4 Annuities and Loans

4.1 Introduction

In previous section, we discussed different methods for crediting interest, and we claimed that compound interest is the "correct" way to credit interest. This section is concerned with valuing a large number of cash flows.

4.2 Loans

Toward the end of the last section we solved some time value of money problems which involved several cash flows. We approached those problems by applying compound interest to each individual cash flow, then we added or subtracted the results. This works well for a small number of payments, but can be very tedious when many payments are involved.

Example 11 (A loan with very few payments). *Jill borrows* $1000 *today. She will repay the loan by making four equal payments over the next year. The payments will be made at the end of every third month. The interest is* 3.2% *APR compounded quarterly. Determine the size of Jill's level payments.*

Solution: Let R denote size of level payments. The discount factor for a single quarter is $\left(1 + \frac{.032}{4}\right)^{-1} = (1.008)^{-1}$. The present value of the payments is then

$$R(1.008)^{-1} + R(1.008)^{-2} + R(1.008)^{-3} + R(1.008)^{-4}$$

Factor out the R and compute:

$$R\left((1.008)^{-1} + (1.008)^{-2} + (1.008)^{-3} + (1.008)^{-4}\right) = R \cdot 3.921262$$

R is to be set so that this payment stream has the same present value as the loan principal, $1000, so $3.921262R = \$1000$ so $R = \$1000/3.921262 = \$255.02$.

Example 12 (Loan with a lot of payments). *Bryan takes out a home loan worth* $250,000 *today. He will repay the loan by making equal payments at the end of each month for the next 30 years. The interest is* 6.0% *APR compounded monthly. Determine the size of Bryan's level payments.*

We can try to set this up like the previous problem. However, the previous problem involved adding four payments together, whereas this problem will involve adding together $30 \cdot 12 = 360$ payments!

A loan involves making payments of equal size at equally spaced intervals of time. If the interest rate remains constant of the entire period of the loan, then we will be able to compute

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the present value of the loan using a simple formula. The advantage of this formula is that valuing a loan with 4 payments or 4000 payments will require about the same amount of computational effort.

Formula for Present Value of a Loan or Annuity

P denotes the principal of a loan (how much was borrowed)

R denotes the payment size

t the number of years (the term of the loan)

r is the nominal interest rate per year

m is the number of conversion periods per year

i is the interest rate per period, so $i = r/m$

n is the number of conversion periods in the term, so $n = t \cdot m$

Then
$$P = R \cdot \frac{1 - (1 + i)^{-n}}{i}$$

Jill's loan, revisited:

Jill borrows $1000 today. She will repay the loan by making four equal payments over the next year. The payments will be made at the end of every third month. The interest is 3.2% APR compounded quarterly. Determine the size of her level payments.

In this case, we have the following:

$$P = 1000$$
$$t = 1$$
$$m = 4$$
$$n = 4 \cdot 1 = 4$$
$$r = 0.032$$
$$i = \frac{0.032}{4} = 0.008$$

We wish to find R. So,

$$1000 = R \cdot \frac{1 - (1.008)^{-4}}{0.008} = R \cdot 3.921262$$

Solving for R gives $R = \frac{1000}{3.921262} = 255.02$, same as what we found without the formula.

The real advantage of the formula will become clear when we revisit Bryan's loan.

Bryan takes out a home loan worth $250,000 today. He will repay the loan by making equal payments at the end of each month for the next 30 years. The interest is 6.0% APR compounded monthly. Determine the size of the level payments.

In this case, we have the following:

$$P = 250,000$$
$$t = 30$$
$$m = 12$$
$$n = 12 \cdot 30 = 360$$
$$r = 0.06$$
$$i = \frac{0.06}{12} = 0.005$$

We wish to find R.

$$250,000 = R \cdot \frac{1 - (1.005)^{-360}}{0.005} = R \cdot 166.79164$$

Solving for R gives

$$R = \frac{250000}{166.79164} = 1,498.88$$

Without the loan formula, we would have had to add together 360 individual payments!

Bryan borrowed $250,000 and paid this back by making 360 payments of $1498.88. But wait, this means he pays back $360 \cdot 1498.88 = 539,596.80$. Doesn't that mean his payments were too large?

NO! The First Principle of Financial Mathematics insists that figure of $539,596.80 is somewhat meaningless, as it is formed by adding together dollar figures at different times.

Does the $539,596.80 have any meaning? Yes. The difference $539,596.80 - $250,000 = $289,596.80 can be interpreted as the interest paid. Think of this as saying "$289,596.80 is the cost of borrowing $250,000 for 30 years."

4.3   Understanding the Present Value of a Loan

So we have a formula involving the present value of a loan, and we can do some computations with it. This is only useful if we understand what it is that we are computing. At the very beginning of the loan, the present value of the loan represents the principal of the loan, namely, how much we borrowed. The next few examples will try to give meaning to the present value of a loan when that present value is valued at a later date.

Example 13 (More on Bryan's Loan). *Exactly 10 years after Bryan took out his home loan, he strikes it rich. First thing he wants to do is pay off the remaining balance on his home loan. How much will he need to pay?*

Solution: He has 20 years remaining on the loan, so $20 \cdot 12 = 240$ payments left. We compute the PV of the loan 10 years into the loan.

$$P = \$1498.88 \frac{1 - (1.005)^{-240}}{0.005} = \$209,214.83$$

This present value represents how much it would cost to pay off the loan in full at that point in time.

Real world finance is never quite this simple, as Bryan may have to put up with early payment fees, etc. We will generally ignore such fees in this course.

What are the advantages to Bryan paying off the loan early?

- Bryan had already made 120 payments of $1,498.88$, so total repaid is $120 \cdot \$1498.88 = \$179,865.60$.

- He then pays $\$209,214.83$ to settle the remainder of the loan.

- Total amount paid by Bryan is $\$209,214.83 + \$179,865.60 = \$389,080.43$.

- He borrowed $\$250,000$, so the total interest charge is $\$389,080.43 - \$250,000 = \$139,080.43$.

- Had he continued to make regular payments for the full term of the loan, his total interest expense would have been $\$289,596.80$

- By paying off the loan early, he saved $\$289,596.80 - \$139,080.43 = \$150,516.37$

Most loans will lock you in at a fixed interest rate for the entire term of the loan. In order to take advantage of changing interest rates, one may need to refinance.

Suppose Dana takes out a home loan with $\$300,000$ principal. She makes payments at the end of each month for 30 years. The interest is 8.4% APR compounded monthly.
(Note that Dana's *monthly* interest rate is $i = \frac{0.084}{12} = 0.007$.)

First, lets determine her payment size.

$$\$300,000 = R \cdot \frac{1 - (1.007)^{-360}}{0.007}$$

so $R = \$2,285.51$

Next, determine the interest charges on this loan.
She pays back $\$2285.51 \cdot 360 = \$822,783.60$. Subtracting the principal leaves $\$522,783.60$. (Notice the interest charge is almost twice the total amount that she borrowed!)

After five years, interest rates drop to 6% APR compounded monthly. Dana wants to refinance[10] her home loan to take advantage of this lower interest rate. How can she do this?

We'll need to determine Dana's outstanding balance after 5 years.

$$\$2,285.51 \cdot \frac{1 - 1.007^{-300}}{0.007} = \$286,225.65$$

She currently owes $\$286,225.65$.

Suppose that, in addition to taking advantage of the lower interest rate, Dana chooses to shorten the term of her loan when she refinances. In particular, she refinances to a 20 year loan with 6% APR compounded monthly. Lets determine her new payment size.

$$\$286,225.65 = R \cdot \frac{1 - 1.005^{-240}}{0.005}$$

so $R = \$2050.61$. Notice that upon refinancing, her payment size decreased AND the number of payments decreased!

Next, lets determine Dana's total interest expense under the re-financing. We need to be sure to include the interest expense of the first 5 years of the original loan!

$$\$2285.81 \cdot 60 + \$2050.61 \cdot 240 = \$137,148.60 + \$492,146.40 = \$629,295$$

is the total repaid. Subtracting principal leaves $\$329,295$ in interest expenses.

Finally, lets determine Dana's savings in interest charges due to refinancing

With out refinancing:	$522,783.60
With refinancing:	$329,295.00
Interest savings:	$193,488.60

Dana saves $\$193,488.60$ by refinancing!

While in theory, it is advantageous to refinance whenever interest rates drop, in practice it is usually only advantageous to refinance if the interest rate changes by a lot. Why? In the real world, refinancing involves extra bank fees and transaction costs.

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[10]What does refinance mean? Basically, she takes out a new loan. The principal of the new loan is equal to the balance of the existing loan. The principal of the new loan is then used to pay off the outstanding balance of original.

4.4   Understanding Loan Payments and Interest

Suppose you take out a loan of $10,000$ at 5% APR compounded annually for an unspecified amount of time. You are required to make a payment at the end of each year, but the size of the payment is up to you. The loan will be settled as soon as the present value of all of your payments is equal to the principal of the loan, where the present value is valued at the beginning of the loan.

At the very least, you should pay back at least $500 each year.

Why? At the end of one year, the outstanding balance has grown to $10,000 \cdot (1.005) = \$10,500.00$. If you pay back $500, your balance drops back down to $10,000, in which case you will never payoff the loan. The outstanding balance (right after payment) will always be $10,000. In this case, your payment size is just large enough to cover the accrued interest. This is referred to as a perpetuity. Payments continue forever.

What happens if you pay back only $400 each year? At the end of the first year, your outstanding balance is $10,500. You pay back $400 leaving you with a balance of $10,100. Your payment was not even large enough to cover that year's interest expense. After another year, the balance grows to $10,100 \cdot 1.05 = \$10,605$, which is reduced to $10,205 after the $400 payment is applied. The outstanding balance is moving in the wrong direction! Each year, you are further in debt!

What happens if you pay back more than the $500? For concreteness, suppose you pay back $800 at the end of each year. Over the course of the first year, the balance of $10,000 grows to $10,500, and after the payment of $800, the balance is knocked down to $9,700. The interest expense in the first year is $500 and the $800 payment can be broken up as

$$\$800 = \$500 + \$300$$

where the $500 is applied towards interest and the $300 is applied towards the principal.

Roughly, how long will it take to repay this loan? Principal fell $300 after the first year's payment. If it falls by same amount each year, then it will require $10,000/300 = 33.3333$ payments.

In actual fact, the loan will be paid off in fewer than 33.33333 payments. We'll see that with each passing payment, more and more of the payment is applied to the principal.

Over the second year the outstanding balance grows from $9,700$ to $9700 \cdot (1.05) = \$10,185$. The interest expense in second year is 5% of $9,700$ or 485. You still make a payment

of \$800, but now only \$485 is eaten up by interest. The remaining \$800 − \$485 = \$315 is applied towards principal.

What happens in the third year? At beginning of third year, outstanding balance is \$10,185 − \$800 = \$9,385. Interest charge over the year is 5% of \$9,385, so \$469.25. Amount towards principal is therefore \$800 − \$469.25 = \$330.75.

For first three years, amount towards principal are \$300, \$315, \$330.75. Look at the consecutive ratios:
$$\frac{315}{300} = 1.05 \text{ and } \frac{330.75}{315} = 1.05$$
Indeed, the amount applied toward principal is an exponential function in terms of the number of payments, and the base of the exponential function is the accumulation factor for this interest rate.

4.5   The Effect of Large Payments

This section investigates the effect of paying more than the requested loan payment amount. Why would you want to pay more? By making larger payments, you will pay off the loan with fewer payments. In fact, we'll see that you can potentially save a lot by making larger payments. For example, if a loan requires 60 payments of \$100 then it should take fewer than 20 payments of \$300 to pay off the loan.

Lets look at a concrete example.

Example 14. *Gil recently purchased a car, and he took out an car loan for* \$7500. *The loan is to be repaid with payments at the end of each month, for 48 months at 2.4% APR compounded monthly. What is Gil's interest expense?*

Lets determine the size of Gil's monthly payments.

Self Check 3. *Determine Gil's monthly payments.*[11]

Now that we know Gil's monthly payments are \$164.03, let's determine Gil's interest expense. He makes 48 payments of \$164.03 in order to pay back \$7500, so his interest expense is

$$48 \cdot \$164.03 − \$7500 = \$373.44$$

Example 15 (More on Gil's loan). *Suppose Gil decides to double the size of his payments, in order to pay off the loan more quickly. How many payments will he need to make?*

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[11]

$$\$7500 = R \cdot \frac{1 − (1.002)^{-48}}{0.002},$$

giving monthly payments of $R = \$164.03$.

His payment size is now $\$164.03 \cdot 2 = \$328.06$. The loan formula now reads

$$\$7500 = \$328.06 \cdot \frac{1 - (1.002)^{-n}}{0.002}$$

Notice the unknown is stuck up in the exponent. Most financial calculators will compute the term of a loan for you, but we'll review the algebra for those who want to see it. First divide both sides by 328.06 to get

$$0.045723 = 1 - 1.002^{-n}$$

Subtract 1.002^{-n} from both sides then subtract 0.045723 from both sides to get

$$1.002^{-n} = 0.954277.$$

Now take the reciprocal of each side:

$$1.002^{n} = 1.04791376.$$

Take logarithms on both sides and use laws of logarithms to get

$$n \cdot \ln(1.002) = \ln(1.047914).$$

Finally, divide both sides by $\ln(1.002)$ and evaluate the natural logarithms:

$$n = \frac{\ln(1.047914)}{\ln(1.002)} = 23.42422$$

Thus, if Gil doubles his payments, then he will need to make 23.42422 payments. (Hmmm... how can he make .42422 of a payment? We'll fix this in a minute, but for the moment assume that if makes sense to make fractional a fractional payment.)

By doubling his payments, Gil pays back $23.42 \cdot \$328.06 = \$7,684.55$ and so he pays $\$184.55$ to interest. Notice that, by doubling the size of his payments, the total time to pay off the loan is less than half that it would be had he made regular payments and the amount he pays to interest is less than half of what he would pay if he had made regular payments.

But how can he make 23.42422 payments? He can't. He will either make 23 or 24 payments. In either case, the size of the last payment won't be the same as the other payments.

Lets determine how much he will owe immediately after the 23rd payment. First, we'll find the time 0 present value of the 23 payments: $\$328.06 \cdot \frac{1-1.002^{-23}}{.002} = \$7367.27$. But the principal is $\$7500$ so $\$7500 - \$7367.27 = \$132.73$ is the time 0 value of the balance remaining immediately after the 23rd payment. Now we need to apply an accumulation factor to push the $\$132.73$ 23 months forward: $\$132.73 \cdot (1.002)^{23} = \$138.97$.

Gil now has two options.

(a.) Gil could settle the loan in 23 payments if his first 22 payments are \$328.06 and his 23rd payment includes the remaining balance[12], \$328.06+\$138.97 = \$467.03 His interest expense is then $22 \cdot \$328.06 + \$467.03 - \$7500 = \$184.35$

(b.) Gil could settle the loan with 24 payments by making 23 payments of \$328.06 and then the 24th payment will be smaller than normal. The remaining \$138.97 will accrue interest for one more month, so his 24th payment would be $\$138.97 \cdot (1.002) = \$139.25$ In this case, his interest expense is $\$328.06 \cdot 23 + \$139.25 - \$7500 = \$184.63$

Why does doubling payments have a super-linear effect on paying off the loan? Look at first month's payment. Balance grew from \$7500 to $\$7500 \cdot (1.002) = \$7515.00$. Regardless of the size of the payment, the first \$15 of the payment will pay off the accrued interest, and the remaining payment is applied towards the principal.

If Gil pays normal payment of \$164.03, then \$15 is applied to interest and the remaining \$149.03 is applied towards principal. The outstanding balance is then \$7350.97. The interest accrued in the second period is then $\$7350.97 \cdot 0.002 = \$14.70$.

If Gil doubles his payment to \$328.06, then as before, the first \$15 is applied towards the interest, and the remaining $\$328.06 - \$15 = \$313.08$ applies towards principal. The outstanding balance is now \$7186.94. The interest accrued in the second period is then $\$7186.94 \cdot 0.002 = \$14.37$, which is \$0.33 less than previously.

Making larger than expected payments causes the principal to shrink more quickly. In turn, this causes the accrued interest per period to shrink more quickly, which in turn means that even more of the next payment will be applied to the principal. Through this feedback mechanism, the effect of large and early payments is greatly amplified.

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[12]This is referred to as a *balloon payment*