CHAPTER TWO

Numbering Systems

Chapter one discussed how computers remember numbers using transistors, tiny devices that act like switches with only two positions, on or off. A single transistor, therefore, can only remember one of two possible numbers, a one or a zero. This isn't useful for anything more complex than controlling a light bulb, so for larger values, transistors are grouped together so that their combination of ones and zeros can be used to represent larger numbers.

This chapter discusses some of the methods that are used to represent numbers with groups of transistors or ***bits***. The reader will also be given methods for calculating the minimum and maximum values of each representation based on the number of bits in the group.

2.1 Unsigned Binary Counting

The simplest form of numeric representation with bits is ***unsigned binary***. When we count upward through the positive integers using decimal, we start with a 0 in the one's place and increment that value until we reach the upper limit of a single digit, i.e., 9. At that point, we've run out of the "symbols" we use to count, and we need to increment the next digit, the ten's place. We then reset the one's place to zero, and start the cycle again.

Ten's place	One's place
	0
	1
	2
	3
	:
	8
	9
1	0

Figure 2-1 Counting in Decimal

Since computers do not have an infinite number of transistors, the number of digits that can be used to represent a number is limited. This would be like saying we could only use the hundreds, tens, and ones place when counting in decimal.

This has two results. First, it limits the number of values we can represent. For our example where we are only allowed to count up to the hundreds place in decimal, we would be limited to the range of values from 0 to 999.

Second, we need a way to show others that we are limiting the number of digits. This is usually done by adding leading zeros to the number to fill up any unused places. For example, a decimal 18 would be written 018 if we were limited to three decimal digits.

Counting with bits, hereafter referred to as counting in binary, is subject to these same issues. The only difference is that decimal uses ten symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) while binary only uses two symbols (0 and 1).

To begin with, Figure 2-2 shows that when counting in binary, we run out of symbols quickly requiring the addition of another "place" after only the second increment.

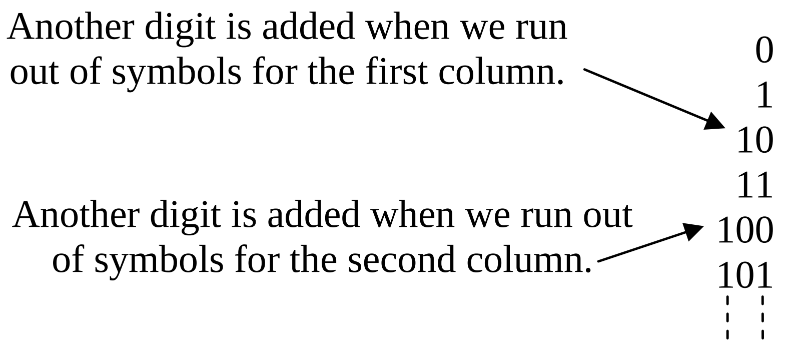

Another digit is added when we run out of symbols for the first column.
Another digit is added when we run out of symbols for the second column.
0
1
10
11
100
101

Figure 2-2 Counting in Binary

If we were counting using four bits, then the sequence would look like: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111. Notice that when restricted to four bits, we reach our limit at 1111, which happens to be the fifteenth value. It should also be noted that we ended up with 2 x 2 x 2 x 2 = 16 different values. With two symbols for each bit, we have 2^n possible combinations of symbols where n represents the number of bits.

In decimal, we know what each digit represents: ones, tens, hundreds, thousands, etc. How do we figure out what the different digits in binary represent? If we go back to decimal, we see that each place can contain one of ten digits. After the ones digit counts from 0 to

9, we need to increment the tens place. Subsequently, the third place is incremented after 9 tens and 9 ones, i.e., 99 increments, have been counted. This makes it the hundreds place.

In binary, the rightmost place is considered the ones place just like decimal. The next place is incremented after the ones place reaches 1. This means that the second place in binary represents the value after 1, i.e., a decimal 2. The third place is incremented after a 1 is in both the ones place and the twos place, i.e., we've counted to a decimal 3. Therefore, the third place represents a decimal 4. Continuing this process shows us that each place in binary represents a successive power of two.

Figure 2-3 uses 5 bits to count up to a decimal 17. Examine each row where a single one is present in the binary number. This reveals what that position represents. For example, a binary 01000 is shown to be equivalent to a decimal 8. Therefore, the fourth bit position from the right is the 8’s position.

Decimal value	Binary value
0	00000
1	00001
2	00010
3	00011
4	00100
5	00101
6	00110
7	00111
8	01000

Decimal value	Binary value
9	01001
10	01010
11	01011
12	01100
13	01101
14	01110
15	01111
16	10000
17	10001

Figure 2-3 Binary-Decimal Equivalents from 0 to 17

This information will help us develop a method for converting unsigned binary numbers to decimal and back to unsigned binary.

Some of you may recognize this as "base-2" math. This gives us a method for indicating which representation is being used when writing a number down on paper. For example, does the number 100 represent a decimal value or a binary value? Since binary is base-2 and decimal is base-10, a subscript "2" is placed at the end of all binary numbers in this book and a subscript "10" is placed at the end of all decimal numbers. This means a binary 100 should be written as 100_2 and a decimal 100 should be written as 100_{10}.

2.2 Binary Terminology

When writing values in decimal, it is common to separate the places or positions of large numbers in groups of three digits separated by commas. For example, 345323745_{10} is typically written $345,323,745_{10}$ showing that there are 345 millions, 323 thousands, and 745 ones. This practice makes it easier to read and comprehend the magnitude of the numbers. Binary numbers are also divided into components depending on their application. Each binary grouping has been given a name.

To begin with, a single place or position in a binary number is called a ***bit***, short for binary digit. For example, the binary number 0110_2 is made up of four bits. The rightmost bit, the one that represents the ones place, is called the ***Least Significant Bit or LSB***. The leftmost bit, the one that represents the highest power of two for that number, is called the ***Most Significant Bit or MSB***. Note that the MSB represents a bit position. It doesn't mean that a '1' must exist in that position.

The next four terms describe how bits might be grouped together.

- ***Nibble*** – A four bit binary number
- ***Byte*** – A unit of storage for a single character, typically an eight bit (2 nibble) binary number (short for binary term)
- ***Word*** – Typically a sixteen bit (2 byte) binary number
- ***Double Word*** – A thirty-two bit (2 word) binary number

The following are some examples of each type of binary number.

Bit	1_2
Nibble	1010_2
Byte	10100101_2
Word	1010010111110000_2
Double Word	$10100101111100001100111011101101_2$

2.3 Unsigned Binary to Decimal Conversion

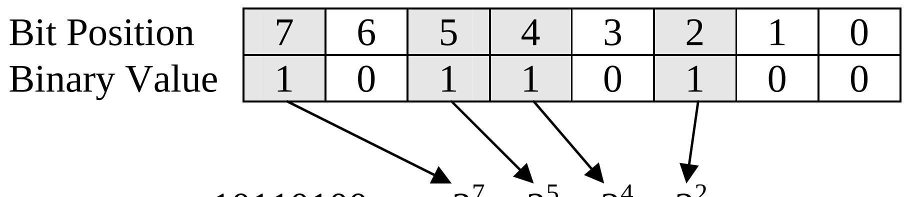
As shown in section 2.1, each place or position in a binary number corresponds to a specific power of 2 starting with the rightmost bit which represents $2^0=1$. It is through this organization of the bits that we will convert binary numbers to their decimal equivalent. Figure 2-4 shows the bit positions and the corresponding powers of two for each bit in positions 0 through 7.

Numbered bit position	7	6	5	4	3	2	1	0
Corresponding power of 2	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Decimal equivalent of power of 2	128	64	32	16	8	4	2	1

Figure 2-4 Values Represented By Each of the First 8 Bit Positions

To begin converting an unsigned binary number to decimal, identify each bit position that contains a 1. It is important to note that we number the bit positions starting with 0 identifying the rightmost bit.

Next, add the powers of 2 for each position containing a 1. This sum is the decimal equivalent of the binary value. An example of this process is shown in Figure 2-5 where the binary number 10110100_2 is converted to its decimal equivalent.



Bit Position	7	6	5	4	3	2	1	0
Binary Value	1	0	1	1	0	1	0	0

$$\begin{aligned} 10110100_2 &= 2^7 + 2^5 + 2^4 + 2^2 \\ &= 128_{10} + 32_{10} + 16_{10} + 4_{10} \\ &= 180_{10} \end{aligned}$$

Figure 2-5 Sample Conversion of 10110100_2 to Decimal

This brings up an important issue when representing numbers with a computer. Note that when a computer stores a number, it uses a limited number of transistors. If, for example, we are limited to eight transistors, each transistor storing a single bit, then we have an upper limit to the size of the decimal value we can store.

The largest unsigned eight bit number we can store has a 1 in all eight positions, i.e., 11111111_2. This number cannot be incremented without forcing an overflow to the next highest bit. Therefore, the largest decimal value that 8 bits can represent in unsigned binary is the sum of all powers of two from 0 to 7.

$$\begin{aligned} 11111111_2 &= 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 \\ &= 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 \\ &= 255_{10} \end{aligned}$$

If you add one to this value, the result is 256 which is 2^8, the power of two for the next bit position. This makes sense because if you add 1 to 11111111_2, then beginning with the first column, 1 is added to 1 giving us a result of 0 with a 1 carry to the next column. This propagates to the MSB where a final carry is passed to the ninth bit. The final value is then $100000000_2 = 256_{10}$.

$$11111111_2 + 1 = 100000000_2 = 256_{10} = 2^8$$

Therefore, the maximum value that can be represented with 8 bits in unsigned binary is $2^8 - 1 = 255$.

It turns out that the same result is found for any number of bits. The maximum value that can be represented with n bits in unsigned binary is $2^n - 1$.

$$\text{Max unsigned binary value represented with n bits} = 2^n - 1 \qquad (2.1)$$

We can look at this another way. Each digit of a binary number can take on 2 possible values, 0 and 1. Since there are two possible values for the first digit, two possible values for the second digit, two for the third, and so on until you reach the n-th bit, then we can find the total number of possible combinations of 1's and 0's for n-bits by multiplying 2 n-times, i.e., 2^n.

How does this fit with our upper limit of 2^n-1? Where does the "-1" come from? Remember that counting using unsigned binary integers begins at 0, not 1. Giving 0 one of the bit patterns takes one away from the maximum value.

2.4 Decimal to Unsigned Binary Conversion

Converting from decimal to unsigned binary is a little more complicated, but it still isn't too difficult. Once again, there is a well-defined process.

To begin with, it is helpful to remember the powers of 2 that correspond to each bit position in the binary numbering system. These were presented in Figure 2-4 for the powers of 2^0 up to 2^7.

What we need to do is separate the decimal value into its power of 2 components. The easiest way to begin is to find the largest power of 2 that is less than or equal to our decimal value. For example if we were converting 75_{10} to binary, the largest power of 2 less than or equal to 75_{10} is $2^6 = 64$.

The next step is to place a 1 in the location corresponding to that power of 2 to indicate that this power of 2 is a component of our original decimal value.

Next, subtract this first power of 2 from the original decimal value. In our example, that would give us $75_{10} – 64_{10} = 11_{10}$. If the result is not equal to zero, go back to the first step where we found the largest power of 2 less than or equal to the new decimal value. In the case of our example, we would be looking for the largest power of 2 less than or equal to 11_{10} which would be $2^3 = 8$.

When the result of the subtraction reaches zero, and it eventually will, then the conversion is complete. Simply place 0's in the bit positions that do not contain 1's. Figure 2-6 illustrates this process using a flowchart.

If you get all of the way to bit position zero and still have a non-zero result, then one of two things has happened. Either there was an error in one of your subtractions or you did not start off with a large enough number of bits. Remember that a fixed number of bits, n, can only represent an integer value up to $2^n – 1$. For example, if you are trying to convert 312_{10} to unsigned binary, eight bits will not be enough because the highest value eight bits can represent is $2^8 – 1 = 255_{10}$. Nine bits, however, will work because its maximum unsigned value is $2^9 – 1 = 511_{10}$.

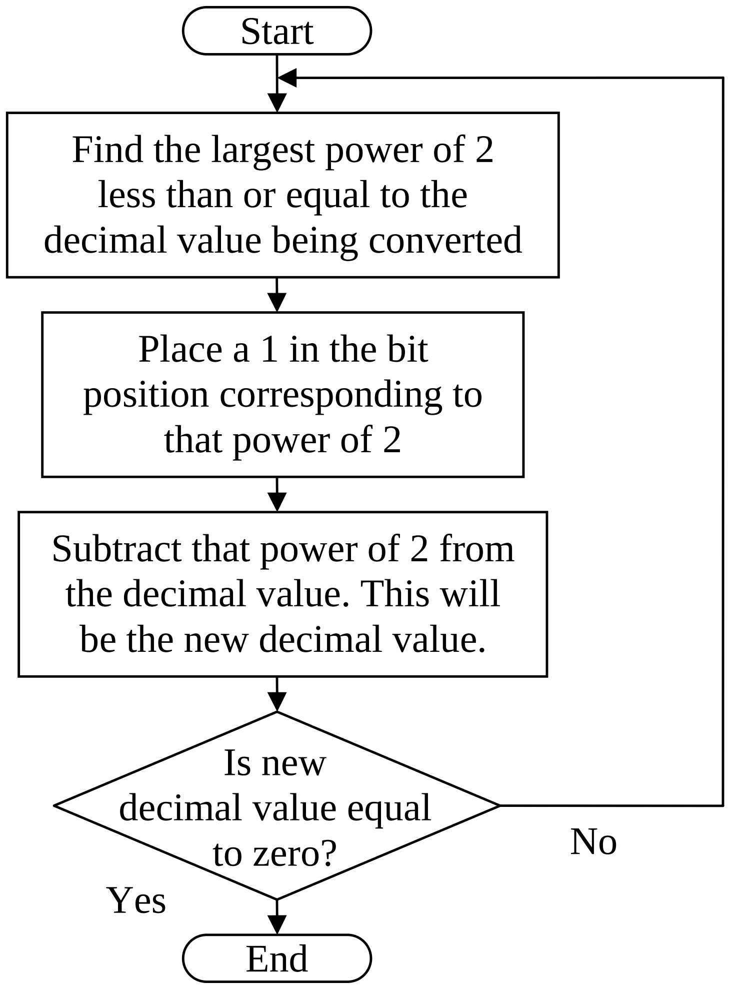

Start
Find the largest power of 2 less than or equal to the decimal value being converted
Place a 1 in the bit position corresponding to that power of 2
Subtract that power of 2 from the decimal value. This will be the new decimal value.
Is new decimal value equal to zero?
No
Yes
End

Figure 2-6 Decimal to Unsigned Binary Conversion Flow Chart

Example

Convert the decimal value 133_{10} to an 8 bit unsigned binary number.

Solution

Since 133_{10} is less than $2^8 - 1 = 255$, 8 bits will be sufficient for this conversion. Using Figure 2-4, we see that the largest power of 2 less than or equal to 133_{10} is $2^7 = 128$. Therefore, we place a 1 in bit position 7 and subtract 128 from 133.

Bit position	7	6	5	4	3	2	1	0
	1							

$$133 - 128 = 5$$

Our new decimal value is 5. Since this is a non-zero value, our next step is to find the largest power of 2 less than or equal to 5. That would be $2^2 = 4$. So we place a 1 in the bit position 2 and subtract 4 from 5.

Bit position	7	6	5	4	3	2	1	0
	1					1		

$$5 - 4 = 1$$

Our new decimal value is 1, so find the largest power of 2 less than or equal to 1. That would be $2^0 = 1$. So we place a 1 in the bit position 0 and subtract 1 from 1.

Bit position	7	6	5	4	3	2	1	0
	1					1		1

$$1 - 1 = 0$$

Since the result of our last subtraction is 0, the conversion is complete. Place zeros in the empty bit positions.

Bit position	7	6	5	4	3	2	1	0
	1	0	0	0	0	1	0	1

And the result is:

$$133_{10} = 10000101_2$$

2.5 Binary Representation of Analog Values

Converting unsigned (positive) integers to binary is only one of the many ways that computers represent values using binary bits. This chapter still has two more to cover, and Chapter 3 will cover even more.

This section focuses on the problems and solutions of trying to map real world values such as temperature or weight from a specified range to a binary integer. For example, a computer that uses 8 bits to represent an integer is capable of representing 256 individual values from 0 to 255. Temperature, however, is a floating-point value with unrealistic upper and lower limits. Can we get a computer to represent a temperature using eight bits? The answer is yes, but it will cost us in the areas of resolution and range.

Another example of analog values is the pattern of sound waves such as that from music. Figure 2-7 represents such a signal.

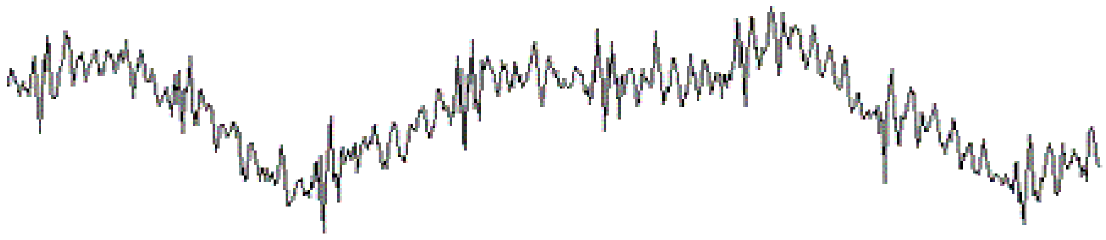


Figure 2-7 Sample Analog Signal of Sound

Remember that a single binary bit can be set to only one of two values: logic 1 or logic 0. Combining many bits together allows for a range of integers, but these are still discrete values. The real world is analog, values represented with floating-point measurements capable of infinite resolution. To use an n-bit binary number to represent analog, we need to put some restrictions on what is being measured.

First, an n-bit binary number has a limited range. We saw this when converting unsigned positive integers to binary. In this case, the lower limit was 0 and the upper limit was 2^n-1. To use n-bits to represent an analog value, we need to restrict the allowable range of analog measurements. This doesn't need to be a problem.

For example, does the typical bathroom scale need to measure values above 400 pounds? If not, then a digital system could use a 10-bit binary number mapped to a range from zero to 400 pounds. A binary 0000000000_2 could represent zero pounds while 1111111111_2 could represent 400 pounds.

What is needed next is a method to map the values inside the range zero to 400 pounds to the binary integers in the range 0000000000_2 to 1111111111_2. To do this, we need a linear function defining a one-to-one mapping between each binary integer and the analog value it represents. To do this, we turn to the basic math expression for a linear function.

$$y = mx + b$$

This function defines m as the rate of the change in y with respect to changes in x and b as the value y is set to when x equals 0. We can use this expression to map a binary integer x to an analog value y.

The slope of this function, m, can be calculated by dividing the range of analog values by the number of intervals defined by the n-bit binary integer. The number of intervals defined by the n-bit binary integer is equal to the upper limit of that binary number if it were being used as an unsigned integer, i.e., 2^n-1.

$$m = \frac{\text{Range of analog values}}{\text{Number of intervals of binary integer}}$$

$$m = \frac{\text{Max analog value - Min analog value}}{2^n - 1} \qquad (2.2)$$

Let's go back to our example of the kitchen scale where the maximum analog value is 400 pounds while the minimum is zero pounds. If a 10-bit binary value is used to represent this analog value, then the number of intervals of the binary integer is $2^{10} - 1 = 1023$. This gives us a slope of:

$$m = \frac{400 \text{ pounds} - 0 \text{ pounds}}{1023 \text{ binary increments}} = 0.391 \text{ pounds/binary increment}$$

That means that each time the binary number increments, e.g., 0110110010_2 goes to 0110110011_2, it represents an increment in the analog value of 0.391 pounds. Since a binary value of 0000000000_2 represents an analog value of 0 pounds, then 0000000001_2 represents 0.391 pounds, 0000000010_2 represents $2 \times 0.391 = 0.782$ pounds, 0000000011_2 represents $3 \times 0.391 = 1.173$ pounds, and so on.

In some cases, the lower limit might be something other than 0. This is important especially if better accuracy is required. For example, a kitchen oven may have an upper limit of 600^oF. If zero were used as the lower limit, then the temperature range $600^o\text{F} - 0^o\text{F} = 600^o\text{F}$ would need to be mapped to the 2^n possible binary values of an n-bit binary number. For a 9-bit binary number, this would result in an m of:

$$m = \frac{600^o\text{F} - 0^o\text{F}}{2^9 - 1} = 1.1742 \text{ degrees/binary increment}$$

Does an oven really need to measure values below 100^{o}F though? If not, a lower limit of 100^{o}F could be used reducing the size of the analog range to 500^{o}F. This smaller range would improve the accuracy of the system because each change in the binary value would result in a smaller increment in the analog value.

$$m = \frac{600^oF - 100^oF}{2^9 - 1} = 0.9785 \text{ degrees/binary increment}$$

The smaller increment means that each binary value will be a more accurate representation of the analog value.

This non-zero lower limit is realized as a non-zero value for **b** in the linear expression y=mx + b. Since **y** is equal to **b** when **x** is equal to zero, then b must equal the lower limit of the range.

$$b = \text{Minimum analog value} \qquad 2.3$$

The final expression representing the conversion between an analog value at its binary representation is shown in Equation 2.4.

$$A_{calc} = \left(\frac{A_{max} - A_{min}}{2^n - 1} * X\right) + A_{min} \qquad (2.4)$$

where:

A_{calc} = analog value represented by binary value
A_{max} = maximum analog value
A_{min} = minimum analog value
X = binary value representing analog value
n = number of bits in binary value

Example

Assume that the processor monitoring the temperature of an oven with a temperature range from 100^{o}F to 600^{o}F measures a 9-bit binary value of 011001010_2. What temperature does this represent?

Solution

Earlier, we calculated the rate of change, m, for an oven with a temperature range from 100^{o}F to 600^{o}F is 500^{o}F ÷ 511 binary increments. Substituting this along with our minimum analog value of 100^{o}F into Equation 2.4 gives us:

$$\text{temperature} = \frac{500}{511}\,^{o}\text{F/binary increment} * \text{binary value} + 100^{o}\text{F}$$

If the processor monitoring the temperature of this oven reads a binary value of 011001010_2, the approximate temperature can be determined by converting 011001010_2 to decimal and inserting it into the equation above.

$$\begin{aligned} 011001010_2 &= 2^7 + 2^6 + 2^3 + 2^1 \\ &= 128 + 64 + 8 + 2 \\ &= 202_{10} \end{aligned}$$

$$\text{temperature} = \frac{500^{o}\text{F}}{511} * 202 + 100^{o}\text{F}$$

$$\text{temperature} = 297.65^{o}\text{F}$$

The value from the above example is slightly inaccurate. The binary value 011001010_2 actually represents a range of values 0.9785^{o}F wide centered around or with a lower limit of 297.65^{o}F. Only a binary value with an infinite number of bits would be entirely accurate. Since this is not possible, there will always be a gap or resolution associated with a digital system due to the quantized nature of binary integer values. That gap is equivalent to the increment or rate of change of the linear expression.

$$\text{Resolution} = \frac{\text{Analog range}}{2^n - 1} \qquad (2.5)$$

Example

Assume that the analog range of a system using a 10-bit analog-to-digital converter goes from a lower limit of 5 ounces to an upper limit of 11 ounces. What is the resolution of this system?

Solution

To determine the resolution, we begin with the analog range.

$$\begin{aligned}\text{Analog range} &= \text{Max analog value - Min analog value}\\ &= \text{11 ounces – 5 ounces}\\ &= \text{6 ounces}\end{aligned}$$

Substituting this range into equation 2.5 and using n=10 to represent the number of bits, we get:

$$\text{Resolution} = \frac{\text{6 ounces}}{2^{10} - 1}$$

$$= \frac{\text{6 ounces}}{\text{1023 increments}}$$

$$= \text{0.005865 oz/inc}$$

If we examine the results of the example above, we see that our system can measure 0 ounces, 0.005865 ounces, 0.011730 ounces, (2 * 0.005865 ounces), 0.017595 (3 * 0.005865 ounces), and so on, but it can never represent the measurement 0.015 ounces. Its resolution is not that good. In order to get that resolution, you would need to increase the number of bits in the binary integer or reduce the analog range.

Example

How many bits would be needed for the example above to improve the resolution to better than 0.001 ounces per increment?

Solution

Each time we increase the number of bits in our binary integer by one, the number of increments in the range is approximately doubled. For example, going from 10 bits to 11 bits increases the number of increments in the range from $2^{10} – 1 = 1023$ to $2^{11} – 1 = 2047$. The question is how high do we have to go to get to a specified resolution? To answer that, let's begin by setting Equation 2.5 to represent the fact that we want a resolution of ***better than*** 0.001 ounces/increment.

$$\text{0.001 oz/inc.} > \frac{\text{6 ounces}}{2^n – 1}$$

Solving for $2^n – 1$ gives us:

$$2^n - 1 > \frac{\text{6 ounces}}{\text{0.001 oz/inc.}}$$

$$2^n - 1 > \quad \text{6,000 increments}$$

By substituting different integers for n into the above equation, we find that n=13 is the lowest value of n for which a resolution better than 0.001 ounces/increment is reached. n=13 results in a resolution of $6 \div 8191 = 0.0007325$ ounces/increment.

2.6 Sampling Theory

The previous discussion of the integer representation of analog values shows how the number of bits can affect the roundoff error of the representation. In general, an n-bit analog-to-digital converter divides the analog range into $2^n - 1$ increments. Figure 2-8 presents four graphs, each with a different number of bits providing different levels of resolution. The figure shows how the addition of a bit can improve the resolution of the values represented by the binary integers.

Earlier, it was mentioned how a computer can only capture a "snap shot" or sample of an analog voltage. This is sufficient for slowly varying analog values, but if a signal is varying quickly, details might be missed. To improve the signal's digital representation, the rate at which the samples are taken, the ***sampling rate***, needs to be increased.

There is also a chance of missing a higher frequency because the sampling rate is too slow. This is called ***aliasing***, and there are examples of it in everyday life.

When riding in a car at night, you may have noticed that at times the wheels of an adjacent car appear to be spinning at a different rate than they really are or even appear to spin backwards. (If you have no idea what I'm talking about, watch the wheels of the car next to you the next time you are a passenger riding at night under street lights.)

The effect is caused by the fact that the light from street lamps actually pulses, a fact that is usually not detectable with the human eye. This pulsing provides a sampling rate, and if the sampling rate is not fast enough for the spinning wheel, the wheel appears to be spinning at a different rate than it really is. Street lights are not necessary to see this effect. Your eye has a sampling rate of its own which means that you may experience this phenomenon in the day time.

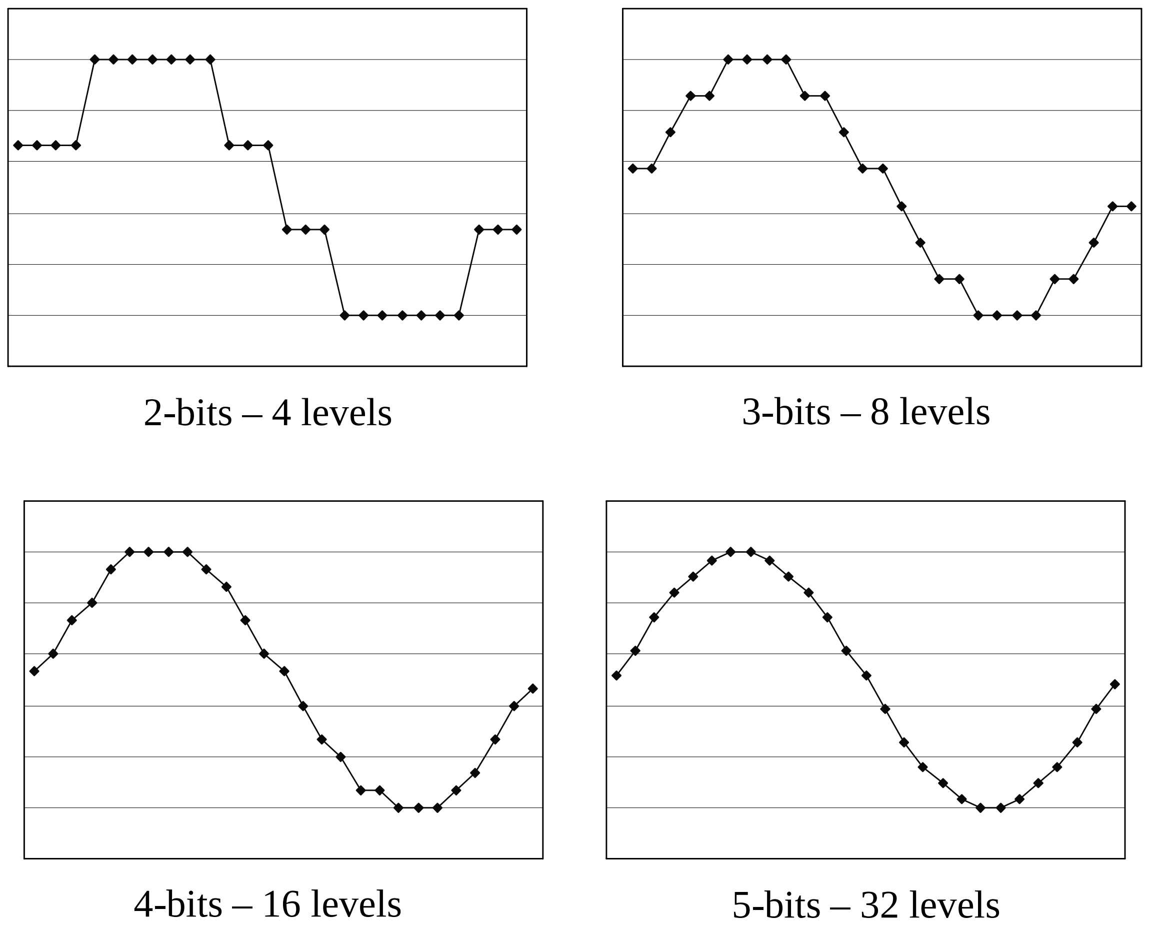


Figure 2-8 Effects of Number of Bits on Roundoff Error

Aliasing is also the reason fluorescent lights are never used in sawmills. Fluorescent lights blink much like a very fast strobe light and can make objects appear as if they are not moving. If the frequency of the fluorescent lights and the speed of a moving saw blade are multiples of each other, it can appear as if the spinning blade is not moving at all.

Both of these examples are situations where aliasing has occurred. If a signal's frequency is faster than the sampling rate, then information will be lost, and the collected data will never be able to duplicate the original.

The graphs in Figure 2-9 show how different sampling rates can result in different interpretations of the collected data, the dark points representing the samples. Note that the bottom-right graph represents a good sampling rate. When the computer reproduces the signal, the choppiness of the reproduction will be removed due to the natural filtering effects of analog circuitry.

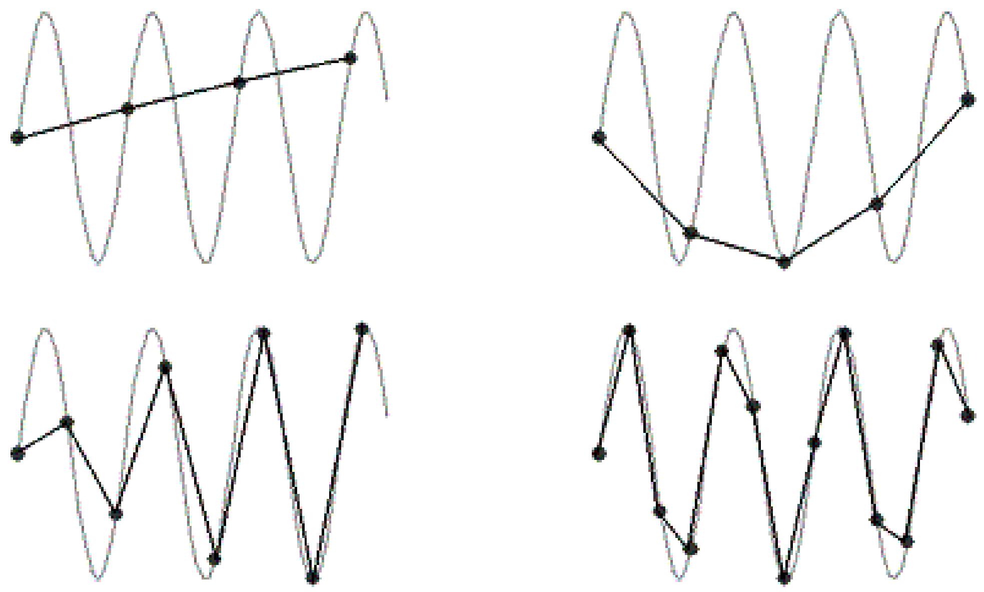


Figure 2-9 Aliasing Effects Due to Slow Sampling Rate

To avoid aliasing, the rate at which samples are taken must be more than twice as fast as the highest frequency you wish to capture. This is called the ***Nyquist Theorem***. For example, the sampling rate for audio CDs is 44,100 samples/second. Dividing this number in half gives us the highest frequency that an audio CD can play back, i.e., 22,050 Hz.

For an analog telephone signal, a single sample is converted to an 8-bit integer. If these samples are transmitted across a single channel of a T1 line which has a data rate of 56 Kbps (kilobits per second), then we can determine the sampling rate.

$$\text{Sampling rate}_{T1} = \frac{56{,}000 \text{ bits/second}}{8 \text{ bits/sample}}$$

$$\text{Sampling rate}_{T1} = 7{,}000 \text{ samples/second}$$

This means that the highest analog frequency that can be transmitted across a telephone line using a single channel of a T1 link is 7,000÷2 = 3,500 Hz. That's why the quality of voices sent over the telephone is poor when compared to CD quality. Although telephone users can still recognize the voice of the caller on the opposite end of the line when the higher frequencies are eliminated, their speech often sounds muted.

2.7 Hexadecimal Representation

It is usually difficult for a person to look at a binary number and instantly recognize its magnitude. Unless you are quite experienced at using binary numbers, recognizing the relative magnitudes of 10101101_2 and 10100101_2 is not immediate (173_{10} is greater than 165_{10}). Nor is it immediately apparent to us that 1001101101_2 equals 621_{10} without going through the process of calculating 512 + 64 + 32 + 8 + 4 + 1.

There is another problem: we are prone to creating errors when writing or typing binary numbers. As a quick exercise, write the binary number $1001011111101101001000111_2$ onto a sheet of paper. Did you make a mistake? Most people would have made at least one error.

To make the binary representation of numbers easier on us humans, there is a shorthand representation for binary values. It begins by partitioning a binary number into its nibbles starting at the least significant bit (LSB). An example is shown below:

The number:	1001011110110100100111					
…can be divided into:	10	0101	1110	1101	0010	0111

Next, a symbol is used to represent each of the possible combinations of bits in a nibble. We start by numbering them with the decimal values equivalent to their binary value, i.e.:

$$0000_2 = 0_{10}$$
$$0001_2 = 1_{10}$$
$$0010_2 = 2_{10}$$
$$: \; : \; :$$
$$1000_2 = 8_{10}$$
$$1001_2 = 9_{10}$$

At 9, however, we run out of decimal characters. There are six more nibbles to label, so we begin using letters: A, B, C, D, E, and F. These represent the decimal values 10_{10}, 11_{10}, 12_{10}, 13_{10}, 14_{10}, and 15_{10} respectively.

$$1010_2 = A$$
$$1011_2 = B$$
$$: \; : \; :$$
$$1111_2 = F$$

Table 2-1 presents the mapping between the sixteen patterns of 1's and 0's in a binary nibble and their corresponding decimal and hexadecimal (hex) values.

Table 2-1 Converting Binary to Decimal and Hexadecimal

Binary	Decimal	Hex
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7

Binary	Decimal	Hex
1000	8	8
1001	9	9
1010	10	A
1011	11	B
1100	12	C
1101	13	D
1110	14	E
1111	15	F

Another way to look at it is that hexadecimal counting is also similar to decimal except that instead of having 10 numerals, it has sixteen. This is also referred to as a base-16 numbering system.

How do we convert binary to hexadecimal? Begin by dividing the binary number into its nibbles (if the number of bits is not divisible by 4, add leading zeros), then nibble-by-nibble use the table above to find the hexadecimal equivalent to each 4-bit pattern. For example:

The number:	1001011110110100100111					
…is divided into:	0010	0101	1110	1101	0010	0111
...which translates to:	2	5	E	D	2	7

Therefore, $1001011110110100100111_2 = 25ED27_{16}$. Notice the use of the subscript "16" to denote hexadecimal representation.

Going the other way is just as easy. Translating $5D3F21_{16}$ to binary goes something like this:

The hexadecimal value:	5	D	3	F	2	1
…translates to:	0101	1101	0011	1111	0010	0001

Therefore, $5D3F21_{16} = 010111010011111100100001_2$.

It is vital to note that computers do not use hexadecimal, humans do. Hexadecimal provides humans with a reliable, short-hand method of writing large binary numbers.

2.8 Binary Coded Decimal

When was the last time you multiplied your house number by 5? Or have you ever added 215 to your social security number? These questions seem silly, but they reveal an important fact about numbers. Some numbers do not need to have mathematical operations performed on them, and therefore, do not need to have a mathematically correct representation in binary.

In an effort to afford decimal notation the same convenience of conversion to binary that hex has, Binary Coded Decimal (BCD) was developed. It allows for fast conversion to binary of integers that do not require mathematical operations.

As in hex, each decimal digit represents a nibble of the binary equivalent. Table 2-2 shows the conversion between each decimal digit and the binary equivalent.

Table 2-2 Converting BCD to Decimal

BCD Nibble	Decimal Digit	BCD Nibble	Decimal Digit
0000	0	1000	8
0001	1	1001	9
0010	2	1010	Invalid
0011	3	1011	Invalid
0100	4	1100	Invalid
0101	5	1101	Invalid
0110	6	1110	Invalid
0111	7	1111	Invalid

For example, the BCD value 0001 0110 1001 0010 equals 1692_{10}.

It is important to note that there is no algorithmic conversion between BCD and decimal. BCD is only a method for representing decimal numbers in binary.

Another item of note is that not all binary numbers convert from BCD to decimal. 0101 1011 0101 for example is an illegal BCD value because the second nibble, 1011, does not have a corresponding decimal value.

There are two primary advantages of BCD over binary. First, any mathematical operation based on a factor of ten is simpler in BCD. Multiplication by ten, for example, appends a nibble of zeros to the right side of the number. All it takes to truncate or round a base-10 value in BCD is to zero the appropriate nibbles. Because of this advantage, BCD is used frequently in financial applications due to legal requirements that decimal values be ***exactly*** represented. Binary cannot do this for fractions as we shall see in Chapter 3.

The second advantage is that conversion between entered or displayed numeric characters and the binary value being stored is fast and does not require much code.

The primary disadvantage is that unless the operation is based on a power of ten, mathematical operations are more complex and require more hardware. In addition, BCD is not as compact as unsigned binary and may require more memory for storage.

BCD can be used to represent signed values too, although there are many implementations. Different processor manufacturers use different methods making it hard to select a standard. One of the easiest ways to represent negative numbers in BCD is to add a nibble at the beginning of the number to act as a plus/minus sign. By using one of the illegal BCD values to represent a negative sign and another to represent a positive sign, BCD values can be made negative or positive. Binary values of 1010, 1100, or 1110 typically mean the number is positive while binary values of 1011 or 1101 mean the number is negative. For example, –1234 in signed BCD would be 1101 0001 0010 0011 0100 while +1234 would be 1100 0001 0010 0011 0100. BCD values preceded with 1111 typically indicate unsigned values.

2.9 Gray Codes

The use of binary counting sequences is common in digital applications. For example, an n-bit binary value can be used to identify the position of a rotating shaft as being within one of 2^n different arcs.

As the shaft turns, a sensor can detect which of the shaft's arcs it is aligned with by reading a digital value and associating it with a specific arc. By remembering the previous position and timing the changes between positions, a processor can also compute speed and direction.

Figure 2-10 shows how a shaft's position might be divided into eight arcs using three bits. This would allow a processor to determine the shaft's position to within $360^o/8 = 45^o$.

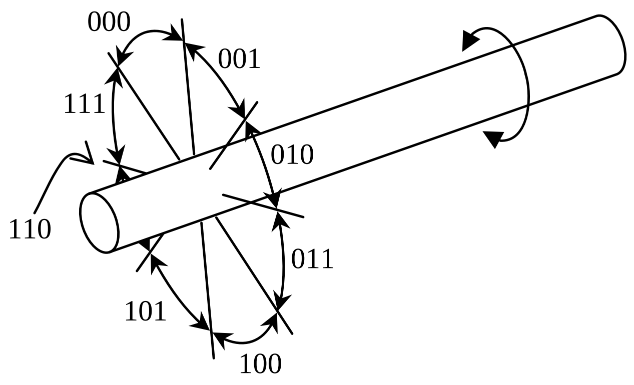


Figure 2-10 Eight Binary Values Identifying Rotating Shaft Position

One type of shaft position sensor uses a disk mounted to the shaft with slots cut into the disk at different radii representing different bits. Light sources are placed on one side of the disk while sensors on the other side of the disk detect when a hole is present, i.e., the sensor is receiving light. Figure 2-11 presents a disk that might be used to identify the shaft positions of the example from Figure 2-10.

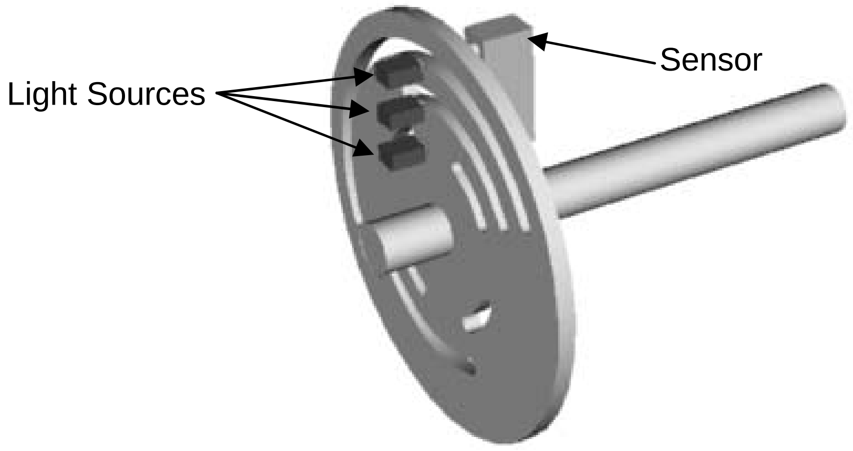


Figure 2-11 Example of a Position Encoder

In its current position in the figure, the slots in the disk are lined up between the second and third light sensors, but not the first. This means that the sensor will read a value of 110 indicating the shaft is in position number $110_2 = 6$.

There is a potential problem with this method of encoding. It is possible to read the sensor at the instant when more than one gap is opening or closing between its light source and sensor. When this happens, some of the bit changes may be detected while others are not. If this happens, an erroneous measurement may occur.

For example, if the shaft shown above turns clockwise toward position $101_2 = 5$, but at the instant when the sensor is read, only the first bit change is detected, then the value read will be $111_2 = 7$ indicating counter-clockwise rotation.

To solve this problem, alternate counting sequences referred to as the ***Gray code*** are used. These sequences have only one bit change between values. For example, the values assigned to the arcs of the above shaft could follow the sequence 000, 001, 011, 010, 110, 111, 101, 100. This sequence is not correct numerically, but as the shaft turns, only one bit will change as the shaft turns from one position to the next.

There is an algorithm to convert an n-bit unsigned binary value to its corresponding n-bit Gray code. Begin by adding a 0 to the most significant end of the unsigned binary value. There should now be n boundaries between the n+1 bits. For each boundary, write a 0 if the adjacent bits are the same and a 1 if the adjacent bits are different. The resulting value is the corresponding n-bit Gray code value. Figure 2-12 presents an example converting the 6 bit value 100011_2 to Gray code.

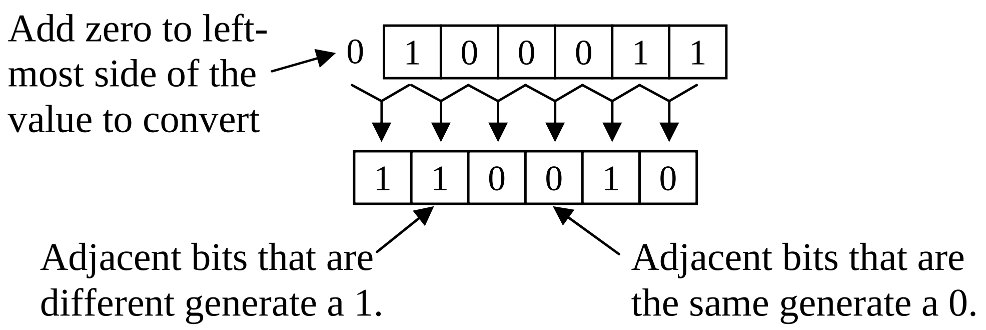

Add zero to left-most side of the value to convert
0
1 0 0 0 1 1
1 1 0 0 1 0
Adjacent bits that are different generate a 1.
Adjacent bits that are the same generate a 0.

Figure 2-12 Conversion from Unsigned Binary to Gray Code

Using this method, the Gray code for any binary value can be determined. Table 2-3 presents the full Gray code sequence for four bits. The shaded bits in third column are bits that are different then the bit immediately to their left. These are the bits that will become ones in the Gray code sequence while the bits not shaded are the ones that will be zeros. Notice that exactly one bit changes in the Gray code from one row to the next and from the bottom row to the top row.

Table 2-3 Derivation of the Four-Bit Gray Code

Decimal	Binary	Binary w/starting zero	Gray Code
0	0 0 0 0	0 0 0 0 0	0 0 0 0
1	0 0 0 1	0 0 0 0 1	0 0 0 1
2	0 0 1 0	0 0 0 1 0	0 0 1 1
3	0 0 1 1	0 0 0 1 1	0 0 1 0
4	0 1 0 0	0 0 1 0 0	0 1 1 0
5	0 1 0 1	0 0 1 0 1	0 1 1 1
6	0 1 1 0	0 0 1 1 0	0 1 0 1
7	0 1 1 1	0 0 1 1 1	0 1 0 0
8	1 0 0 0	0 1 0 0 0	1 1 0 0
9	1 0 0 1	0 1 0 0 1	1 1 0 1
10	1 0 1 0	0 1 0 1 0	1 1 1 1
11	1 0 1 1	0 1 0 1 1	1 1 1 0
12	1 1 0 0	0 1 1 0 0	1 0 1 0
13	1 1 0 1	0 1 1 0 1	1 0 1 1
14	1 1 1 0	0 1 1 1 0	1 0 0 1
15	1 1 1 1	0 1 1 1 1	1 0 0 0

2.10 What's Next?

In this chapter, we've covered the different methods of representing values, specifically positive integers, using digital circuitry. In addition to counting integers, the issues surrounding the conversion of analog or "real world" values to digital were examined along with some of the problems encountered when sampling. Finally, two methods of binary representation were presented: hexadecimal and BCD.

Chapter 3 examines the special needs surrounding the digital representation of addition, subtraction, and floating-point values. It also introduces the operation of the processor in handling some arithmetic functions.

Problems

1. What is the minimum number of bits needed to represent 768_{10} using unsigned binary representation?
2. What is the largest possible integer that can be represented with a 6-bit unsigned binary number?
3. Convert each of the following values to decimal.
   a) 10011101_2 b) 10101_2 c) 111001101_2 d) 01101001_2
4. Convert each of the following values to an 8-bit unsigned binary value.
   a) 35_{10} b) 100_{10} c) 222_{10} d) 145_{10}
5. If an 8-bit binary number is used to represent an analog value in the range from 0_{10} to 100_{10}, what does the binary value 01100100_2 represent?
6. If an 8-bit binary number is used to represent an analog value in the range from 32 to 212, what is the accuracy of the system? In other words, if the binary number is incremented by one, how much change does it represent in the analog value?
7. Assume a digital to analog conversion system uses a 10-bit integer to represent an analog temperature over a range of -25^{o}F to 125^{o}F. If the actual temperature being read was 65.325^{o}F, what would be the closest possible value that the system could represent?
8. What is the minimum sampling rate needed in order to successfully capture frequencies up to 155 KHz in an analog signal?
9. Convert the following numbers to hexadecimal.
   a) 1010111100101100011_2
   b) 10010101001001101001_2
   c) $0110110100101001100 1_2$
   d) 101011001000010_2
10. Convert each of the following hexadecimal values to binary.
    a) $ABCD_{16}$ b) $1DEF_{16}$ c) 8645_{16} d) $925A_{16}$

11. True or False: A list of numbers to be added would be a good candidate for conversion using BCD.
12. Determine which of the following binary patterns represent valid BCD numbers (signed or unsigned). Convert the valid ones to decimal.
    a.) 1010111100101100011
    b.) 10010101001001101001
    c.) 01101101001010011001
    d.) 11000110010000010000
    e.) 1101100101110010
    f.) 111100010010010101101000
    g.) 10101100100010
13. Convert the decimal number 96404_{10} to BCD.
14. Create the 5-bit Gray code sequence.