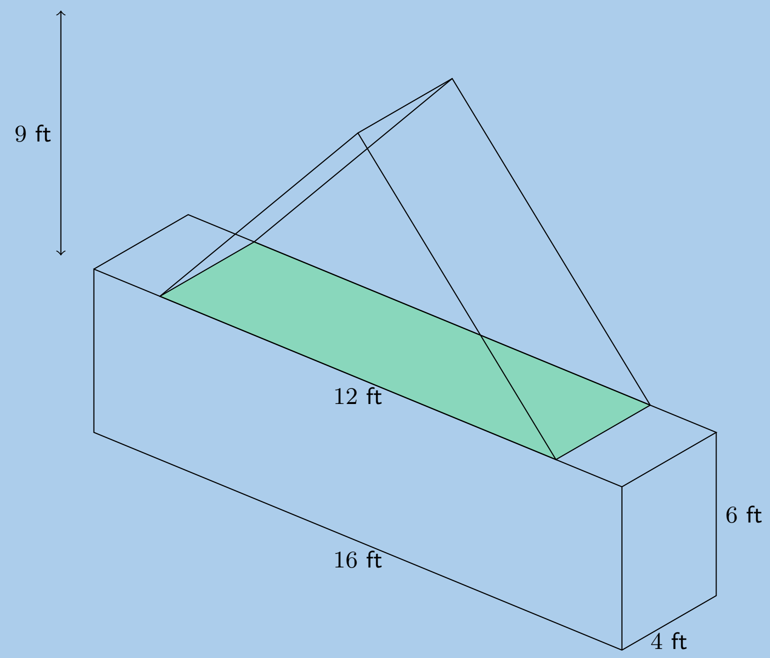
$$\begin{aligned} V &= \pi r^2 h + \frac{4}{3}\pi r^3 \\ &= \pi(6)^2(11) + \frac{4}{3}\pi(6)^3 \\ &= 684\pi \\ &= 2148{,}85 \text{ m}^3 \end{aligned}$$

For the surface area of the two half spheres we can use the surface area of a sphere. For the cylinder we need to exclude the area of the two circles from our calculation since these are covered up by the two half spheres. The surface area is:

$$\begin{aligned} \text{surface area } &= 2\pi r h + 4\pi r^2 \\ &= 2\pi\,(6)\,(11) + 4\pi\,(6)^2 \\ &= 276\pi \\ &\approx 867{,}08 \text{ m}^2 \end{aligned}$$

Therefore the volume and surface area are: $2148{,}85 \text{ m}^3$ and $867{,}08 \text{ m}^2$ respectively.

c)



Solution:

This shape consists of a triangular prism and a rectangular prism. The volume is:

$$\begin{aligned} V &= \frac{1}{2}bhH + lbh \\ &= \frac{1}{2}(12)(9)(4) + (4)(6)(16) \\ &= 600 \text{ ft}^3 \end{aligned}$$

For the surface area we need to exclude the base of the triangular prism as well as part of the top of the rectangular prism.

We first need to calculate the slant height for the triangular prism:

$$\begin{aligned} h_s^2 &= 9^2 + 4^2 \\ &= 97 \\ h_s &= \sqrt{97} \end{aligned}$$

Now we can calculate the surface area of the triangular prism. Remember that we do not need to include the base in our calculation so we only have 2 triangles and 2 rectangles.

$$\begin{aligned} \text{surface area } &= 2\left(\frac{1}{2}bH\right) + 2\left(bh_s\right) \\ &= (12)(9) + 2(12)(\sqrt{97}) \\ &= 108 + 24\sqrt{97} \end{aligned}$$

For the rectangular prism we can calculate the full surface area and then subtract the base of triangular prism from this.

$$\begin{aligned} \text{surface area } &= 2(bh) + 2(bl) + 2(hl) - \text{base triangular prism} \\ &= 2(16)(6) + 2(16)(4) + 2(6)(4) - (12)(4) \\ &= 256 \end{aligned}$$

Now we can add the two surface areas together to get the total surface area:

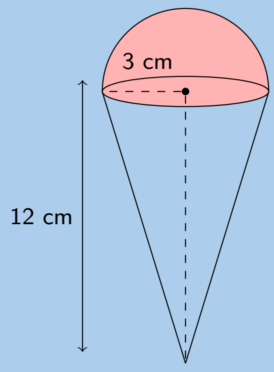
$$\begin{aligned} \text{surface area } &= 256 + (108 + 24\sqrt{97}) \\ &\approx 600{,}37 \text{ ft}^2 \end{aligned}$$

The volume and surface area are 600 ft^3 and 600,37 ft^2 respectively.

23. An ice-cream cone (right cone) has a radius of 3 cm and a height of 12 cm. A half scoop of ice-cream (hemisphere) is placed on top of the cone. If the ice-cream melts, will it fit into the cone? Show all your working.

Solution:

We can draw a quick sketch of the problem:

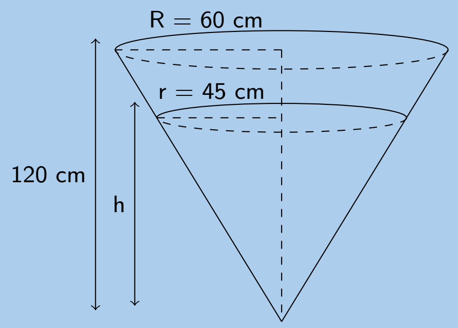


Now we can calculate the volume of the cone and the volume of the ice-cream. The scoop of ice-cream is a half sphere and so the volume of this is half the volume of a sphere.

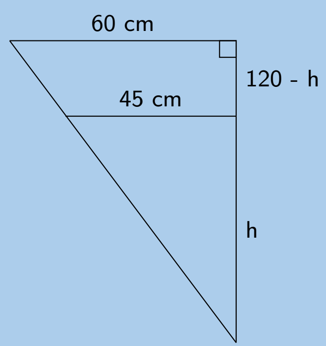
$$\begin{aligned} V_{\text{cone}} &= \frac{\pi(3)^3 12}{3} \\ &= 36\pi \\ &\approx 113{,}1 \\ V_{\text{scoop}} &= \frac{4}{6} \times \pi(3)^3 \\ &= 18\pi \\ &\approx 56{,}5 \end{aligned}$$

Yes, the ice-cream will fit into the cone if it melts since the volume of the ice-cream is less than the volume of the cone.

24. A receptacle filled with petrol has the shape of an inverted right circular cone of height 120 cm and base radius of 60 cm. A certain amount of fuel is siphoned out of the receptacle leaving a depth of h cm.



a) Show that $h = 90$ cm.
Solution:
We can draw the following two triangles based on the information in the figure:



These two triangles are similar triangles. They are both right-angled and share a common angle. Therefore we can use the ratio of the sides to find h:

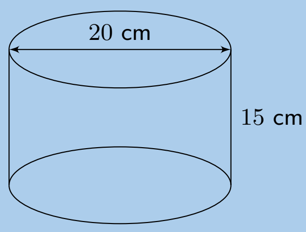
$$\begin{aligned} \frac{h}{120} &= \frac{45}{60} \quad \text{(similar triangles)} \\ \therefore h &= \frac{(45)(120)}{60} \\ &= 90 \text{ cm} \end{aligned}$$

b) Determine the volume of fuel that has been siphoned out. Express your answer in litres if 1 l = 1000 cm^3
Solution:
The volume of fuel that has been siphoned out is the total volume of fuel minus the volume of fuel left. The volume of a cone is $\frac{4}{3}\pi r^2 H$. From the previous question we have the vertical height for both cones.

$$\begin{aligned}
\text{Volume siphoned out } &= \frac{1}{3}\pi R^2 H_{\text{start}} - \frac{1}{3}\pi r^2 H_{\text{end}} \\
&= \frac{1}{3}\pi(60)^2(120) - \frac{1}{3}\pi(45)^2(90) \\
&= 144\ 000\pi - 60\ 750\pi \\
&\approx 261\ 537{,}59 \text{ cm}^3 \\
&\approx 261{,}5 \text{ l}
\end{aligned}$$

25. Find the **volume** and **surface** area of the following prisms.

a)



Solution:

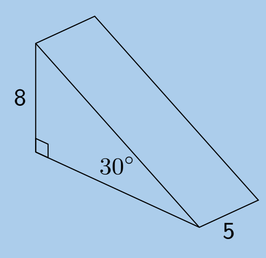
We are given the diameter of the cylinder. The radius is half the diameter.

$$\begin{aligned}
V &= \pi r^2 h \\
&= \pi(10)^2(15) \\
&= 15\ 000\pi \\
&\approx 47\ 123{,}89 \text{ cm}^3
\end{aligned}$$

$$\begin{aligned}
A &= 2\pi rh + 2(\pi r^2) \\
&= 2(\pi(10)(15)) + 2(\pi(10)^2) \\
&= 300\pi + 200\pi \\
&\approx 1570{,}80 \text{ cm}^2
\end{aligned}$$

Therefore the volume and surface area are 47 123,89 cm^3 and 1570,80 cm^2 respectively.

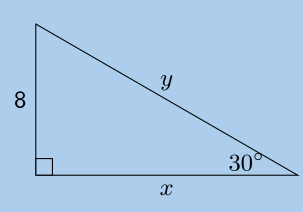
b)



Solution:

This is a triangular pyramid. We are given the vertical height as well as an angle. Since it is a right-angled triangle we can use trigonometry to help us find the missing length.

We redraw the triangle we are interested in:



Now we can calculate x (the slant height) and y (the base):

$$\begin{aligned}\frac{x}{8} &= \tan 30^\circ \\ x &= 8 \tan 30^\circ\end{aligned}$$

$$\begin{aligned}\frac{8}{y} &= \sin 30^\circ \\ \frac{8}{\sin 30^\circ} &= y\end{aligned}$$

Now we know all the lengths we need to know to calculate the volume.

$$\begin{aligned}V &= (\frac{1}{2}bh) \times H \\ &= (\frac{1}{2}(8)(8 \tan 30^\circ)) \times 5 \\ &= 160 \tan 30^\circ \\ &\approx 92{,}38\end{aligned}$$

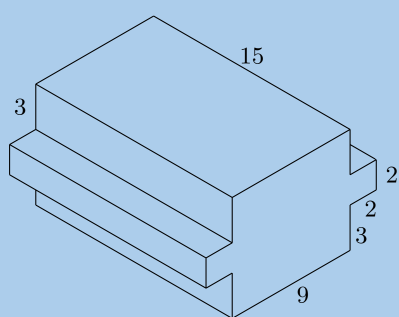
And the surface area is:

$$A = 2(\frac{1}{2}bh) + (H \times h_s) + (H \times h_s) + (H \times b) + (H \times h)$$

$$\begin{aligned}&= 2(\frac{1}{2}(8)(8 \tan 30^\circ) + (5 \times \frac{8}{\sin 30^\circ}) \\ &+ (5 \times 8 \tan 30^\circ) + (5 \times 8) \\ &= 64 \tan 30^\circ + \frac{40}{\sin 30^\circ} + 40 \tan 30^\circ + 40 \\ &\approx 180{,}04\end{aligned}$$

The volume and surface area are: 92,38 and 180,04 respectively.

c)



Solution:

Let: $L = 9$, $B = 8$, $H = 15$, $l = 2$, $b = 2$ and $h = 15$.

We can view this shape as three rectangular prisms. Two of the three prisms are exactly the same. The volume is therefore:

$$\begin{aligned}V &= 2(lbh) + LBH \\ &= 2((2)(2)(15)) + ((8)(9)(15)) \\ &= 120 + 1080 \\ &= 1200\end{aligned}$$

For the surface area we have several different rectangles. Each of the smaller prisms has 5 exposed rectangles. The larger rectangular prism has 4 rectangles that are not covered up by the smaller prisms. The remaining two rectangles are partly covered up by the smaller prisms and so can be considered as 4 separate rectangles.

We will start by finding the surface area of one of the smaller prisms:

$$\begin{aligned}A_{\text{smaller prism}} &= 2(bl) + 2(bh) + lh \\ &= 2((2)(2)) + 2((2)(15)) + (2)(15) \\ &= 98\end{aligned}$$

For the larger prism we get:

$$\begin{aligned}A_{\text{larger prism}} &= 2(BH) + 2(BL) + 4(Hx) \\ &= 2((8)(15)) + 2((9)(15)) + 4((15)(3)) \\ &= 690\end{aligned}$$

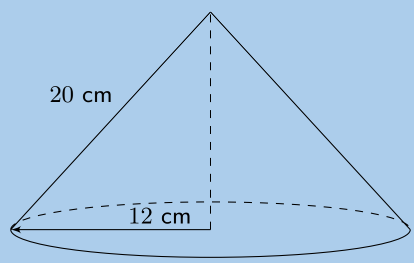
Therefore the total surface area is:

$$\begin{aligned}A &= 98 + 98 + 690 \\ &= 886\end{aligned}$$

The volume and surface area are 1200 and 886 respectively.

26. Determine the volume of the following:

a)



Solution:

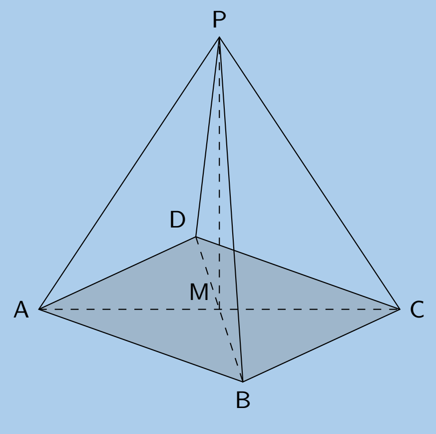
We first need to find the vertical height (H):

$$\begin{aligned}H &= \sqrt{(20)^2 - (12)^2} \\ &= 16\end{aligned}$$

$$\begin{aligned}V &= \frac{1}{3}\pi r^2 H \\ &= \frac{1}{3}\pi (12)^2 (16) \\ &= 768\pi \\ &\approx 2412{,}743\end{aligned}$$

b)

$ABCD$ is a square, $AC = 12$ cm, $AP = 10$ cm.



Solution:
We first find the vertical height:

$$
\begin{aligned}
H &= \sqrt{(10)^2 - (6)^2} \\
&= 8
\end{aligned}
$$

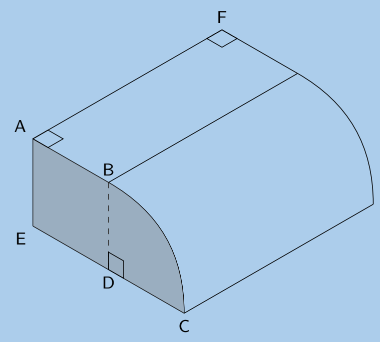
We also need to find the length of the side of the square. To do this we note that triangle ABC is a right-angled isosceles triangle. So we can find the length of the side of the square using the theorem of Pythagoras:

$$
\begin{aligned}
AC^2 &= AB^2 + BC^2 \\
12^2 &= 2(AB^2) \\
77 &= AB^2 \\
\therefore AB &= \sqrt{77}
\end{aligned}
$$

Now we can find the volume:

$$
\begin{aligned}
V &= \frac{1}{3}\pi b^2 H \\
&= \frac{1}{3}\pi(77)(8) \\
&= \frac{616}{3}\pi \\
&\approx 645{,}07 \text{ cm}^3
\end{aligned}
$$

27. The prism below has the following dimensions:
$AB = 4$ units, $EC = 8$ units, $AF = 10$ units. BC is an arc of a circle with centre D. $AB \parallel EC$.



a) Explain why BD, the radius of the arc BC, is 4 units.
Solution:
Since D is the centre of the circle $BD = DC$ (they are both radii of the arc).
$AB \parallel EC$ and BD joins AB and EC, therefore $AB = ED = 4$ units.
We also know that $EC = 8$ units and since $EC = ED + DC$, $DC = 4$ units. Therefore BD is 4 units.

b) Calculate the area of the shaded surface.
Solution:
We have just calculated that $BD = 4$. We also know that $AB = ED = 4$ and so $ABDE$ is a square ($AB \parallel EC$).
This means that we have the area of a square plus one quarter the area of a circle.
The total area is:

$$
\begin{aligned}
A &= AB^2 + \frac{1}{4}\pi r^2 \\
&= (4)^2 + \frac{1}{4}\pi(4)^2 \\
&= 16 + 4\pi \\
&= 28{,}57 \text{ units}^2
\end{aligned}
$$

c) Find the volume of the prism.

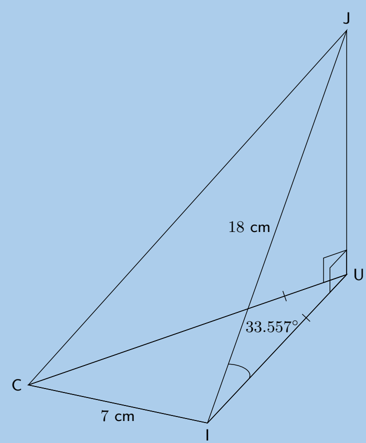
Solution:

The area of the shaded piece is the area of the base. For the volume we know that we can calculate the volume by multiplying the area of the base and the height.

$$\begin{aligned} V &= \text{area of base} \times \text{height} \\ &= (16 + 4\pi) \times (10) \\ &= 285{,}664 \text{ units}^3 \end{aligned}$$

You can also calculate the volume using the volume of a rectangular prism and one quarter of the volume of a cylinder.

28. A cooldrink container is made in the shape of a pyramid with an isosceles triangular base. This is known as a tetrahedron. The angle of elevation of the top of the container is 33,557°. $CI = 7$ cm; $JI = 18$ cm.



a) i. Show that the length UI is 15 cm.

ii. Find the height JU (to the nearest unit).

iii. Calculate the area of $\triangle CUI$.
Hint: construct a perpendicular line from U to CI

iv. Find the volume of the container

Solution:

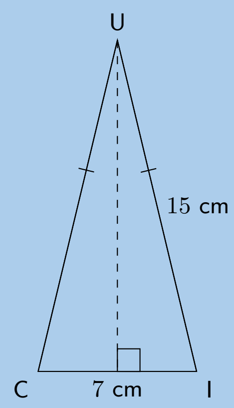
i. $\triangle UIJ$ is a right-angled triangle. We can use trigonometry to help us find UI. In this case we will use the cosine ratio as we have the hypotenuse (JI) and are looking for the adjacent side (UI).

$$\begin{aligned} \cos 33{,}577^\circ &= \frac{UI}{JI} \\ UI &= 18 \cos 33{,}577^\circ \\ &= 15 \text{ cm} \end{aligned}$$

ii. $\triangle UIJ$ is a right-angled triangle. We can use trigonometry to help us find JU. In this case we will use the sine ratio as we have the hypotenuse (JI) and are looking for the opposite side (JU).

$$\begin{aligned} \sin 33{,}577^\circ &= \frac{JU}{JI} \\ JU &= 18 \sin 33{,}577^\circ \\ &= 10 \text{ cm} \end{aligned}$$

iii.



We first find h:

$$\begin{aligned} h &= \sqrt{15^2 - 3{,}5^2} \\ &= 14{,}586 \end{aligned}$$

Now we can find the area:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(3{,}5)(14{,}586) \\ &= 25{,}526 \text{ cm}^2 \end{aligned}$$

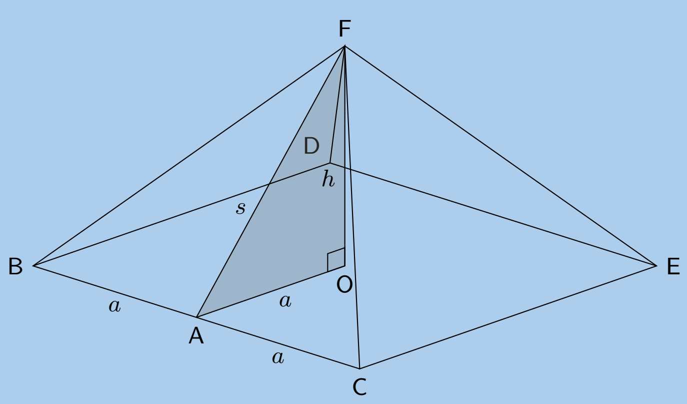
iv.

$$\begin{aligned} V &= \frac{1}{3} \times \frac{1}{2}(bh) \times JU \\ &= \frac{1}{3} \times (25{,}526)(9{,}950) \\ V &= 84{,}661 \text{ cm}^3 \end{aligned}$$

b) The container is filled with the juice such that an 11,85% gap of air is left. Determine the volume of the juice.
Solution:
To find the volume of the juice we need to multiply the total volume of the container by the percentage of juice in the container.

$$\begin{aligned} V_j &= V \times (1 - 0{,}1185) \\ &= 0{,}8815(84{,}661) \\ V_j &= 74{,}626 \text{ cm}^3 \end{aligned}$$

29. Below is a diagram of The Great Pyramid.
This is a square-based pyramid and O is the centre of the square.



$BA = AC = a$ and $OF = h =$ height of the pyramid. The length of the side of the pyramid $BC = 755{,}79$ feet and the height of the pyramid is 481,4 feet.

a) Determine the area of the base of the pyramid in terms of a.
Solution:

$$\begin{aligned} A &= b^2 \\ &= (2a)^2 \\ &= 4a^2 \end{aligned}$$

b) Calculate $AF(= s)$ to 5 decimal places.
Solution:

$$BC = 2a$$

$$\begin{aligned} AF &= \sqrt{a^2 + h^2} \\ &= \sqrt{(0{,}5BC)^2 + (OF)^2} \\ &= \sqrt{(377{,}895)^2 + (481{,}4)^2} \\ &= 612{,}00538 \text{ feet} \end{aligned}$$

c) From your calculation in question (b) determine $\frac{s}{a}$.
Solution:

$$\begin{aligned} \frac{s}{a} &= \frac{612{,}00538}{377{,}895} \\ &= 1{,}620 \end{aligned}$$

d) Determine the volume and surface area of the pyramid.
Solution:

$$\begin{aligned} V &= \frac{1}{3}\pi b^2 H \\ &= \frac{1}{3}\pi (2a)^2 h \\ &= \frac{1}{3}\pi (755{,}79)^2 (481{,}4) \\ &\approx 91\ 661\ 532{,}5 \text{ feet}^3 \end{aligned}$$

$$\begin{aligned} A &= b(b + 2h_s) \\ &= 2a(2a + 2(s)) \\ &= 755{,}79(755{,}79 + 1224{,}0176) \\ &\approx 1\ 551\ 425{,}432 \text{ feet}^2 \end{aligned}$$

The volume and surface area are: 91 661 532,5 feet^3 and 1 551 425,432 feet^2 respectively.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GTJ	2. 2GTK	3. 2GTM	4. 2GTN	5. 2GTP	6. 2GTQ
7. 2GTR	8. 2GTS	9. 2GTT	10. 2GTV	11. 2GTW	12. 2GTX
13. 2GTY	14. 2GTZ	15a. 2GV2	15b. 2GV3	16. 2GV4	17. 2GV5
18. 2GV6	19a. 2GV7	19b. 2GV8	19c. 2GV9	19d. 2GVB	20. 2GVC
21. 2GVD	22a. 2GVF	22b. 2GVG	22c. 2GVH	23. 2GVJ	24. 2GVK
25a. 2GVM	25b. 2GVN	25c. 2GVP	26a. 2GVQ	26b. 2GVR	27. 2GVS
28. 2GVT	29. 2GVV				

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CHAPTER 14

Probability

- This chapter covers the use of probability models to compare relative frequency to theoretical probability. Venn diagrams are introduced and used to answer probability problems. Intersection, union, mutually exclusive events and complementary events are all introduced.
- The difference between theoretical probability and relative frequency should be carefully explained.
- The terminology and usage of language in this section can be confusing, especially to second-language speakers. Discuss terminology regularly and emphasise the careful reading of questions.
- Union and intersection symbols have been included, but "and" and "or" is the preferred notation in CAPS.
- Make sure to outline the differences between "and", "or", "only" and "both". For example, there may be no difference between tea **and** coffee drinkers and tea **or** coffee drinkers in common speech but in probability, the "and" and "or" have very specific meanings. Tea **and** coffee drinkers refers to the intersection of tea drinkers with coffee drinkers, i.e. those who drink both beverages, while tea **or** coffee drinkers refers to the union, i.e. those who drink only tea, those who drink only coffee and those who drink both.

14.1 Theoretical probability

Exercise 14 – 1:

1. A learner wants to understand the term "event". So the learner rolls 2 dice hoping to get a total of 8. Which of the following is the most appropriate example of the term "event"?
   - event set $= \{(4;4)\}$
   - event set $= \{(2;6);(3;5);(4;4);(5;3);(6;2)\}$
   - event set $= \{(2;6);(6;2)\}$

   Solution:
   We recall the definition of the term "event":
   An event is a specific set of outcomes of an experiment that you are interested in.
   Therefore the most appropriate example of the term "event" is event set $= \{(2;6);(3;5);(4;4);(5;3);(6;2)\}$.
2. A learner wants to understand the term "sample space". So the learner rolls a die. Which of the following is the most appropriate example of the term "sample space"?
   - $\{1;2;3;4;5;6\}$
   - $\{H;T\}$
   - $\{1;3;5\}$

   Solution:
   We recall the definition of the term "sample space":
   The sample space of an experiment is the set of all possible outcomes of all experiment.
   Therefore the most appropriate example of the term "sample space" is $\{1;2;3;4;5;6\}$
3. A learner finds a 6 sided die and then rolls the die once on a table. What is the probability that the die lands on either 1 or 2?
   Write your answer as a simplified fraction.
   Solution:

$$\begin{aligned} n(E) &= \text{number of outcomes in the event set} = 2 \\ n(S) &= \text{number of possible outcomes in the sample space} = 6 \end{aligned}$$

   Finally, we calculate the probability:

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

   Therefore, the probability that the die lands on either 1 or 2 $= \frac{1}{3}$.

4. A learner finds a textbook that has 100 pages. He then selects one page from the textbook. What is the probability that the page has an odd page number?
   Write your answer as a decimal (correct to 2 decimal places).
   Solution:

$$\begin{aligned} n(E) &= \text{number of outcomes in the event set} = 50 \\ n(S) &= \text{number of possible outcomes in the sample space} = 100 \end{aligned}$$

   Finally, we calculate the probability:

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{50}{100} \\ &= 0{,}50 \end{aligned}$$

   Therefore, the probability that the page has an odd page number = 0,50.
5. Even numbers from 2 to 100 are written on cards. What is the probability of selecting a multiple of 5, if a card is drawn at random?
   Solution:
   There are 50 cards. They are all even.
   All even numbers that are also multiples of 5 are multiples of 10: $(10,\ 20, \ldots, 100)$.
   There are 10 of them.
   Therefore, the probability of selecting a card that is a multiple of 5 is $\frac{10}{50} = \frac{1}{5}$.
6. A bag contains 6 red balls, 3 blue balls, 2 green balls and 1 white ball. A ball is picked at random. Determine the probability that it is:

   a) red
      Solution:

$$\begin{aligned} n(E) &= 6 \\ n(S) &= 12 \end{aligned}$$

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{6}{12} \\ &= \frac{1}{2} \end{aligned}$$

   b) blue or white
      Solution:

$$\begin{aligned} n(E) &= 3 + 1 = 4 \\ n(S) &= 12 \end{aligned}$$

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{4}{12} \\ &= \frac{1}{3} \end{aligned}$$

   c) not green
      Solution:

$$
\begin{aligned}
n(E) &= 6+3+1 = 10 \\
n(S) &= 12
\end{aligned}
$$

$$
\begin{aligned}
P(E) &= \frac{n(E)}{n(S)} \\
&= \frac{10}{12} \\
&= \frac{5}{6}
\end{aligned}
$$

d) not green or red
Solution:

$$
\begin{aligned}
n(E) &= 3+1 = 4 \\
n(S) &= 12
\end{aligned}
$$

$$
\begin{aligned}
P(E) &= \frac{n(E)}{n(S)} \\
&= \frac{4}{12} \\
&= \frac{1}{3}
\end{aligned}
$$

7. A playing card is selected randomly from a pack of 52 cards. Determine the probability that it is:

a) the 2 of hearts
Solution:

$$
\begin{aligned}
n(E) &= 1 \\
n(S) &= 52
\end{aligned}
$$

$$
\begin{aligned}
P(E) &= \frac{n(E)}{n(S)} \\
&= \frac{1}{52}
\end{aligned}
$$

b) a red card
Solution:
Half the deck is red and half the deck is black.

$$
\begin{aligned}
n(E) &= 26 \\
n(S) &= 52
\end{aligned}
$$

$$
\begin{aligned}
P(E) &= \frac{n(E)}{n(S)} \\
&= \frac{26}{52} \\
&= \frac{1}{2}
\end{aligned}
$$

c) a picture card
Solution:
There are 3 picture cards in a suit and 4 suits.

$$n(E) = 12$$
$$n(S) = 52$$

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{12}{52} \\ &= \frac{3}{13} \end{aligned}$$

d) an ace

Solution:

There are 4 aces in a pack.

$$n(E) = 4$$
$$n(S) = 52$$

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{4}{52} \\ &= \frac{1}{13} \end{aligned}$$

e) a number less than 4

Solution:

For each suit of 13 cards there are 3 cards less than 4: A, 2 and 3.

$$n(E) = 12$$
$$n(S) = 52$$

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{12}{52} \\ &= \frac{3}{13} \end{aligned}$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GVY 2. 2GVZ 3. 2GW2 4. 2GW3 5. 2GW4 6. 2GW5
7. 2GW6

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14.2 Relative frequency

Exercise 14 – 2:

1. A die is tossed 44 times and lands 5 times on the number 3.
   What is the relative frequency of observing the die land on the number 3? Write your answer correct to 2 decimal places.

Solution:

Recall the formula:

$$f = \frac{p}{t}$$

Identify variables needed:

$$\begin{aligned} p &= \text{number of positive trials} = 5 \\ f &= \text{total number of trials} = 44 \end{aligned}$$

Calculate the relative frequency:

$$\begin{aligned} f &= \frac{p}{t} \\ &= \frac{5}{44} \\ &= 0{,}11 \end{aligned}$$

Therefore, the relative frequency of observing the die on the number 3 is 0,11.

2. A coin is tossed 30 times and lands 17 times on heads.
   What is the relative frequency of observing the coin land on heads? Write your answer correct to 2 decimal places.
   Solution:
   Recall the formula:

$$f = \frac{p}{t}$$

Identify variables needed:

$$\begin{aligned} p &= \text{number of positive trials} = 17 \\ f &= \text{total number of trials} = 30 \end{aligned}$$

Calculate the relative frequency:

$$\begin{aligned} f &= \frac{p}{t} \\ &= \frac{17}{30} \\ &= 0{,}57 \end{aligned}$$

Therefore, the relative frequency of observing the coin on heads is 0,57.

3. A die is tossed 27 times and lands 6 times on the number 6.
   What is the relative frequency of observing the die land on the number 6? Write your answer correct to 2 decimal places.
   Solution:
   Recall the formula:

$$f = \frac{p}{t}$$

Identify variables needed:

$$\begin{aligned} p &= \text{number of positive trials} = 6 \\ f &= \text{total number of trials} = 27 \end{aligned}$$

Calculate the relative frequency:

$$\begin{aligned} f &= \frac{p}{t} \\ &= \frac{6}{27} \\ &= 0{,}22 \end{aligned}$$

Therefore, the relative frequency of observing the die on the number 6 is 0,22.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'. 1. 2GW8 2. 2GW9 3. 2GWB

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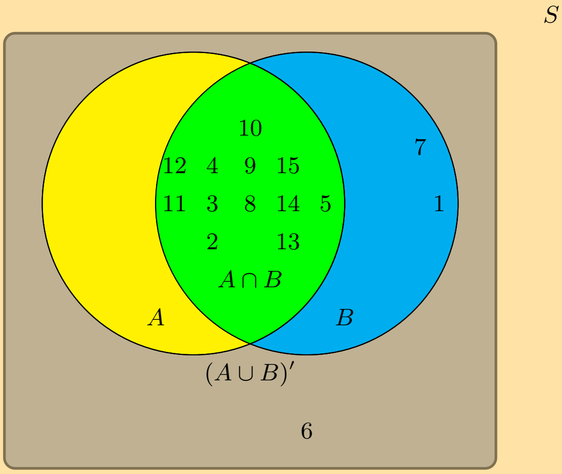
14.3 Venn diagrams

You can use an online tool such as this one to generate Venn diagrams.

Exercise 14 – 3:

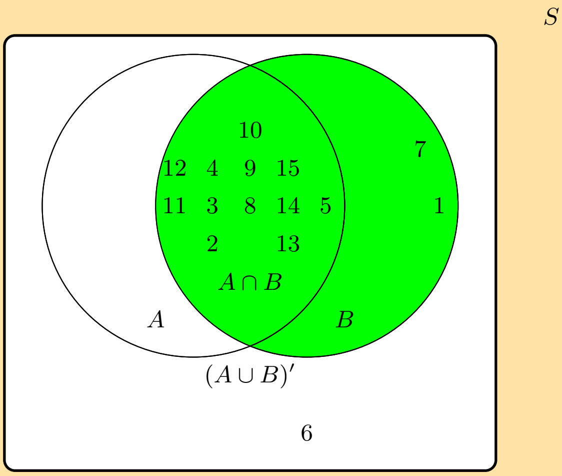
1. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \epsilon \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of B. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of B?

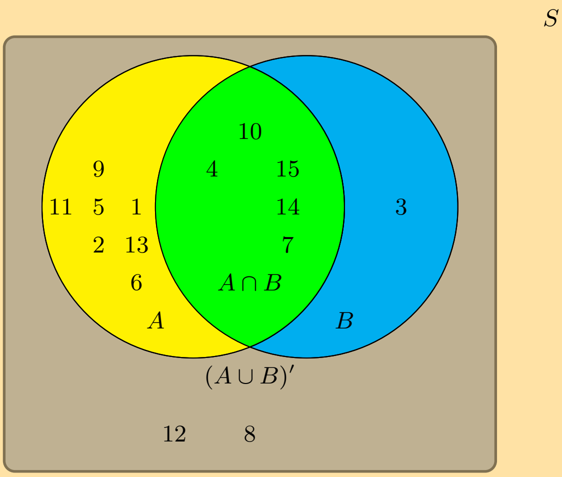
- $\{2; 3; 4; 5; 8; 9; 10; 11; 12; 13; 14; 15\}$
- $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$
- $\{1; 6; 7\}$
- $\{6\}$

Solution:
The event set B can be shaded as follows:



Therefore the event set $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$ best describes the event set of B.

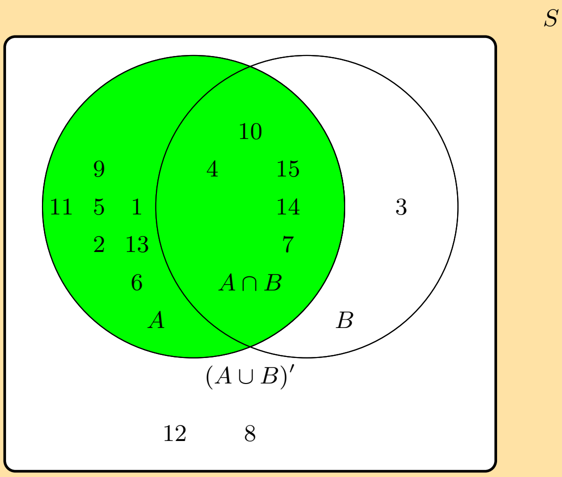
2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of A. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of A?

- $\{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 14; 15\}$
- $\{3; 8; 12\}$
- $\{3; 4; 7; 10; 14; 15\}$
- $\{1; 2; 4; 5; 6; 7; 9; 10; 11; 13; 14; 15\}$
- $\{4; 7; 10; 14; 15\}$

Solution:
The event set A can be shaded as follows:



Therefore the event set $\{1; 2; 4; 5; 6; 7; 9; 10; 11; 13; 14; 15\}$ best describes the event set of A.

3. Pieces of paper labelled with the numbers 1 to 12 are placed in a box and the box is shaken. One piece of paper is taken out and then replaced.

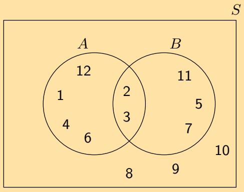
   a) What is the sample space, S?
   Solution:
   $S = \{n : n \in \mathbb{Z},\ 1 \leq n \leq 12\}$ or $S = \{1; 2; \ldots; 12\}$.

   b) Write down the set A, representing the event of taking a piece of paper labelled with a factor of 12.
   Solution:
   $A = \{1; 2; 3; 4; 6; 12\}$.

   c) Write down the set B, representing the event of taking a piece of paper labelled with a prime number.
   Solution:
   $B = \{2; 3; 5; 7; 11\}$.

d) Represent A, B and S by means of a Venn diagram.

Solution:



e) Find:

i. $n(S)$

ii. $n(A)$

iii. $n(B)$

Solution:

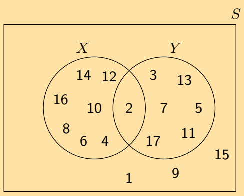
i. 12

ii. 6

iii. 5

4. Let S denote the set of whole numbers from 1 to 16, X denote the set of even numbers from 1 to 16 and Y denote the set of prime numbers from 1 to 16. Draw a Venn diagram depicting S, X and Y.

Solution:



5. There are 71 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41, those who take History is 36, and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.

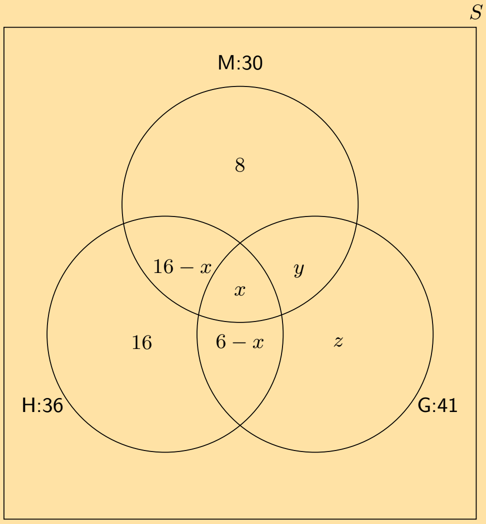
a) Draw a Venn diagram to illustrate all this information.

Solution:

We are told that 16 learners take Maths and History. Out of these 16 learners some take Geography as well and some do not.

We are also told that 6 learners take Geography and History. Out of these 6 learners some take Maths as well and some do not.

Let the number of learners who take Maths, History and Geography $= x$. Then we can draw the Venn diagram as follows:



b) How many learners take Maths and Geography but not History?

Solution:

In the above Venn diagram the number of learners who take Maths and Geography but not History is indicated by y. To find y we first need to determine x.

To find x we note that the total number of learners who take History is equal to the sum of each of the following:

- The number of learners who take History only: 16
- The number of learners who take History and Maths but not Geography: $16 - x$
- The number of learners who take History and Geography but not Maths: $6 - x$
- The number of learners who take all three subjects: x

$$\begin{aligned}
36 &= 16 + (16 - x) + (6 - x) + x \\
&= 16 + 16 - x + 6 - x + x \\
&= 38 - x \\
\therefore x &= 2
\end{aligned}$$

Now we can find y using the same method as to find x. This time we will use the total number of learners who take Maths.

$$\begin{aligned}
30 &= 8 + (16 - x) + x + y \\
&= 8 + 14 + 2 + y \\
&= 24 + y \\
\therefore y &= 6
\end{aligned}$$

Therefore 6 learners take Maths and Geography but not History.

c) How many learners take Geography only?

Solution:

Now we need to find z. We will use the total number of learners who take Geography to find z.

$$\begin{aligned}
41 &= (6 - x) + x + y + z \\
&= 4 + 2 + 6 + z \\
&= 12 + z \\
\therefore z &= 29
\end{aligned}$$

Therefore 29 learners take Geography only.

d) How many learners take all three subjects?

Solution:

When we drew the Venn diagram we let x be the number of learners that take all three subjects. We calculated x in the first question. Therefore 2 learners take all three subjects.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GWD 2. 2GWF 3. 2GWG 4. 2GWH 5. 2GWJ

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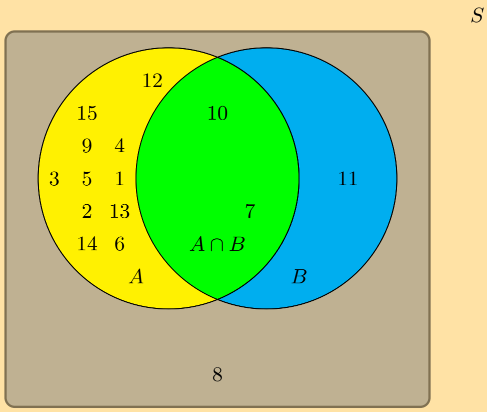
14.4 Union and intersection

Exercise 14 – 4:

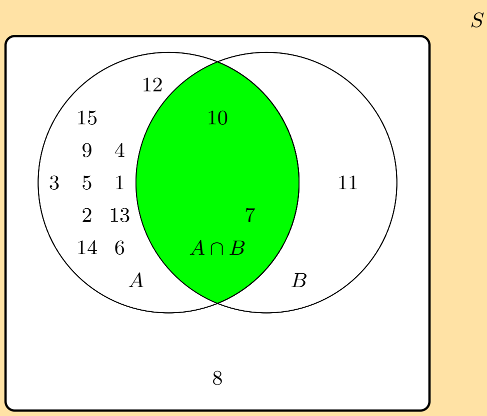
1. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of the intersection between event set A and event set B, also written as $A \cap B$. They get stuck, and you offer to help them find it.
Which set best describes the event set of $A \cap B$?

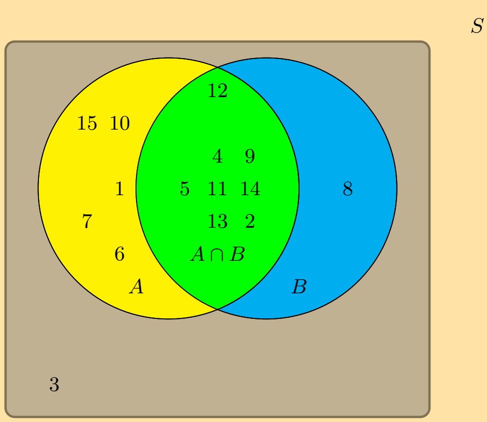
- $\{7; 10; 11\}$
- $\{1; 2; 3; 4; 5; 6; 7; 9; 10; 11\}$
- $\{1; 2; 3; 4; 5; 6; 7; 9; 10\}$
- $\{7; 10\}$

Solution:
The intersection between event set A and event set B, also written as $A \cap B$, can be shaded as follows:



Therefore the event set $\{7; 10\}$ best describes the event set of $A \cap B$.

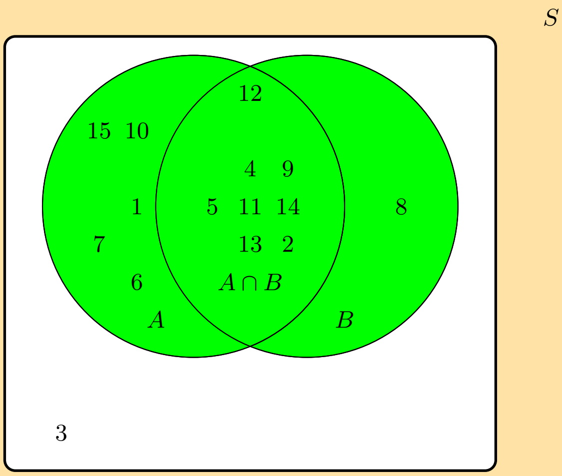
2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$
They are asked to identify the event set of the union between event set A and event set B, also written as $A \cup B$. They get stuck, and you offer to help them find it.
Which set best describes the event set of $A \cup B$?

- $\{1; 6; 7; 10; 15\}$
- $\{1; 2; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$
- $\{2; 4; 5; 9; 10; 11; 12; 13; 14\}$
- $\{3\}$

Solution:
The union between event set A and event set B, also written as $A \cup B$, can be shaded as follows:



Therefore the event set $\{1; 2; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$ best describes the event set of $A \cup B$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'. 1. 2GWK 2. 2GWM

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14.5 Probability identities

Exercise 14 – 5:

1. A group of learners is given the following event sets:

Event Set A	1	2	5	6

Event Set B	3

Event Set $A \cap B$	empty

The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 6\}$.
They are asked to calculate the value of $P(A \cup B)$. They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal places.
Solution:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Identify variables needed:

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} = \frac{4}{6} = 0{,}67 \\ P(B) &= \frac{n(B)}{n(S)} = \frac{1}{6} = 0{,}17 \\ P(A \cap B) &= \frac{n(A \cap B)}{n(S)} = \frac{0}{6} = 0 \end{aligned}$$

Calculate $P(A \cup B)$:

$$\begin{aligned} P(A \cup B) &= P(B) + P(A) - P(A \cap B) \\ &= (0{,}17) + (0{,}67) - (0) \\ &= 0{,}83 \end{aligned}$$

Therefore, the value of $P(A \cup B)$ is 0,83.

2. A group of learners is given the following event sets:

Event Set A	1	2	6

Event Set B	1	5

Event Set $A \cup B$	1	2	5	6

The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 6\}$.
They are asked to calculate the value of $P(A \cap B)$. They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal value.
Solution:

$$P(A \cup B) = P(A) + P(B) - P(A \cup B)$$

Make $P(A \cap B)$ the subject, and we get:

$$P(A \cap B) = P(B) + P(A) - P(A \cup B)$$

Identify variables needed:

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} = \frac{3}{6} = 0{,}5 \\ P(B) &= \frac{n(B)}{n(S)} = \frac{2}{6} = 0{,}33 \\ P(A \cup B) &= \frac{n(A \cup B)}{n(S)} = \frac{4}{6} = 0{,}67 \end{aligned}$$

Calculate $P(A \cap B)$:

$$\begin{aligned} P(A \cap B) &= P(B) + P(A) - P(A \cup B) \\ &= (0{,}33) + (0{,}5) - (0{,}67) \\ &= 0{,}17 \end{aligned}$$

Therefore, the value of $P(A \cap B)$ is 0,17.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'. 1. 2GWP 2. 2GWQ

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14.6 Mutually exclusive events

Exercise 14 – 6:

State whether the following events are mutually exclusive or not.

1. A fridge contains orange juice, apple juice and grape juice. A cooldrink is chosen at random from the fridge. Event A: the cooldrink is orange juice. Event B: the cooldrink is apple juice.
   Solution:
   We are choosing just one cooldrink from the fridge. This cooldrink cannot be both an orange juice and an apple juice. Therefore these two events are mutually exclusive.
2. A packet of cupcakes contains chocolate cupcakes, vanilla cupcakes and red velvet cupcakes. A cupcake is chosen at random from the packet. Event A: the cupcake is red velvet. Event B: the cupcake is vanilla.
   Solution:
   We are choosing just one cupcake from the packet. This cupcake cannot be both a red velvet cupcake and a vanilla one. Therefore these two events are mutually exclusive.
3. A card is chosen at random from a deck of cards. Event A: the card is a red card. Event B: the card is a picture card.
   Solution:
   We are choosing just one card from the deck. This card can be both a red card and a picture card. Therefore these two events are not mutually exclusive.
4. A cricket team plays a game. Event A: they win the game. Event B: they lose the game.
   Solution:
   The cricket team can either win the game or lose the game. They cannot simultaneously win and lose the game. Therefore these two events are mutually exclusive.
   Note that a tie game does not count as either a win or a loss. In a tie neither team can be said to have won the match.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GWR 2. 2GWS 3. 2GWT 4. 2GWV

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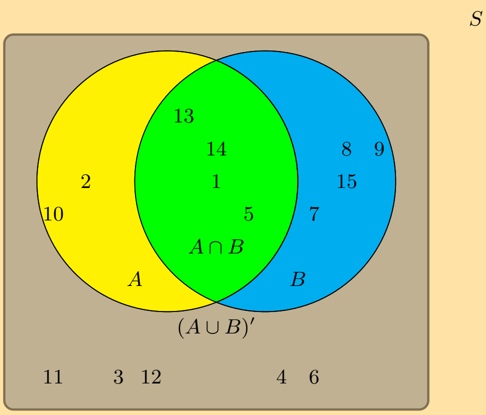
14.7 Complementary events

Exercise 14 – 7:

1. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.

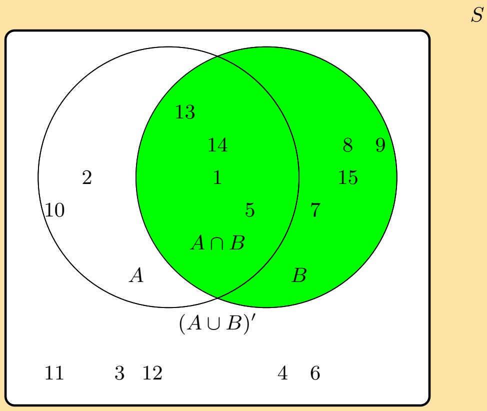
They are asked to identify the complementary event set of B, also known as B'. They get stuck, and you offer to help them find it.

Which of the following sets best describes the event set of B'?

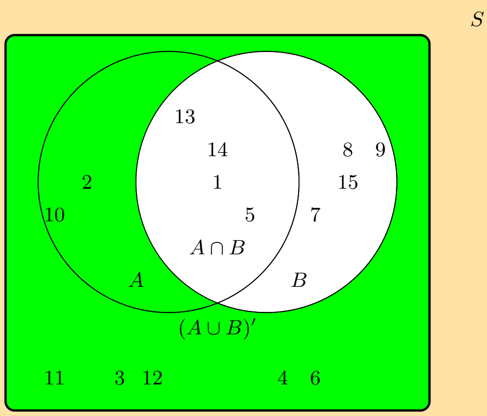
- $\{1; 5; 13; 14\}$
- $\{2; 3; 4; 6; 10; 11; 12\}$
- $\{3; 4; 6; 11; 12\}$

Solution:

The event set B can be shaded as follows:

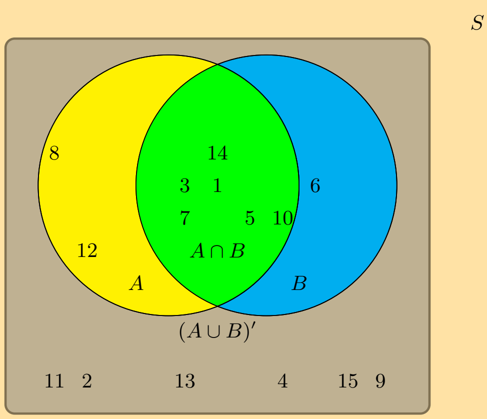


The complementary event set B' can be shaded as follows:



Therefore the event set $\{2; 3; 4; 6; 10; 11; 12\}$ best describes the complementary event set of B, also known as B'.

2. A group of learners are given the following Venn diagram:

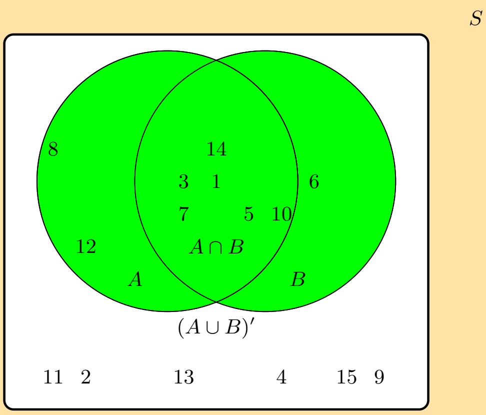


The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.

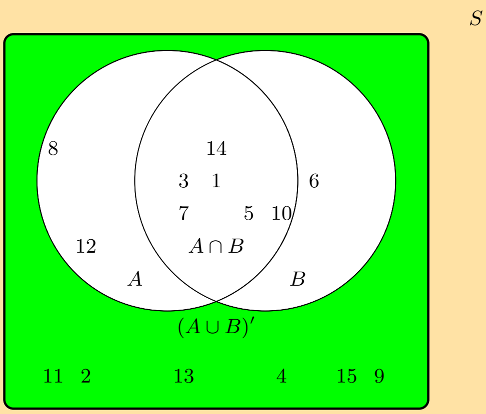
They are asked to identify the complementary event set of $(A \cup B)$, also known as $(A \cup B)'$. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of $(A \cup B)'$?

- $\{2; 4; 9; 11; 13; 15\}$
- $\{1; 3; 5; 6; 7; 8; 10; 12; 14\}$
- $\{6; 8; 12\}$

Solution:
The event set $(A \cup B)$ can be shaded as follows:

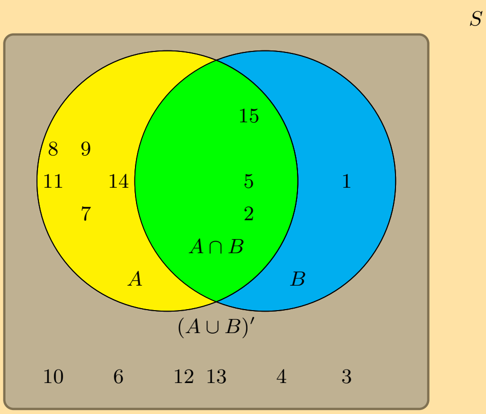


The complementary event set $(A \cup B)'$ can be shaded as follows:



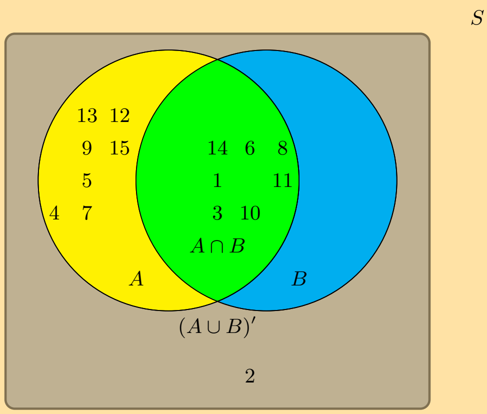
Therefore the event set $\{2; 4; 9; 11; 13; 15\}$ best describes the complementary event set of $(A \cup B)$, also known as $(A \cup B)'$.

3. Given the following Venn diagram:



The sample space can be described as $\{n : n \epsilon \mathbb{Z},\ 1 \leq n \leq 15\}$.
Are $(A \cup B)'$ and $A \cup B$ mutually exclusive?
Solution:
We recall the definition of the term "mutually exclusive":
Two events are called mutually exclusive if they cannot occur at the same time.
The event set for $(A \cup B)'$ is: $\{3; 4; 10; 12; 13\}$
The event set for $A \cup B$ is: $\{1; 2; 5; 6; 7; 8; 9; 11; 14; 15\}$
The question we must ask: Can they occur at the same time?
By observing both sets, we can identify the following overlapping event set: $\{\}$ or $\varnothing$.
Therefore, yes, the event sets $(A \cup B)'$ and $A \cup B$ are mutually exclusive in this example.

4. Given the following Venn diagram:



The sample space can be described as $\{n : n \epsilon \mathbb{Z},\ 1 \leq n \leq 15\}$.
Are A' and B' mutually exclusive?
Solution:
We recall the definition of the term "mutually exclusive":
Two events are called mutually exclusive if they cannot occur at the same time.
The event set for A' is: $\{2\}$
The event set for B' is: $\{2; 4; 5; 7; 9; 12; 13; 15\}$
The question we must ask: Can they occur at the same time?
By observing both sets, we can identify the following overlapping event set: $\{2\}$
Therefore, no, the event sets A' and B' are not mutually exclusive in this example.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GWW 2. 2GWX 3. 2GWY 4. 2GWZ

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14.8 Chapter summary

This video summarises the concepts covered in this chapter. Note that some of the examples used in this video may not be suited to all learners.

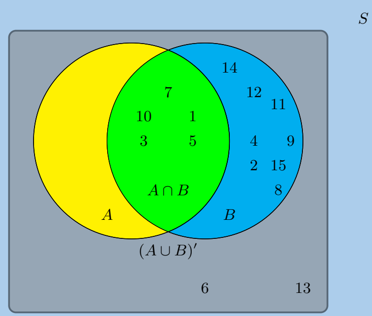
End of chapter Exercise 14 – 8:

1. A learner wants to understand the term "outcome". So the learner rolls a die. Which of the following is the most appropriate example of the term "outcome"?
   - A teacher walks into the class room.
   - The die lands on the number 5.

- The clock strikes 3 pm.

Solution:
We recall the definition of the term "outcome":
An outcome of an experiment is a single result of that experiment
Therefore the most appropriate example of the term "outcome" is: the die lands on the number 5.

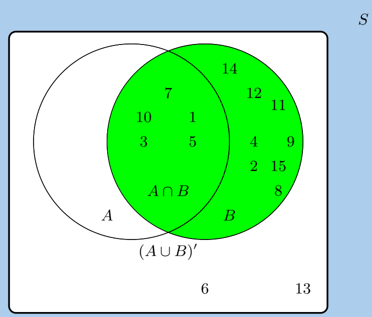
2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of B. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of B?

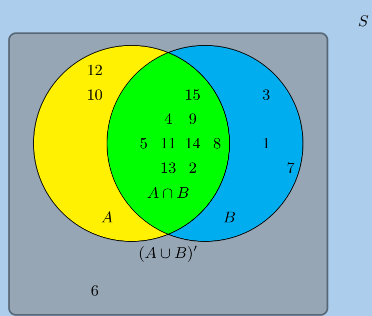
- $\{6; 13\}$
- $\{1; 3; 5; 7; 10\}$
- $\{2; 4; 6; 8; 9; 11; 12; 13; 14; 15\}$
- $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$

Solution:
The event set B can be shaded as follows:



Therefore the event set $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$ best describes the event set of B.

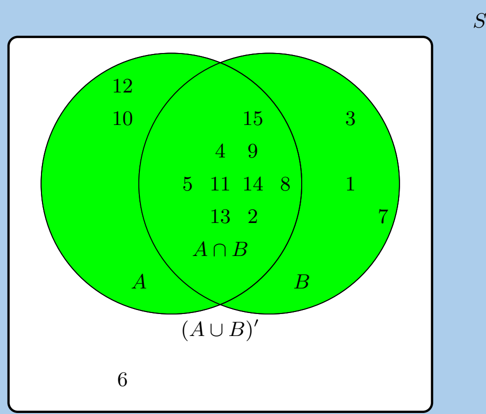
3. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of the union between event set A and event set B, also written as $A \cup B$. They get stuck, and you offer to help them find it.
Write down the event set that best describes $A \cup B$.

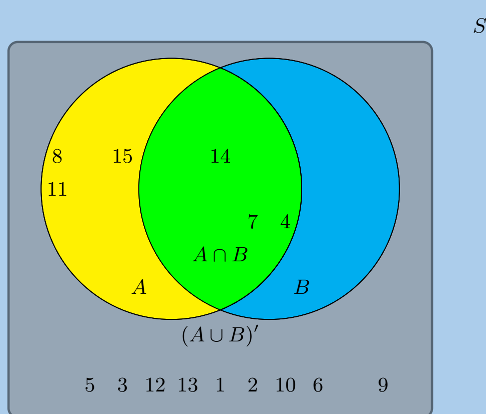
Solution:

The union between event set A and event set B, also written as $A \cup B$, can be shaded as follows:



Therefore the event set $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$ best describes the event set of $A \cup B$.

4. Given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.

Are $A \cup B$ and $(A \cup B)'$ mutually exclusive?

Solution:

We recall the definition of the term "mutually exclusive":

Two events are called mutually exclusive if they cannot occur at the same time.

The event set for $A \cup B$ is: $\{4; 7; 8; 11; 14; 15\}$.

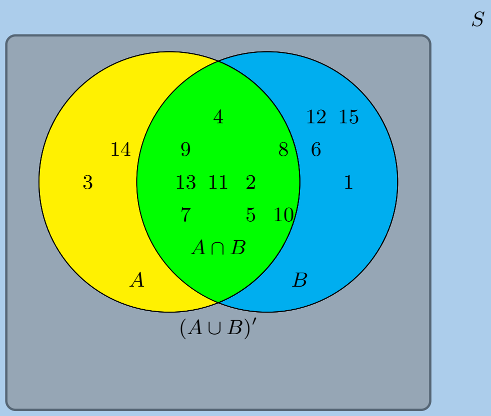
The event set for $(A \cup B)'$ is: $\{1; 2; 3; 5; 6; 9; 10; 12; 13\}$.

The question we must ask: Can they occur at the same time?

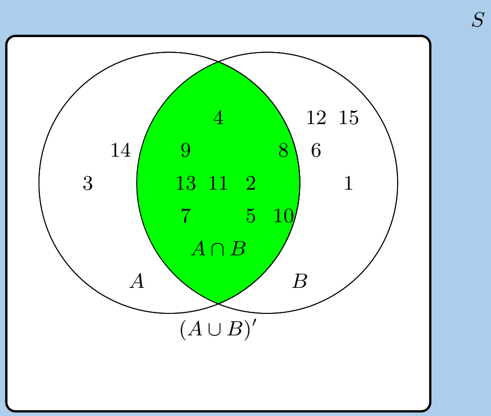
By observing both sets, we can identify the following overlapping event set: $\{\}$

Therefore, yes, the event sets $A \cup B$ and $(A \cup B)'$ are mutually exclusive in this example.

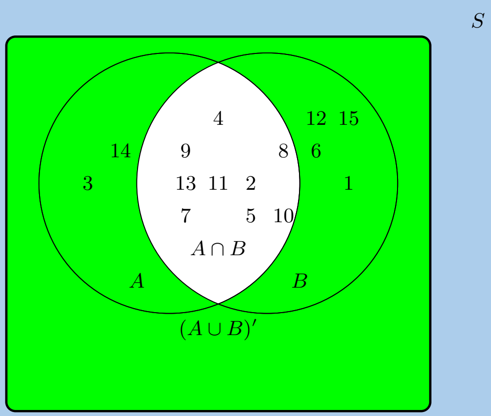
5. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \epsilon \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the complementary event set of $(A \cap B)$, also known as $(A \cap B)'$. They get stuck, and you offer to help them find it.
Write down the set that best describes the event set of $(A \cap B)'$.
Solution:
The event set $(A \cap B)$ can be shaded as follows:



The complementary event set $(A \cap B)'$ can be shaded as follows:



Therefore the event set $\{1; 3; 6; 12; 14; 15\}$ best describes the complementary event set of $(A \cap B)$, also known as $(A \cap B)'$.

6. A learner finds a deck of 52 cards and then takes one card from the deck. What is the probability that the card is a king?
   Write your answer as a decimal (correct to 2 decimal places).
   Solution:

$$n(E) = \text{number of outcomes in the event set} = 4$$
$$n(S) = \text{number of possible outcomes in the sample space} = 52$$

Finally, we calculate the probability:

$$\begin{aligned} P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{4}{52} \\ &\approx 0{,}08 \end{aligned}$$

Therefore, the probability that the card is a King $\approx$ 0,08.

7. A die is tossed 21 times and lands 2 times on the number 3.
   What is the relative frequency of observing the die land on the number 3? Write your answer correct to 2 decimal places.
   Solution:
   Recall the formula:
   $$f = \frac{p}{t}$$
   Identify variables needed:
   $$p = \text{number of positive trials} = 2$$
   $$f = \text{total number of trials} = 21$$
   Calculate the relative frequency
   $$\begin{aligned} f &= \frac{p}{t} \\ &= \frac{2}{21} \\ &= 0{,}10 \end{aligned}$$
   Therefore, the relative frequency of observing the die on the number 3 is 0,1.
8. A coin is tossed 44 times and lands 22 times on heads.
   What is the relative frequency of observing the coin land on heads? Write your answer correct to 2 decimal places.
   Solution:
   Recall the formula:
   $$f = \frac{p}{t}$$
   Identify variables needed:
   $$p = \text{number of positive trials} = 22$$
   $$f = \text{total number of trials} = 44$$
   Calculate the relative frequency:
   $$\begin{aligned} f &= \frac{p}{t} \\ &= \frac{22}{44} \\ &= 0{,}50 \end{aligned}$$
   Therefore, the relative frequency of observing the coin on heads is 0,50.
9. A group of 45 children were asked if they eat Frosties, Strawberry Pops or both. 31 children said they eat both and 6 said they only eat Frosties. What is the probability that a child chosen at random will eat only Strawberry Pops?
   Solution:
   $$45(\text{all}) - 6(\text{only Frosties}) - 31(\text{both}) = 8(\text{only Strawberry Pops})$$
   $$\therefore \frac{8}{45} = 0{,}18$$
10. In a group of 42 learners, all but 3 had a packet of chips or a cooldrink or both. If 23 had a packet of chips and 7 of these also had a cooldrink, what is the probability that one learner chosen at random has:

a) both chips and cooldrink

Solution:

$\frac{7}{42} = \frac{1}{6}$

b) only cooldrink

Solution:

Since $42 - 3 = 39$ learners had at least one, and 23 learners had a packet of chips, then $39 - 23 = 16$ learners only had a cooldrink.

$\frac{16}{42} = \frac{8}{21}$

11. A box contains coloured blocks. The number of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

A block is selected randomly. What is the probability that the block will be:

a) purple

Solution:

Before we answer the questions we first work out how many blocks there are in total. This gives us the sample space: $n(S) = 24 + 32 + 41 + 19 = 116$.

The probability that a block is purple is:

$$\begin{aligned} P(\text{purple}) &= \frac{n(E)}{n(S)} \\ &= \frac{24}{116} \\ &= 0{,}21 \end{aligned}$$

b) purple or white

Solution:

The probability that a block is either purple or white is:

$$\begin{aligned} P(\text{purple} \cup \text{white}) &= P(\text{purple}) + P(\text{white}) - P(\text{purple} \cap \text{white}) \\ &= \frac{24}{116} + \frac{41}{116} - 0 \\ &= 0{,}56 \end{aligned}$$

c) pink and orange

Solution:

Since one block cannot be two colours the probability of this event is 0.

d) not orange

Solution:

We first work out the probability that a block is orange:

$$\begin{aligned} P(\text{orange}) &= \frac{32}{116} \\ &= 0{,}28 \end{aligned}$$

The probability that a block is not orange is:

$$\begin{aligned} P(\text{not orange}) &= 1 - 0{,}28 \\ &= 0{,}72 \end{aligned}$$

12. A small nursery school has a class with children of various ages. The table gives the number of children of each age in the class.

Age	3 years old	4 years old	5 years old
Male	2	7	6
Female	6	5	4

If a child is selected at random what is the probability that the child will be:

a) a female

Solution:

We calculate the total number of pupils at the school: $6 + 2 + 5 + 7 + 4 + 6 = 30$.

The total number of female children is $6 + 5 + 4 = 15$.

The probability of a randomly selected child being female is:

$$\begin{aligned} P(\text{female}) &= \frac{n(E)}{n(S)} \\ P(\text{female}) &= \frac{15}{30} \\ P(\text{female}) &= 0{,}5 \end{aligned}$$

b) a 4 year old male

Solution:

The probability of a randomly selected child being a 4 year old male is:

$$\begin{aligned} P(\text{male}) &= \frac{7}{30} \\ &= 0{,}23 \end{aligned}$$

c) aged 3 or 4

Solution:

There are $6 + 2 + 5 + 7 = 20$ children aged 3 or 4. The probability of a randomly selected child being either 3 or 4 is: $\frac{20}{30} = 0{,}67$.

d) aged 3 and 4

Solution:

A child cannot be both 3 and 4, so the probability is 0.

e) not 5

Solution:

This is the same as a randomly selected child being either 3 or 4 and so is 0,67.

f) either 3 or female

Solution:

The probability of a child being either 3 or female is:

$$\begin{aligned} P(3 \cup \text{female}) &= P(3) + P(\text{female}) - P(3 \cap \text{female}) \\ &= \frac{8}{30} + \frac{15}{30} - \frac{6}{30} \\ &= 0{,}56 \end{aligned}$$

13. Fiona has 85 labelled discs, which are numbered from 1 to 85. If a disc is selected at random what is the probability that the disc number:

a) ends with 5

Solution:

The set of all discs ending with 5 is: $\{5; 15; 25; 35; 45; 55; 65; 75; 85\}$. This has 9 elements.

The probability of drawing a disc that ends with 5 is:

$$\begin{aligned} P(5) &= \frac{n(E)}{n(S)} \\ P(5) &= \frac{9}{85} \\ P(5) &= 0{,}11 \end{aligned}$$

b) is a multiple of 3

Solution:

The set of all discs that are multiples of 3 is:

$\{3; 6; 9; 12; 15; 18; 21; 24; 27; 30; 33; 36; 39; 42; 45; 48; 51; 54; 57; 60; 63; 66; 69; 72; 75; 78; 81; 84\}$.

This has 28 elements.

The probability of drawing a disc that is a multiple of 3 is:$P(3_m) = \frac{28}{85} = 0{,}33$.

c) is a multiple of 6

Solution:

The set of all discs that are multiples of 6 is: $\{6; 12; 18; 24; 30; 36; 42; 48; 54; 60; 66; 72; 78; 84\}$. This set has 14 elements.

The probability of drawing a disc that is a multiple of 6 is: $P(6_m) = \frac{14}{85} = 0{,}16$.

d) is number 65

Solution:

There is only one element in this set and so the probability of drawing 65 is: $P(65) = \frac{1}{85} = 0{,}01$.

e) is not a multiple of 5

Solution:

The set of all discs that are a multiple of 5 is: $\{5; 10; 15; 20; 25; 30; 35; 40; 45; 50; 55; 60; 65; 70; 75; 80; 85\}$. This set contains 17 elements. Therefore the number of discs that are not multiples of 5 is: $85 - 17 = 68$.

The probability of drawing a disc that is not a multiple of 5 is: $P(\text{not multiple of } 5) = \frac{68}{85} = 0{,}80$.

f) is a multiple of 3 or 4

Solution:

In part b), we worked out the probability for a disc that is a multiple of 3. Now we work out the number of elements in the set of all discs that are multiples of 4: $\{4; 8; 12; 16; 20; 24; 28; 32; 36; 40; 44; 48; 52; 56; 60; 64; 68; 72; 76; 80; 84\}$. This has 28 elements.

The probability that a disc is a multiple of either 3 or 4 is:

$$\begin{aligned} P(3_m \cup 4_m) &= P(3_m) + P(4_m) - P(3_m \cap 4_m) \\ &= \frac{1}{3} + \frac{28}{85} - \frac{1}{3} \times \frac{28}{85} \\ &= 0{,}55 \end{aligned}$$

g) is a multiple of 2 and 6

Solution:

The set of all discs that are multiples of 2 and 6 is the same as the set of all discs that are a multiple of 6. Therefore the probability of drawing a disc that is both a multiple of 2 and 6 is: 0,16.

h) is number 1

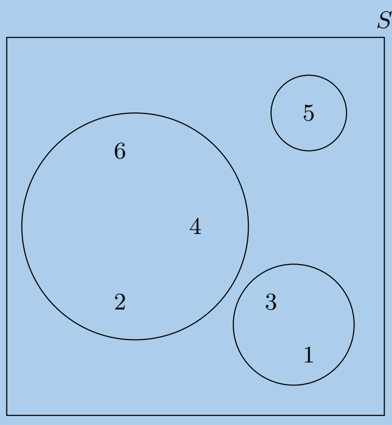
Solution:

There is only 1 element in this set and so the probability is 0,01.

14. Use a Venn diagram to work out the following probabilities for a die being rolled:

a) a multiple of 5 and an odd number

Solution:



For a die being rolled the sample space is $\{1; 2; 3; 4; 5; 6\}$

There is only one possibility here: the die lands on a 5. Therefore the probability is: $\frac{1}{6}$.

b) a number that is neither a multiple of 5 nor an odd number

Solution:

There is only one multiple of 5 in the sample space, this is an odd number. Therefore the set of numbers that are neither a multiple of 5 nor an odd number is: $\{2; 4; 6\}$. Therefore the probability is: $\frac{3}{6} = \frac{1}{2}$.

c) a number which is not a multiple of 5, but is odd

Solution:

There is only one multiple of 5 in the sample space, this is an odd number. Therefore the set of numbers that are not a multiple of 5 but is an odd number is: $\{1; 3\}$. Therefore the probability is: $\frac{2}{6} = \frac{1}{3}$.

15. A packet has yellow sweets and pink sweets. The probability of taking out a pink sweet is $\frac{7}{12}$. What is the probability of taking out a yellow sweet?

    Solution:

    $1 - \frac{7}{12} = \frac{5}{12}$

16. In a car park with 300 cars, there are 190 Opels. What is the probability that the first car to leave the car park is:

    a) an Opel

    Solution:

    $\frac{190}{300} = \frac{19}{30}$

    b) not an Opel

    Solution:

    $1 - \frac{19}{30} = \frac{11}{30}$

17. Nezi has 18 loose socks in a drawer. Eight of these are plain orange and two are plain pink. The remaining socks are neither orange nor pink. Calculate the probability that the first sock taken out at random is:

    a) orange

    Solution:

    $\frac{8}{18} = \frac{4}{9}$

    b) not orange

    Solution:

    $1 - \frac{4}{9} = \frac{5}{9}$

    c) pink

    Solution:

    $\frac{2}{18} = \frac{1}{9}$

    d) not pink

    Solution:

    $1 - \frac{1}{9} = \frac{8}{9}$

    e) orange or pink

    Solution:

    $\frac{1}{9} + \frac{4}{9} = \frac{5}{9}$

    f) neither orange nor pink

    Solution:

    $1 - \frac{5}{9} = \frac{4}{9}$

18. A plate contains 9 shortbread cookies, 4 ginger biscuits, 11 chocolate chip cookies and 18 Jambos. If a biscuit is selected at random, what is the probability that:

    a) it is either a ginger biscuit or a Jambo

    Solution:

    Total number of biscuits is $9 + 4 + 11 + 18 = 42$.

$$\begin{aligned} \frac{4}{42} + \frac{18}{42} &= \frac{22}{42} \\ &= \frac{11}{21} \end{aligned}$$

    b) it is not a shortbread cookie

    Solution:

$$\begin{aligned} 1 - \frac{9}{42} &= 1 - \frac{3}{14} \\ &= \frac{11}{14} \end{aligned}$$

19. 280 tickets were sold at a raffle. Jabulile bought 15 tickets. What is the probability that Jabulile:

    a) wins the prize

Solution:

$\frac{15}{280} = \frac{3}{56}$

b) does not win the prize

Solution:

$1 - \frac{3}{56} = \frac{53}{56}$

20. A group of children were surveyed to see how many had red hair and brown eyes. 44 children had red hair but not brown eyes, 14 children had brown eyes and red hair, 5 children had brown eyes but not red hair and 40 children did not have brown eyes or red hair.

a) How many children were in the school?

Solution:

The following possibilities exist for the hair and eye colour surveyed. Each of these is mutually exclusive:

- A child has brown eyes and red hair.
- A child has brown eyes but not red hair.
- A child has red hair but not brown eyes.
- A child does not have red hair or brown eyes.

Since we are given the total number of children in each of these four groups we can add these together to get the total number of children: $44 + 14 + 5 + 40 = 103$.

b) What is the probability that a child chosen at random has:

i. brown eyes

ii. red hair

Solution:

i. $\frac{19}{103}$

ii. $\frac{58}{103}$

c) A child with brown eyes is chosen randomly. What is the probability that this child will have red hair?

Solution:

$\frac{14}{(14+5)} = \frac{14}{19}$

21. A jar has purple sweets, blue sweets and green sweets in it. The probability that a sweet chosen at random will be purple is $\frac{1}{7}$ and the probability that it will be green is $\frac{3}{5}$.

a) If I choose a sweet at random what is the probability that it will be:

i. purple or blue

ii. green

iii. purple

Solution:

i. Same as not green: $1 - \frac{3}{5} = \frac{2}{5}$

ii. $\frac{3}{5}$

iii. $\frac{1}{7}$

b) If there are 70 sweets in the jar how many purple ones are there?

Solution:

$\frac{1}{7} \times 70 = 10$

c) $\frac{2}{5}$ of the purple sweets in (b) have streaks on them and the rest do not. How many purple sweets have streaks?

Solution:

$10 \times \frac{2}{5} = 4$

22. Box A contains 3 cards numbered 1, 2 and 3.
Box B contains 2 cards numbered 1 and 2.
One card is removed at random from each box.
Find the probability that:

a) the sum of the numbers is 4.

Solution:

$S = \{(1,1); (1,2); (2,1); (2,2); (3,1); (3,2)\}$

$$P = \frac{n(E)}{n(S)}$$
$$P = \frac{2}{6}$$
$$\therefore P = \frac{1}{3}$$

b) the sum of the two numbers is a prime number.

Solution:

$$P = \frac{n(E)}{n(S)}$$
$$P = \frac{4}{6}$$
$$\therefore P = \frac{2}{3}$$

c) the product of the two numbers is at least 3.

Solution:

$$P = \frac{n(E)}{n(S)}$$
$$P = \frac{3}{6}$$
$$\therefore P = \frac{1}{2}$$

d) the sum is equal to the product.

Solution:

$$P = \frac{n(E)}{n(S)}$$
$$P = \frac{1}{6}$$

23. A card is drawn at random from an ordinary pack of 52 playing cards.

a) Find the probability that the card drawn is:

i. the three of diamonds

ii. the three of diamonds or any heart

iii. a diamond or a three

Solution:

i.

$$P = \frac{n(E)}{n(S)}$$
$$P = \frac{1}{52}$$

ii.

$$P = \frac{n(E)}{n(S)}$$
$$= \frac{14}{52}$$
$$P = \frac{7}{26}$$

iii.

$$P = \frac{n(E)}{n(S)}$$
$$P = \frac{16}{52}$$
$$P = \frac{4}{13}$$

b) The card drawn is the three of diamonds. It is placed on the table and a second card is drawn. What is the probability that the second card drawn is not a diamond.
Solution:

$$\begin{aligned} P &= \frac{n(E)}{n(S)} \\ &= \frac{39}{51} \\ &= \frac{13}{17} \end{aligned}$$

24. A group of learners is given the following event sets:

Event Set A	3	4

Event Set B	2	4	5

Event Set $A \cup B$	2	3	4	5

The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 6\}$
They are asked to calculate the value of $P(A \cap B)$. They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal value.
Solution:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Make $P(A \cap B)$ the subject, and we get:

$$P(A \cap B) = P(B) + P(A) - P(A \cup B)$$

Identify variables needed:

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} = \frac{2}{6} = 0{,}33 \\ P(B) &= \frac{n(B)}{n(S)} = \frac{3}{6} = 0{,}5 \\ P(A \cup B) &= \frac{n(A \cup B)}{n(S)} = \frac{4}{6} = 0{,}67 \end{aligned}$$

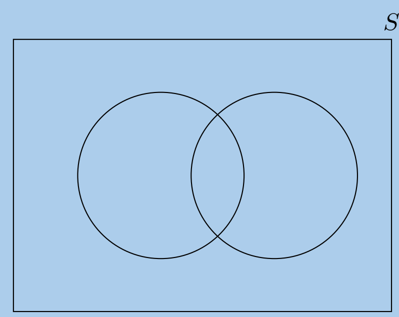
Calculate $P(A \cap B)$:

$$\begin{aligned} P(A \cap B) &= P(B) + P(A) - P(A \cup B) \\ &= (0{,}5) + (0{,}33) - (0{,}67) \\ &= 0{,}17 \end{aligned}$$

Therefore, the value of $P(A \cap B)$ is 0,17.

25. For each of the following, draw a Venn diagram to represent the situation and find an example to illustrate the situation.

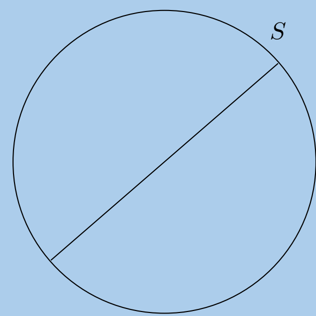
a) a sample space in which there are two events that are not mutually exclusive
Solution:



An example is drawing a card from deck of cards. We can draw a red card that is also a picture card.

b) a sample space in which there are two events that are complementary

Solution:

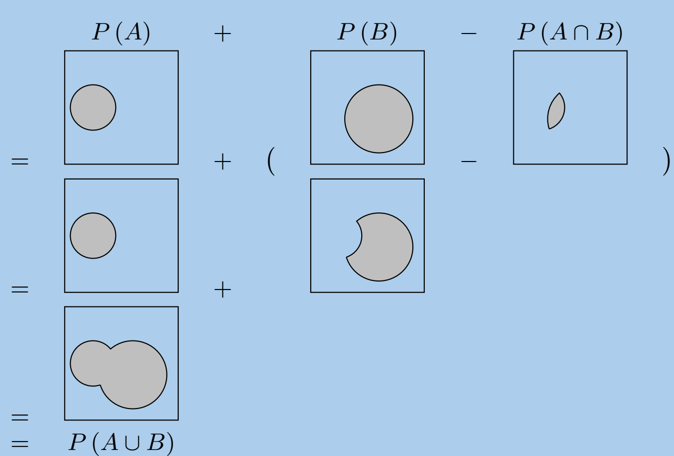


An example is rolling a die. The event of rolling an even number is complementary to the event of rolling an odd number.

26. Use a Venn diagram to prove that the probability of either event A or B occurring (A and B are not mutually exclusive) is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Solution:



27. All the clubs are taken out of a pack of cards. The remaining cards are then shuffled and one card chosen. After being chosen, the card is replaced before the next card is chosen.

a) What is the sample space?

Solution:

{deck without clubs}

b) Find a set to represent the event, P, of drawing a picture card.

Solution:

$P = \{J; Q; K$ of hearts, diamonds or spades$\}$

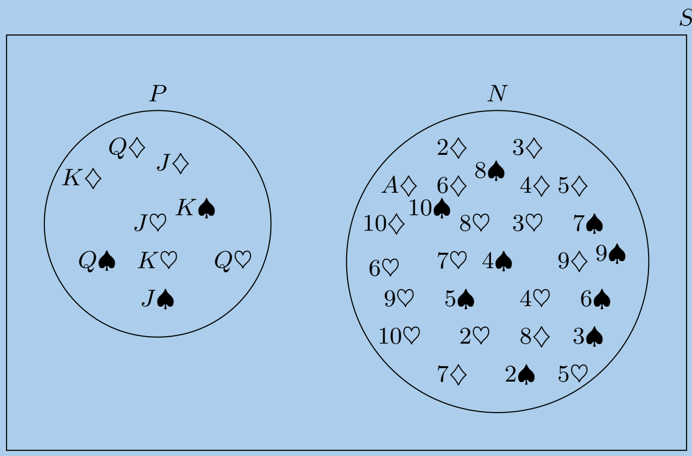
c) Find a set for the event, N, of drawing a numbered card.

Solution:

$N = \{A; 2; 3; 4; 5; 6; 7; 8; 9; 10$ of hearts, diamonds or spades$\}$

d) Represent the above events in a Venn diagram.

Solution:



e) What description of the sets P and N is suitable? (Hint: Find any elements of P in N and of N in P.)

Solution:

Mutually exclusive and complementary.

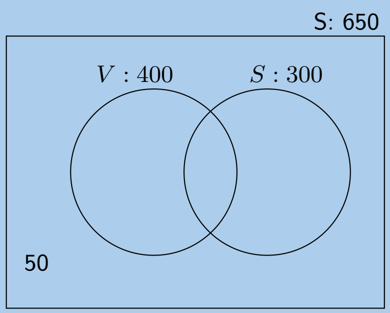
28. A survey was conducted at Mutende Primary School to establish how many of the 650 learners buy vetkoek and how many buy sweets during break. The following was found:

- 50 learners bought nothing
- 400 learners bought vetkoek
- 300 learners bought sweets

a) Represent this information with a Venn diagram.

Solution:

The following Venn diagram represents the given information. However we can calculate more information from this that will help us answer the second part of this question.



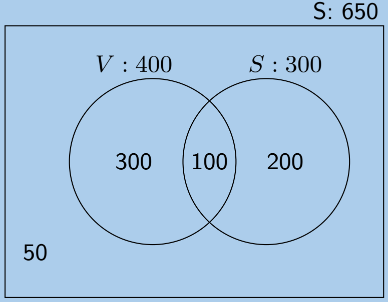
We note the following information:

- 400 learners bought vetkoek, some of these also bought sweets.
- 300 learners bought sweets, some of these also bought vetkoek.
- Of the total number of learners, 50 did not buy anything, so $650 - 50 = 600$ bought either vetkoek or sweets or both.

Let the number of learners who bought vetkoek only be v, the number of learners who bought sweets only be s and the number of learners who bought both be b. Now we note the following:

$$\begin{aligned} 600 &= v + s + b \\ \text{But } v + b &= 400 \\ \therefore 600 &= 400 + s \\ \therefore s &= 200 \\ \text{Also } s + b &= 300 \\ \therefore b &= 100 \\ \therefore v &= 600 - s - b \\ &= 300 \end{aligned}$$

We can fill this in on the Venn diagram:



b) If a learner is chosen randomly, calculate the probability that this learner buys:

i. sweets only
ii. vetkoek only
iii. neither vetkoek nor sweets
iv. vetkoek and sweets
v. vetkoek or sweets

Solution:

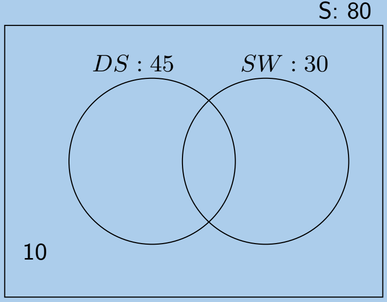
i. $\frac{200}{650} = 30{,}8\%$
ii. $\frac{300}{650} = 46{,}2\%$
iii. $\frac{50}{650} = 7{,}7\%$
iv. $\frac{100}{650} = 15{,}4\%$
v. $\frac{600}{650} = 92{,}3\%$

29. In a survey at Lwandani Secondary School, 80 people were questioned to find out how many read the Sowetan, how many read the Daily Sun and how many read both. The survey revealed that 45 read the Daily Sun, 30 read the Sowetan and 10 read neither. Use a Venn diagram to find the percentage of people that read:

a) only the Daily Sun

Solution:

The following Venn diagram represents the given information. However we can calculate more information from this that will help us answer the problem.



We note the following information:

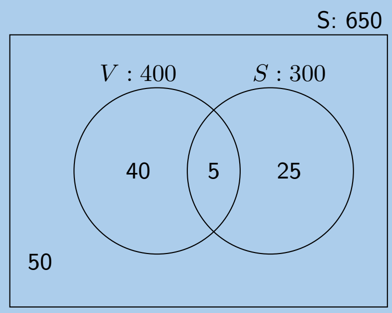
- 45 people read the Daily Sun, some of these also read the Sowetan.

- 30 people read the Sowetan, some of these also read the Daily Sun.
- Of the total number of people questioned 10 did not read either newspaper, so $80 - 10 = 70$ read neither.

Let the number of people who read the Daily Sun only be d, the number of people who read the Sowetan only be s and the number of people who read both be x. Now we note the following:

$$\begin{aligned} 70 &= d + s + x \\ \text{But } d + x &= 45 \\ \therefore 70 &= 45 + s \\ \therefore s &= 25 \\ \text{Also } s + x &= 30 \\ \therefore x &= 5 \\ \therefore d &= 70 - s - x \\ &= 40 \end{aligned}$$

We can fill this in on the Venn diagram:



$\frac{40}{80} = 50\%$.

b) only the Sowetan

Solution:

$\frac{25}{80} = 31{,}25\%$.

c) both the Daily Sun and the Sowetan

Solution:

$\frac{5}{80} = 6{,}25\%$.

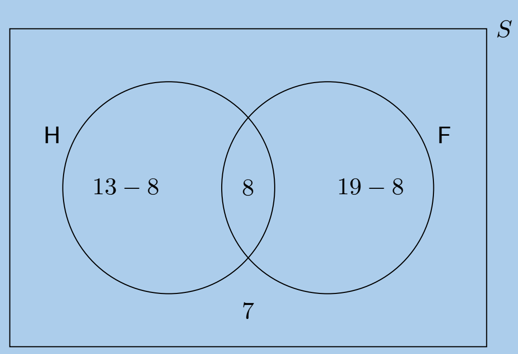
30. In a class there are

- 8 learners who play football and hockey
- 7 learners who do not play football or hockey
- 13 learners who play hockey
- 19 learners who play football

How many learners are there in the class?

Solution:

Let H and F be the learners who play hockey and football respectively.



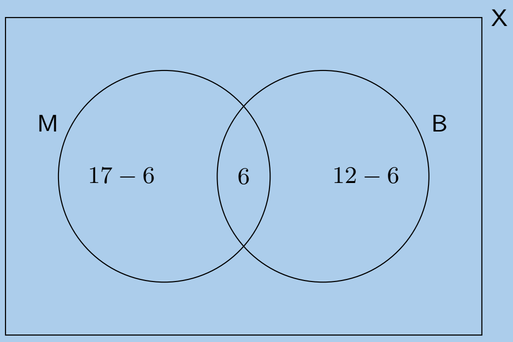
$$
\begin{aligned}
N &= 7 + 5 + 8 + 11 \\
&= 31
\end{aligned}
$$

31. Of 36 people, 17 have an interest in reading magazines and 12 have an interest in reading books, 6 have an interest in reading both magazines and books.

a) Represent the information in a Venn diagram.

Solution:

Let B and M be the people who read books and magazines respectively.



b) How many people have no interest in reading magazines or books?

Solution:

11 people only read magazines, 6 people only read books and 6 people read both.

The total number of people is 36. Therefore we can find the number of people who have no interest in reading magazines or books:

$$36 - 11 - 6 - 6 = 13$$

c) If a person is chosen at random from the group, find the probability that the person will:

i. have an interest in reading magazines and books.
ii. have an interest in reading books only.
iii. not have any interest in reading books.

Solution:

i.

$$
\begin{aligned}
P &= \frac{n(E)}{n(S)} \\
P &= \frac{6}{36} \\
P &= \frac{1}{6}
\end{aligned}
$$

ii.

$$
\begin{aligned}
P &= \frac{n(E)}{n(S)} \\
P &= \frac{6}{36} \\
P &= \frac{1}{6}
\end{aligned}
$$

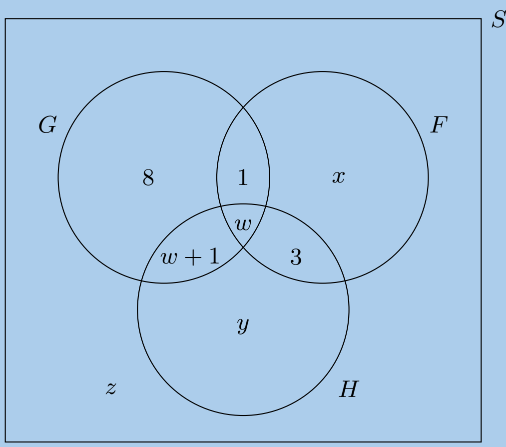
iii.

$$
\begin{aligned}
P &= \frac{n(E)}{n(S)} \\
P &= \frac{36 - 12}{36} \\
P &= \frac{24}{36} \\
&= \frac{2}{3}
\end{aligned}
$$

32. 30 learners were surveyed and the following information was revealed from this group:

- 18 learners take Geography
- 10 learners take French
- 6 learners take History, but take neither Geography nor French.

In addition the following Venn Diagram has been filled in below:
Let G be the event that a learner takes Geography.
Let F be the event that a learner takes French.
Let H be the event that a learner takes History.



a) From the information above, determine the values of w, x, y and z.
Solution:

$$y = 6$$

$$\begin{aligned} 8 + (w+1) + w + 1 &= 18 \\ 2w &= 18 - 10 \\ w &= 4 \end{aligned}$$

$$\begin{aligned} 10 &= x + 1 + w + 3 \\ 10 &= x + 8 \\ x &= 10 - 8 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} z &= 30 - (8 + 1 + w + w + 1 + x + 3 + y) \\ z &= 30 - (13 + 8 + 2 + 6) \\ z &= 30 - 29 \\ z &= 1 \end{aligned}$$

Therefore $w = 4$, $x = 2$, $y = 6$ and $z = 1$.

b) Determine the probability that a learner chosen at random from this group:

i. takes only Geography.
ii. takes French and History, but not Geography.

Solution:

i.

$$\begin{aligned} P &= \frac{n(E)}{n(S)} \\ &= \frac{8}{30} \\ &= \frac{4}{15} \end{aligned}$$

ii.

$$
\begin{aligned}
P &= \frac{n(E)}{n(S)} \\
&= \frac{3}{30} \\
&= \frac{1}{10}
\end{aligned}
$$

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GX3	2. 2GX4	3. 2GX5	4. 2GX6	5. 2GX7	6. 2GX8
7. 2GX9	8. 2GXB	9. 2GXC	10. 2GXD	11. 2GXF	12. 2GXG
13. 2GXH	14. 2GXJ	15. 2GXK	16. 2GXM	17. 2GXN	18. 2GXP
19. 2GXQ	20. 2GXR	21. 2GXS	22. 2GXT	23. 2GXV	24. 2GXW
25a. 2GXX	25b. 2GXY	26. 2GXZ	27. 2GY2	28. 2GY3	29. 2GY4
30. 2GY5	31. 2GY6	32. 2GY7			

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CHAPTER 15

Investigations and projects

15.1 Assignment: functions

1. Draw the graphs of each of the following by means of point-to-point plotting (or use technology).

a) $y = x^2$	b) $y = 2x$	c) $y = 2 + x$	d) $y = \frac{1}{x}$	e) $y = x^2 + 2$
f) $y = -2x^2$	g) $y = 2\left(\frac{1}{x}\right)$	h) $y = -\frac{1}{x} + 2$	i) $y = -3x - 4$	j) $y = -\frac{1}{x-1} + 2$
k) $y = -5$	l) $y = \frac{1}{x+3} - 2$	m) $y = \frac{x}{3}$	n) $y = -\frac{2}{x}$	o) $y = (x+1)^2$
p) $y = x^2 - 2x - 8$	q) $y = -2x + 3$	r) $x^2 + y^2 = 25$		

2. Now indicate whether it is a straight line, parabola, hyperbola or any other function by filling the algebraic equations in the correct column below:

Straight line					
Parabola					
Hyperbola					
Other					

3. What do the equations representing straight line graphs have in common?
4. What do the equations representing parabolas have in common?
5. What do the equations representing hyperbolas have in common?
6. Write down three other expressions that make: straight lines, parabolas and hyperbolas.
7. Is the graph of $y = 3x^2 + 2x + 1$ a line, a parabola or some other shape? Explain.
8. Is the graph of $y = 3x^3 + 2x + 1$ a line, a parabola or some other shape? Explain.
9. What do you notice about the graphs of the following equations: $y = \frac{1}{x}$, $y = \frac{1}{x} + 2$, $y = x^2$, $y = x^2 + 2$. Make a conjecture about the effect of '+2'.
10. What do you notice about the graphs of the following equations: $y = \frac{1}{x}$, $y = 2\frac{1}{x}$, $y = x^2$, $y = 2x^2$. Make a conjecture about the effect of '$\times$2'.

For teachers:

This assignment will be marked according to the following rubric:

	6 - 8	4 - 5	2 - 3	0 - 1
Accuracy of graphs (if technology is used learners must provide printouts of all graphs)	Accurate and correct throughout	Almost all correct	Some correct	Mostly incorrect
Completion of table		All entries correct	Most entries correct	Incomplete with errors
Observations made	Clear and correct explanations given for all conclusions	Clear explanations given but conclusions incomplete	Did not clearly explain the reasoning	No attempt or has given a vague description or incomplete conclusions
Correctness of expressions for linear, parabolic and hyperbolic functions		Accurate and correct	Almost all correct	No attempt or many errors
Punctuality			Deadline met	Deadline not met or negotiated deadline met
Presentation				Acceptable or not acceptable

15.2 Investigation: Number patterns

Investigate one of the two options outlined below and then write up a full description of the work you did. You need to show a significant number of special cases, including some which might produce exceptions or results which are different from the rest. You should be able to make some conjectures from what you discover from the special cases you investigate. Write these conjectures as clearly as possible and then attempt to prove them.

There are two options for this investigation. Choose only one of the two options.

Option 1: An interesting sequence

The first term of a sequence $T_1 = 2$ and the second term $T_2 = 5$.

Calculate the next 6 terms using the following rule: $T_n = \frac{T_{n-1}+1}{T_{n-2}}$.

Explain what it means if a sequence 'starts to recur'. Does the sequence start to recur?

Investigate other sequences where you choose the first two terms and then use the same rule to calculate subsequent terms. For the first two terms, try whole numbers, integers, fractions, numbers which are equal...

Make some conjectures and attempt to prove them.

Option 2: Reverse, subtract, reverse, add

Follow the rules used in this example for other three digit numbers:

1. Take any three digit number, e.g: 378
2. Reverse it: 873
3. Take the smaller number from the bigger number: $873 - 378 = 495$
4. Reverse the difference: 594
5. Add this reversed difference to the original difference: $594 + 495 =$ 1089

Investigate whether digits could be equal; how many different options there could be; whether 0 could be used as on or more of the digits...

Any conjecture(s)? Attempt to prove it (them).

15.3 Project: Finance

Section A: exchange rates

The table below shows the average Rand (R)/ US dollar ($) exchange rate from 2000 to 2007. The figure given under the column "Exchange Rate" is how many Rands were required to get one US Dollar.

Year	Exchange rate
2000	6,94
2001	8,58
2002	10,52
2003	7,57
2004	6,45
2005	6,37
2006	6,78
2007	7,06

1. In 2006 a book published in America cost $ 15,00. How much would you expect to pay for the book in South Africa?

2. You go to the bookshop and see the book on the shelf. It costs R 136,00. Is this more or less than you expected? Suggest some reasons for the difference between the price you expected to pay and the marked price of the book.
3. In which year would you expect the book to cost the least? Substantiate your answer.
4. In 2007, a t-shirt made in South Africa costs R 95,00. The shirt is exported to America. How much will it cost in $?
5. If you were running a clothing factory where the clothes were made from imported American cotton, in which year would the exchange rate have been best for you and in which year would it have been worst? Explain.
6. If you were running a business that exported South African chocolates to America, in which year would the exchange rate have been the best for you and in which year would it have been the worst? Explain.
7. The banks only give a few exchange rates in the same format as the table above, i.e. how many Rand you need to buy one unit of foreign currency ($, Euros (€) and British Pounds (£)). All other exchange rates are quoted in the amount of foreign currency that can be purchased with R 1,00. In 2007, the Rand/Australian dollar (A$) exchange rate was R 1,00 to A$0,17. If you went to the bank with R 100, how many Australian dollars would you be able to get?
8. How much would you need in R to get A$ 1? Is this a better or worse exchange rate than the Rand/Dollar exchange rate in 2007? Explain your answer.
9. Work out the exchange rate between Rands and US Dollars in the same format as the exchange rate for Australian dollars, i.e. work out how many US dollars you can get for R 1,00.
10. Using the 2007 exchange rates for Australian dollars and US dollars, work out how many Australian dollars you would need to get 1 US dollar.
11. In December 2007, the price of crude oil, which is used to make petrol, was $ 92,00 per barrel. How much did South Africa pay for a barrel of oil and how much did Australia pay for a barrel of oil?
12. In 2007, the exchange rate between the Rand and the British pound was R 14,13 to 1€. The Rand/US dollar exchange rate was R 7,06 to 1$. Estimate the exchange rate between the US dollar and the British pound.
13. You are visiting America and decide to buy a Big Mac burger for lunch. It costs $ 3,20. What is the Rand equivalent of the Big Mac Burger. Use the 2007 exchange rate given in the table above.
14. While you are eating your Big Mac, you compare what you have just paid in Rands for your burger with what you would have paid in South Africa (R 15,50). Is the Big Mac more or less expensive in America?
15. Experiment with exchange rates, using any method you think is suitable, to work out a Rand/Dollar exchange rate that would result in the price of a Big Mac in America being equivalent to the price that you pay for a Big Mac in South Africa. Does the official Rand/Dollar exchange rate undervalue or overvalue the Rand?

Note: the last three questions that you have answered represent a simplified version of what is known as "Burgenomics" or the "Big Mac Index". Burgenomics is used by economists all over the world to compare exchange rates and determine whether currencies are undervalued or overvalued. You can find out more about Burgenomics on the internet.

Section B: inflation and interest

- In 2000 a pair of track shoes costs R 450. The inflation rate is 5,4%. What would you expect the shoes to cost in 2007?
- The table below gives the actual annual inflation rates from 2000 to 2007. Use this table to work out the 2007 price of the track shoes. How much does this compare with your answer in 1? Explain why your answers are different.

Year	Inflation rate (%)
2000	5,4
2001	5,7
2002	9,2
2003	5,8
2004	1,4
2005	3,4
2006	4,7
2007	7,7

You decide to buy a cellphone costing R 890. You have saved R 320 towards the cost of the cellphone and your parents have agreed to lend you the balance at 8,5% per year simple interest. You will pay back your parents in equal monthly payments for a period of 2 years. Work out your monthly repayments.

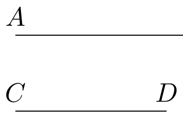
Instead of borrowing money, you decide that you are going to save for two years and then buy your new cellphone.

1. You invest the R 320 that you have in a special savings account which pays 8% per annum intereste, compounded monthly. Calculate the balance in this account after 2 years.
2. After six months you deposit R 170 into an ordinary savings account at 7,5% interest per annum, compound six monthly. Six months later, you deposit R 160 and six months later you deposit R 170 into the same account. Calculate the balance in this account at the end of two years.
3. During the two-year period that you are saving, the rate of inflation is 6,7% per annum. What will the cost of the cellphone be at the end of the two years?

4. Using your results determine whether or not you will have enough money to pay cash for the cellphone.

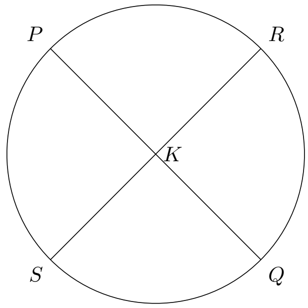
15.4 Assignment: Shape, space and measurement

1. On a sheet of unlined paper, construct line $AB = 9$ cm. Now construct line DC so that DC is parallel to AB and equal to AB. Join B to C and A to D.



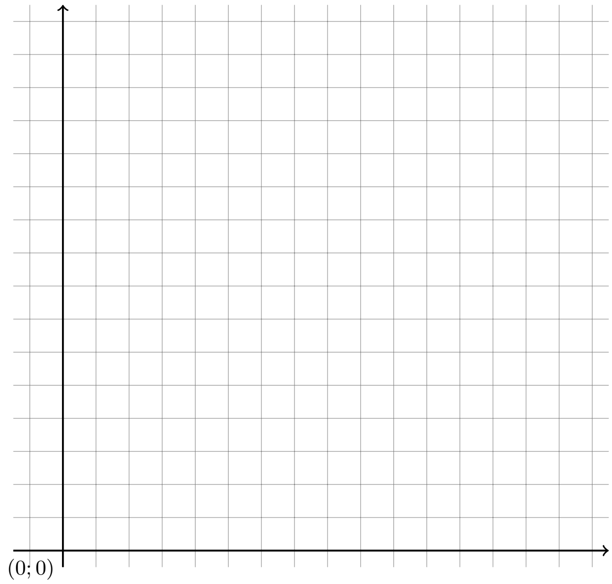
   a) What type of quadrilateral is $ABCD$?
   b) Write down four conjectures about $ABCD$ that involve either equal lines or equal angles.
   c) Confirm each conjecture by measuring the lines or angles and write down the measurements.

2. Using a compass, construct a circle with a radius of 5 cm. Label the centre of the circle K.
   a) Draw any two diameters of the circle and label them PQ and RS as shown in the diagram.



   b) How long is PQ? Explain how you know this without measuring PQ.
   c) Which other lines equal PK? Give a reason for this.
   d) Join points P, R, Q and S. What type of triangle is $\triangle PKR$? Give a reason.
   e) What type of triangle is $\triangle PKS$? Give a reason.
   f) Prove that $S\hat{P}R = 90°$.
   g) Are there any other angles that are 90°? If so, name them.
   h) Do you think that you have proved that quadrilateral $PQRS$ is a rectangle? Explain.

3. On the grid provided, plot points $F(1; 1)$ and $G(6; 1)$.



a) What is the length of FG?

b) Plot point E so that FE is the same length as FG and the coordinates of E are integers, but FE is **not** parallel to the y-axis. What are the coordinates of E? Explain the method you used to plot E.

c) Plot point H so that $EFGH$ is a rhombus. What are the coordinates of H?

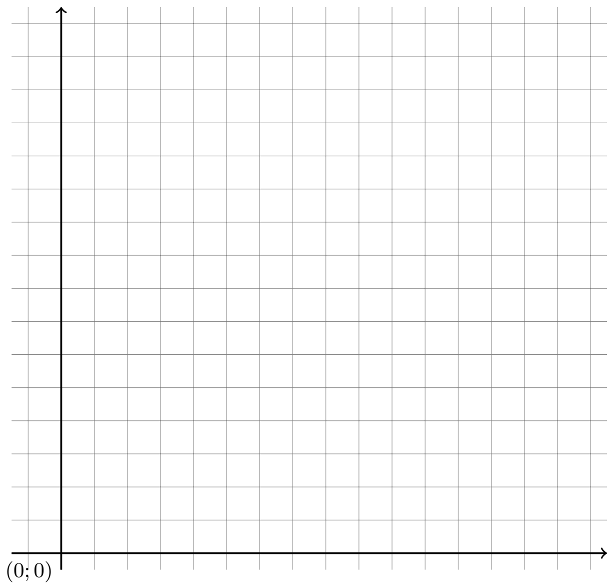
d) Which property of a rhombus did you use to draw $EFGH$?

e) Draw the diagonals FH and DE. Using coordinate geometry, prove that the diagonals bisect each other.

4. What is the definition of a regular polygon? Using your definition of a regular polygon, decide which of the following are regular polygons. In each case, you must state which requirements of your definition are true (if any) and which are not true (if any). Make use of diagrams to illustrate your answers.

   a) Scalene triangle
   b) Rhombus
   c) Isosceles trapezium
   d) Isosceles triangle
   e) Square
   f) Kite
   g) Parallelogram
   h) Equilateral triangle

5. On the grid paper, plot the points $A(-1;2)$, $B(0;-5)$ and $(4;7)$. Join the points.



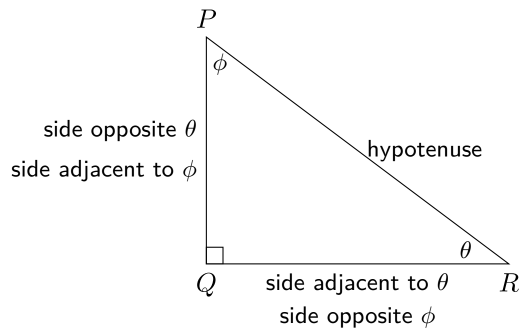
   a) What type of triangle is $\triangle ABC$?
   b) Prove the conjecture you have made in the previous question.
   c) Note: the coordinates of A, B and C are all integers and no integer is used more than once. Use this rule to answer the following question.
      On the grid, plot points J, K, L and M so that $JKLM$ is a kite. Give the coordinates of the four points you have plotted.
   d) Prove that $JKLM$ is a kite.

15.5 Investigation: Trigonometry

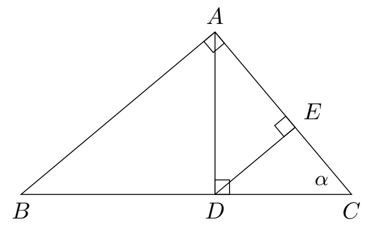
You are reminded of the definitions of sine, cosine and tangent, abbreviated as $\sin$, $\cos$ and $\tan$:

In the right angled triangle below:

$$\sin\theta = \frac{\text{side opposite }\theta}{\text{hypotenuse}} = \frac{PQ}{PR} \qquad \sin\phi = \frac{\text{side opposite }\phi}{\text{hypotenuse}} = \frac{QR}{PR}$$
$$\cos\theta = \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{QR}{PR} \qquad \cos\phi = \frac{\text{side adjacent to }\phi}{\text{hypotenuse}} = \frac{PQ}{PR}$$
$$\tan\theta = \frac{\text{side opposite }\theta}{\text{side adjacent to }\phi} = \frac{PQ}{QR} \qquad \tan\phi = \frac{\text{side opposite }\phi}{\text{side adjacent to }\phi} = \frac{QR}{PQ}$$



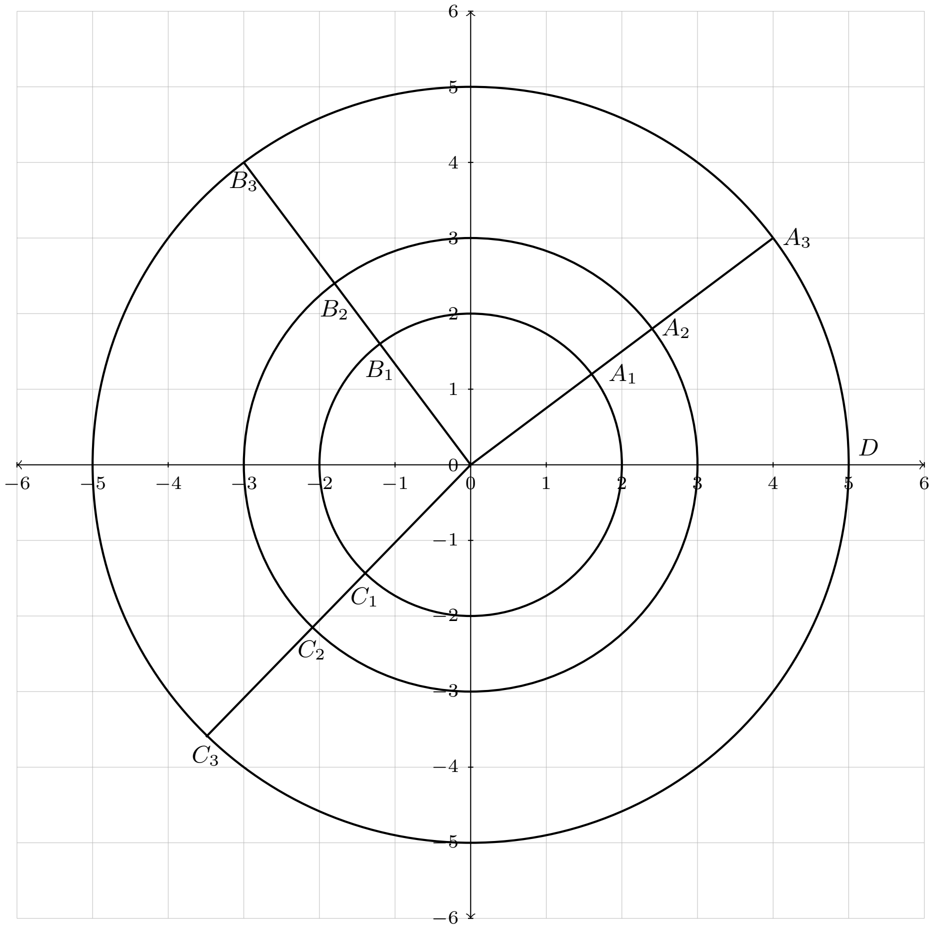
Task 1



1. Name all the similar triangles in the sketch above.
2. Given that $BD = 8$ units, $DC = 4$ units and $AD = 4\sqrt{2}$ units, calculate the lengths of all the other line segments in the sketch. Leave your answers in surd form.
3. Express $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ in as many different ways as possible.
   For example $\sin\alpha = \frac{BA}{BC} = \frac{AD}{AC} = \ldots$
4. Given that $A\hat{B}C = \beta$, express $\sin\beta$, $\cos\beta$ and $\tan\beta$ in as many different ways as possible.

Task 2

The three circles have radii 2, 3 and 5 units.



1. Read, as accurately as possible, the co-ordinates of the points marked: $A_1, A_2, A_3, \ldots C_3$ and hence complete the following table (work correct to 1 decimal place):

	x-coordinate	y-coordinate	$\frac{x}{r}$	$\frac{y}{r}$	$\frac{y}{x}$
A_1					
A_2					
A_3					
B_1					
B_2					
B_3					
C_1					
C_2					
C_3					

2. Write any observations about what you read and calculated using the co-ordinates of the nine points.

Task 3

1. Measure $A\hat{O}D$, $B\hat{O}D$ and $C\hat{O}D$ and use your calculator to determine the sine, cosine and tangent of each of these three angles.
2. How, if at all, are the ratios determined in the previous question related to the values in the table completed in task 2?
3. Use your calculator to investigate:
   a) the maximum and minimum values (if they exist) of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any values of θ (try multiples of 10°)
   b) the value(s) of $\sin^2\theta + \cos^2\theta$ for at least 5 values of θ (use $\theta = 0°$, some positive values and some negative values)

c) the values of $\sin\theta$, $\cos\theta$, $\frac{\sin\theta}{\cos\theta}$ and $\tan\theta$ for at least 5 values of θ

4. Write any observations about the results you obtained through your calculator work.

15.6 Project: Introduction to data handling

In this assignment we wish to investigate how people get to school in the morning. What is the most popular mode of transport and is there a correlation between their distance from school and the duration of their journey?

In order to do this, you need to collect information from the learners at your school. Before collecting data, it is important to be sure that you know what questions you want to be able to answer so that you collect all the data you need. You also need to consider your sample size quite carefully. It needs to be big enough to represent the population adequately but small enough to make it measurable. The greater the amount of data you collect, the more accurate your deductions will be.

Part 1

For each individual you will need to record the following:

1. What mode of transport they use to get to school (this could be walk, cycle, bus, car, train, other). If they use a combination of transport, for example they walk to the bus stop, catch the bus and then walk to school, record the mode that represents the majority of their journey in terms of time.
2. How far away from school they live
3. How long a typical journey to school takes

Part 2

1. Determine the modal form of transport for your dataset and both the median and mean for the distance travelled and time taken.
2. Explain why finding the mean and median form of transport is not sensible.
3. Explain why it might be best to group the time taken and the distance travelled into intervals before finding the mode of these two sets of data.
4. Decide on reasonable intervals for your data for the time taken to get to school. Group your data and represent it graphically using a histogram.
5. From your histogram, determine what the modal interval is for time taken to get to school.
6. Attempt to give reasons for certain transport methods being favoured over others in relation to the area in which your school lies and how it might differ from other schools in different areas.

Part 3

A scatter plot is good for determining if there is any correlation between two different sets of data. There are three different types of correlation that may occur: positive correlation, where one set of data increases as the other set increases; negative correlation, where one set of data decreases as the other increases; or no correlation, where the datasets seem unrelated.

You need to draw a scatter plot to investigate the correlation between the distance from school and the time it takes to travel that distance. In order to do this, you need to label the x-axis according to one dataset and the y-axis according to the second dataset. You then plot a point for each individual in your sample. The more points that are plotted on the graph, the clearer the relationship becomes.

Using your graph, determine if there is a correlation between the distance people travel to school and the time it takes them to get there. Motivate your answer making reference to your scatter diagram and attempt to find reasons for the correlation, or the lack thereof.

Part 4

Represent the data you have on the mode of transport used when travelling to school in a pie chart. Be sure to write a key and inidicate the percentages of the sample group on your diagram.

Part 5

Using all the information you have collected and represented in different ways, write a paragraph describing the transport habits of learners at your school.

VERSION 1.1 CAPS

GRADE 10 MATHEMATICS TEACHER'S GUIDE

WRITTEN BY SIYAVULA AND VOLUNTEERS

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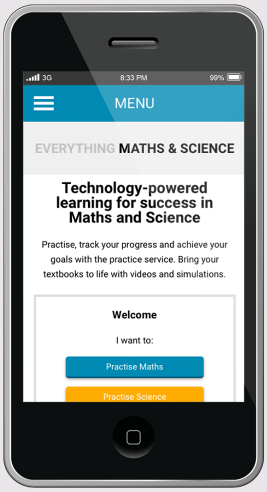

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