Challenge Math:
Exciting Mathematical Enrichment Explorations for Elementary Students

for my family:
Stephen, Sam, and Maggie

Introduction for Teachers

Why Do Challenge Math Groups?

Children learn best when they are taught at (or slightly above) a level they are ready for. As soon as a classroom of children has more than one child in it, there are a range of abilities, not just in mathematics, but in everything.

Enrichment pullout groups for the children who are ready for more advanced topics in math have many benefits:

- The children studying the advanced topics get to see mathematics as exciting, vibrant, and creative instead of thinking that math is always something that requires memorization, speed, and no creativity. In actuality, that's the exact opposite of what the study of mathematics is all about. In weekly pullouts with interesting, meaty questions, the students come alive and look forward to "playing math games" (where they're actually learning complex ideas and stretching their brains) every week.
- Having students work in groups (as opposed to handing your bright students a workbook to work on when the classroom material isn't challenging enough) with other children ready for advanced material shows them that mathematics is not a solitary discipline -- mathematics is exciting and vibrant and creative and fun. Students learn that being good at mathematics is not a dirty little secret to hide from their peers, but that others in their class also find comfort in symmetry and joy in patterns.
- The students who are not ready for the advanced topics can get more instruction time at their own level with a different parent volunteer who works with them on what they are ready to learn.
- The lucky parents who get to direct a challenge math group get to feel useful and connected to their children's lives. They'll learn the names and faces and personalities of their child's classmates. And most importantly, they will show their child how important his/her education is to them. Children will take more seriously what their parents show by example are important.
- You will have an hour each week to focus on the child or children who you think needs more attention.

How to Use This Book

Encourage your parent volunteers to read the Introduction to this book, perhaps give them some suggestions about what you will be teaching in class, but after that, give them some latitude to decide which lessons best fit their own interests and that of their group. Encourage the volunteers to USE this book: encourage them to make notes in the book of what they thought worked or how they might change the lesson for the next year. The book will become more useful to you as you acquire notes and ideas of the parents of your students over the years.

You might want to pick out a quarter's worth of lessons at the beginning of the term, and take all the materials needed by your parent volunteers for those lessons and put them in a box so that when the parent picks up the children in the classroom, it's a habit for one of the students to take the box with the group. This keeps pencils (and the games that can result from a group of students carrying pencils) and other distractions from hindering the beginning of the lesson, and allows the parents to bring out supplies at the right moments.

Introduction

The Carleton College Challenge Mathematics Curriculum Project

When my children were in our local public elementary school, their classrooms were a typical mixture of abilities and interests; some students could reliably count to 100 or read simple sentences in kindergarten, whereas other students were struggling to perform these tasks a year, or even two, later. Whole classrooms were not differentiated by ability, but instead there were regular, weekly pull-outs for reading and mathematics which would group kids more by what they were developmentally prepared for. Those weekly Challenge Math pull-outs were often run by parent volunteers, many of whom, including myself, had not been trained in teaching mathematical concepts to elementary school students, and were not often given a curriculum to follow. My own background in mathematics, however, made it easier for me to come up with ideas for the content of the lessons, I would imagine, than for some of the other parents.

After eight years of volunteering in the elementary school while my children passed through it's doors, I was pondering one day what the college mathematics majors in my classes who were interested in education could do for a senior capstone experience. That's when the Carleton College Challenge Mathematics Curriculum Project was born. For each of the next two years I led four senior math majors through this service-learning curriculum project. Each Carleton student went to Bridgewater Elementary each week and ran a 45-minute Challenge Math group, with five or six students (the same group of students for the whole year), and then wrote up lesson plans for the activities. By the end of each year, they had created a book of lesson plans from which parent volunteers could run future Challenge Math groups.

This is a compilation of their work in large part, with some of my favorite projects from my own Challenge Math groups thrown in.

What is Challenge Math?

Simply one type of student enrichment program in mathematics, Challenge Math offers to students a glimpse of mathematics as a subject they won't recognize – not adding or multiplying, or recognizing shapes, but asking questions both big and small and reasoning logically, an opportunity to work at their individual developmental level with like-minded peers, a chance to see mathematics as fun, interesting, lively, and useful, and a preview of the light at the end of the arithmetic tunnel. These Challenge Math groups do not need to serve only the brightest students in the classroom; they can serve any group of like-ability students. You want to work with like-ability students so that there is no one student answering all the questions or directing the others; you want to create a forum for better conversation and logical discussions of the ideas.

Our teaching of Challenge Math was inspired by the paradigm shift in mathematics education that changed the question from "What is the right answer?" to "Why is the answer right?" By leading the students through questions and not lecturing to them, they have the opportunity to own the material in a way that is not possible by them just listening to a teacher lecture.

Bob and Ellen Kaplan and their style of teaching (see their book *Out of the Labyrinth*) heavily influenced the pedagogical ideals of my students. The fundamental idea behind the Kaplans' style of teaching is that the students should discover the math on their own. This presents many challenges for the teacher, whose natural instinct is to tell the students the solutions to the problems. Even after practicing this style for a while, it is still difficult to steer the class in the right direction without directly handing the students the answers. Although you're not answering questions, your role, beyond giving the students the question to start discussion, is as a guide toward discovery, not as a bestower of truth. Have faith that your students will surprise you by thinking through problems and working to find the answers.

Your year of Challenge Math presents you with a unique opportunity to inspire a group of students. Throughout your lessons, your goal is not to replace their classroom curriculum, but rather to supplement it with explorations into various areas of mathematics. Perhaps the most important gift that a good mathematical education can give to a student is the ability to logically approach a problem with confidence. You're in the wonderful position where you don't have a goal to reach by the end of the year as their classroom teachers do; you have the opportunity to let them explore and be creative, all the while developing logical skills that will serve them the rest of their lives.

Realities of Pull-outs

For a Challenge Math group to be successful any given week, the students need to be ready to learn and be in a good environment. The students need to want to be there. Some weeks a child may try to get attention by keeping you from making a good learning environment. You should discuss in advance with your classroom teacher what to do if a student doesn't want to be in your Challenge Math group that week. It's always good to be able to give the child a choice, like "You may do this activity with us, or you may work on a worksheet quietly at your desk in your classroom; it's your choice."

In your first Challenge Math pull-out of the year, set a good tone. The puzzles, problems, and questions in these lessons are interesting and fun on their own. If you encourage or allow the students to get physically wild on your first day, that will set the expectation for that behavior for weeks to come. Your classroom teacher probably already set down guidelines about quiet and respectful behavior; don't ask them immediately to show off their finest soccer move in the hallway. Find a quiet place to work (even if it's a corner of a hallway, or an empty cafeteria), and reward student behavior that allows the students to focus and concentrate.

Keep in mind that there are times of the academic year (especially near impending breaks) when students are too "antsy" to sit down and concentrate. That doesn't mean that they aren't able to consider challenging mathematical questions, however. There are some of these lessons designed for the students to solve mathematical puzzles by moving around.

Mathematics for All

We all know the story: mathematics is the gateway to many advanced degrees and highly-respected (read "well-paying") jobs, but too often a small gap in ability discourages some students from working hard to understand the underlying principles of mathematics, which then makes the ability gap grow. With Challenge Math, none of the students have seen the topics before, and the students are put in like-ability groups, so the disparity doesn't exist, and students work to their potential. Upper elementary and middle school is the time when many girls report being socialized away from mathematics and the sciences, but in Challenge Math they hear encouragement and positive reinforcement; they'll hear early on that they're capable of success in Challenge Math.

The Lessons

The lessons may be used in any order; on the first page of each lesson is a note if there are any prerequisites or suggested next lessons. Any given lesson may stretch over two or more given Challenge Math sessions; that's completely up to you. If you find your students interested in a particular area of mathematics, you may decide to explore other lessons in that area. If you do stretch a lesson over more than one day, remember to take a few minutes at the beginning of subsequent lessons to remind the students what they did before to lead up to it; or, better yet, ask the students to remind you.

Level: Each lesson indicates on the top of the page with a number of stars what mathematical knowledge is required. This does not mean that if your students are in fourth grade you should look for lessons marked with four stars. Even lessons that only rely on counting can have something to offer all students; more mathematically mature students will just be able to take the ideas further or work with less help. Instead use these stars as a guide if your students are at the beginning of their elementary education; choosing a lesson where they need to have multiplication secure may be too challenging for them. The star levels:

1. Counting is secure: students understand a one-to-one relationship between objects to count and the counting numbers

2. Addition is secure: students understand not only that 3+5=8, but also that that means when a group of three objects and a group of five objects are combined, the result is a group of eight objects.
3. Multiplication is secure: students understand not only that 3x4=4x3=12, but also that that means that three groups of four and four groups of three are the same size and are size twelve.
4. All arithmetical operations are secure and they are understood.
5. Ready for abstraction: the students understand that we can let a symbol, like x or a box represent a variable – a quantity that changes.

Mathematical Diversions: Sometimes the students will surprise you by discovering something much faster than you had imagined, or you'll find it's near the end of a quarter and they are unable to concentrate. There is a section at the back of this book with suggestions for mathematical games or puzzles which could be used to fill time at the end of a lesson or could be turned into a lesson by asking good, leading questions.

Acknowledgements

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Your Role in this Process

When I was pregnant with my first child, part of my vast reading about educating newborns was information about what the baby could understand and do right after birth. I was struck by the doctors who said that a baby recognizes the sound of his mother's voice and that after birth, the baby turns toward the sound that he's heard for the past nine months in utero. I saw a doctor reporting on this phenomenon – he demonstrated holding the baby, carefully cradled in his two hands with the baby's head in his right hand and bottom in his left, near his mother immediately after birth and asks the mother to call out to her child. He said that most babies naturally turn their heads in the direction of the mother, and the mother-child bonding begins immediately. "What about those babies who don't naturally turn their heads?" asked the interviewer, "their mothers must be devastated." "Oh, no, that's easy because they're small," answered the doctor, as he gently twisted his right hand a few degrees to show that a baby under his care would turn toward the mother's voice "naturally," with his help if necessary. He knew that helping nature along in the formation of that mother-child bond was most beneficial.

Young children are naturally curious about all things, including mathematics. It's years of mathematical drudgery and being told that there is some magic correct answer that they're not getting, or not getting fast enough, that turns them away from mathematics. The students need time, space, interesting questions, permission to be creative, and encouragement to allow themselves to enjoy mathematics. You play an important role in their discovery, though. You need to be this doctor and realize how important an early student-math bond is, and if the students don't naturally turn their heads, or their brains, toward the ideas, give them a gentle little nudge.

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P.S. You won't know all the answers to the questions you or your students raise; just encourage exploration, discovery, and conversation, have fun, and be positive!

Numbers:
You Can Count on Me

1. Number Buddies

Every number between 1 and 9 has a buddy; what's a buddy? A buddy is the number's friend who helps it add to 10.

For example, 7's buddy is 3 because 7 + 3 = 10.
And 6 is 4's buddy because 6 + 4 = 10.
What's 5's buddy? Let them think about this: 5 is his own buddy because 5 + 5 = 10.

Once they get the idea of number buddies, take it up a notch and give them a new definition for buddies: two numbers between 1 and 19 are buddies if they help each other add to 20.

Now 1's buddy isn't 9 anymore, it's 19 since 1 + 19 = 20.
Do more examples.
Ask them if any number is his own buddy. (Yes, 10 is.)
Show them how every number except 10 has exactly one other buddy.
Make a table of number-buddy pairs.

Once they feel comfortable about number buddies to 20, take the big step and talk to them about number buddies to 100. That is, two numbers are buddies if they help each other add to 100.

For example, 17 and 83 are buddies since 17 + 83 = 100.

These may take quite a bit of work to figure out; give them paper and pencil and time. Have them challenge each other with number-buddy-to-100 puzzles.

If time, ask them what happens if 0 wants to play the game, too. What is 0's number buddy?

Number Buddies to 100 is just another way of talking about making change from a dollar, but in making change you need an additional step because if 83's number buddy is 17, then the students need to know that 17 cents is a dime, a nickel, and two pennies. Spend some time making change and practicing this idea.

Later in the year, this can be brought back again any time you have an extra five minutes to do math.

Introduction:

Complicated addition or subtraction problems (and making change) are made much easier by having a deep understanding of numbers that add to 10, and numbers that add to 100.

Objectives:

- To promote facility with addition and subtraction.

Materials Used:

- Scratch paper
- Pencils
- About $5 in change, either real or fake, in all types of coins

Taking it Further:

You can play this game with fractions for students who have seen fractions before. A fraction (between 0 and 1) has a number buddy that makes it add to 1. For example, 1/3 has a buddy of 2/3 and 7/27 has a buddy of 20/27.

2. Our Friend the Number Line

What is a number line? Ask the students to tell you. It's a line that we draw that has a place for every real number on it, and it helps us keep track of numbers, to keep them in order.

Draw a line 100 cm long on a long piece of paper. Don't put little tick marks on it, just draw the line. Tell the students that this is your number line for keeping track of the numbers from 0 to 100. Now put a dot on the left-hand end of the line and say that's where 0 is. And put a dot on the right-hand end of the line and say that's where 100 is.

Tell the students that you want their help putting all the other counting numbers on the number line from 0 to 100. Have them point where they think 17 should be, or 42, or 91, or 3. Let them discuss this some among themselves and try to figure out how to place the numbers. It's likely they will be far off in their estimates of where to place the numbers; that's okay, let them discuss it with each other.

After a while, ask them where 50 should go. Probably with some conversation, they'll agree it should go in the middle, and you help them place a dot (whether or not you measure it) in the middle. Now ask them again where other numbers go, like 17, 42, 91, or 3. Did this help them place the numbers?

Ask them if they have other ideas about how to place the numbers on the line. Let them experiment and think and try to get a good number line from 0 to 100.

If you did the lesson on Number Buddies, ask them if they see anything special about number buddies to 100. (If you locate both of them on the number line, the distance the smaller one is from 0 should be the same as the distance the bigger one is from 100.) Have them try to explain why that happens.

Let them draw their own number lines with time remaining, and they don't have to stop the line at 0 and 100; they can continue in either direction as they are comfortable.

Introduction:

To get a good sense of the relationship between numbers, students should learn to imagine the numbers on a number line. This exploration will allow the students to understand better how numbers are placed on a number line.

Objectives:

- To introduce the idea of a number line.
- To work on understanding the relationships and relative sizes of numbers

Materials Used:

- A large sheet of paper (at least 1 meter long)
- Scratch paper
- Pencils or markers

3. What Numbers Do You Know?

To start with, ask the students that of all the numbers they worked with frequently, what is the biggest number they have seen. Then ask them what they think the biggest number is.

The students might come up with many different answers, from a "zillion" to a googolplex. Some might even venture to say infinity.

Now's the time to talk to them about infinity. Is it an actual number? Where would it go on the number line? Remember, all real numbers have a home on the number line. Infinity is not a real number, it's an idea. The number line stretches on forever, so if you try to put it on the number line, wherever you put it, there are numbers bigger than it, which can't be. Infinity is bigger than all numbers.

Explain that numbers keep going forever because if you think at some point that you've hit the biggest number, you can always add one to it and get something bigger.

See if the kids know about negative numbers. What's the smallest number? The kids should see after the previous discussion that there is no smallest number. They should see with a number line that -17 is smaller than -7, for example.

Ask them to draw a number line and put down some numbers on either side of zero.

Next, focus on a smaller region of the number line, say zero to one. Ask them if there are any numbers in this interval. See what answers they come up with. Ask them to locate these in the interval. For example, ask them to place 1/2, 1/3, 1/4, etc. Do they know approximately where they go? Talk to them about how they know where to place these. In other words, why is 1/5 smaller than 1/4? Talking about dividing a cake is helpful sometimes -- which is bigger, 1/3 of a cake or 1/4 of a cake? (That is, if you need to divide a cake equally between 3 people or between 4 people, which group of people would you rather be in? Why?)

Introduction:

Students will enjoy discussing some very big numbers and all students love the idea of infinity. Understanding that infinity is not a real number is important for your students.

Objectives:

- To have a conversation about infinity not being "a number".
- To talk about numbers being arbitrarily large or small.

Materials Used:

- Large writing surface
- Colored markers

Fun Fact:

A googol is 10^{100}. That's a 1 followed by 100 zeroes.

A googolplex is 10^{googol}.
That's a 1 followed by a googol zeroes.

There are roughly 10^{80} elementary particles (protons, neutrons, etc.) in the universe, so if you tried to write a googolplex as 1000000... you would need more than a googolplex of atoms to do it, so we know it's impossible to write it out!

4. Fairly Dividing Cake Using Reciprocals

If you have a big cake and a friend over who's going to split the cake with you, how much of the cake does each person get? Half, but we write that 1/2 because the 1 cake gets divided into 2 equal pieces.

How about if three of you want to split the cake? You'd each get 1/3 of the cake. Is 1/3 bigger or smaller than 1/2? Discuss. Explain that in cooking, sometimes we measure ingredients using a measurement called a cup (show them the cup). But what do cooks do if they need less than a cup? They can use reciprocals!

How many 1/2 cups are in one cup? Pour beans into the 1/2 cup and dump it into the empty cup. Repeat to see that there are two 1/2 cups in a cup. How about 1/3 cup? How about 1/4 cup? Do many experiments to see which one is bigger (pouring from a bigger cup to a smaller one spills the beans). What if we had 1/5 cup? How many would it take to make a cup? How does it compare to, say, 1/3 cup?

Have them make an educated guess about two reciprocals with large denominations, say 1/27 and 1/48. Which one is bigger? Explain in words why. Explain in terms of sharing a cake with 27 or 48 people why.

Now get out the paper plates and let each one of the students cut apart a paper plate along the lines you have drawn. Have them write the correct fraction on each of the fraction pieces they have cut apart. Have them make the same comparisons they were making before with the beans: how many 1/4s does it take to make 1/2? Which is bigger, 1/6 or 1/3? With remaining time, have them draw their own circles and cut them into reciprocals.

Introduction:

Introduced in What Numbers Do You Know?, reciprocals are tricky for many students to understand since larger denominators mean smaller reciprocals. This lesson helps solidify their understanding of reciprocals using tactile objects.

Objectives:

- To get practice and facility with using common reciprocals: 1/2, 1/3, 1/4, 1/8
- To be able to compare the size of two reciprocals

Materials Used:

- 6-10 plain paper plates or circular pieces of paper; draw lines to divide the first in halves, the second in thirds, the third in fourths, etc.
- Measuring cups of sizes 1, 1/2, 1/3, 1/4
- 2 cups of dried beans
- A big bowl for the beans
- Markers
- Scissors
- Scratch Paper

Fun Fact:

Just as there is no biggest real number, there is no smallest reciprocal. For any teeny, tiny little number written as a reciprocal, there's always one smaller (closer to zero).

5. Systematic Counting

The two pictures below are of a triangle made up of little triangles and of a square made up of little triangles. Start with the big triangle. Challenge the students to count how many triangles they see.

After a while, they may have counted all the smallest size triangles that they see, but hopefully someone noticed that the whole image is a triangle as well.

Once someone notices that there are different sizes of triangles, ask them again to count all the different triangles of any size that they see. Then the challenge becomes counting them in some systematic way that will make us confident we have them all.

Allow them time to experiment and try to convince each other that they have them all.

There are 13 triangles altogether: 9 that use 1 small triangle, 3 that use 4 small triangles, and 1 that uses all 9 small triangles.

Next try to count how many squares you see in the picture of the big square. (In this one, some of the squares are tilted!) Give them time and let them discuss. (There are a total of 72 squares: 16 1x1, 9 2x2, 4 3x3, and 1 4x4 that are upright; 24 1x1, 13 2x2, 4 3x3, and 1 4x4 that are tilted.)

Introduction:

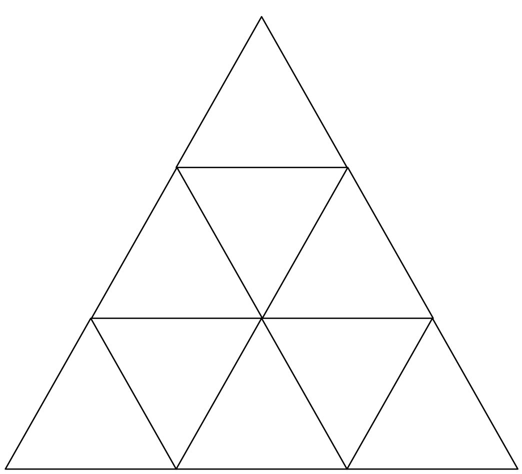
A recurring theme in mathematics is that of logically organizing your work, your thoughts, your justifications so that you can convince others of your validity. This is an introduction to that idea where students need to convince each other that they have found all the shapes they are counting.

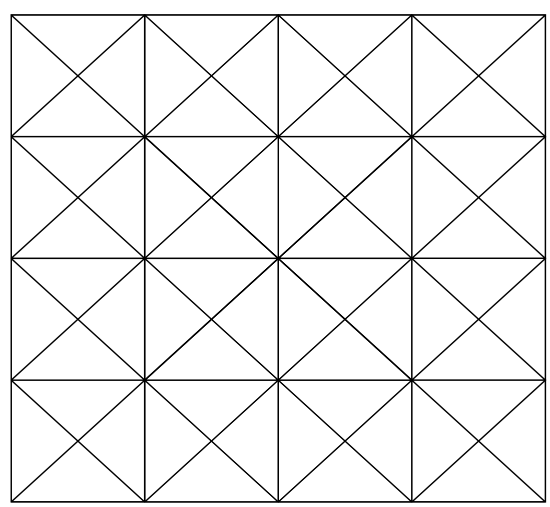
Objectives:

- To appreciate the importance of systematically doing things

Materials Used:

- One copy of the triangle and one of the square for each child
- Scratch paper
- Pencils





6. Negative Numbers

Find a nice long wall and tape the numbered pieces of paper to it, evenly spaced, each a step apart, to form a number line. Put the "+" sign on beyond the 10, and the "-" sign on beyond the -10. Ask the students to add some small positive numbers, such as 5+3 by doing the following: have a student stand under five, then take three steps down the positive number line (towards the "+"), hence ending up under 8. Let each student try such a problem with small positive numbers whose sum is less than 10.

Now ask the students how they would find 5-3. Help them discover that 5-3 is really starting at five and moving three steps is the Other Direction, which happens to be the negative direction (towards the "-"). What, then, is 3-5? If you start at three and take five steps in the negative direction, you end up at -2! What does that mean? Suppose you had three pieces of candy, but you needed to give someone else five. Even after you handed over the three pieces, you'd still owe someone two more!

Suppose now that you want to find -5-3. You start at negative five and walk three steps in the negative direction to end up at -8.

What then, do you do with 3-(-5)? It's clear that you start at three, but then you're supposed to walk in the negative direction -5 steps. That is, you're supposed to walk in the negative direction backwards 5 steps, which is just the same as walking in the positive direction 5 steps. So 3-(-5)=8.

Let them play with this idea a long while before you ask them: What is -4-(-8)? (The answer is -4+8=4.) What is -2-3? (Answer: -5) What is -3-(-7)? (Answer: 4)

If they are having great fun with it, ask them what they think: 3-(-(-7)) is. (Starting at 3, you need to turn around three times from the positive direction, ending facing in the negative direction and take 7 steps, which lands you on -4.)

Introduction:

Addition and subtraction become more complicated when using negative numbers. This exploration gives the students practice to become more facile with adding and subtracting negative numbers.

Objectives:

- To understand a visual explanation of negative numbers and how they work with the operations addition and subtraction

Materials Used:

- The numbers -10, -9, ..., -1, 0, 1, ..., 9, 10 written big and bold, each on a separate piece of paper
- A large piece of paper with a "+" sign and one with a "-" sign
- Scratch paper and pencils
- Tape

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7. Place Value Game

Start by writing two different three-digit numbers, like 224 and 259, and asking the students which one is bigger and why. Repeat this several times with different choices of numbers. Now, as a whole group, deal out three cards from one of the decks. If you can put the cards in any order, which order makes the largest number? Discuss how you know.

When they seem to understand the idea of making the largest number from three cards, have them break up into pairs. Each person in the pair takes three cards (face down) from the top of their deck, turns them over, and makes the largest number they can from them. The student with the largest number wins that round. Continue through the whole deck.

Now shuffle the decks again, and make the game a little harder. This time, each pair of students has a deck, face down, in front of them. They take turns drawing one card and placing it face-up in front of them in one of three positions: the ones, tens, or hundreds position. Once placed, the card can't be moved. After they've each drawn three cards, the student with the larger number wins.

After they have had some time to play this game (perhaps going through the deck a couple of times), draw them back together and have a conversation about what a possible strategy might be (if a number 4 or smaller is drawn, put it in the remaining open space furthest to the right; if a number 7 or larger is drawn, put it in the remaining open space furthest to the left) and how it might fail (if a 4, 3, then 2 are drawn -- our strategy makes 234, but 432 is a bigger number).

With extra time, play the same game with students taking turns drawing cards from a shuffled deck, trying to make the largest eight-digit number they can. Try, too, with three or four players using the same deck. As the number of digits and the number of players get larger, it begins to be important to count the types of cards which have already been played. Cards that players have in front of them are clearly no longer in the deck. How does that change things?

Introduction:

Students will quickly learn the importance of place values when they compete to create the largest number.

Objectives:

- To appreciate the roles of place values in numbers

Materials Used:

- A shuffled deck of cards, with the face cards and 10s removed, for each pair of students
- Paper and pencil

8. You're Worth a Million

How much is $1,000,000? Ask your students to discuss this and estimate the number of things they could buy for a million dollars. How long could they live on it? How far could they travel on it? Could they start their own business? Could they get an entirely new wardrobe? Depending on what their dreams are, it might be quite easy to spend and it might be quite difficult.

Have each student pick one dream way of spending money (unlike real life, saving it is not allowed!). This could be a group project, as desired. Some possibilities could include:

A trip for all your friends to France for a week
A new wardrobe for you and your family
A new home for your family
Building a community zoo
Establishing a homeless shelter in your community
Starting a new line of education in your school district
Throwing a big party for your school

Whatever the dream is, make sure that you list all of the things that cost money, including clothes, books, furniture, animals, salaries, etc. Use catalogs or interviews or the internet to find the costs of things, and do your best to spend $1,000,000 -- it's a one-time offer and money not spent will disappear. Also try not to go over because that money comes out of your pocket.

What can you buy? Create a poster announcing your new product or plan and a detailed list of expenses.

Introduction:

How much is a million dollars? In this lesson (which can stretch over several days), students should be encouraged to spend $1,000,000, keeping track of how it's spent.

Objectives:

- To appreciate the cost of goods
- To understand the size of one million dollars

Materials:

- Scratch paper and pencils
- Perhaps catalogs or internet access, depending on what the students' interests are--you may want to ask them about their interests a week in advance to bring the appropriate materials for this lesson
- Posterboard
- Markers

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9. Mean vs. Median

Two different types of averages can be calculated using some data and they often given quite different answers. To help the students understand these averages, start by asking them what they know about averages. Some students might say "I'm average height" or "My mom says I'm above average in reading."

When we're calculating the average, we are giving some indication of the center of a bunch of numbers. For example, an average height is the center of a collection of heights. What do we mean by "center"? There are two answers to that question, and which one we choose tends to depend on the data we have. We just want to familiarize the students with the two kinds.

The **mean** of a collection of numbers is just the sum of the numbers divided by the number of numbers in the collection. For example, the mean of (5. 2, 1, 5, 7) is (5+2+1+5+7)/5, or 4. Sometimes the mean will not be a whole number; that's fine. Have them calculate a bunch of different means.

The **median** of a collection of numbers is just the middle number when they're lined up in increasing order. From our example before, we would line up the numbers (1, 2, 5, 5, 7) and take the middle one, or 5. If we have an even number of numbers in our collection, like (2, 4, 4, 5, 8, 9), the mean is calculated the same way, (2+4+4+5+8+9)/6=5.33, but there is no middle number for the median (both 4 and 5 are in the middle) so we take the number halfway between the two middle numbers, or 4.5, as the median.

For practice, deal out 4-8 cards for each student and ask them to find both the mean and median of their cards. Can they think of an example where the mean and the median would be equal? ((1, 2, 3, 4, 5) is an example.) Can they think of an example where they're greatly different? ((1, 2, 3, 4, 500) is an example.)

End by having the kids find the average of their heights (in inches) using both kinds of averages. What happens when a professional basketball player's height is added in the mix?

Introduction:

The word "average" usually means one of two different things: the mean or the median. In this lesson the students will explore the difference between these two types of average.

Objectives:

- To be introduced to the idea of an average and two ways that it is calculated
- To understand why the two types of averages are different

Materials Used:

- Scratch paper and pencils
- Several decks of playing cards with face cards removed
- Calculators

Fun Facts:

An average beaver can cut down two hundred trees a year.

Cats average 16 hours of sleep a day, more than any other mammal.

Cows produce, on average, 40 glasses of milk each day.

On average, a man's beard grows nearly 30 feet during his lifetime.

10. Counting on Planet Quarto

There's something special in our number system about the number ten. Consider as you're counting from 1 to over 1000 what happens: when you hit 10 (or 20 or 30) you start counting over again, but you keep track of how many tens you have as well. When you hit 100 (which is 10x10), you start counting over again, but now you have to keep track of how many tens and how many hundreds. When you hit 1000 (which is 10x10x10) you start counting over again, but you need to keep track of how many tens, hundreds, and thousands you have, and this continues.

Why is ten so special? Perhaps because we have ten fingers. Consider the creatures on Planet Quarto who only have four fingers. How do they count? They can only use the numbers 0, 1, 2, and 3. Discuss this for a while with the students. Eventually someone will suggest (or you should lead them to see) that there will be something special about four. When we hit four, we'll start over counting again, but this time we'll need to keep track of how many fours we have. Then what happens? Let the students explore with paper or counters, however they would like to think about it. Give them some time; this is difficult.

The next neat thing happens when we get to 16 (which is 4x4): then we'll need to start counting all over again, but we'll need to keep track of the fours and the sixteens. Then when we get to 64, we'll start counting over, but we'll need to keep track of the fours, the sixteens, and the sixty-fours.

How would we write such numbers? Just as with Base-10, where we have a ones place which can include numbers from 0 to 9, then a tens place which can include numbers from 0 to 9 indicating how many tens we have, in Quarto-Counting (or Base-4 counting), we have a ones place which can hold numbers 0 to 3, then a fours place which can hold numbers 0 to 3, then a sixteens place which can hold numbers 0 to 3, etc. How do we write the numbers in Quarto? 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, 101, 102, 103, 110, etc., which are the numbers from one to twenty.

Introduction:

Students are used to counting in Base 10 and would have no reason to even consider other bases, but understanding how technology works requires us to understand counting and arithmetic in other bases.

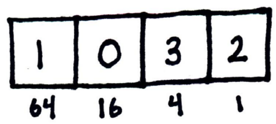
Objectives:

- To understand that there are ways to write numbers other than Base 10
- To learn how to add 1 to a number in Base 4

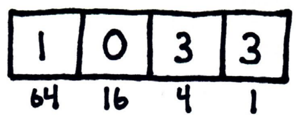
Materials:

- Scratch paper and pencils
- Some small counters, like small plastic colored chips

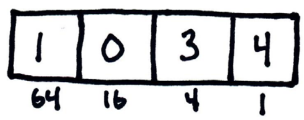
Another way to think about this is to draw boxes for each of the place values in our Planet Quarto numbers. That is, consider the four-digit number if Quarto-Counting, 1032. (By the way, that number corresponds to 1 sixty-four, 0 sixteens, 3 fours, and 2 ones, which is 1(64)+0(16)+3(4)=2(1)=78) We can draw squares on our paper to represent the different place values, and put counters in the squares to represent 1032 such as:



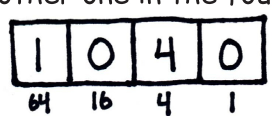
Then adding one to this, is the same as increasing the number of counters in the ones box to 3:



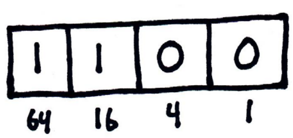
Add one more, and we want to increase the number of counters in the ones box to 4:



But we're not allowed to have four counters in the ones box, so we must exchange those four ones for one four and put a another one in the fours box.



Oh, but now that's a problem because we can't have four counters in the fours box. So we're going to need to exchange the 4 fours for 1 sixteen, and put one new marker in the sixteens box. That gives us 1100.



Add one more, and that's simple, we have plenty of room in the ones box, so it's 1101, and we continue on.

Keep adding one at a time and every few additions stop to figure out what the new number you've created is. For example, 1101 is 1 sixty-four, 1 sixteen, and 1 one. That means 1101 on Planet Quarto is the same as 81 on Planet Earth.

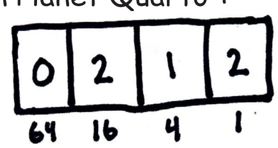
11. Adding on Planet Quarto

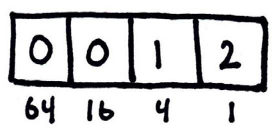
Now that the students know how to count on Planet Quarto (or in Base 4), let's have them add two numbers that are both written in Base 4. Recall that when we're adding two multi-digit numbers in Base 10, whenever a particular column has more than ten in it, we must carry. Remind the students of this and ask them what they think happens in Base 4.

In Base 4, whenever a column has 4 or more in it, we must carry. For example, let's add 3012 and 12. Notice that these numbers don't have any digits that are larger than 3 since they are numbers on Planet Quarto. We can line up the numbers from the right, just as in Base 10, like this:

212
+12

Or think about them in the place-value boxes like we did in "Counting on Planet Quarto":





Now adding in the ones column we see we have 2+2=4 counters, but whenever four counters get together in the same box they explode and shoot one counter over to the box to the left. (This explosion is like carrying in usual Base 10 addition.) So those 4 counters in the ones box explode, throwing one counter into the fours box and leaving nothing in the ones box. What do we have in the fours box? 212 had one four, 12 had one four, and we got one more four from the ones box explosion, giving us 3 fours, which isn't enough to explode, so we write down 3 in the fours box.. Then to the 2 in the sixteens box, we don't have any more to add, so it stays 2, which gives us the sum 230.

Is this correct? We might be more confident if we could check our work by switching everything to Base

Introduction:

Following up on "Counting on Planet Quarto", we can now talk to the students about how to add two numbers that are written in a base different than Base 10, for example in Base 4. This lesson could take several sessions.

Objectives:

- To understand that there are ways to write numbers other than Base 10
- To learn how to do simple addition in Base 4
- To expand this understanding to include other bases.

Materials:

- Scratch paper and pencils
- Some small counters, like small plastic colored chips
- Calculator

Taking it Further:

Watch James Tanton's YouTube videos on "Exploding Dots".

10 again. What is 212 in Base 10? Well, 212 is 2 sixteens, 1 four, and 2 ones, which is 2(16)+1(4)+2(1)=38. What is 12 in Base 10? 1(4)+2(1)=6. What is 230 in Base 10? 2(16)+3(4)+0(1)=44, which is 38+6 in Base 10.

Now it's time to just practice Planet Quarto addition:
Can you find 103+10? (Answer 113) 131 + 11? (Answer 202) 212+113? (Be careful in this last one: the ones box will have five counters in it, which is the same as one counter in the fours box and one in the ones box. (Answer 331)

Here are some more to practice, with their answers:
1) 111+111=222
2) 131+12=203
3) 213+10=223
4) 213+11=230
When they get more confident, try 212+222, which should give them 1100.

What's so special about Planet Quarto? Nothing! What about our friends on Planet Octavo where they each have 8 fingers? They can only use the numbers 0, 1, 2, 3, 4, 5, 6, and 7 when they write numbers, and when they get to 8, they need to write that as 10. Write out the first 50 numbers counting in Base 8.

Then try adding some numbers in Base 8, where we must carry every time a box gets bigger than or equal to 8. For example, if you want to add 135 and 117, when you add in the ones box you have 12 counters, but then eight of them explode, shooting one counter into the eights box and leaving the other 4 counters in the ones box unharmed. Then you have 5 in the eights box, which causes no explosion, and you have 2 in the sixty-fours box. Thus the sum is 254.

Here are some addition problems to practice on, with their answers:
1) 123+406=531
2) 374+202=576
3) 375+102=500

What do these numbers mean? Do some practice on converting those addition problems back to Base 10.

What about other bases? Can you do Base 3 arithmetic? You bet! You'll only have 0, 1, and 2 in your numbers, but that's fine. Can you do Base 7? Yes! How about Base 16? This is very hard to think about, but it's very useful in computer science to work some things in Base 16. How does that work? Well, now your digits can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. If you have the Base 16 number 12F and you want to add 5, what happens? F counters + 5 counters gives you 20 counters in the ones box, so sixteen of them immediately explode, adding one counter to the sixteens box and four remaining safely in the ones box. When the dust settles, 12F+5=134. Give it a try!

12. Binary: Talk like a Computer

Computers are amazing. They can do additions and multiplications of really big numbers in the blink of an eye. How do they do that?

If you can, show the kids a desktop computer (not a laptop) and have them identify the various parts, with your help as needed. There is the monitor (screen), the keyboard, the mouse, and a box that is connected to the other parts which is where the actual computations are done (sometimes called "the computer" because it's where computations take place).

Inside the box is a little chip, called the "processor" which does calculations at lightning speed. The processor, for example, is what calculates big addition and multiplication problems. How does it work so fast? It does its calculations by sending and receiving electrical impulses. How does it take electricity and interpret it as a number?

How Loud am I Talking?

The first step in answering this question is to suggest the idea that you measure how much electricity is coming into the processor. Ask the students if they've ever had to rate something on a scale and what sort of scale they should use -- most will suggest a scale of 1 to 10. Now tell them that you are going to say a word (it doesn't matter which word) and that they should each rate how loud you were talking on a scale of 1 to 10. Say something that falls in the mid-range. You should get a variety of answers. This demonstrates how hard it is to accurately interpret electricity as a number if we use a scale of 1 to 10. Now tell them to use a scale from 0 to 1 where 0 is not-talking and 1 is talking to rate your loudness. Say a word and have them guess, and they should all say 1. Don't speak and have them guess and they should all say 0. Repeat this demonstration until they see that now there is no problem.

So the processor will use just 0s and 1s to convert electricity into a number: 1 means that it is receiving electricity, and 0 means it's not. But how do we describe really big numbers when we just have 0s and 1s?

Introduction:

Let's use our new knowledge gained from "Counting on Planet Quarto" and "Adding on Planet Quarto" to understand a little about how computers work.

Objectives:

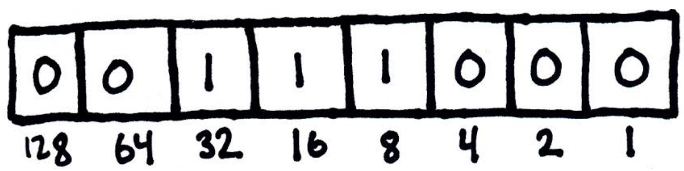
- To understand a little better how computers work
- To practice students' understanding of different bases by doing arithmetic in Base 2, or Binary.

Materials:

- Scratch paper and pencils
- Some small counters, like small plastic colored chips
- Calculator

Let them think about this a little. They already know how to represent any number in Base 4 or Base 3 or Base 16. We could use Base 2!

Base 2 is also called Binary, and we can convert any number we want into Base 2. Let's try it: What is 56 in Base 2? If we label the boxes from right to left with the powers of 2, so there's a one's box, a twos box, a fours box, an eights box, a sixteens box, a thirty-twos box, and a sixty-fours box (we could keep going), then we know that our number 56 has 0 sixty-fours in it, but it does have 1 thirty-two (in fact it's 24 more than 32), so it also has 1 sixteen in it and 1 eight. So written in Base 2, 56 is really 0111000. (The zero on the left is really unnecessary, just as any "leading zeroes" in a Base 10 number are unnecessary.)



Now practice converting numbers from Base 10 to Binary and back again. Get good at it so that next lesson you can learn a card trick to amaze your friends.

13. Binary Fitch

The students are going to learn a trick that uses the idea of binary arithmetic to amaze an audience. First, do the trick as one large group so that everyone understands, then allow the students to work in pairs to perfect their showmanship for an audience.

The trick works like this: two people perform the trick in front of an audience. One of these two (we'll call her the "friend") leaves the room during the first part of the trick. The second one of the two (we'll call him the "setter") asks an audience member to pick any five cards from a standard deck of cards, and explains that he'll use his amazing mental powers to communicate one of these cards to his friend. Actually, he's going to use binary numbers to communicate the card to his friend!

When the friend leaves the room, and an audience member chooses any six cards from the deck, the setter takes the six cards, looks at them, and immediately hands back to the audience member one of the cards that doubles up a suit, remembering the card he handed back. (There are only four suits in the deck: spades, hearts, diamonds, clubs, so if he's holding six cards, there must be two of one suit. There might be more than one pair doubled up, say if you have five diamonds and a club, but then you can hand back any one of the repeated suit.) The setter then explains that he will use his amazing mental powers to communicate that card just handed back to his friend.

The setter then sets the cards down on a table in front of him with the card in his hand of the same suit as the target card being the first he sets down, say it's a heart. Now the setter still has five cards in his hand, and he needs to communicate to his friend the value of the target card. The friend will already know its suit is a heart when she returns by the suit that is laid down on the left.

Let's turn the values of the cards all into numbers: an ace we'll say is 1, a jack is 11, a queen is 12, a king is 13, and all the number cards say their value. Now the setter needs to communicate a number between one and thirteen to the friend. Suppose the target card is

Introduction:

We will practice binary arithmetic by learning a card trick based on a card trick created by William Fitch Cheney.

Objectives:

- To practice understanding binary numbers
- To learn a card trick to amaze their friends

Materials:

- Scratch paper and pencils
- One deck of cards for every two students (jokers removed)
- Ideally, an audience to watch the trick after it is learned

a queen. Then the setter needs to communicate a 12 to the friend. How does he do that? He thinks of what 12 would be in binary: 1100, and he puts any leading zeroes (zeroes to the left) that he needs to end up with a number with four places in it. (Ours has four places, but if he were trying to communicate the number 3, which is 11 in binary, he'd think 0011.)

Now if he thinks of his cards as having a "1" side (face up) and a "0" side (face down), he's all set! He can use the cards remaining in his hand, in any order, and set them down on the table next to the suit card: face up, face up, face down, face down. (Or, for dramatic flair, the setter can put all four cards face up to begin with, then say, "Let's make this a little harder for her", and he switches the appropriate ones to face down.

When the friend is called back to the room, she sees the suit on the left, and four cards indicating 1100, then just needs to quickly convert back to Base 10, to get the number 12 and she announces the queen of hearts, to the amazement of all. Ta da!

Ask the students if they were the friend what the following sets of cards would describe:

2 of hearts, up, down, down, up
5 of spades, down, down, down, up
7 of diamonds, down, up, up, down

The trick works best if it's well-practiced and well-presented, so the students should divide into pairs and the leader should hand random collections of five cards to each pair for them to practice.

Finally, don't call the trick "Binary Fitch" to others because the word "binary" may give them a clue as to how it's performed.

Answers:
2 of hearts. up, down, down, up corresponds to 9 of hearts
5 of spades, down, down, down, up corresponds to ace of spades
7 of diamonds, down, up, up, down corresponds to 6 of diamonds

★★★★

14. Place Value Game: Other Bases

The goal of this game is to make the largest number you can. Each player has to make his or her own three-digit number, so each player takes turns obtaining numbers and putting them in the position of their choice within the number.

For example, if you are practicing Base 4 numbers, use the deck of cards with jokers (representing 0s), aces (representing 1s), 2s, and 3s. (All other cards should be removed from the deck.) Now students take turns drawing cards and each time they obtain a 0, 1, 2, or 3, they need to place the number in one of the three positions in front of them: the ones place, the fours place, or the sixteens place. Once they put the number in a place, they are not allowed to move it. Each student should take turns drawing a number and placing it until all three positions have been filled. Who has the largest number?

Make sure that each student translates her Base 4 number to a Base 10 number. Hopefully they will see that if the number looks bigger in Base 4, it will be bigger in Base 10.

Don't tell the students, but lead them to the revelation that bigger numbers are better in higher place-value spots, and that if you get a small number, it is better to put it in the ones place, where it has the least effect on the size of the number.

To practice binary, let the students flip a coin and call heads "1" and tails "0".

Practice different bases and, for more challenge, four-digit or longer numbers.

Introduction:

This game, with only one strategy to learn and a large element of luck, can reinforce place-value thinking and can help practice different bases. This game can be made as easy or as difficult as you want, depending on your use of bases.

Objectives:

- To practice understanding numbers in different bases
- To reinforce the concept of place-value even in different bases

Materials:

- Scratch paper and pencils
- Coins to flip if practicing binary
- Deck of cards with the jokers if practicing other bases

★★★★

15. Prime Numbers: Building Blocks

What is a prime number? A prime number is a whole number, greater than or equal to 2, that has only itself and 1 as divisors.

Is 2 a prime number? The only things that divide evenly into 2 are 1 and itself, so yes, 2 is prime.

Is 3 prime? The only numbers that divide evenly into 3 are 1 and 3 itself, so yes, 3 is prime.

How about 4? It's true that 1 and 4 both divide evenly into 4, but so does 2. So is 4 prime? No.

Let the students play with this idea and talk about the numbers all the way to 10. They should find that the primes are 2, 3, 5, and 7. Hopefully they will notice that any time you have an even number (other than 2), it's not prime because to be an even number means that 2 divides into it evenly.

Now set the students loose on their charts of numbers, and have them color in the numbers that are not prime, leaving the numbers that are prime alone.

After giving them some time to work, ask them what they're noticing. Someone might notice, if they didn't earlier, that no even number other than 2 is prime. Also, any multiple of 3 other than 3 itself (like 6, 9, 12, 15, etc) is not prime. How about any multiple of 5? We know 5 is prime, but 10, 15, 20, 25, 30, etc are not prime because they have 5 as a divisor.

Once completed, do they notice any patterns? (There really aren't any.) Are there more primes in the first half of the chart or the second half? Are there any twin primes? (Those are prime numbers that are separated by two, such as 11 and 13. Are there more?)

An especially inspired group of students may wish to make a chart of all the numbers from 1 to 1000 and find how many primes there are.

Introduction:

Prime Numbers are building blocks in mathematics in the sense that any number can be written as the product of prime numbers. Prime numbers are very important in mathematics, to help us see the relationship between numbers.

Objectives:

- To learn what a prime number is
- To find all prime numbers between 1 and 100.

Materials:

- Scratch paper and pencils
- One copy for each student of a chart listing the numbers from 1 to 100, where the numbers 1 to 10 are in the first row, 11 to 20 are directly below them, etc.

Fun Fact:

Mathematicians know there are infinitely many primes, but they still don't know if there are infinitely many pairs of twin primes or not; the open question is called the Twin Primes Conjecture.

16. Clock Arithmetic

On a standard clock, there are only twelve numbers, the numbers 1, 2, 3, ..., 12. We go completely through all the hours twice each day, and we designate which time through we've gone by a.m. and p.m. For the moment let's think of "12" as "0". That is, we're going to think of pasting the number "0" over the "12" on our clock.

"Military time" is a different way of keeping track of the hours in the day, and it has hours from 0, 1, 2, ... 23. The morning hours: 0, 1, 2, .., 11 are the same on both clocks, but when the military clock says 15, what time is it? Well, we only have the numbers 0, 1, ..., 11 on our clock, so when the military clock says 12, ours will say 0. When the military clock says 13, ours will say 1. When the military clock says 14, ours will say 2. When the military clock says 15, ours will say 3, so it's 3 p.m.

We would say the numbers 15 and 3 are the same in clock arithmetic, as are the pairs 12 and 0; 13 and 1; 14 and 2. Can you find other pairs that are the same?

Now instead of our usual clock that marks the hours from 0, ..., 11, what if we lived on a different planet where the hours on their six-hour clock were marked by 0, ..., 5? Seven hours after the clock started at 0, what would it say? (1)What about 11 hours after the clock started? (5) What about the 23rd hour after the clock started? How do you figure this out? Well, since on this clock 6 is the same as 0, we can just cast out all the 6s from the number 23. Let's see, 23 = 6+6+6+5. Casting out the 6s means that 23 is equivalent to 5 on our clock. What about 82? (4)

Now let's try adding two numbers. If I tell you that on our different planet, we started the clock and 17 hours went by, then 15 more. Now what does the clock say? 32 hours have gone by in total, and 32 = 6+6+6+6+6+2, so our clock would say 2.

Try this with other clocks where the numbers are 0, 1, 2, ..., 9 or even 0, 1!

Introduction:

Clock arithmetic, or by its more mathematical name, modular arithmetic, is simply a kind of addition (and multiplication) where you have a finite number of numbers to work with. This will be good practice for students thinking outside the box.

Objectives:

- To help the students think outside the box, creatively
- To understand that there are other, legitimate, ways to do arithmetic
- To let the students practice hands-on with modular arithmetic

Materials:

- Scratch paper and pencils

Fun Fact:

This clock arithmetic works with days of the week. Let Sunday be 0 and Monday be 1, etc. There are 7 days in the week, so 7 is the same as 0. Then if you start a trip on a Tuesday (2) and are gone for 72 days, what day do you come home? 2+72=7(10)+4, so on Thursday!

17. In Search of Pi

Discuss with the students what makes a circle a circle. How does it differ from an oval? After they have given several descriptions, if they haven't quite been able to say that all the points on a circle are the same distance from the center, help them see that.

Find many examples of circles within your school. These could be coffee cups, trash cans, support beams, the equator on a globe, the distance around an orange -- come up with many different examples. Now for all of these circles (or as many as are practical), you want to record the same information: the circumference of the circle (that is, the distance around, like you're wrapping a belt around the circle), and the diameter of the circle (that is, the distance across the circle, where the diameter would go through the center of the circle; it's also the furthest distance any two points can be from each other on the circle). These can be measured in inches, or for small items, in centimeters or millimeters. However, if you measure an object's circumference in cm, use cm to measure its diameter as well. Different objects can be measured in different units.

Make a large chart together of the different things you've measured, their circumferences and their diameters. Now, after you've gathered all the data, use a calculator to take the circumference of each object divided by its diameter. What do you see? Regardless of how large the object is, you should see that this ratio is always around 3 (or if you measured very accurately, around 3.14). This number that you're approximating is the number pi, and it is the ratio of the circumference to the diameter of **any** circle.

If some of your numbers didn't quite give a ratio near 3, talk to your students about why that might be. Was something mis-measured? Did they change units? Use this as a teachable moment about the importance of collecting good information to make good decisions.

For fun, how many digits of pi can you memorize? Some people have memorized hundreds for the fun of it!

Introduction:

Pi (pronounced like "pie") is the name of a very special number in mathematics. It is the ratio of the circumference of a circle to its diameter, and it's independent of the size of the circle you use.

Objectives:

- To learn the interesting relationship between any circle's circumference and its diameter
- To approximate a value for pi

Materials:

- Scratch paper and pencils
- Rulers
- A tape measure
- String
- A calculator

Fun Fact:

Some numbers, like 3.1 can be written as a fraction of integers (3.1=31/10); those are called rational numbers and we can tell from their decimal form if they can be written as a fraction because the decimal form will stop (like 3.1) or repeat forever (like 3.13131...). Pi is an "irrational" number which means that it cannot be written as a fraction of two integers, and it's decimal form never stops or repeats: Pi=3.141592653589793238...

18. Challenge Math's Biggest Problem

Write the following problem on a white board:

2 + 4 * 3

And ask the students to tell you what the answer is. Some students may think of it as (2+4)*3=18, and others may think of it as 2+(4*3)=14. If both answers aren't given to you, you should suggest them both. Ask which one is correct.

Either could be correct, but it was decided long ago that when parentheses aren't provided in a problem with multiple operations, that the order the operations should be performed in is:

Parentheses
Exponents
Multiplication/**D**ivision
Addition/**S**ubtraction

which means that there is one correct solution to the above problem, and that is to do the multiplication first, so 2+4*3=2+12=14.

How can we remember the order of operations? All we need to do is remember PEMDAS, and we can remember the order. How do we remember PEMDAS? Spend a few minutes having each student come up with his or her own mnemonic device, such as the standard "Please Excuse My Dear Aunt Sally," or the less standard "Please Emphasize Math -- Duhmost Awesome Subject."

Now we know that we can solve problems with multiple operations, such as: $2+4^2-6*3$. There are no parentheses, so the first thing we do is the exponent, and we get 2+16-6*3. Then we do multiplication/division to get 2+16-18. Finally we do addition/subtraction to get 2+16-18=0.

Lay out the big piece of paper on the floor and have the students take turns writing Challenge Math's Biggest Problem. This will be one long order of operations problem, so they'll need to make sure there's lots of variety in the operations they use. Encourage them especially to use parentheses and exponents, since those are what most students forget to use. The problem should fill the entire paper from short side to short side, and will look something like the following:

Introduction:

Upper elementary students will be taught the order for performing multiple arithmetic operations, usually abbreviated PEMDAS. This lesson provides reinforcement for when the order of operations is taught in class.

Objectives:

- To learn the Order of Operations
- To write their own mnemonic device for PEMDAS
- To gain substantial practice in performing operations in the correct order

Materials:

- Scratch paper and pencils
- One long piece of paper (ask the teacher if the school keeps butcher paper for banners, and cut a piece that's 10 feet long
- One marker per student
- One white board and marker

$4^2 + 5 * 2 * (13 - 12) - 6 * 5 + ((24 / 6) * 2^3 + 7) + (20 / (7 - 3)) * 2^2 + 6 * 2 * (7 + 2) / 2$

Have the students take turns slimming down this problem by writing simpler versions of this problem beneath it. Make sure all the other students are watching so that no mistakes are made.

Notice when you start with parentheses, you can simplify (13-12) quite easily, but $((24/6)*2^3+7)$ can't be easily reduced. Within those parentheses you need to apply the order of operations again. First you do the parentheses to get $4*2^3+7$, then exponents to get 4*8+7, then multiplication/division to get 32+7, then addition/subtraction to get 39.

After all parentheses have been removed, you should have:

$4^2 + 5 * 2 * 1 - 6 * 5 + 39 + 5 * 4 + 6 * 2 * 9 / 2$

Next comes exponents, so it reduces to:

16 + 5 * 2 * 1 - 6 * 5 + 39 + 5 * 4 + 6 * 2 * 9 / 2

Then comes multiplication/division, so it reduces to:

16 + 10 - 30 + 39 + 20 + 54

Finally, we do addition/subtraction, and the answer is:

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If there's time remaining have them write more math problems, making them longer each time.

Ask the teacher if there's room on the classroom wall to display Challenge Math's Biggest Problem!

19. Algebra and the Golden Rule

Cup your hands together like you're holding something precious, but secret. Peek a little in between your fingers and say, "I'm holding a number, and I won't tell you what it is. But when I add 3 to it, I get 8. What's the number?"

Many students will immediately tell you that the number is 5, but make them work hard to explain to you exactly how they know that. Some may say that they subtracted 3 from 8. Why did they do that?

You can introduce the idea of an unknown (the number in your hand), by writing down a way to describe mathematically what you said above:

$$\square + 3 = 8$$

What we need to do is to get rid of the 3 on the left-hand side so that we get "box" all by itself. But an equals sign is a very important symbol in mathematics; perhaps the most important symbol of all. It means that the things on either side of the equals sign are equivalent. At this point you can choose to stop and have a discussion about the mathematical sentence "2+3=6." Is it true? Is it false? Why is it not true?

Since we need to respect the equals sign, we can't just change one side of an equation and not change the other; if we want to subtract 3 from the left, we also have to subtract it from the right:

$$\square + 3 - 3 = 8 - 3$$

$$\square = 5$$

This is the **Golden Rule of Algebra**: anything you do to one side of an equation, you must do to the other.

Now ask the students to think about the equation

$$\square -17 = 152.$$

In this case they'll need to add 17 to both sides to see what box is. ($\square$=169)

Let's make the original equation a little more difficult.

$$2*(\square-17) = 152$$

What can we do with this equation to find out what box is? There is another convenient way to write the given

Introduction:

The Golden Rule of Algebra gives students a way to solve an equation for some unknown variable. The lesson is an easy introduction to the ideas of Algebra.

This lesson follows Challenge Math's Biggest Problem.

Objectives:

- To learn the Golden Rule of Algebra
- To solve algebraic puzzles

Materials:

- Scratch paper and pencils
- Small white boards and markers for all, optional
- A large white board or black board for you to write problems on

Fun Fact:

It's easy for students to check their own answers in these puzzles because they can just take what they believe to be the true value of the unknown and substitute it back into the original equation to see if the equation is true! It's a very good idea for the students to learn how to check their own answers, without relying on someone else's answer key.

equation, by distributing the 2, which we should consider: 2*box -34 = 152. So when we find a solution to one equation it should be a solution to the other equation as well since they're the same equation! Let the students discuss how they would solve either one of these equations. After they've proposed some ideas, help them see that in the first version of the equation, we would need to divide both sides by 2 then add 17 to both sides. In the second version we would need to add 34 to both sides, then divide both sides by 2. In either case, we'll see box = 93. In the first version of the equation why can't we add 17 to both sides first? We can! Then we'd see

$$2*(\square - 17) + 17 = 152 + 17$$

but that doesn't help us get box all by itself because the 17s on the left-hand-side don't cancel.

Here's another example, where instead of a box, an x is used to denote an unknown number:

$$(x - 4) * 12 = 24$$

Let the students discuss the correct way to approach this one. Eventually they should see that they need to divide both sides by 12. Then they should add 4 to both sides, to see x=6.

Once they understand the idea of solving for an unknown, then they just need practice. Here are a list of equations for them to explore, with the answers listed parenthetically. How many can they solve?

1. x+4=7 (x=3; Make them explain how they solved even the easy ones they can "see" the answer to. If they "guess and check" that's very good; but they also need to understand how to do it using the Golden Rule of Algebra.)
2. x/2=4 (x=8)
3. 7*x + 1=8 (x=1)
4. (x/2)-4=2 (x=12)
5. 24/x=2 (x=12; Here you first multiply both sides by x, then divide both sides by 2.)
6. (75-x)/10=7 (x=5; After multiplying both sides by 10, then add x to both sides and subtract 70 from both sides.)
7. 2*(18-x)=4 (x=16)
8. 7*x=49 (x=7)
9. 35/x=7 (x=5)
10. (77+x)/7=11 (x=0; Yes, 0 is a number, too!)
11. (x+7)-2=20 (x=15)
12. (22-x)/7=3 (x=1)
13. (3+x)*4=12 (x=0)
14. ((x/2)-2)*2=10 (x=14; To both sides first divide by 2, then add 2, then multiply by 2.)
15. (2*x-3)/4+5=10 (x=11.5; To both sides first subtract 5, then multiply by 4, then add 3, then divide by 2.)

Estimation and Approximation: Making Smart Guesses

1. Lengths: Walking Across Town

We're going to try measuring some distances in student paces. What's a pace? It's the length of a comfortable step as one is walking. Discuss what potential problems could arise using paces for our measurement. (Students have different size steps. Some students may try to run, increasing the size of their steps.) Explain that we want to be scientists and try to minimize the errors that we make, then have the students practice taking steps so that they're uniform in size.

First, let's estimate the numbers of paces there will be down one designated hallway in the school. It could be that the hallway is lined with lockers, or artwork evenly spaced on the wall, or large tiles on the floor, and you notice that there are two lockers in one pace, or two paces between pieces of art, or one pace is about one tile on the floor. Then you just need to count the lockers (or artwork or tiles) and multiply appropriately for your answer. Check your answer by trying to actually pace out the whole hallway.

Now, estimate the number of paces between two distant places within the building, such as the cafeteria and the gymnasium. Since you know how many paces are in the hallway, you could estimate how many hallways there are of the same length between the two places (perhaps four such hallways, or 4 1/2). Then you can just multiply the number of hallways times the number of paces in each hallway.

When the students understand what you are doing to make these estimates, take them outside to measure the longest dimension of the school building in paces. You may want to have each of the students pace it off themselves, and discuss the results. They won't be the same, but you're just trying to get a good estimate. How do the students want to resolve this? Perhaps a number that's far off could be discarded, and the others could be averaged. Now look at a map that includes the length of the school, which you know in paces, and some other destination. Following the roads, estimate how many paces it takes to get to your destination!

Introduction:

Students who learn to get good at approximations will be much better about not making careless mistakes in calculations later on --they will develop an intuition for what an answer should be.

Objectives:

- To give the students practice in approximating a linear distance.

Materials:

- Scratch paper and pencils
- Map from a phone book or from online showing the outline of your elementary school and some other point of interest about a mile or two away
- A calculator
- Permission to walk outside the building

Fun Idea:

Using what you learned in this lesson and a map of the city, can you use your approximate number of paces between your school and your chosen destination to estimate how many paces there are across the entire city?

2. Areas: Papering Your Principal's Office

If all the students in your school wanted to draw a square picture for your principal for Principal's Day, would she have enough space on her walls to hang them all? Discuss this idea and see what sorts of answers the students come up with and why.

To start this lesson, make sure the students understand what "area" is. Ask them their ideas and direct them into understanding that length allows us to compare the sizes of two one-dimensional things. Then ask how we would compare the sizes of two two-dimensional things, like two pieces of paper.

We could measure the lengths of their sides, but what we want to know is how much paper there is. Is there more paper in a piece that is 3x7 or is there more paper in a piece that is 4x5? Show them the two pieces and have them figure out how we can measure. Let them discuss and lead them to see that we could just count how many little graph paper squares there are in each of the shapes, and we see that 3x7 has 21, while 4x5 has 20. Point out to them that 3x7=21 and 4x5=20, and show them that the 3x7 shape really has 3 groups of 7, so calculating its area by multiplying 3x7 makes sense.

Cut out several other rectangular shapes from graph paper and have them find and compare the areas.

Now taking the large prepared squares into the principal's office, choose one wall to start with. Figure out how many squares would fit left to right and how many would fit up to down. Have them draw that rectangle on their graph paper. Do the same with the other three walls. (You can decide whether or not to make adjustments for the windows and door. These are just approximations.) Then you can leave the office and calculate the area of the four walls of the principal's office.

Is there enough room for every student in school to draw her a picture and have them hung on her wall?

Introduction:

Students will continue to think about estimations by investigating the number of sheets of square typing paper it would take to wallpaper their principal's office.

Objectives:

- To give the students an introduction to area
- To give the students practice in approximating areas

Materials:

- Scratch paper and pencils
- Graph paper
- A 3x7 shape and a 4x5 shape cut out of the graph paper
- Several sheets of scratch paper cut into squares whose side length is 8 1/2 inches
- A calculator
- Permission to use the principal's or someone's office; an empty classroom would work also

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3. Areas: How Big is Your State?

How big is the state you live in? Suppose your state is Minnesota, with very rough boundaries and you want to know how much land is in Minnesota compared with, say, Wisconsin. How do you find out the area of Minnesota? We can estimate it!

Have the students discuss how they can estimate the size of Minnesota. Maybe one way would be to put a big rectangle around Minnesota and see how big that is. That would tell us a number that's an overestimate for the size. Maybe we could put a rectangle inside Minnesota and get an underestimate for the size. How can we do better?

Bring out the graph paper and ask if they have ideas how they could use that. Give them some time to explore and think. Help lead them to the idea of tracing the state onto their graph paper, then they can count how many little squares are in the state. This won't be a simple multiplication problem, unless your state is a rectangle! They will need to count the little squares. And their approximation will be better if they count the squares on the border that are only half in the state as 1/2. Let them think about how best to do it.

Once we know how many little squares are in Minnesota, then we need to figure out how big a little square is in terms of the map. That's where the scale comes in. Suppose one little graph paper square actually is 32 miles x 32 miles. They should believe then, if we had little tiny squares of one mile on each side, there would be 32x32 of them in one of your graph paper squares. So there are 32x32=1024 square miles in one graph paper square. Now you just multiply by the number of graph paper squares they counted, and you have an estimate for the size of your state.

To stretch them further, pull out a map of the entire United States and have them conjecture which is the biggest state of all.

Introduction:

Continuing their study of areas, students will learn to approximate an area with uneven boundaries -- one that is not a perfect rectangle.

Objectives:

- To give the students practice in calculating areas
- To get some experience using approximation techniques to approximate something they can't measure directly.

Materials:

- Scratch paper and pencils
- Graph paper
- Cutout image of your state with a scale indicating distances on it; the image should be as large as possible, but still fit on a normal piece of typing paper
- A calculator

4. Volumes: How Big is the Gym?

How big is your gym? This is a different question than asking how long a hallway is or how big the size of a wall is. Discuss how it differs.

We're trying to somehow indicate to others not only that the gym is long or that it has a big wall, but that it is big in three dimensions: length, width, and height. Before we measured a long line in steps and a large area by seeing how many little squares fit into it. What should we do now?

Review with them (or explain, if they haven't seen before) that the volume of a big rectangular box (like a shoe box, or the gym) is measured by multiplying the lengths of the three dimensions together, length x width x height. You can demonstrate on the Rubik's cube: how many little cubes does it take to make up the big cube? If you were to disassemble it, how many would be on the bottom row? (3x3=9), and the middle and the top row? (All 9) So all together, we'd have 27, which is 3x3x3 little cubes in the big cube.

Now imagine that we're going to find the volume of the gym, but we don't have any measuring devices. What should we measure with? How about kids! Take the median height of students in your group -- say Maggie's height is the median height. Then we'll measure the gym in cubes that are the size of Maggie on each side.

Have them visualize and describe (or act out) how big such a cube would be. Get them to visualize how many such cubes would fit in the gym. How would we figure it out? Let them think a while and realize that we need to measure the three dimensions of the gym in Maggies.

One way to do this is to have Maggie (and hopefully there are other kids about Maggie's height) lay down head-to-toe across the length of the gym. Then repeat across the width of the gym. Getting the height of the gym can be a little more challenging, but many gyms are made of cinder block or something similar and the students can figure out how many cinder blocks there are in one Maggie and estimate from there.

Introduction:

Continuing their study of estimation, students will approximate a volume.

Objectives:

- To give the students an introduction to the idea of volume
- To get some experience using approximation techniques to approximate a volume

Materials:

- Scratch paper and pencils
- A calculator
- A Rubik's cube, if possible
- Access to the school gym (or some other large room) for about 20 minutes

Once you have the three estimates, your group can leave the gym and begin the calculation. Have them convince you that they're correct when they give you a number. What are the units on the number? Feet cubed? Yards cubed? When they answer Maggies cubed, ask them to explain again how big that is.

Now what if we change the problem to ask how many kindergartners cubed we could fit in the gym? Suppose Maggie is twice the height of the kindergartner. Does the volume of the gym change? (No!) Is the number of Kindergartens cubed that fits in the gym bigger or smaller than the number of Maggies cubed? (Bigger) Is it twice as much? (No)

If the Kindergarten is exactly half the height of Maggie, how much does the number change? Let them discuss this a while until they can convince you that the number would be eight times bigger because the number is twice as large in each dimension. When you multiply all those twos together, you get eight times as large.

Now what about if you measured in cubic shoes, where you select one shoe from someone in your group, say Adriano. How many of Adriano's shoes are equal to one Maggie height? Suppose it's five. How does this change the number you calculate? (It's 5x5x5=125 times bigger.) Does this make sense? Does the gym get bigger? Have them visualize one Maggie cubed box. How many Adriano's-shoe-cubed boxes fit inside it? (5x5x5=125), so for each of Maggie-cubed boxes, 125 shoe-cubed boxes fit inside. That's why we need to multiply the number of Maggie-cubed boxes by the number 125 to see how many shoe-cubed boxes fit inside.

With any time remaining let them ask about other size cubes (perhaps an adult who is twice Maggie's height) , and see what you learn.

5. Volumes: How Big is the Gym II

How big is your gym? Is it large enough to hold every student in your school? How about every student in your school district? How about every person who lives in your city? Have the students write down their estimates, and keep hold of those.

Let's imagine that we want to know how many people will fit in the gym, if they're crammed in tightly (or stacked like cord wood), from floor to ceiling. This is not a practical question, because if we really had people stacked like cord wood in the gym, they would have no room to move and the people on the bottom of the stack would get squished! But we're just going to play with this idea anyway.

Have them discuss for a while how they would go about figuring this out. First of all, they'd need to decide on the size of a standard person. They could decide to answer the question about how many of one of them they could fit in the gym, say Sam, or pick some other size they all agree on. Now they don't want to measure in Sam-cubes.

Instead, they need to imagine, to visualize, Sams piled in the gym, in a regular fashion (say all laying down with their heads in the same direction). How many will fit?

Let them discuss how this would work. They need to measure the gym in one direction in lengths of Sams -- that is head-to-toe, and in the other direction widths of Sams -- that is shoulder-to-shoulder. They can do this (and will probably want to) by physically laying across the floor. Finally, they will need to find how many Sams (laying on top of each other) will fit in the height of the gym. One way to measure this, if the gym is built of bricks, is to see how many bricks thick Sam is lying down. Then figure out how many would stack up.

Once this information is gathered, you can leave the gym and do your calculations. What did you find out? Was the approximation you found as a team close or far from the guess you made at the beginning? What's the difference between a guess and an approximation?

Introduction:

Continuing their study of volumes, students will have fun with the idea of packing students tightly into the gym, and they'll likely be surprised by what they discover.

Objectives:

- To get some more experience using approximation techniques to approximate a volume when not using cubes to fill a space

Materials:

- Scratch paper and pencils
- A calculator
- Access to the school gym (or some other large room) for about 20 minutes

6. Estimating Numbers

How many shoes are there in your school right now? Let the students work on this question for a while without saying anything. There may be shoes on feet, shoes in gym lockers, spare shoes for teachers in their closets, shoes in the lost-and-found. Estimate the number.

Now pull out the jar and ask them to estimate the number of M&Ms in the jar. Have them take their time and write down their individual estimates on pieces of paper. Then encourage them to discuss a sound way of estimating the number. They should consider such things as if the sides of the jar are straight, how many thicknesses of M&Ms would fit from the bottom to the top of the jar if they were neatly piled on top of each other. They should look at the bottom of the jar to get an idea of how many M&Ms would fit flat against the bottom. A rough estimate is to take the number that would fit across the bottom and multiply by the number that can be stacked neatly in a column. Is that an over- or under-estimate? After there has been good conversation and revised estimates, count the M&Ms and see what they learned. How does it compare with their original estimates? Why the differences?

What if we wanted to estimate the number of drops of water that are in an Olympic-sized swimming pool? That's a big number! How big? Have them make educated guesses -- that is, estimates.

Would it help to know that there are about 600,000 gallons of water in the pool? How does that change your estimates? Let them talk for a while about what they might need, and then show them you brought a teaspoon measure. Would that help? Someone will suggest that you find out how many drops of water in a teaspoon, then how many teaspoons in a gallon. Use the faucet or eyedropper to estimate how many drops of water fill a teaspoon. Then estimate how many teaspoons fill a cup (there are 48, but they might not know that). There are 16 cups in a gallon. Now put it all together, and use the calculator if you like to estimate the number of drops in an Olympic pool!

Introduction:

How many marbles are in the jar? Have you ever been faced with this game? To win a prize you must be the closest to the correct answer. The winner is usually the person who can estimate the best.

Objectives:

- To give the students practice in approximating a number of things.

Materials:

- Scratch paper and pencils
- A calculator
- A jar full of M&Ms or beans or marbles or marshmallows packed tightly (so the students can turn it upside down and they don't move)
- A paper bag to keep the jar in before it's time to use it
- A teaspoon measuring spoon and a measuring cup
- Access to a faucet which can be made to dispense water a drop at a time OR an eyedropper and a cup of water

Logical Thinking Puzzles

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1. Story Time

What follows is a series of word problems for the students to think about and solve. (If you disliked or were anxious about word problems when you were in school, you should try your best not to pass that along to the students. Give them encouragement, and they'll do the rest.)

Here are some tips for each question:

1. Read the entire question before you begin. Get a picture of it in your head.
2. Identify the relevant information (the numbers that matter) and underline them.
3. Identify what you are supposed to do with the numbers (add, subtract, multiply, etc.).
4. Write out the equation you're trying to solve. (A common mistake here is for children to take two numbers and the operation they identified, say addition, and add the two numbers. That isn't necessarily what the questions suggests we should do. The students should explain why the equation they write down makes sense in the problem.)

Groups of two or three students work well for problems like these. They can catch each others' mistakes, while not solving the problem for everyone.

Answers:

1. 4+5+6=15 cards
2. Need 6*7=42 cards; they need two more cards
3. 3+4-2=5 cards
4. Frank gets 15-11=4
5. 18/3=6 each
6. Mrs. F will take 13-7=6 cards; Seth will have 16-6=10.
7. Hot dog first: 2*(8-3)=10; Friend first: 2*8-3=13
8. Red: 14-3=11; Green: 6+5=11; Total: 11+11=22.
9. 3*1+4*2+6*3=3+8+18=29 dimes, which is $2.90
10. 544+136=680 lockers
11. 1000-360*1-175*2-30*3=1000-360-350-90=200 left
12. 2400-500-200-800-300=600 in the bank

Introduction:

Being able to read and understand word problems is crucial to being able to use math in the real world. Rarely are equations placed in front of you already fulled formed; usually one must find the equation before being able to answer a problem.

Objectives:

- To give the students experience translating words to an equation to solve

Materials:

- Scratch paper and pencils
- A calculator

Turning Words Into Math

1. Pedro has four cards. Franco has five cards. Adriano has six cards. How many cards do they have together?
2. There are forty cards The Challenge Math group, which has six people, wants to play a certain game. Each person needs seven cards. Are there enough cars to play the game? How many more cards do we need?
3. Kenneth was walking down the street one day when he found three cards lying on the ground and picked them up. Further down the road he found another four cards and picked them up as well. On his way home, Kenneth accidentally dropped two cards. How many cards does Kenneth have when he gets home?
4. There are fifteen cards on the table. Frank and Sarah get to split the cards, but Sarah picks how many to take first. She decides to take eleven cards. How many cards does Frank get?
5. Russel, Laura, and Peter are about to play a game. The game has eighteen cards, and each player has to have an equal number of cards to play. How many cards does Russel get? Laura? Peter?
6. Mrs. Fairbanks wants to have thirteen cards, but she only has seven cards. Seth has sixteen cards. How many cards will Mrs. Fairbanks take from Seth? How many will Seth have left?
7. Stephen is at an arcade and wants to buy a hot dog. He has eight tickets in his pocket. A hot dog costs three tickets. Stephen also needs to see his friend who says he will double Stephen's tickets. How many cards will Stephen have if he buys a hot dog first? How many tickets will Stephen have if he meets his friend first and then buys a hot dog? What should he do?
8. Jane likes to collect pens. She has fourteen red pens and six green pens. She trades three red pens for five of her friend's green pens. How many of each color pen does she have now? How many pens in total?
9. Ben likes little candies. He decides to buy three red candies that cost 1 dime each, four blue candies that cost 2 dimes each, and six yellow candies that cost 3 dimes each. How much does Ben pay for his candies?
10. There are five hundred and forty-four students at an elementary school. Each student has one locker for themselves. There are one hundred and thirty-six lockers empty. How many lockers does the school have altogether?
11. Bridgewater Elementary orders one thousand cookies for parents and their kids for an afternoon event. Three hundred and sixty people have one cookie, one hundred and seventy-five people have two cookies, and thirty people have three cookies each. How many cookies were left after the event?
12. Brian's paycheck last month was two thousand four hundred dollars. He first has to pay for rent (five hundred dollars) and food (two hundred dollars). He decides to buy a TV which costs eight hundred dollars as well as a DVD player which costs three hundred dollars. Brian puts the rest in the bank. How much did Brian put in the bank?

2. Secret Code

This activity involves some set-up before you see the students. Use the masking tape to create a 5x5 grid like the one pictured here. Label start and finish with a permanent marker, and put enough X's on the floor next to the grid for all but one student in your group to stand on. Label the first X with "up next" and the second X with "in the hole."

The Secret Code is the path, determined by you, from start to finish using steps up (U), to the right (R), to the left (L), and down (D). A sample code for the grid above is URRDRRULULULDLUURDRURDRU. You can check for yourself that this path leads a student from the start box through the grid, not necessarily in the most direct path, and ends in the finish box. The goal of the activity is for the students to discover the correct path from start to finish using trial and error.

Each student has a designated place to stand at all times during this activity. One student will be in the grid, and the remaining students must stand on one of the X's to the side. The student in the grid begins by standing in the start box and then steps to one of the other boxes available. In this case, standing on the

Introduction:

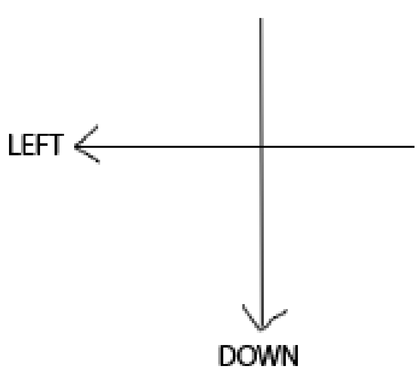
Learning math involves observation, practice, and trial-and-error. This fun little game involves all of those.

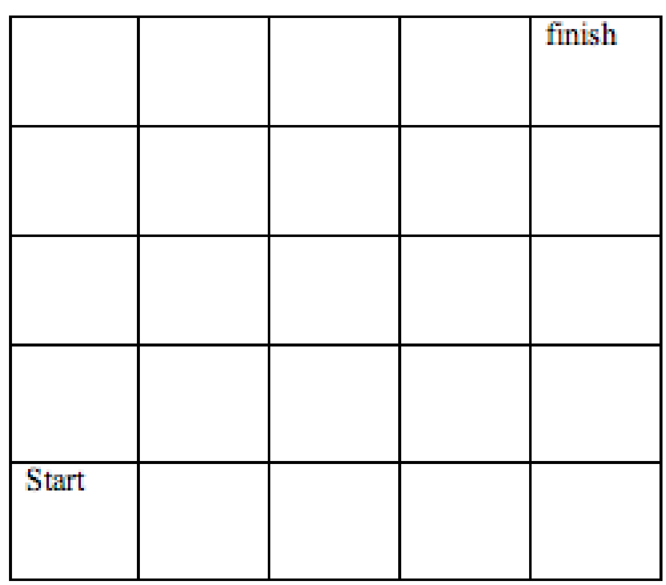
Objectives:

- To learn to observe well
- To get comfortable with trial and error
- To work together as a team

Materials:

- Scratch paper and pencils
- Masking tape
- 1 Permanent marker
- Time to set up a grid on the floor before Challenge Math using the masking tape





X ("up next") X ("in the hole") X X

start box, the student may step up or to the right. If the student chooses the correct move, you will make a 'ding' noise to indicate that they can go on. The student proceeds making choices to move up, down, left or right, until they make a move that is not a part of the secret code. When a mistake is made, you will make a 'buzzer' noise indicating the end of their turn. Each time the student in the grid gets buzzed, they go to the last X and everyone in line moves up one, with the player 'up next' entering the grid.

The following are rules you should communicate to the students before starting:

1. There is NO TALKING during this activity.
2. Boxes on the grid may be visited more than once in the secret code.
3. The secret code may lead you away from the finish box at times, circling back or scooping down; the longer (and harder) the code is, the less direct the route will be.
4. Only moves up, down, left, or right are allowed in the grid; diagonal steps are not allowed.

It will likely take around ten minutes for your group to crack the first secret code. When they do, give them a harder one with more moves involved. After doing the activity twice, pause and get the group together to discuss and answer the following questions:

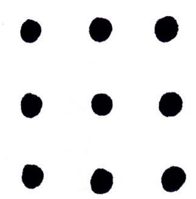
- What is hard about the activity? (Likely they will answer remembering the path.)
- What strategy did you use individually? (Did they pay attention only when it was their turn? Did they watch others and learn from them? Trial-and-error and building on what's come before is how a lot of mathematics is learned.)
- Was it easier to remember the moves when they were watching others or doing it themselves? (Point out that physically walking through the grid uses a type of muscle memory in their bodies, and not just in their brains.)
- What strategy did you use as a group? (Perhaps they tried a way of non-verbal communication to help each other along. Congratulate them on finding ways to think outside the box.)
- What would make the secret code even harder? (Perhaps diagonal steps or stepping two boxes at a time.)

With time remaining, let them come up with their own codes and have others try to guess them.

3. Thinking Outside the Box

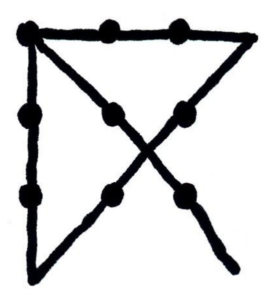
Students seem to enjoy the challenge of this puzzle, even though it's a problem that many adults cannot solve. Spoiler Alert: The answer is below, so don't show the students this page!

The problem, often referred to as the "9-dot problem" starts with a 3x3 grid of dots as below. The challenge to the student is this: connect all 9 dots with 4 straight lines WITHOUT picking up your pencil.



This sounds like an easy task at first, but the children will quickly notice that it's pretty hard, and some might even say "impossible."

Allow them to work together or alone as they choose. If, after a substantial bit of time, they need a hint, suggest that they "think outside the box" or that the four lines don't need to end on the dots or that the lines don't need to stay within the box that the dots are in.



Introduction:

This little brain teaser involves logic, problem solving, and the value of perseverance.

Objectives:

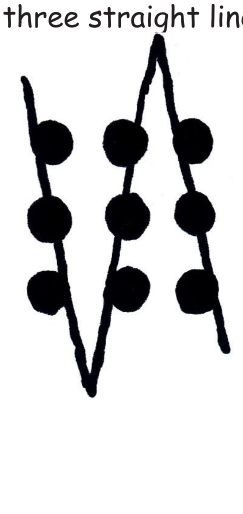
- To practice perseverance

Materials:

- Scratch paper and pencils

Fun Fact:

If the dots are fat enough, it's possible to do this puzzle with only three straight lines!



4. The Farmer's Problem

Start the group off with a puzzle:

A farmer has to bring a wolf, a sheep, and a bag of corn to the market. The problem is that there's a river between his farm and the market that he has to use a canoe to get across. In the canoe, he can only fit one of the three things he is bringing to the market at one time, plus himself. There is a problem, though. If at anytime he leaves the wolf and the sheep alone together, the wolf will eat the sheep. And if the sheep and the corn are alone together, the sheep will eat the corn. The corn is safe with just the wolf. How should he bring his things to the market?

Have the students talk it over or act it out to see which of the three objects they think the farmer should bring across the river first. After a short discussion they should realize that the farmer can ONLY take the sheep first, otherwise something will get eaten.

After the sheep has crossed with the farmer and the farmer has canoed back, the sheep will be on the side of the river closest to the market, and the wolf and corn and farmer will be on the opposite side. Now what should he take across? (Let them discuss it first, but either one will actually give you a solution in the end.) Imagine they say to take the wolf across.

This is where the problem arises. If he leaves the wolf and sheep together on the side closest to the market, the sheep will be eaten. At this point, the students need time to explore all the possibilities.

The tricky part is that the sheep now has to come back with the farmer so the wolf doesn't eat it. Now the wolf is on the side closest to the market and the sheep and corn are on the side with the farmer. At this point the farmer needs to take the corn across the river to the wolf, where it will be safe. Now all the farmer has to do is to go pick up the sheep and bring him across to where he can now take all three safely to market.

Introduction:

This classic puzzle is great practice for logical thinking, discussions and working as a team to come up with the right answer.

Objectives:

- To practice trial and error thinking
- To work together as a team

Materials:

- Scratch paper and pencils
- Three objects (stuffed animals, photos, etc) to represent a wolf, a sheep, and a sack of corn

★★★

5. The Pet Shop Puzzle

Pose this question to the students:

One day Stephen wanted to go buy some pets. He had saved up $100, and he had his heart set on buying 100 pets with his money. At the pet store, he saw that dogs cost $5, cats cost $1, and hamsters cost $.25. He wanted to buy at least one of each. Is there any way that he can spend exactly $100 to buy exactly 100 pets and have at least one of each?

Pose this problem to the students, post the sign listing the prices so they don't forget, and turn them loose on it. They may pose creative solutions like buying a dog, a cat, 98 hamsters, and donating or losing the rest of the money. However, this problem can be solved, so keep them focused and working on it.

If after fifteen or twenty minutes they seem stuck, point out that if the solution involved one hamster, the total cost would end in $.25. In fact, for the dollar amount to be even, Stephen must buy his hamsters four at a time.

Here's one way to reason through it, but before you tell them, give them much time to work through it on their own. Persevering when they're frustrated is a valuable habit to learn. Stephen needs at least 1 dog, 1 cat, and 4 hamsters, which is a total of 6 pets for $7. We want to end up having the number of animals and the number of dollars even, and now we have spent more dollars than we have number of animals. What do we do about that? Buy four more hamsters! Then we have 10 pets for $8, so we have more pets than money spent. What do we do about that? Buy a dog!

If you continue with that reasoning, you'll end up seeing that after you've bought 6 dogs and 32 hamsters and 1 cat, you've spent $39 and have 39 animals. Then you can just buy another 61 cats and you'll have 100 animals for $100!

Introduction:

This little brain teaser involves logic, problem solving, and the value of perseverance.

Objectives:

- To use logic to reason through this puzzle

Materials:

- Scratch paper and pencils
- A small sign saying:
  Dogs $5
  Cats $1
  Hamsters $.25

6. Pico, Fermi, Bagels

This is a simple game to teach the students, but there is interesting and complex logic the students will need to learn to play it well.

To begin the game, choose a three-digit number with no repeating digits, say 489. (The number 484 would not be okay because it has repeating 4s.)

To determine your number, the students take turns guessing three-digit numbers, and you respond to their guesses in a certain way. If none of the three digits in the students guess are in your number, you say "Bagel." If one of the digits the student guesses is in your number, but is not in the correct place, you say, "Pico." (If two or three of the digits the student guesses are in your number, but none are in the correct places, you say, "Pico Pico" or "Pico Pico Pico.") Finally, if one (or two or three) of the digits is in your number and is in the correct place, you say, "Fermi" (or "Fermi Fermi" or "Fermi Fermi Fermi"). Note that if you respond with three "Fermi"s, this means that the student has guessed the secret number! It's also possible that, for example, one digit of the student's guess is correct but in the wrong place, another digit is correct and in the right place, and the third digit is not in your number at all. Then you would respond "Pico Fermi." Note that you only say "Bagel" when none of the digits of the student's guess are in your secret number.

Now for an example. Let's say you chose the number 489.

Guess	Response	Reason
362	Bagel	no digit is correct
820	Pico	the 8 is in the wrong place
418	Pico Fermi	the 8 is in the wrong place and the 4 is in the correct place
518	Pico	the 8 is in the wrong place
487	Fermi Fermi	the 4 and the 8 are in the correct place
489	Fermi Fermi Fermi	

Once the students begin to understand the rules of the game, they can take turns thinking up a secret number,

Introduction:

This classic strategy game requires little to play, but it requires concentration and logical reasoning to play well. After your students learn this game, it's a great one to play on days when you have ten minutes extra time to fill.

Objectives:

- To practice using logical reasoning to figure out a puzzle

Materials:

- Scratch paper and pencils

and you can include yourself in the guessing. You can also play with four or five digit numbers, as long as you don't repeat digits.

After you've played a couple of rounds with the students, it's a good time to start thinking about guessing strategies. To begin with, if the students have not already begun writing down each guess and its response, ask them how they might better remember what numbers have been guessed before, and suggest that they start keeping track.

Now each time a student makes a guess at the secret number, ask him or her to explain to the group his or her logic for making that guess. Let's look at the example from before to determine the logic behind the guesses. In quotes to the right are possible student responses.

Guess	Response	Logic
362	Bagel	"I just guessed three numbers."
820	Pico	"Since nothing was right the first time, I guessed three new digits."
418	Pico Fermi	"Only one digit from 820 was correct, so I kept the 8 in my guess and I put it in a new spot, and then I tried two more digits we hadn't tried before."
518	Pico	"One digit from 418 was in the correct place, so in my guess I kept two digits the same in the same places and I changed the third one."
487	Fermi Fermi	"The 4 must be in the correct place since there were no Fermi's in my 518 guess, but there was one in my 418 guess. And either the 8 goes in the second place or the 1 goes in the third place, but not both since there was only one Pico in 518."
489	Fermi Fermi Fermi	"Either the 7 or the 8 must be in the correct place in 487, but not both. So if the 8 is in the correct place, then the third place must be a number we've never guessed before."

As each student explains their reasoning behind their guesses, encourage the other students to ask questions and even to disagree, as long as they do it respectfully. Also, once one or two students understand the logic behind a certain guess, there's value in having them explain their understanding to the other students.

7. The Revenge of Pico, Fermi, Bagels

In order to solve this challenge, the students need to already be familiar with Pico, Fermi, Bagels.

Here is the problem. You walk in on a game of Pico, Fermi, Bagels already in progress and you see that four guesses have already been made, and you see what scores they received. Can you solve the puzzle and announce the answer without any more guesses?

The four guesses are:

Guess #1:	6152	Pico Fermi
Guess #2:	4182	Pico Pico
Guess #3:	5314	Pico Pico
Guess #4:	5789	Fermi

What's the hidden four-digit number? Reason through it and you should be able to figure it out!

If the students want they can work individually, or they can work at the white board together. Give them time to work and encouragement, and see what they come up with.

How do we begin to approach this? Let's figure out which one of the numbers in the first guess is in the right place (if we can). Is it the 1 or the 2? No, because in either case we'd have a Fermi in Guess #2. Is it the 5? No, because if it was, we'd have a Pico in Guess #4. So it must be the 6. (Does this make sense with the other grades? Check! Yes, it does, and that's good because otherwise the scorer would have made a mistake (which happens!).) Now we know the number is 6 _ _ _.

The number starting with 6 means that 5 cannot be in the number at all because if it was, there would be a Pico on Guess #4. So the Pico in Guess #1 is either the 1 or the 2, but not both.

Then in Guess #2, one of the Picos is for the 1 or 2 and the other Pico is for the 4 or 8. If the 8 in Guess #2 is the Pico, the 8 is Guess #4 is in the same spot, so it should get a Pico too, which it does not. Therefore, it's the 4 in Guess #2 that gets the Pico.

Introduction:

Once your students understand Pico Fermi Bagels, here's a little challenging puzzle to make them test their reasoning skills to find the correct solution.

Objectives:

- To practice using logical reasoning to figure out this puzzle
- To discover game-playing strategies

Materials:

- Scratch paper and pencils
- One white board and marker

What do we know so far? A 6 begins our number, there is a 4 in our number (but not in the first or fourth spot since it never earned a Fermi), and there is either a 1 or a 2 in our number. Furthermore, we know that 5 and 8 are not in our number. Furthermore, from Guess #3, the 4 earned a Pico, so either 1 or 3 (but not both) is in our number. And from Guess #4, either 7 or 9 is in our number, but not both.

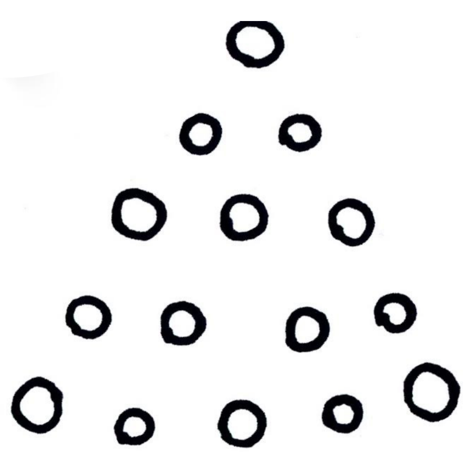
So our four-digit number starts with a 6 and has a 4, a 1 or 2, a 1 or 3, and a 7 or 9. The only way to fit that many numbers in is if 1 is in our number. It can't be in the second spot because Guess #2 didn't have a Fermi, and it can't be in the third spot because Guess #3 didn't have a Fermi. Thus, the 1 is in spot number four: our number looks like 6 _ _ 1, where the other two digits are a 4, and either a 7 or a 9. Of course, it can't be the 9 because the fourth spot is taken, so the 9 can't be a Fermi as in Guess #4. Therefore our number has to be 6741.

8. A Jumping Game

Start with markers on fourteen of the fifteen circles, leaving the circle at the top blank.

One can move the markers by jumping one adjacent marker. The jumped marker is then removed. Take turns jumping adjacent markers until no more jumps are possible. How many markers remain? Try to minimize the number of markers remaining when no more jumps are possible. It's possible to get down to just a single marker left on the board at the end of the game. Can you do it?

Discuss possible strategies with the students and see which ones work.



Introduction:

This is an adaptation of a classic wooden board puzzle which can provide hours of entertainment. Get them started and see how they do!

Objectives:

- To practice using logical reasoning
- To discover game-playing strategies

Materials:

- Scratch paper and pencils
- One sheet for each student containing an outline of fifteen circles laid out in a triangular shape
- 14 pennies or small plastic disks per student

Combinatorics:
Making Life Count

1. Ways to Count Change

Mathematicians need to think in an organized fashion to solve mathematical problems, and it's never too young to begin teaching this structured way of looking at things.

Begin by telling the students they need to pay you five cents. How many ways can they do this? They could give you a nickel, or they could give you five pennies, so there are two ways to pay five cents.

How about ten cents? Let them think about it a while. There are four ways (1 dime, 2 nickels, 1 nickel and 5 pennies, 10 pennies). Help them write it down in a logical manner. An easy way to approach this is to start from the biggest coin to the smallest coin in this way: a dime is the biggest coin you can use to make ten cents, and there's only one way to use a dime. Now the next biggest coin is a nickel. How many ways can you use a nickel to make ten cents? Well, once you use a nickel, you still need to come up with five more cents, which we saw above can be done by a nickel or five pennies. So there are two ways to use a nickel. Now the biggest coin you can use is a penny. How many ways are there to pay ten cents with just pennies? One: ten pennies, and there you have all four ways.

How many ways can you pay fifteen cents? (There are 6 ways: 1 dime and 1 nickel, 1 dime and 5 pennies, 3 nickels, 2 nickels and 5 pennies, 1 nickel and 10 pennies, 15 pennies.)

How many ways can you pay twenty-five cents? Now this gets much more interesting! Let them work on it. If they need help, ask them how many ways they can make twenty-five cents with a quarter. Then how many using dimes as the biggest coin. (Help them see that if they use a dime, then they still need to make 15 more cents, but they saw above they could do that 6 ways.) Then how many using nickels as the biggest coin. Then using pennies as the biggest coin.

It turns out that there are 13 ways of doing it. Can you find them all? Can you find an organized way of writing them all down?

Introduction:

Combinatorics is a field of mathematics that involves counting the number of ways something can be done. Here we are going to count the number of ways we can make change.

Objectives:

- To practice using coins
- To practice making change
- To get a simple example of combinatorics
- To talk about thinking in an organized fashion

Materials:

- Scratch paper and pencils
- Coins, if desired

1 quarter
2 dimes, 1 nickel
2 dimes, 5 pennies
1 dime, 3 nickels
1 dime, 2 nickels, 5 pennies
1 dime, 1 nickel, 10 pennies
1 dime, 15 pennies
5 nickels
4 nickels, 5 pennies
(Note here that 3 nickels and 1 dime would be logical, but we've already written down all the combinations using dimes above, so we don't write it down here.)
3 nickels, 10 pennies
2 nickels, 15 pennies
1 nickel, 20 pennies
25 pennies

Try counting the number of ways to make other amounts: 17 cents or 43 cents.

For a special challenge, count the number of ways using pennies, nickels, dimes, and quarters that you can make a dollar. There are 242 -- it would be best to write them down in an organized list!

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2. The Problem with Handshakes

Begin by asking the students the following question:

If there are 20 people at a party and everyone shakes each others' hands, how many handshakes are there in total?

Let the students discuss this for a while. They might want to try shaking each others' hands and counting. Encourage them to count in a systematic way. (One way they might do this is to count how many hands they each shake, then add them up. However, when Sam and Maggie shake hands, that should count as one handshake, not two.)

If they need a suggestion, have them start with just two people. How many handshakes would there be? (1) How about three people? (3) How about four people? (6)

Now imagine a party of ten people and a new person shows up. How many handshakes have to happen for the new person? Discuss it. (10) So every time a new person is added, the new person needs to shake everyone's hand who was already at the party. Consider one person standing alone in a room, and one by one guests arrive. The number of handshakes needed would be 1+2+3+...+19.

It's always good to have several ways of approaching the same problem. In this case, here's another way to think about it: If there are 20 people at the party standing in a circle, the first person needs to shake 19 hands. The second person already shook hands with the first person, so she needs to shake 18 hands. The third person already shook hands with the first two, so he needs to shake 17 hands, and so on. This means the total number of handshakes would be 19+18+17+16+...+1.

How many is 1+2+....+19? That's an interesting question itself. Look how 1+19=20, and 2+18=20, and 3+17=20. Write out the numbers in the sum, and grouping them this way makes it very easy to see the sum is 190.

Introduction:

Mathematicians often solve a problem by asking an easier problem first. Step-by-step, the answer is created from easier problems. This puzzle is perfect for that sort of analysis.

Objectives:

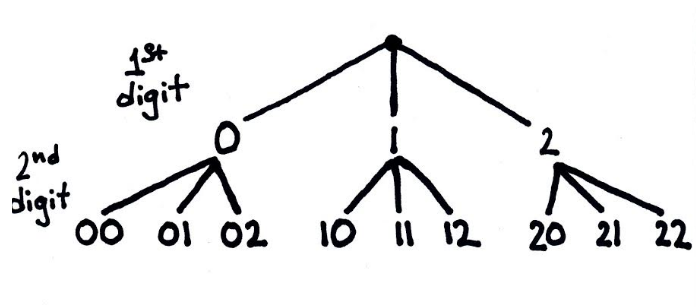
- To continue asking ourselves "How many?"
- To think in an organized fashion
- To build up our answer by answering an easier question first.

Materials:

- Scratch paper and pencils

3. Counting Locker Codes

Begin by asking the students how many different 2-digit locker combinations they can get by using the numbers 0 though 3. Give them some time to think about this and write some down. Encourage them to write the numbers in increasing order so that they remember every one. They should get 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, for a total of sixteen different combinations. Another way of seeing this is to look at the "tree" of possibilities:



Most dial combinations have three numbers in their code. How many different combinations of three numbers can you make with the numbers on the dial being 0 through 3? Let them work on this, and perhaps they can write them all down. The first digit can be 0, 1, 2, or 3. The second digit can be 0, 1, 2, or 3. The third digit can be 0, 1, 2, or 3. There are four choices for each of the digits, so we end up with 4*4*4=64 different possible combinations. Writing them down from smallest to largest, one would have 000, 001, 002, 003, 010, 011, 012, 013, 020, 021, 022, 023, 030, 031, 032, 033, 100, 101, and so on. Again, we could look at a tree like above to see the 64 different options.

Try different examples using small numbers for the different numbers on the dial and the different number of numbers in the combination.

When they're ready, point out that the numbers on dial combinations get much bigger. How about the numbers on our dial go from 0 to 39? Now how many combinations can be made for the lock? Would there be enough so that every student in the school gets a different code? Give the students time to think about this, and they'll realize that there are 40 choices for each of the three numbers in the combination, so

Introduction:

This lesson is an exercise in thinking about how many different combinations we can get from a given group of numbers. This is a good lesson to do early in your lessons about combinatorics because it's a good way for students to think about the problems.

Objectives:

- To practice their counting skills

Materials:

- Scratch paper and pencils

40*40*40=64000 different combinations. Would they like to draw a tree for this example? (Let them start if they want, but it's unlikely that they'll finish!)

If there's time and they need a bit more of a challenge, point out that most dial combinations don't let the first and second number of the combination be fewer than three spaces from each other on the dial. (For example, if the first number of the combination is 17, the second number can't be 15, 16, 17, 18, or 19.) Similarly with the second and third number in the combination. And if the second number is 38, then the third number can't be 36, 37, 38, 39, or 00, because the numbers wrap around on the dial.

Now how many combinations can be made using three numbers on a dial which has the numbers 0 through 39? (This just changes the problem a little, actually. For the first digit, there are still 40 choices. For the second digit now there are just 35 choices, and once the second digit is chosen, there are only 35 for the third, so 40*35*35=49000, still plenty for every student to have her own.)

4. A Large Wardrobe in Little Space

Start this lesson by asking the students the following question:

Your teacher, Ms. Katy, secretly lives at the school. She keeps some clothes in her closet to change her outfit every day. Luckily she never gets any of them dirty. She has five shirts of different colors, three pairs of pants of different colors, and two pairs of shoes (brown and black). If every day she puts on one shirt, one pair of pants and one pair of shoes, how many different outfits can she make?

Let the students explore this problem on their own for a while. Perhaps they will want to draw the different articles of clothing and make lines connecting them to count as one outfit. Perhaps they could try to write the outfits down systematically, such as counting all outfits that start with the blue shirt, then all the outfits that start with the green shirt. They will see some patterns when they do this.

How many outfits can she make? The answer is 5*3*2=30. Now suppose she adds some scarves to her outfits to add more color. She buys six different patterned scarves. Now how many outfits can she make? (For each of the 30 outfits from before, she can wear the polka-dot scarf. Also, for each of the 30 outfits she can wear the striped scarf. Or the checked scarf. Or any of the other scarves.) So for each of the 30 outfits she can wear any one of the six scarves, for a total of 6*30=180 different outfits. That's nearly enough to get her through the whole school year!

With time remaining have the students estimate how many shirts, pants, and shoes they each have at home. How many different outfits can they make? What other accessories do they have which can make more outfits?

Introduction:

This is a simple counting example that has many practical uses.

Objectives:

- To practice using counting arguments

Materials:

- Scratch paper and pencils

★★★

5. The Milk Groups

Pose the following question to the students:

Ms. April likes for the twenty students in her class to have a carton of milk each day at break time. She sends three of her students down to the cafeteria to ask for the cartons of milk, but she likes to mix things up a bit and wants to send a different group of students down every day. Will she have enough different groups of three students in her class of twenty students to send a different group every day of the school-year?

(Keep in mind that the order that the students are chosen to be in the group doesn't matter: the group consisting of Greg, Sarah, and Sally is the same group no matter whose name is called first, second, and third.)

They will discover this is a difficult problem. Have them try to figure out how many groups of three students they can make from a smaller class of, say, six students. Let them try this simpler problem for a while. If there are six (or more) in your group, they can try physically moving in and out of groups of three to try to figure it out.

If you imagine the six children in the simpler problem are named A, B, C, D, E, and F, then the groups they can make, written in a systematic fashion, are ABC, ABD, ABE, ABF, ACD, ACE, ACF, ADE, ADF, AEF, BCD, BCE, BCF, BDE, BDF, BEF, CDE, CDF, CEF, DEF.

The students could similarly name the twenty students in Ms. April's class and write down all possible three-student groups. If they wish to try, let them. (They might become discouraged, though, because the answer is 1140!)

Instead, let's try making the problem easier in a different way to solve it. Instead of making the number of students in the class smaller, let's suppose that it DID matter in which order Ms. April called the students' names. (Instead of being a group of three students,

Introduction:

This is a challenging lesson which highlights for the students the differences between counting where order of the objects matters (students lined up at the door) or order doesn't matter (groups of students working together on a project).

Objectives:

- To practice thinking systematically
- To practice breaking the problem down into a simpler question
- To practice counting things
- To see the difference between ordered and unordered groups

Materials:

- Scratch paper and pencils

now you can think of the three students lining up at the doorway before they go to get the milk: there's a first person, a second person, and a third person.) How many ways can she do that? Does this change the problem? How would you count that?

Surprisingly, this makes the problem easier. How many students could Ms. April call to be first in line? Well there are twenty students in the class, so 20. How many could she call to be second in line? There's already one person at the door, so there are 19 names left that she could call to be second. How about third? There are 18 different names she could call to be third. For each choice of student to be first in line, she has 19 choices of the second student, and for each of those, she has 18 choices for third student. How many does she have all together? (20*19*18=6840 ways!)

(If your students have trouble understanding why we should multiply those numbers, try a simpler example, such as choosing three students to line up from a class of four. They should see that there are four choices for the first person, three for the second once the first in chosen, and two for the third once the first and second are chosen. They can actually have people in your group line up, then they'll see they get 4*3*2=24 different lines.)

But when we started the problem we didn't care about order. We don't care if the three students going to get the milk are Greg, Sarah, and Sally or Sarah, Greg, and Sally. How many times did this (or any) group of three students get counted in our list of 6840 different lines? They appeared as:

Greg, Sarah, Sally
Greg, Sally, Sarah
Sarah, Greg, Sally
Sarah, Sally, Greg
Sally, Greg, Sarah
Sally, Sarah, Greg

That group of three students appeared six times. Is there anything special about these three students? No, the same would be true for any group of three students -- the same group of three students would appear six times. Therefore each group of students actually gets counted six times in the 6840 number, so we need to divide that number by 6 to see how many groups of students there are. That number is 6840/6=1140.

Are there enough different groups to get milk for Ms. April's class each day of the school year? Yes! In fact, different groups of students from the same class could go get the milk for nearly all of elementary school!

6. Students in a Line

How many ways are there for 3 students to line up in a line? Have your students discuss/recall the answer from The Milk Groups lesson. There are 6 ways: ABC, ACB, BAC, BCA, CAB, CBA (if the students are named A, B, and C).

There are three people who can be first; once that person is chosen, two can be second, and after the first and second are chosen, that leaves just one choice for the last place.

Now let's take one of the possible three-person lines from our list, say BCA, and count the number of ways that a fourth person, D, can budge in the line. Let the students come up with: DBCA, BDCA, BCDA, BCAD. There are four ways for the fourth person to join in. Now how many possible ways are there for 4 students to line up? EACH of the ways listed above has four possible ways for student D to budge in, so there are 6*4=24 different possible lines. (If the students aren't sure that this makes sense, have them write down the possible lines.)

Another way to think about a line of four students is like we did with a line of three: there are four choices for the first person in line. Once that person is chosen, there are three choices for next in line, two choices for the third position, and only one choice (that is, we have nothing to choose) for the fourth position. That means there are 4*3*2*1=24 different four-person lines.

What about five-person lines? (They may begin to see a pattern. If they don't, go through the same two types of analysis that we did with four-person lines.) The number of ways is 5*4*3*2*1=120.

What about six-person lines? Or what if 10 students were to line up in order at the door? How many ways can that happen? (10*9*8*7*6*5*4*3*2*1 which is a very big number -- try this on a calculator!) What if your entire class lined up in order at the door -- how many ways can that happen?

Introduction:

In this lesson we discuss some basic principles in combinatorics -- the number of ways students can line up, and the idea of the factorial. This is a good lesson after The Milk Groups.

Objectives:

- To practice counting ways to do something

Materials:

- Scratch paper and pencils
- A calculator

Fun Fact:

Mathematicians have a short-hand way of writing 10*9*8*7*6*5*4*3*2*1, it's 10! where the exclamation point is read "factorial." In general, n! is the product of all the whole numbers from n down to 1.

7. A Combinatorial MATH MESS

How many ways are there to rearrange the letters in the word MATH? The result need not be a word, just a combination of letters. This is like the letters in the word MATH are all lining up at the door. There are four choices for the first letter, three for the second, two for the third, and one for the last. There are 24 ways to write those down.

But now let's make the problem a little harder. How many different ways are there to write down the letters in the word MESS? This word has a repeated letter, so if we were to list all 24 ways like we did for MATH, we would have ESSM and ESSM both listed (as well as SMSE and SMSE, and other pairs). In fact, every combination of letters is written down twice. If we were to color-code the Ss, then the words would be different, such as ES(blue)S(green)M and ES(green) S(blue)M, but since the letters aren't colored, the words are the same.

So how many different ways are there to write down the letters in MESS? There are 24 if the two Ss were distinct, but since each word gets written down twice, there are 24/2=12 distinct ways.

What about MMSS? Now the Ss can be written in two different orders and the M's can be written in two different orders, so if the letters are indistinguishable (unless they're colored, for example), there are 24/(2*2)=6 different ways to write down those four letters distinctly. Try it!

How many different ways can you write down the letters in BOOKKEEPING? There are 11 letters, so if they were all distinct, they could line up in 11*10*9*8*7*6*5*4*3*2*1 (also written as 11!, see the Fun Fact on Students in a Line), which is a very large number. (Put it in your calculator and see!) But the letters are not all distinct. There are two Os which have two different orders, two Ks which have two different orders, and two Es which have two different orders. Thus there are 11!/(2*2*2) distinct ways to write the letters in BOOKKEEPING.

Introduction:

Continuing on with the idea of Students in a Line, in this lesson we count how many different combinations of letters we can make out of "math" or "mess".

Objectives:

- To practice counting ways to do something

Materials:

- Scratch paper and pencils

Fun Fact:

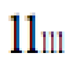
How many distinct ways are there to write the letters in AARDVARK? There are two orders in which to write the R's, but there are six orders in which to write the three A's. So there are 8!/(2*6)=3360 ways. Do you want to try to write them all down?

8. Cracking the Postal Code

Thinking back to what we did in A Combinatorial MATH MESS, how many distinct ways are there to write the letters in the word BEEP? (There are 4!/2, which is 4*3*2*1/2=12. Why not 24? Because the two E's are indistinguishable.)

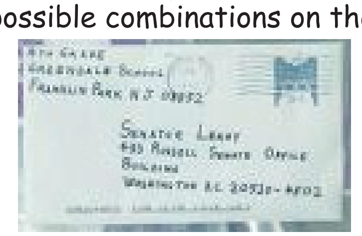
Now how many distinct ways are there to write down the letters in XXYYY? Let them work on this a while, and help them if they get stuck. They could just try writing down all possibilities and see how many they get. There are five letters, so 5! ways to write them in order, but the X's are indistinguishable, so we need to divide by two since there are two ways to write the X's down in order. Then there are six ways (or 3!) to write the Y's down in order, and the Y's are all indistinguishable, so we need to divide by six. That means 5!/(2*6)=10.

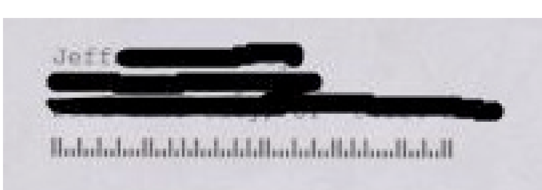
How many different ways are there to write down a combination of two long lines and three short lines?



Students could try writing down combinations, or they can use what they just did. Writing combinations of 2 X's and 3 Y's is the same as writing combinations of 2 long lines and 3 short lines.

At this point pass out the envelopes and ask them to find one of the ten possible combinations on the front of the letter.





Introduction:

This is a good lesson after A Combinatorial MATH MESS. This lesson shows them an example of using combinatorics to help crack the postal code.

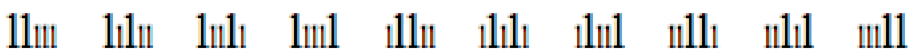
Objectives:

- To see a real-life example of combinatorics

Materials:

- Scratch paper and pencils
- 5 or 6 envelopes that have been through the mail, preferably arriving at places with different zip codes
- One whiteboard and marker

They'll see right away that their lines make up the stamped code (somewhat resembling a bar code) at the bottom of the envelope. Each group of three short lines and two long lines corresponds to a digit between 0 and 9. Have them work in partners to figure out which combination corresponds to which digit between 0 and 9. Make a table of the correspondence. The ten different combinations are:



Here are some sample postal codes for different zip codes:

After everyone agrees on which bars correspond to which digit, refer back to the envelopes and check that the codes at the bottom turn out to be the zip code written in the address on the envelope.

Congratulations! You've cracked the postal code!

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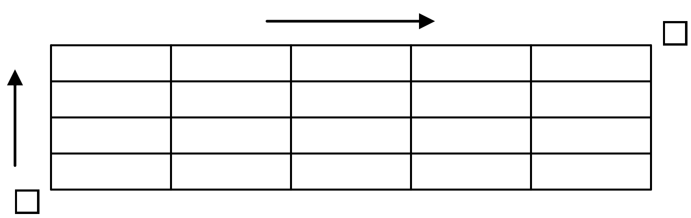
9. A Circuitous Route to School

Draw a map, specifically a grid of streets running noth-south and east-west about five streets by six streets. Draw a house on the southwest corner and a school in the northeast corner. Add more buildings, parks, parking lots, and anything else the kids think of on other corners. Explain that the streets are all one-way streets: the north-south streets are all one-way north, and the east-west streets are all one-way east. Put arrows on the map so that they remember.



Now state this problem to them: We live in this house, and every day we have to go to school through our town. We can walk on any of the streets, but we have to go the direction of the arrows. The problem is that we get bored if we go the same way every time. How many days can we go without being bored? That is, how many different ways are there to get to school?

Let the students try to figure it out. Some will find paths. Make sure their paths are correct, and maybe keep a tally of the ones you've found. They will soon realize that not only is it very difficult to remember which paths they've found before, but also the number of possible paths seems nearly endless. At this point, they may come up with the idea, or you may have to suggest it, that a good way to work on the problem is to do a smaller one first, to see if you can come up with a good strategy.

Start on the blocks very close to home, and when you're sure you've counted all the possible paths to get there, write that number at the intersection on the map.

It's interesting to note that all the southern-most and western-most intersections only have one way each to arrive there. For the southern-most intersections, a person would need to leave home and walk due east;

Introduction:

In this lesson, students discover a process of addition that makes a seemingly difficult problem quite simple. Finding simple and elegant solutions to hard problems is one of the joys of mathematics.

Objectives:

- To think about alternative approaches to a problem
- To practice breaking down a problem into an easier one

Materials:

- Scratch paper and pencils
- A large sheet of paper for a map

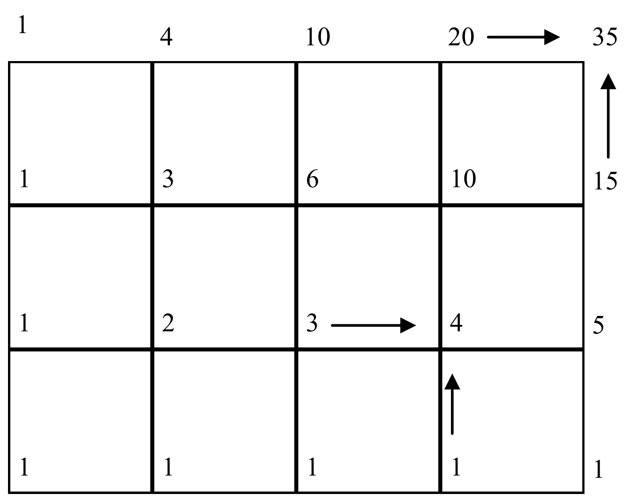
Fun Fact:

- No matter which path you take from home to school walking along the streets on this map, you will always go the exact same distance!

there are no choices, so there's only one path to each of those. Similarly, to get to the western-most intersections, a person would need to leave home and walk due north; there are no choices, so there's only one path to each of those. Now you can put 1s along all of those border intersections.

Once you've counted all the routes on the blocks nearest the house, propose a different way of thinking about the problem if the students haven't suggested it already. Notice that for any intersection on the map, to get to that point you MUST have arrived from the west or from the south (because of the one-way direction of the streets). Help the students gain the key insight to solving the problem: that to find the number of ways to get to a specific corner, you simply have to find the number of ways to get to the corner immediately south of it and the corner immediately west of it, and then add the number of paths to those corners together.

For example, in the diagram below, we see that 4 on the intersection one block north and three blocks east of home comes from the sum of the 3 to its left and the 1 below it. Similarly, the 35 in the northeast corner of this little map comes from the sum of the 20 to its left and the 15 below it.



Now you have what you need to complete the entire map and figure out how many ways there are to walk to school. Are there enough to keep from being bored?!

Draw similar maps and figure out the number of paths on those.

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10. Pascal's Special Triangle

Pascal's Special Triangle is a triangle of numbers formed in a special way. Begin at the top with a 1, and for each row below that, a number is formed by adding the numbers that are in the row immediately above it to the right and to the left (where any blank spaces are considered to be zero, which is what causes the outside of the triangle to be made up of 1s, since above each 1 on the outside is a blank space and a 1). The first few rows of the triangle look like this:

```
        1
       1 1          (1+1=2)
      1 2 1         (2+1=1+2=3)
     1 3 3 1        (3+1=4; 3+3=6)
    1 4 6 4 1
```

The students will have a lot of fun creating this triangle on a big sheet of paper, and they should enjoy the symmetry they see in the entries (like 14641).

After they have written down the first 8 or 10 (or more!) rows, ask them to look at the row sums. The first row sum is 1; the second is 1+1=2; the third is 1+2+1=4, and the pattern continues with 8, 16, 32, 64... Ask the students what the next number in the pattern is, and to predict the sum of the rows without adding up the numbers directly. Can they see why the row sum doubles each time? Give them some time, and have them explain to you again how the numbers in the table are created. If they aren't seeing it themselves, lead them to an explanation of why the sum of the row doubles for each succeeding row. (Each number in a row is used exactly twice to sum to a number in the subsequent row.

The second pattern for the students to consider is called the "hockey stick" pattern: if you sum numbers along a diagonal, starting with any 1 along an edge and continuing as far as you want, the sum will be one row down in the opposite direction from the one the diagonal was following, such as 1+3+6+10=20 in the diagram below.

Introduction:

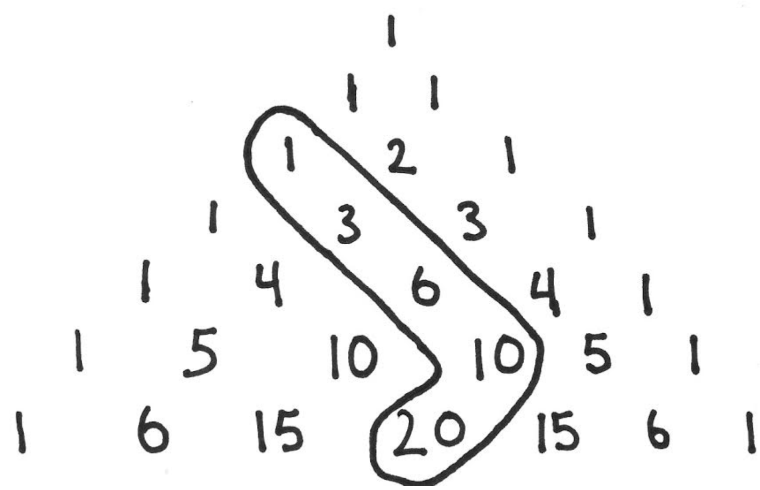
Pascal's Triangle is a concept that students will see in later mathematics courses; it is basic and important in the mathematical counting arguments, and is filled with interesting patterns. In this lesson the students will discover the concept of Pascal's Triangle and will discover some fun patterns within it.

Objectives:

- To learn about Pascal's Triangle
- To find interesting patterns within it

Materials:

- Scratch paper and pencils
- Large sheet of paper



Why does this happen? They can perhaps get an idea for why this happens by looking at the sum at each stage: 1+3=4, which is next to the 6, so 4+6=10, and then 10+10=20.

If you've talked about Milk Groups, the students will appreciate that the number of three-student groups made from a class of 20 students appears in this table! Just go to the row that starts 1 20... then go to the fourth (3+1) entry in it. This works for any size class and any size milk group!! That is, if you have 8 students in your class and you want to know how many different 5-member groups you can make from the 8 students, just go to the row that begins 1 8... and look at the sixth (5+1) number! That's much easier than what we did before.

What's so special about Pascal? This is a very useful triangle, but perhaps with time remaining your students can come up with their own triangles with special properties. Maybe they want to put something other than 1s along the edges, or perhaps they want to subtract or multiply to generate the numbers in the triangle. Or maybe it's not a triangle at all! Let them create their own number "triangles" and look for special properties in theirs.

11. Football Scores

Ask your students the following question:

Suppose you walk into your living room and you see that football is on the television, and you ask your mom what the score is. If she answers 3 to 1, would you believe her? No, because 1 is not a possible score in football. What scores would you believe?

Here are the various ways you can score in football:

Touchdown, 6 points
Touchdown + Extra Point, 7 points
Touchdown + 2-point Conversion, 8 points
Field Goal, 3 points
Safety, 2 points

Is 10 a possible score? Sure, here are just four ways to earn a score of 10. (Are there others?)
1 touchdown + 1 field goal
1 touchdown + extra point + 1 safety
1 touchdown + 2-point conversion + 1 safety
5 safeties

Back to our question now, what scores are possible? Give them time to work on this. Once they realize that all scores other than 1 are possible, give them some moderately high number, like 20, and ask them to write down all possible ways that 20 could be scored. They can try writing it in a systematic manner, such as:
8+8+2+2
8+7+3+2
8+6+6
8+6+3+3
8+6+2+2+2
8+3+3+3+3
8+3+3+2+2+2
8+2+2+2+2+2+2
7+7+6
7+7+3+3
7+7+2+2+2
7+6+3+2+2
7+3+3+3+2+2
7+3+2+2+2+2+2
6+6+6+2

Introduction:

For students who are football fans, this is a particularly engaging way of practicing some combinatorics -- different ways of adding to the same number.

Objectives:

- To practice writing in an organized fashion

Materials:

- Scratch paper and pencils

6+6+3+3+2
6+3+3+3+3+2
6+3+3+2+2+2+2
6+2+2+2+2+2+2+2
3+3+3+3+3+3+2
3+3+3+3+2+2+2+2
3+3+2+2+2+2+2+2+2
2+2+2+2+2+2+2+2+2+2

Now suppose that safeties and two-point conversions are no longer allowed in our game. The only points that can be earned are 3, 6, or 7. What scores are possible now? This takes a little more time to think about. Have them make a chart. They should see, after a little time, that 4, 5, 8, and 11 are no longer possible, but since 12, 13, and 14 are possible, they can always add 3s to those numbers to make any other number. So every other number is possible. Make them convince you this time!

To extend this even further, make up some game and the possible scores in that game. What if your new game had only 2 and 4 as possible points to score? Now what scores are possible? What about 3 and 5? Now what scores are possible?

★★

12. Nim Number Game

To start the game, put 13 pennies in a pile between the two players. Players will alternate turns choosing to take away 1, 2, or 3 pennies at a time. The winner is the player who takes the last penny or pennies. Have the students play many times and try to develop a strategy to win. It is possible, you can tell them, to win every time if you're the first player.

Give them some time to try to figure out the winning strategy for the first player. If they become frustrated, you can make the game easier by starting with 7 pennies in the middle instead. See if they can develop a winning strategy for the first player starting with 7 pennies. Then use that idea to expand to the 13-penny game.

If no one finds the strategy, help them see that the game is basically played in groups of four pennies. Thirteen pennies is three groups of four with one left over. The starting player, if she wants to win, must always take one coin to start the game. Then there are three groups of four coins left. The starting player can always control the groups of four from here on out because no matter what the second player takes (1, 2, or 3 coins), the starting player can ensure that the number of coins left in the middle is a multiple of four. (If the second player takes 3, for instance, the starting player should take 1 on her second turn. If the second player takes 2, the starting player should take 2. If the second player takes 1, the starting player should take 3.) Now there are two groups of four coins left.

Follow the same strategy with the remaining two groups of four to get to one group of four. Then whatever the second player takes (1, 2, or 3), the starting player can take the last penny/pennies to win.

This can be played with any number of pennies that isn't a multiple of four. If is IS a multiple of four, the second player now has the advantage and can use the strategy above to guarantee a win.

Try starting with different numbers of pennies.

Introduction:

Nim is a classic game (dating from ancient times) that has several different variations to play. Most of the rules are common and all of them have a basic strategy to follow to guarantee a win for the first player.

Objectives:

- To learn this class game of strategy

Materials:

- Approximately 100 objects such as pennies, toothpicks, or colored counters,

Taking it Further:

Another version of the Nim game goes like this: There are three distinct piles of objects, with 3 objects in the first pile, 4 in the second, and 5 in the third. Two players take turns taking objects. Each player, on his turn, must remove at least one object, and may remove any number of objects, provided they come from the same heap. The last player to remove an object wins.

13. Adding Up

Ask the opening question, "How many ways can we add two numbers to get 5? They should pretty easily come up with all the ways to add two numbers to get 5 (5+0, 4+1, 3+2). After initial brainstorming, make it clear that we are only adding positive numbers. (If we allowed adding 0, which is not a positive number, we could have 5+0, 5+0+0, 5+0+0+0, etc, and there would be infinitely many answers.)

Do we count both 3+2 and 2+3? We could, but mathematicians only count 3+2 because they put the numbers in descending order. If they wanted to count both, imagine the trouble they would have writing all the possibilities for 2+2+1+1+1 when counting the partitions of 7.

When students list all the ways of writing the number 5, they should come up with the following partitions of 5; there are seven in all:
5 , 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1

Next let them work on partitions of 6 for a while, and make sure they're writing all the partitions in descending order. (There are 11 in total: 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.)

One way to organize your list of partitions of 6 is to write down all partitions that contain 6 as the biggest number, or "part." (There's 1.) Then all partitions that contain 5 as the biggest part. (1) Then all the partitions that contain 4 as the biggest part. (2) Then 3 as the biggest part. (3) When you're counting how many contain 2 as the biggest part, you know the partition can only contain 2s and 1s because 2 in the biggest part, so you don't need to count 2+4 because that's already been counted when we counted partitions with 4 as the biggest part.

With time remaining, let them try partitions of 7.

Introduction:

How many different ways can you add up whole numbers and get the number 7? (Such a sum would be called a "partition" of 7.) The answer to this is easy to understand, but involves some complex mathematical ideas.

Objectives:

- To practice counting things

Materials:

- Scratch paper and pencils

★★

14. Patterns of Partitions

We will begin this lesson by reviewing the ideas of Adding Up by finding the partitions of 8. Give the students some time to work on this. Using the idea of starting with the biggest part we are allowed in our sum, we find 22 ways:

8
7+1
6+2
6+1+1
5+3
5+2+1
5+1+1+1
4+4
4+3+1
4+2+2
4+2+1+1
4+1+1+1+1
3+3+2
3+3+1+1
3+2+2+1
3+2+1+1+1
3+1+1+1+1+1
2+2+2+2
2+2+2+1+1
2+2+1+1+1+1
2+1+1+1+1+1+1
1+1+1+1+1+1+1+1

Help your students find an organized way to write the list, like the one above.

Now, ask the students: How many ways are there to add to 8 if your highest number is 1? How about if your highest number is 2? Or 3? Do this for all numbers from 1 to 8.

1 is the highest: 1+1+1+1+1+1+1+1, or 1 way
2 is the highest: 2+2+2+2, 2+2+2+1+1, 2+2+1+1+1+1,
2+1+1+1+1+1+1, or 4 ways
3 is the highest: 3+3+2, 3+3+1+1, 3+2+2+1,
3+2+1+1+1, 3+1+1+1+1+1, or 5 ways
4 is the highest: 4+4, 4+3+1, 4+2+2, 4+2+1+1, 4+1+1+1+1, or 5 ways
5 is the highest: 5+3, 5+2+1, 5+1+1+1, or 3 ways

Introduction:

Following up on the ideas of Adding Up, an interesting fact about partitions is that the number of partitions of 7 that have 3 different parts is the same as the number of partitions of 7 whose biggest part is 3. This generalizes to say that the number of partitions of n that have k different parts is the same as the number of partitions of n whose biggest part is k. This has a nice visual proof which we will describe in Ferrers Diagrams.

Objectives:

- To practice the ideas of Adding Up
- To play more with partitions

Materials:

- Scratch paper and pencils

6 is the highest: 6+2, 6+1+1, or 2 ways
7 is the highest: 7+1, or 1 way
8 is the highest: 8, or 1 way

Now ask the question: How many ways are there to add up to 8 using just 1 number? Or 2 numbers? Or 3 numbers? Do this for all numbers from 1 to 8.

1 number: 8, or 1 way
2 numbers: 7+1, 6+2, 5+3, 4+4, or 4 ways
3 numbers: 6+1+1, 5+2+1, 4+3+1, 4+2+2, 3+3+2, or 5 ways
4 numbers: 5+1+1+1, 4+2+1+1, 3+3+1+1, 3+2+2+1, 2+2+2+2, or 5 ways
5 numbers: 4+1+1+1+1, 3+2+1+1+1, 2+2+1+1+1, or 3 ways
6 numbers: 3+1+1+1+1+1, 2+2+1+1+1+1, or 2 ways
7 numbers: 2+1+1+1+1+1+1, or 1 way
8 numbers: 1+1+1+1+1+1+1+1, or 1 way

The students probably recognized halfway through that the answers that we get to the two questions are the same! Ask them if they think this is something special for partitions of 7. If they are interested, verify that this pattern works with smaller numbers as well.

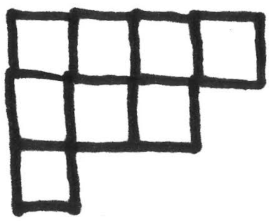
Why do they believe this happens? (They probably won't be able to find a convincing answer to this, but having them think about the question is a good idea.)

★★

15. Ferrers Diagrams

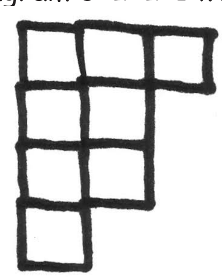
Mathematicians like to take some idea, like that of partitions of integers, and be able to visualize it in a way that allows them to see and understand better what is happening.

A Ferrers Diagram represents a partition by representing each part (or summand) of the addition problem by a row with that number of blocks, with the summands decreasing in size. For example, 4+3+1 is represented as:



First, get out 8 blocks. Ask the students to tell you one way to add positive integers to get to 8, that is, have them tell you a partition of 8. Then organize the blocks (as above) according to what they say. Do this several times until they understand how you are constructing these diagrams, and then have them make their own to reinforce how it is done. Remember to have them put the rows in decreasing order, just as the way we list the partitions when we're writing them all down.

At some point a student may notice (especially if you are sitting in a circle, and thus they have different points of view on the diagrams) that viewed from one direction a partition may be different than if it is viewed in another direction. Those two views are partitions of the same number because the number of little blocks in the partition (8, in our case) hasn't changed. For example, the diagram 3+2+2+1 would look like:



Introduction:

This lesson continues on with the ideas in Adding Up and Patterns of Partitions. A useful way to visualize partitions is called a Ferrers Diagram. Using this visual technique, students will be able to understand why the pattern found in Patterns of Partitions works.

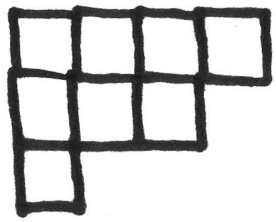
Objectives:

- To practice the ideas of Adding Up and Patterns of Partitions
- To play more with partitions

Materials:

- Scratch paper and pencils
- Square or rectangular blocks (Scrabble tiles turned upside down work well)

But, from Daniel's point of view, the diagram is 4+3+1:



Once they understand this, find all of the partitions of 8 and figure out their corresponding diagrams. Put arrows between partitions that look like each other when viewed from a different angle. Remember, diagrams of partitions always have row lengths that remain the same or decrease as you go from top to bottom, just like partitions always have the parts listed in decreasing order. All the diagrams for the partitions of 8 will pair up. (If you are looking at the partitions of 7, instead of the partitions of 8, there will be one diagram that is its own pairing.)

Now comes the thinking and proving part. Ask the students what each row represents, and then ask them what each column represents. These are difficult questions. Perhaps have them recall what you did in Patterns of Partitions, in finding the patterns between the partitions that had a certain number as the biggest part and the partitions that had that same number of parts.

With coaxing, they will see that the top row of the diagram is the biggest part used, and that the number of rows is the number of parts (or summands). Then they will see that diagrams that look the same from different angles simply have these numbers switched. And there we have it, the proof that these numbers (the number of parts of the partition and the size of the largest part of the partition) will always be the same!

16. Looking for Patterns in Sequences

A *sequence of numbers* is an ordered list of numbers. The list can be finite, such as 1, 2, 3, or it can be infinite, such as 2, 4, 6, 8, ... Here, the "..." indicates that the sequence goes on forever. The numbers in a sequence don't all have to be positive numbers or whole numbers or increasing numbers, and numbers within a sequence can repeat. Any list of numbers you write down is a sequence. Have your students write down examples of sequences.

Now consider sequences that follow some sort of pattern. Ask your students in the sequence 2, 4, 6, 8, ... which numbers come next (10, 12, 14, 16), and they will probably quickly see that you're adding by two each time. In the sequence 1, 4, 7, 10, 13, ... it may take them a moment longer to realize that you are adding by three each time.

Other examples of sequences whose patterns they can try to find include:

2, 6, 2, 6, 2, ... (flip-flop between the two numbers)

1, -1, 1, -1, 1, -1, ... (flip-flop OR multiply by -1 each time)

5, 5, 5, 5, 5, ... (5 forever)

1, 1/2, 1/3, 1/4, 1/5, ... (the reciprocals of the numbers 1, 2, 3, 4, ...)

3, 6, 9, 12, 15, ... (add 3 each time)

1, 3, 2, 4, 3, 5, 4, 6, ... (add 2, subtract 1, add 2, subtract 1, ...)

1, 1/2, 1/4, 1/8, 1/16, ... (divide by 2 each time)

3, 6, 12, 24, 48, ... (multiply by 2 each time)

What about the sequence 1, 3, 6, 10, 15, 21, ...? Can they figure out the next numbers? Give them time to write down the sequence and think about it a while. If some students see the pattern quickly, give the others a chance to think before they're told. The same

Introduction:

Pattern finding is an important part of mathematics: in this lesson, students practice finding the next number in a given sequence, and develop techniques for more difficult sequences.

Objectives:

- To define sequence and create examples of sequences
- To look for patterns in numbers

Materials:

- Scratch paper and pencils

number is not being added or subtracted each time in this pattern. Instead, consider the differences between consecutive numbers. Write small between each pair of consecutive numbers what number is added to the previous one to get the next one. Those small numbers you're writing are 2, 3, 4, 5, 6, ... We recognize those! Now we see that the next few terms in our sequence are 28, 36, 45, 55.

There's a famous sequence called the Fibonacci Sequence that starts 1, 1, 2, 3, 5, 8, 13,... What number is next? They should see that it's not the same number being added or subtracted or multiplied each time. In fact, if they look at the differences of consecutive terms, they will see 0, 1, 1, 2, 3, 5, 8, .. the sequence itself with a 0 at the beginning! In this sequence, then, each number (except for the first two) can be found by adding the previous two terms. For example, 2=1+1, 3=1+2, 5=2+3, 8=3+5, etc.

Taking it Further:
We say that a sequence *converges* if it gets closer and closer to some number forever. And then we say a sequence *diverges* if it fails to converge. So every sequence is either convergent or divergent.

Of the examples given in this lesson, only three of them converge. Which three? Can the students figure out which ones they are?

5, 5, 5, 5, ... converges to 5
1, 1/2, 1/3, 1/4, 1/5, ... converges to 0
1, 1/2, 1/4, 1/8, 1/16, ... converges to 0

All of the other sequences in this lesson diverge. Note that they don't have to go to infinity to diverge. The sequence 2, 6, 2, 6, 2, 6, ... doesn't go to infinity, but it diverges because it fails to converge.

A Bit of Sequence Nonsense:
On a lighter and less mathematical note, another neat sequence is the "Look-and-Say Sequence." Tell your students what the sequence is called, and then show it to them and see if they can figure out the pattern based on the title. The sequence is:

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ...

To generate the next number in the sequence, simply say aloud the numbers in the previous term. For example, the first "1" is said aloud as "one 1", which you then write as "11" to make the next term. Then the "11" is read aloud as "two 1s" or "21." Then the "21" is read as "one 2, one 1" or "1211." Then the "1211" is read as "one 1, one 2, two 1s" or "111221." Notice that whenever there are more than one of a number next to each other, they are counted in a group. For example, if there was a number with "111" in it, it is read as "three 1s" or "31." Ask the students to figure out the next few numbers in the sequence.

17. Fibonacci's Bees

The Fibonacci sequence shows up a lot in nature and can be used to model mathematical relationships. In this lesson we will look at two occurrences of the Fibonacci sequence. Begin by asking one student to remind everyone what the Fibonacci Sequence is (1, 1, 2, 3, 5, 8, 13, 21, ...).

Now read aloud to the students (or explain in your own words): If a female bee lays an egg and a male does not fertilize it, it will hatch a male. If a female bee lays an egg and it is fertilized by a male, it will hatch a female. So all female bees have two parents (a female and a male) and all male bees have one parent (a female). The illustration below may be helpful for visualizing this:

The most important thing here is to make sure everybody understands that female bees have two parents and male bees have one parent. Now consider a male bee and his ancestors. (We're talking about ancestors (parents and grandparents), not offspring (children and grandchildren).)

Show the students how to draw just the first couple of generations of a male bee's family tree. Start the male bee at the bottom center since you're impressing upon the students that you're going back (up) in time. Then have them draw as many generations as will fit on a big sheet of paper. Let's look at the number of bees in each generation. In the example below there are seven generations. Keep a table of the number of male bees, female bees, and total bees in each generation. Look at the numbers in the column that gives the total number of bees in each generation. What do you notice about these numbers? Let them see that they are the Fibonacci Sequence!

Introduction:

As a follow-up to the Looking for Patterns in Sequences lesson, this lesson focuses on the Fibonacci Sequence and two interpretations of it.

Objectives:

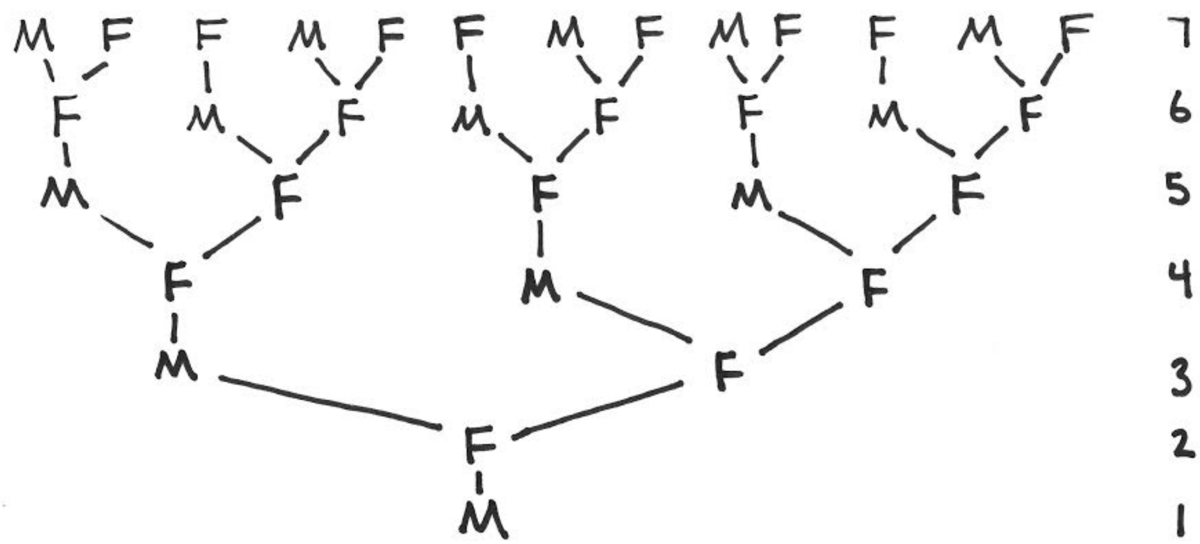
- To have fun with the Fibonacci Sequence

Materials:

- Scratch paper and pencils
- A big piece of paper or access to a big black or white board
- Sheets of graph paper (at least one each)

Fun Fact:

Next time you are holding a flower with petals, count the number of petals it has -- it's often a Fibonacci number!



GEN.	MALE	FEMALE	TOTAL
7	5	8	13
6	3	5	8
5	2	3	5
4	1	2	3
3	1	1	2
2	0	1	1
1	1	0	1

Why does this happen? Open up a discussion about answers to this question, keeping in mind that the answer is somewhat complicated. It might be a good idea to give the students five minutes of silent thinking time before starting this discussion. They probably won't be able to explain why the Fibonacci sequence occurs, but they should be able to come up with some good starting thoughts.

Let's see if we can see why the totals are Fibonacci numbers.

Have the students notice that in any given generation (row of the family tree), all the bees in that generation have a female parent (but not all will have male parents) and all the female parents in the generation above come from one of those bees. So to count the female bees in any given generation, you can just count the number of total bees one generation lower.

Next notice that for any generation, all the male bees are grandfathers of one of the bees two generations lower, and all of the bees in that two-generations-lower generation have male grandfather bees. So to count the male bees in any given generation, you can just count the number of total bees two generations lower.

Bringing it all together, the total number of bees in a given generation is the number of female bees in that generation plus the number of male bees in that generation. But that's just the sum of the total number of bees in the generation one lower and the total number of bees in the generation two lower. That is, it's the sum of the two previous total number of bees. But that's how the Fibonacci Sequence is defined, as the sum of the two numbers before it!

18. Fibonacci's Spiral

Before meeting with the students, draw the largest rectangle possible on a graph paper where the sides are consecutive Fibonacci numbers, such as 21x34 or 34x55 or 55x89, depending on the size of your graph paper. Make the same large rectangle on each sheet of graph paper. For this discussion, let's say the largest you can draw in 34x55. Have them put the papers in front of them so the rectangles are wide and short.

When you give the students the graph paper, ask them to find the dimensions of the rectangle by counting how many little squares it is in each dimension. Do they recognize these numbers? Ask them to recall the Fibonacci Sequence and write out the first ten or so terms on their paper, not in the rectangle. Point out that their dimensions are Fibonacci numbers.

Ask them to draw the largest square that they can within this rectangle. (It should be 34x34.) They could draw it anywhere within the rectangle, but encourage them to draw it all the way to one side. Have them color in the square with some color, carefully not coloring outside the square.

Now there's a colored square and a smaller white rectangle. What are the dimensions of the rectangle? They could count again, or they could realize that one of the dimensions (the 34) didn't change, and the other dimension is 55-34=21. Oh, again the dimensions are Fibonacci numbers. Ask them to draw the largest square they can in the white rectangle, this time at the top of the rectangle. (It should be 21x21.) Have them color that square a different color.

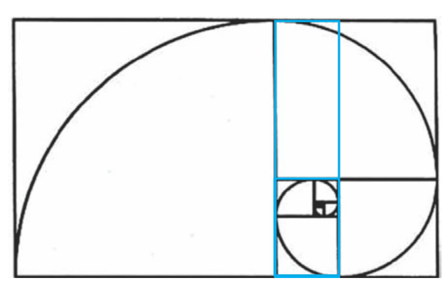
The white rectangle remaining should be 21x13, still Fibonacci numbers. Draw in the largest square on the right side of the white rectangle and color it in. The remaining white rectangle should be 13x8. Draw the largest square on the bottom of the white rectangle and color it in. Continue, coloring in the largest square on the left, on the top, on the right, on the bottom, until it's not possible to see the rectangles any more. Connect the outermost points where consecutive squares meet to form a beautiful spiral!

Introduction:

Another follow-up to Looking for Patterns in Sequences that uses the Fibonacci Sequence is in drawing a beautiful spiral using the ideas of the Fibonacci Sequence. This spiral is found in nature in the shape of nautilus shells.

Materials:

- Scratch paper and pencils
- A photo of a nautilus shell, optional
- A particular rectangle drawn on graph paper, at least one sheet each
- Colored markers



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19. Fibonacci's Fraction

Ask someone to recall what the Fibonacci Sequence is and to write down the first thirteen or fourteen numbers in it. (The last few may take a few minutes to calculate.)

We want to think about the ratio of consecutive Fibonacci numbers. That is, use calculators to calculate:
2÷1=2
3÷2=1.5
5÷3=1.66667
8÷5=1.6
13÷8=1.625
21÷13=1.61538
34÷21=1.61905
55÷34=1.61765
89÷55=1.61818
144÷89=1.61798
233÷144=1.61806

It's amazing that these numbers are getting closer and closer (converging) to some number. Most sequences that you would look at, if you took the sequence of consecutive ratios, they wouldn't converge. What is this special number they are limiting on? It's called the Golden Ratio, and it's approximately 1.618033989.

Can we find the Golden Ratio occurring in nature? How about in our own bodies? Have students work in pairs to take four measurements on each student (it doesn't matter whether the measurements are in inches or centimeters, as long as all measurements are in the same units). For each student, measure
1. Total height
2. Height from the floor to the belly button
3. Arm length from elbow fold to fingertips
4. Arm length from elbow fold to wrist crease
And for each student calculate
Total height/Belly-button height and
Arm length to fingertips/Arm length to wrist.
What do you notice? These ratios should be approximately the Golden Ratio.

On the image of the Vitruvian Man, can the students find other Golden Ratios?

Introduction:

The final example using the Fibonacci Sequence from Patterns in Sequences is this discussion of the Golden Ratio. This topic lends itself well to doing research and a presentation -- there are many simple, good resources on the golden ratio with many images. Wikipedia would be a good starting point.

Objectives:

- To look at another example of the Fibonacci Sequence in nature

Materials:

- Scratch paper and pencils
- An image of Da Vinci's Vitruvian Man
- Yardsticks or tape measures
- Calculators

20. The Catalan Numbers

In Looking for Patterns in Sequences, we explored how one might form a sequence by taking the previous term or terms and add (or subtract, or multiply by) another number. Some sequences are patterns, but they aren't so easy to spot. The Catalan Numbers are a perfect example.

Here are four different exercises to do which involve counting different things. The students could break into four different groups with each group working on an exercise. These exercises will take a while, and need concentration by the students. Another way to approach them is to work on each of the exercises as one large group, and spread them out over several sessions. What the students will discover (you should not reveal this in advance) is that all four exercises have exactly the same answers, and those answers are the Catalan Numbers. The joy and mystery is in the discovery that from four very different sounding exercises, the same numbers arise.

In the pages that follow are the answers to these four exercises, and it shows how the Catalan Numbers arise. You should be sure to read through the solutions before you get the group working on them so that you are clear on the statement of the problems.

That Catalan Numbers are:
1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, ... (There is good information about the Catalan Numbers on Wikipedia.)

The five exercises:

1. Triangulating polygons. First draw regular convex polygons (a regular convex polygon has all the same length sides and all the same angles) with 3, 4, 5, 6, and 7 sides, like this:

Introduction:

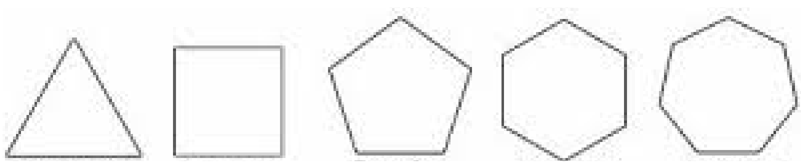
This lesson, which follows on Looking for Patterns in Sequences, could be spread over several days. It investigates a very special sequence of numbers called the Catalan Numbers by looking at a variety of interesting things that can be counted.

Objectives:

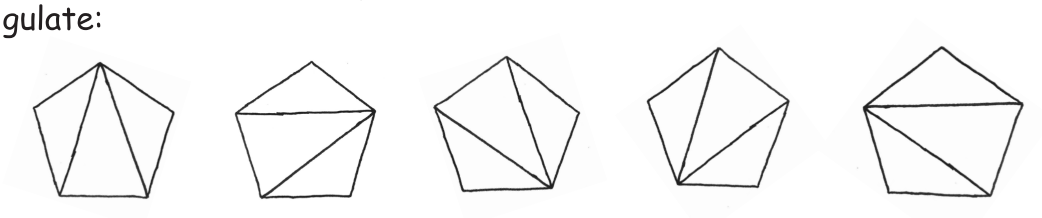
- To learn about the Catalan Numbers
- To practice counting a variety of objects in an organized way

Materials:

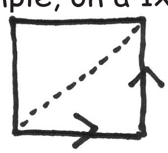
- Scratch paper and pencils



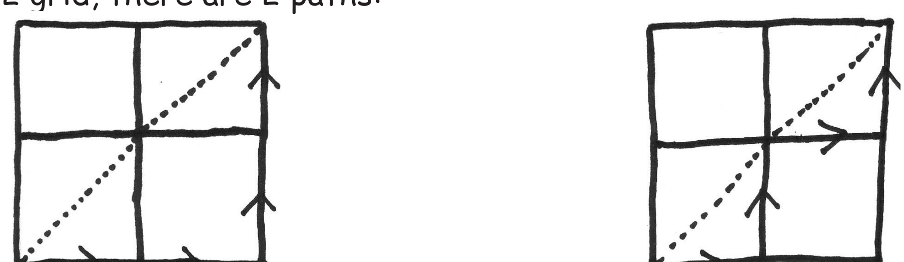
Then count the number of ways to triangulate each shape. To triangulate a shape with more than three sides means to draw a line that connects two of the vertices. Continue connecting vertices with non-intersecting lines until there are no more lines that you can draw without crossing each other. The polygon will now be split into triangles. For example, when there are five sides, there are five ways to triangulate:



2. Catalan Paths. Given a grid (like graph paper) that is n squares by n squares, we want to count the number of ways that you can get from the bottom left-hand corner to the top right-hand corner using only moves up and to the right, and never going above the diagonal. For example, on a 1x1 grid, the only path is:

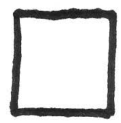


and on a 2x2 grid, there are 2 paths:

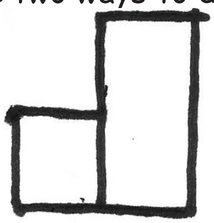


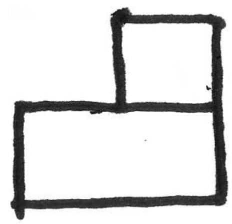
3. Staircases. For each staircase step the task is to find the number of possible ways to divide the shape into a number of rectangles equal to the number of steps it has. For example, you must divide the 2-step staircases into 2 rectangles, while the 3-step staircases must be divided into 3 rectangles. (Remember that a square is a rectangle.)

For example, there is only one way to divide the 1-step staircase:



But there are two ways to divide the 3-step staircase:





4. The Order of Multiplication. We know how to multiply two numbers, but if you need to multiply more than two, you just group the numbers so that you can multiply them two at a time. If you have n different numbers that need to all be multiplied together , there are several different ways you can group the multiplication. Here we want to count the number of ways the terms in a product can be grouped.

For example, when you have two numbers a and b that you're multiplying, they could be grouped in just one way:
(ab)

But when you have three numbers that you're multiplying, there are two ways:
(a(bc)), ((ab)c)

Now find how many groupings you can do the groupings for more factors.

Have students keep track of the number of ways for each of these tasks. You all should see the same sequence appear!

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21. Ready, SET, Go

The cards in the game SET are designed to each have four different attributes: they each have a symbol or several of the same symbols (diamond, oval, or wavy), they each have a number of symbols (either 1, 2, or 3), they each have a color (either red, purple, or green) and they each have a shading (solid, shaded, or empty). There is one of each type of card in a SET deck. (How many cards are there? There are three choices for each of the four attributes, so there are $3*3*3*3=3^4=81$ different cards.)

If the students have not played the game SET before, introduce it to them now.

Now spread all the cards out on a table so that the students can see them. You choose two of the cards randomly off the table, and ask the students how many different cards you could put with the two cards you chose to make a set. (It shouldn't take them long to realize that there's only one card that will go with any two chosen randomly.)

Now pull just one card off the table and ask them to figure out how many different sets that one card belongs to. Let them work on this a while. (For the second card in the set, any one of the remaining 80 cards on the table would work, and once that second card is selected, there's only one choice for the third card. But this is including every set twice (One, two, three is the same as one, three, two) So there are 80*1/2=40 sets that the given card is a member of.)

Now, how many different sets could be made from the 81 cards? Although the order of the cards in a set does not matter (there isn't a first card), just like in Milk Groups, it's easier to suppose there is. Suppose the sets were going to line up at the door. There are 81 cards that could be

Introduction:

The game of SET is a fantastic game to work on pattern recognition with children. Here we are going to use the SET cards to do some simple counting.

An idea from Milk Groups reappears in this lesson to help with a counting problem.

Objectives:

- Given two SET cards, to count the number of sets that can be formed
- Given one SET card, to count the number of sets that can be formed

Materials:

- Scratch paper and pencils
- A game of SET

first in line. Once one is chosen first, there are 80 cards that could be chosen for second, and then the third card is the one in the deck that makes a set. So if we want to order the sets, there are 81*80*1=6480 different ordered sets. But we don't want ordered sets, we just want to know how many sets there are. For each set of three cards, in how many ways can they be ordered? There are 3 choices for the first card, leaving 2 choices for the second card, and 1 choice then for the third. So there are 3*2*1=6 different orders for any set of cards That means that every set of cards got counted 6 times, so the real number of sets of cards are 6480/6=1080.

Taking it Further:

If you get to look at all the cards and you are given time to think, what is the largest collection of cards you can put together that does NOT contain a set? The students can try this as a group, or work individually to come up with the largest collection they can. (It has been proven that the largest such collection is 20.)

Graphs:
A Picture's Worth a Thousand Words

1. Are you IN or OUT?

Use masking tape to draw two big circles (they don't need to be perfect circles -- they don't even need to be circles, as long as the objects have insides and outsides) on the floor so that they overlap. Each circle should be about eight feet or more across.

Ask the students to come up with examples of things that they have sorted into categories before. Have they sorted toys by type, books by who they belong to, or socks by size. Have them list some examples.

Now explain to the students that you want to sort them into categories by some defining characteristic. Indicate one of the two circles on the floor, and tell them that everyone who has red on their shirt should stand inside the circle, and everyone else should stand outside the circle. Make sure everyone understands.

Distribute the hats and gloves so that only some students have hats and some students have gloves and make sure that some students have both and some have neither. You can play, too!

Now have them all get out of the circle. Tell the students who are wearing hats to stand in the first circle. Once everyone gets a mental picture of what that looks like, tell them that anyone wearing gloves needs to stand in the second circle. There should be some students that need to stand in the overlap of the two circles because they're wearing both a hat and gloves. There should be some students who are in neither circle because they're wearing neither. And there should be students wearing one or the other who are only standing in one circle, not both. Have them draw on paper a picture of what this looks like.

Now have them draw on their paper two overlapping circles, and pick two categories, like "wearing a red shirt" and "wearing brown shoes", and have them put students' names in those circles according to what they're wearing. Try other examples.

Introduction:

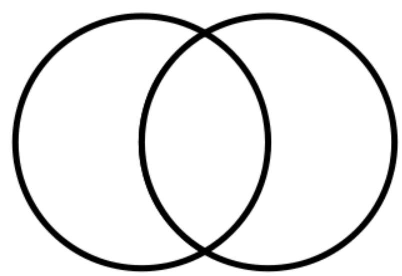
Venn Diagrams are a way to visualize categories and decide who (what) falls into the various categories.

Objectives:

- To learn what a Venn Diagram is

Materials:

- Scratch paper and pencils
- Masking tape
- Hats and gloves



Taking it Further:

What happens if the categories are "wearing a red shirt" and "wearing a shirt"? (One circle is inside the other one!) Or how about "wearing a red shirt" or "wearing a blue shirt"? (Now the circles don't overlap at all because no one can wear both!)

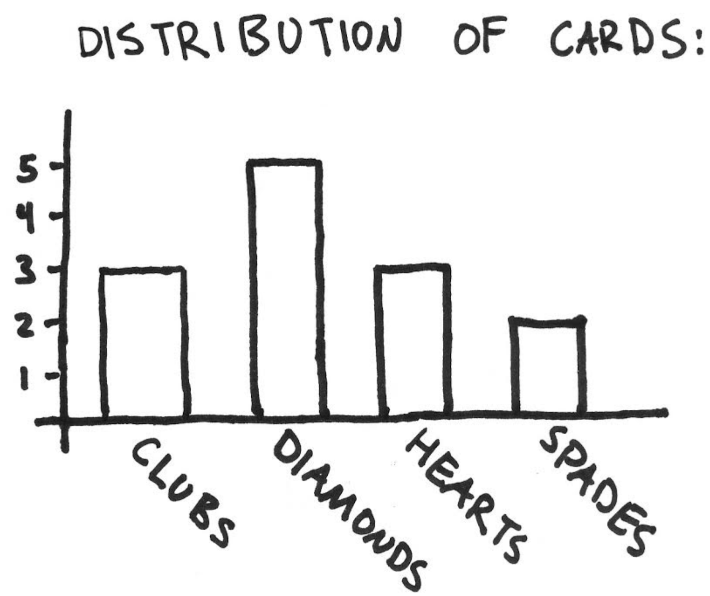
2. Bar Charts

A Bar Chart is a type of graph that helps us understand some information. To make a chart, we first need data to put into our graph.

Shuffle the deck of cards and deal the cards out to the students. (They don't have to all get the same number.) Have the students count the number of each suit that they have, for example: 3 Clubs, 5 Diamonds, 3 Hearts, 2 Spades.

Now how can we graphically display this data, so that if we looked at the picture we could recover the information put into the chart? There are various graphical displays that would work, but today we're going to talk about Bar Charts.

Show them how to make a Bar Chart, and ask them to take time (and use their rulers) to make their own. We put the four categories: Clubs, Diamonds, Hearts Spaces along the bottom, and above each of those, we draw a rectangle that indicates the number of each suit we had. For our example, a chart would look like:



Introduction:

Descriptive Statistics is an important area of mathematics, and one that the general consumer of mathematics uses daily when reading a newspaper or reading a report.

Objectives:

- To learn what a bar chart is and how to create one

Materials:

- Scratch paper and pencils
- Playing cards
- Rulers

Make sure they put a reasonable title on their chart and that the numbers along the vertical axis are evenly spaced.

Gather some more data for them to graph. Ask the students to count, as a group, the total number of brothers and the total number of sisters that they have. Graph that data as a Bar Chart with two bars.

Finally, ask them to graph the total numbers of pets that they have, broken into categories. How many: Cats? Dogs? Birds? Fish? Rodents? Graph it!

3. Percentages and Pie Charts

Begin this lesson by talking to the students about what a percent is. A percent is a number that tells you what proportion of an object you're talking about. For example 5 percent is just telling us "5 parts of 100" (literally, per cent, or per 100). What is 30% of a doughnut? If we were to divide the doughnut into 100 equal pieces, we would have 30 of those 100 pieces.

Have them practice some basic percentages: What is 50%? It's 50 parts of 100, so it's 1/2. What's 25%? It's 25 parts of 100 (or you can think of 25 cents of 100 cents, or a dollar), so it's 1/4. What's 10%? It's 10 parts of 100 (or 10 cents of a dollar), so it's 1/10.

Can you have percentages less than 0 or greater than 100? No, neither, because you can take neither -3 parts of 100 nor 120 parts of 100. Percentages can only be between 0 and 100. (Of course a newspaper could talk about the stock market being down 3% and write that as -3%, but the negative there just means that something is decreasing by 3% of its original value.) Percentages can have decimals: 3.79% makes sense. It means 3.79 parts of 100. We won't worry about decimal percentages for now.

A Pie Chart is a way of showing percentages that make them very clear. A Pie Chart is simply a circle that is thought to contain 100% (or all) of the object we're talking about, and we simply color in the percentage we're interested in.

For example, what is the color blend of M&Ms milk chocolate candies? In 2008, the M&Ms website stated it was: 24% cyan blue, 20% orange, 16% green, 14% bright yellow, 13% red, 13% brown. We can put this information in a Pie Chart. We draw a circle and imagine it contains all the M&Ms made in 2008. Of these, we know 24% are blue, so we sketch what is 24 of the 100 parts and label it. Then we do the rest of the chart. This can be easily shown in a pie chart like:

Introduction:

Percentages are a part of everyday life that students need to understand. To interpret sales on goods, interest payments, taxes, tips, and so much more, students need a good grounding in percentages.

Objectives:

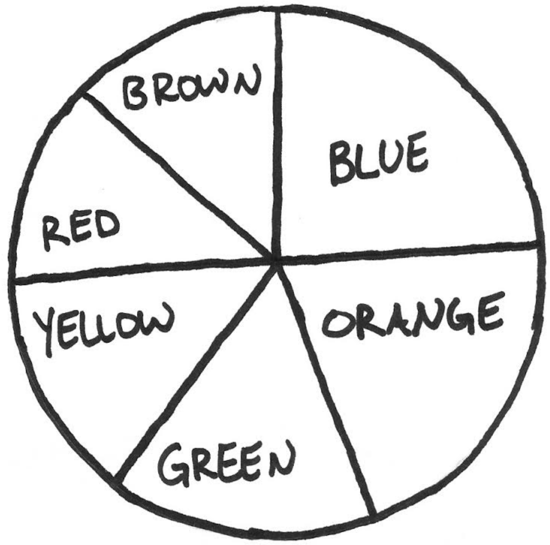
- To learn what a percentage is
- To understand how to depict percentages in a pie chart

Materials:

- Scratch paper and pencils
- Something to trace around to draw circles about 4" in diameter

Fun Fact:

Something about percentages that the students can then graph.



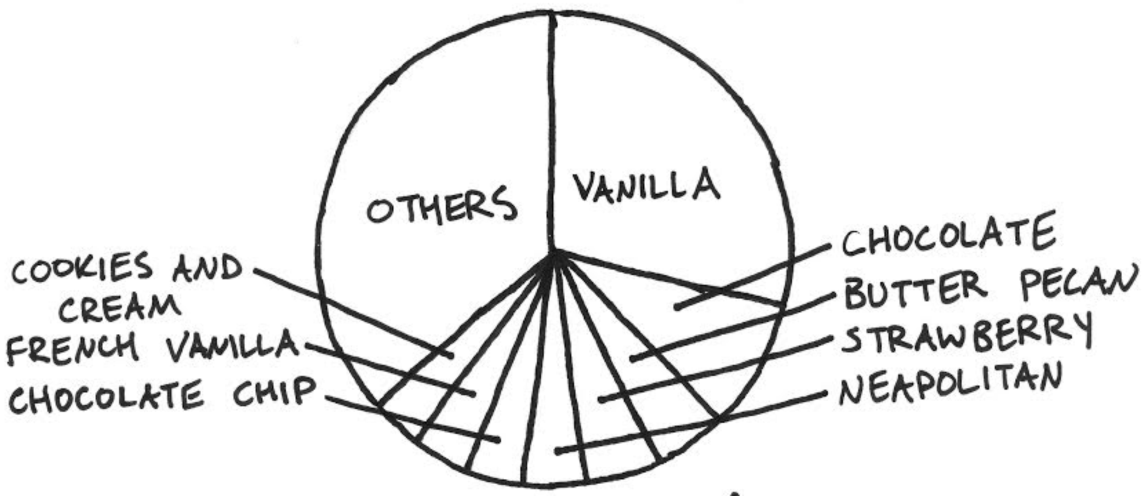
Now have them try making a pie chart. What are the most popular ice cream flavors? According to the International Ice Cream Association (that sounds like a fun place to work), the top flavors are:

1. Vanilla, 29%
2. Chocolate, 8.9%
3. Butter pecan, 5.3%
4. Strawberry, 5.3%
5. Neapolitan, 4.2%
6. Chocolate chip, 3.9%
7. French vanilla, 3.8%
8. Cookies and cream, 3.6%

All others, 35.6%

Now, what does our circle contain? It contains everyone's favorite ice cream flavors. So we MUST have "All others" included because otherwise, someone's favorite doesn't get counted. We also know that 100% of the favorite ice cream flavors must be represented, and if we didn't include "All others", our percentages would only add up to 64.4%.

Spend some time with the students approximating the graph of the top ice cream flavors. It should look approximately like:



If you have enough time, have the students ask their classmates what their favorite ice cream flavors are, and make a pie chart of that information. How do the two charts compare?

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4. Function Machines

Introduce the students to the idea of a machine (you can just describe it in words, or you can draw something on paper) which has a singular purpose: it takes something as input, and, after a bit of churning, produces a different something as output. Tell the students that you know the way the machine works, so they can try to feed it inputs, and you will tell them what pops out in the end. Their job is to figure out the function of the machine.

The lesson starts with this simple Human Function Machine game to introduce the students to the idea of a function. You, the leader, will be the function. Think of an action or type of behavior (for example, folding arms) that your function will produce. Assign one student to be the guesser, then go around a corner so that the guesser can't see you. Have each non-guessing student come around the corner and then tell them to perform your action or behavior when they walk back out. The guesser's job is to figure out what your function does. Here are some examples of functions to try:

- the folded arms function
- the sitting down function
- the silence function
- the crazy function
- the somersault function

For an added bit of exploration, try making your input take two pieces of input (people) and having them perform some sort of action together as the output. For example the holding hands function. Also try the "Dani Function" where no matter who goes around the corner, Dani (a member of your group) always comes out.

Now propose to the students that you can represent different mathematical operations using these human functions. Give the example of the "Plus Karina Function" where every input goes around the corner alone and comes back with Karina. Ask for volunteers who think they have a good representation of functions involving subtraction and multiplication. While you can't actually split apart any of your human inputs, talk hypothetically about what a "divide by two" function would

Introduction:

This is an introduction to the idea of a function. Students will begin thinking about relationships between numbers and doing basic arithmetic in their heads without thinking of them as worksheet-style problems.

Objectives:

- To learn about functions
- To practice simple arithmetic operations without writing them down

Materials:

- Scratch paper and pencils
- Colored markers if you wish to draw the machines

look like.

Now let's switch to having numbers as both input and output. Tell them that you are the machine (but you no longer have to be around the corner), and they should feed some input into you. When they shout out numbers, you perform your operation and tell them the output. You can start by adding small numbers to the input. In order to write examples of functions down here, we will use the letter x to represent the input, and y will represent the output. When they figure out your machine, create a new machine for them to guess. If they want to come up with their own way of writing down a function machine, they may invent a symbol of their own to represent the input.

You could use functions like:
y=x+1 ("add 1" is the function)
y=x+3 ("add 3")
y=x-2 ("subtract 2")
y=x-5 ("subtract 5")

As the problem gets harder, one good way for them to keep track of what they've heard for input and output is to make a table, like:

x	y
3	1
16	14
-5	-7
2	0
-1	-3

For more challenging functions, make a change in a different place value, such as:
y=x+10 or y=x+100
Or try:
y=2*x (which students might think of as "x+x")
y=10-x (which students may think of as "the input plus the output equals 10")
y=100-x ("the input plus the output equals 100")
y=0 (or you could use any constant; this demonstrates that the machine doesn't really have to take into account the input. Some students may see this as "the number minus itself")

With any time remaining, let the students come up with their own.

5. Graphing Equations

When you are working with a function machine, or the equation describing that machine, what the machine does to the input to produce the output may be difficult to discover; it's always good to have a way to visualize what you're thinking about. A graph of the function is a nice way to look at and be able to think about the function.

Consider the function machine y=x+1. We saw a table of inputs and outputs looks like:

x (input)	y (output)
-3	-2
-2	-1
-1	0
0	1
1	2
2	3
3	4

We can graph this on a pair of axes, one horizontal (the x-axis or the input axis) and one vertical (the y-axis or the output axis). On a sheet of graph paper, darken a horizontal line and a vertical line on the paper, so that the two lines cross near the middle of the paper. Where they cross at the center is the 0 value on each axis.

To graph (-3,-2), count 3 units to the left of center, and 2 units down. Similarly plot all the other points in our table of inputs and outputs. It looks like the points fall along a line. Should we connect the dots into a line? Yes, because we see that if we enter some input in between the numbers we've already used, say our input is 2.7, then the output is 3.7, and it continues to fall along the line of dots.

Let's gather some data to plot. To do this, take the students outside or to the gym where they can be loud. Have the students pair up; they are going to count each other doing jumping jacks. For each pair of students,

Introduction:

Students will learn how to visualize a function by plotting points on an x-y coordinate axis.

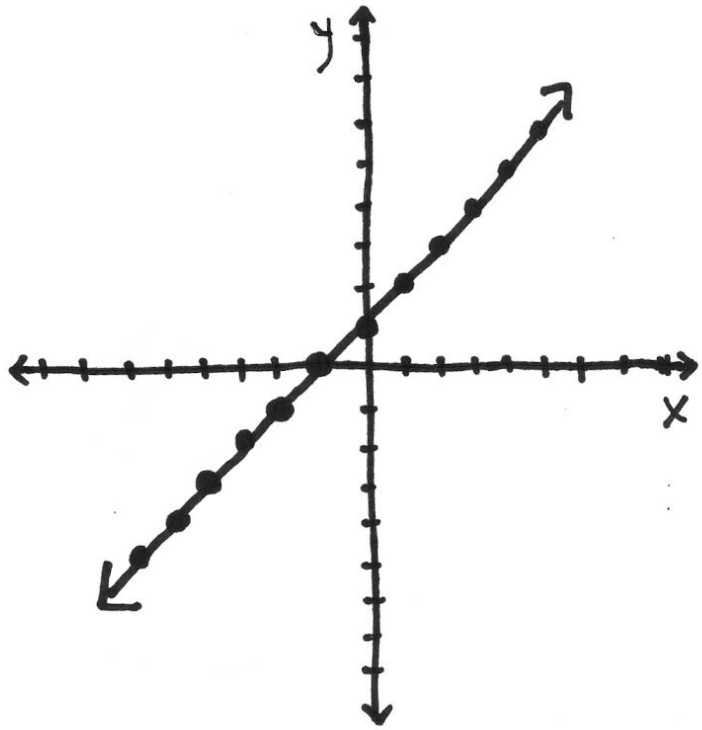
This lesson follows nicely on Function Machines

Objectives:

- To learn how to make a table of values of a function
- To plot some points of a function

Materials:

- Scratch paper and pencils
- Graph paper
- A straight edge (or ruler)
- Permission to leave the school for a few minutes or to go into the gym
- A watch with a second hand
- Sidewalk chalk, optional



name one member A and one member B. For ten seconds that you time, have the As count how many jumping jacks the Bs can do in 10 seconds. Begin a table with this data in it for each B student. For example:

Time (seconds)	Number of Jumping Jacks
10	11

Now, time all the B students for 20 seconds, 30 second, 40 seconds, 50 seconds, and 60 seconds. Each time, have their partners record how many jumping jacks they can do in that period of time. (Set some standard for what counts as a jumping jack so that students don't try flailing arms to get their counts high.) Their tables may now look like:

Time (seconds)	Number of Jumping Jacks
10	11
20	20
30	28
40	35
50	42
60	50

Do this for all the A students, also, then head back inside. (Of course, if it's a gorgeous day, you could draw your own graph paper on the sidewalk with sidewalk chalk and proceed outside.)

Have each student draw a pair of axes on their graph paper. Have them label the x-axis (the horizontal one) "Time" and the y-axis (the vertical one) "Number of Jumping Jacks." Next, have each student plot the points from their own data. Now look at the graphs and ask the students if they can learn anything from the graphs. Most likely, the students' graphs will not be in a straight line, they will level off. This shows that the students got tired as they had to do jumping jacks for longer time periods.

Try plotting, or graphing, other functions. Try y=3*x+1. Make a table of values of input and output, and then plot those points on a pair of axes. What do you see? (It's a line going up and to the right, and it goes through (0,1).) Repeat this process with y=x-1, y=-x, y=2*x+1, and y=-2*x-3, taking one function at a time. (They are all lines; the first and third heading up, and the second and fourth heading down.) For something more challenging, suppose your function is $y=x^2$. Make a table of values of input and output, and then plot those points on a pair of axes. What does the graph look like? (It's a parabola, or a giant U shape.)

6. Manipulating Graphs

Begin by having one of the students recall how to graph the two functions $y=x$ and $y=x^2$. Have them make tables of inputs and outputs, and have them plot the points and connect them with smooth curves. (The first is a line; the second is a parabola, or a giant U.) Keep these main two functions on paper or on the board in front of everyone for the rest of the lesson.

Now use the same method of analysis (making a table of values, plotting points on a pair of axes, connecting the points with a smooth curve) to see what happens when we place a negative sign in front of the inputs, that is, investigate, separately, $y=-x$ and $y=-x^2$. What did the negative sign do to the graphs? (It flips them over the horizontal axis.)

What happens if we add a number to or subtract a number from the original function? This time, have the students graph the two functions, separately, $y=x+2$ and $y=x^2+2$. Have them explain how these graphs compare to the original two. Have them conjecture what they will see when they graph $y=x-5$ and $y=x^2-5$. Is there conjecture right? (Both graphs should be moved down 5 units.)

This is a bit more difficult, but have them graph $y=(x-3)$ and $y=(x-3)^2$. (We substituted x-3 in for x this time.) Also have them graph $y=(x+4)$ and $y=(x+4)^2$. How did these graph change? Have them come up with an explanation. (Putting x+n (where n is positive) in for x moves the graph n units to the left, and putting x-n (where n is positive) in for x moves the graph n units to the right.) Now they should conjecture what they will see when they graph $y=x+5$ and $y=(x-5)^2$. (Please note that shifting the line $y=x$ three units up and shifting $y=x$ three units left looks exactly the same.)

If there's time remaining, graph $y=x^3$ (where $x^3=x*x*x$). Then have the students conjecture and draw graphs for $y=-x^3$, $y=x^3-2$, $y=(x+5)^3$.

Introduction:

This lesson will show students how they can manipulate functions to change the appearance of their graphs.

This lesson follows on Graphing Equations.

Objectives:

- To practice graphing a function on coordinate axes
- To understand what can happen to a function when small changes are made to it

Materials:

- Scratch paper and pencils
- Graph paper
- Chalk if you wish to work on a blackboard

Probability:
What are the Odds?

★★★★

1. Beans in a Bowl

Let's begin by understanding what probability means. Probability is the chance (or the odds or the likelihood) that an event will happen. What are the odds that when we're selecting a single card from a shuffled deck, it is the 7 of diamonds? What is the chance that the next student to walk around the corner is wearing a blue shirt? What is the likelihood that the lottery ticket I bought last night is the winning ticket? We often call the event we're interested in happening a "success."

To understand probability, let's start with an easier example. Let's put five blue beans (or candies or markers) and five red beans in the same bowl. Mix them around. Ask the students what the odds are of pulling out a red bean if they were to pull out one bean without looking. They should see that the odds are the same for pulling out a red bean as a blue bean, so it's 50%. How else can we see that? There are 10 different beans we could pull out (those are the different "outcomes" of the experiment), and 5 of them are successes (they're red beans). So the odds of us pulling out a red bean are 5/10=.5=50%.

Does a 50% probability mean that if we pull out one bean it's going to be half red and half blue? No! There's no such bean in our bowl. However, it does tell us that we're just as likely to pull out red as blue. Have the students pull out some beans and see what colors they get. (Make sure they put the bean back each time before the next person pulls out one; otherwise, you're changing your denominator!) Repeat this experiment 20 times and see what colors get pulled out. It should be about 50--50. (Will it be exactly 10 red beans and 10 blue beans drawn in 20 draws? Probably not, but the fraction of red beans should get closer to 50% as the number of times we repeat the experiment increases. Try it!)

Now repeat this activity with different numbers of red and blue beans. Consider 8 blue and 4 red beans in our bowl. Now are the odds of pulling out a red bean higher, lower, or the same as before? They're lower of course because there are fewer red beans than blue

Introduction:

Probability is the study of odds or the likelihood of some event happening. A good understanding of probability will help the students understand which of several outcomes is more likely to happen.

The field of probability is used extensively by actuaries and insurance companies and casinos.

Objectives:

- To define some of the terms in beginning probabilities
- To calculate a few simple probabilities

Materials:

- Scratch paper and pencils
- Colored objects that are identical in shape (for example, colored candies, but they will be well-handled!)--ten of each of several different colors
- An opaque bowl (or hat) to put the objects in

beans. How much lower? Well, there are now only 4 successes, but there are 12 different outcomes (12 different beans that could be pulled from the bowl). That means the odds of pulling a red are 4/12=33.3%.

What are the odds of pulling a blue out in this case? There are 8 successes then, and 12 outcomes still, so 8/12=66.7%

Notice that the odds that you pull a red OR a blue bean out of the bowl gives you 12 successes with 12 outcomes, so 12/12=100%. This makes sense because there are only two different color beans in the bowl, so you have to pull out one of the two colors. Every time you play you would have success. Notice that the odds of pulling a red plus the odds of pulling a blue add up to 100%.

What's the largest probability that can happen? Let the students think about this a while and discuss it. (100% because you can't have more successes than you have possible outcomes.) What's the smallest probability that can happen? Discuss. (0% because you could have no successes; for example the probability of pulling a yellow bean out of a bowl with only red and blue beans in it.)

Now let's make the probabilities a bit more interesting. In your bowl place 3 red beans, 3 blue beans, and 2 yellow beans. Ask the students to discuss if they are equally likely to pull out the different colors of beans. That is, what's the probability of pulling out a blue? Is it the same as pulling out a red or pulling out a yellow? They're not all the same. Now for red there are 3 successes out of 8 outcomes (the 8 different beans in the bowl), so the probability of pulling out a red is 3/8. The probability of pulling out a blue is 3/8, but the probability of pulling out a yellow is only 2/8 because there are fewer yellow beans in the bowl. Then ask them what the probability is of pulling out a bean of some color from the bowl. (100%) What's the probability of pulling out a green bean? (0%) Try pulling a bean out of the bowl, record the color, then replace the bean into the bowl. Repeat this experiment 100 times. What do they see? They probably won't get the exact proportions that they expect, but it should be somewhat close. Try pulling one bean at a time out and recording its color 100 more times. Are the probabilities closer?

If time permits try other combinations of colors of beans in the bowl and ask what the probabilities are. Try:

2 blue, 2 red, 2 yellow
3 blue, 2 red, 1 yellow
3 blue, 3 red, 3 yellow, 1 green

2. Probability with Coins

If you ask a student, after doing Beans in a Bowl, what the probability is of flipping heads when flipping a coin, she should answer 50--50 (or 50% or 1/2, all of which are correct) because success for us is flipping a head, and there are two possible outcomes, so 1/2.

What is the probability of flipping two heads in a row? The student might again answer 50%, but this time the answer is only 25%. Why? One way to show this is to list all the possible outcomes of tossing two coins (HH, TT, HT, TH). We see there are 4 total outcomes, only one of which is our success. Thus, the probability is 1/4. Another way to think about this is by drawing a tree diagram. like we did when we counted locker codes.

A tree diagram also shows all possible outcomes and how many of the desired events we are looking for. Next you can ask what the probability of not flipping two heads in a row is. There are two ways to think about this. First, we can consider all four possible outcomes (HH, TT, HT, TH), and we see that 3 of them are now a success for us, so the probability is 3/4. Another way to think about this is to realize that either we flip two heads in a row or we don't. Since the probability of flipping two heads in a row is 1/4, then the rest of the time we must get not-two-heads, or 3/4 of the time.

Now challenge the students to flip 6 heads in a row. Have them flip a coin 6 times and record how many of the 6 flips were heads. Repeat this 100 times. Probably not many were all heads! Why? Have them think about how many outcomes are possible by listing them or drawing a tree (there are 64) and how many are all heads (1), so the probability is 1/64, or a less than 2% chance!

Introduction:

Besides the examples of pulling colored beans from a bowl, coins are one of the easier examples for getting the students comfortable with probability.

This lesson follows after Beans in a Bowl.

Objectives:

- To practice calculating probabilities
- To learn about a tree diagram to think about probabilities

Materials:

- Scratch paper and pencils
- Pennies, at least one per student

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3. Probability with Dice

Starting with 1 die, ask what the probability is of rolling, say, a 5. Again counting all the outcomes (1, 2, 3, 4, 5, 6), there are 6 outcomes but only one success (rolling the 5), so the probability is 1/6.

With just one die we can ask different questions to get the students thinking. What's the probability of rolling a 5 or higher? (2/6) What's the probability of rolling an odd number? (3/6) What's the probability of rolling anything but a 5? (5/6 since a success could now be rolling a 1, 2, 3, 4, 6)

Moving on to 2 dice, you can propose the question: If I roll both dice, what's the probability that the two dice added up will equal 7? Let them discuss this for a while. To start with, go through all the possibilities of what the two dice could come up. There are 36 different possibilities, all equally likely. (1,1; 1,2; 1,3; 1,4; 1,5; 1,6; 2,1; 2,2; 2,3; 2,4; 2,5; 2,6; 3,1; 3,2; 3,3; 3,4; 3,5; 3,6; 4,1; 4,2; 4,3; 4,4; 4,5; 4,6; 5,1; 5,2; 5,3; 5,4; 5,5; 5,6; 6,1; 6,2; 6,3; 6,4; 6,5; 6,6) The students should notice that the sums of those pairings can be anything from 2 (1+1) to 12 (6+6), but the SUMS are not equally likely. In fact, have them group the 36 different rolls that can happen by what sums those rolls give, They should see that there are two ways to roll a sum of 3, but six ways to roll a sum of 7. What sum is the most likely to occur? (7 with probability 6/36) What sum is least likely to occur? (2 and 12 are equally likely, with probability 1/36 each.)

What's more likely, a sum of 7 or a sum of anything but 7? (Anything but 7 has a probability of 30/36, so that's more likely.) What's the probability of rolling a sum of 11 or 12? (There are 3 ways to roll an 11 or 12, so 3/36.) What's the probability of rolling a sum of 4 or lower? (There are 6 ways to roll a sum of 2, 3, or 4, so the probability is 6/36.) What's the probability of rolling an odd sum? (There are 18 ways to roll a 3, 5, 7, 9, or 11, so the probability is 18/36.) What's the probability of rolling an even sum? (You could just count them, or you could notice that you either have to roll an odd sum or an even sum, so the probability is 1-18/36=18/36.) What's the probability of a sum of 0?

Introduction:

Dice may prove a little tricker to work with than coins since now there are six sides instead of two, but they still provide many useful examples in probability.

This lesson follows Probabilities with Coins.

Objectives:

- To practice calculating probabilities
- To see how dice are different than coins in computing probabilities

Materials:

- Scratch paper and pencils
- Dice, at least one pair per student

Taking it Further:

How many different ways are there to roll three dice? Let the students think about this a while. To roll three dice, you must first roll two (there are 36 ways to do that), and for each of those 36 ways, there are 6 ways to roll the third die, so there are 36*6=216 ways. Now, what's the probability that the same number appears on all three dice? (6/216)

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4. Let them Roll Some More

An easy way to keep track of all possibilities that can occur when rolling two dice is to make a 6x6 table, where the rows indicate the roll on the first die and the columns indicate the roll on the second die. Have the students make a 6x6 table and label the entries with pairs of numbers to indicate what the rolls of the two dice would be, like:

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

They should check with their neighbor to see that their tables agree.

Now ask them some probability questions that they can use their tables to answer. If they would like, they could use colored pencils to color in all the entries in the table that would be a "success" for each event.

1. What's the probability of getting a sum of 5 on the two dice? (4/36)
2. What's the probability of rolling at least one 2? (That would be rolling one 2 OR two 2s.) (11/36)
3. What's the probability of getting a sum of 5 on the two dice AND rolling at least one two? (2/36)

If they colored in the successes for the first question with red, and the successes for the second question with blue, then the successes for the third question are now in purple!

Give them plenty of time to ask each other probability questions about rolling two dice, using ANDs and ORs.

Introduction:

Continuing with examples of probabilities using dice, this lesson considers more complicated combinations of several events and computes their probabilities.

This lesson follows Probabilities with Dice.

Objectives:

- To practice calculating probabilities
- To work with probabilities of events that are combined with AND or OR

Materials:

- Scratch paper and pencils
- Dice, at least one pair per student
- Colored pencils

5. Keep on Rolling

To begin this lesson, ask the students to recreate the 6x6 chart of all possibilities for rolling two dice that we created in Let them Roll Some More.

Have them review the previous lesson by asking them to find the probability of rolling a 5 AND a sum of 10. (1/36) Ask them other questions reviewing probabilities with OR and AND.

Now ask them the following: If I rolled a 3 on the first die, what is the probability of getting a sum of 8 when I roll the second die? Let them discuss this a while. This is called a "conditional probability" because we're looking for a probability conditioned on "If I rolled a 3 on the first die."

To solve this problem, we look at the condition -- rolling a 3 on the first die -- and we have just those six events as outcomes. There are only 6 outcomes now. Then we consider that we want a sum of 8. When we roll a 3 on the first die, there's only one way to get a sum of 8, that is a success, and that is by rolling a 5 on the second die. So the probability is 1/6.

What if we ask: Given the sum is 10, what's the probability that the first roll is a 4? Now our outcomes are any pairs of rolls whose sum is 10. There are three. Our success is the first roll being a 4, so the probability is 1/3. What is we ask: Given the sum is 10, what's the probability that the first roll is a 2? If we look at the possible outcomes, there are no pairs of rolls where the first roll is a 2, so the probability is 0/3=0.

Here are some more practice problems to discuss:

1. If your sum is an even number, what is the probability of one of the dice being a 2? (5/18)
2. If your sum is 4, what is the probability that you rolled two 2s? (1/3)
3. If your first roll is a 2, what's the probability that the sum will be even? (3/6)

With the time remaining, take turns asking each other conditional probabilities.

Introduction:

Taking the examples of probabilities using dice a little bit further, we can talk about conditional probabilities.

This lesson follows Let them Roll Some More.

Objectives:

- To practice calculating probabilities
- To learn what a conditional probability is

Materials:

- Scratch paper and pencils
- Dice, at least one pair per student
- Colored pencils

6. Let's go Fishing!

We are going to do some fishing of our colored fish. Let's begin by "stocking" the "lake" (or putting colored crackers in the bowl) with 10 purple, 10 green, and 10 red.

Now start fishing (that is, have a student pull a fish out of the lake without looking). Before looking at the fish, have the students guess what color it is. What does probability tell us? (The three colors are all equally likely, with chance 10/30 of being pulled out.)

Suppose you pull out a green one, but you really wanted a purple one. If you toss the green one back in the lake and go fishing again for the desired purple fish, the probability of pulling a purple fish remains 10/30. Suppose the next fish you pull out is red, still not the desired purple fish. When you throw the red one back in, the probability of pulling out a purple fish remains at 10/30.

Suppose instead, though, you had not thrown the blue or the red fish back in the lake. Would the probabilities have changed? Give the students some time to think about this and suggest answers. Yes, the probabilities would have changed. When you pulled out the green one, if you didn't return it to the lake, the odds of getting a purple one are 10/29 on the next turn. Which odds are better for getting a purple fish, 10/30 or 10/29? (10 in 29 are better odds than 10 in 30.) So we can improve our odds by leaving the green fish out.

What if we leave the red fish out of the lake as well; how does that change the odds? Now there are still 10 purple fish in the lake, but there are only 28 fish in the lake remaining, so the odds are now 10/28.

If we are still unlucky, even after leaving the green and the red fish out of the lake, and we pull two more greens and two more reds in our next four pulls, now what are the odds of getting a purple? (10/24, which are much better than 10/30).

Try some other resampling experiments (fishing more) where you do or do not put the fish back in the lake.

Introduction:

To further develop the concept of probability, in this lesson we look at how probabilities or chances will change with the replacement or non-replacement of objects after sampling.

This lesson follows Beans in a Bowl.

Objectives:

- To practice calculating probabilities
- To learn the difference between replacement and non-replacement in sampling

Materials:

- Scratch paper and pencils
- Fish-shaped crackers of various colors
- An opaque bowl or hat

7. Counting on Independence

To begin, ask the students to write down 50 imaginary flips of a two-sided coin (without actually flipping a coin). When they're done they should have a 50-item list with either "heads" or "tails" as each item. Now ask them to actually flip a coin 50 times and to write down the results. Ask them to compare their two lists, and then encourage them to discuss their findings with each other.

They may find that when they actually flipped the coin, sometimes they got six, seven, eight, or more heads or tails in a row. But on their list of imaginary flips most students list no more than four or five heads or tails in a row.

When students make their imaginary lists they often assume that if they flipped heads one time, then they are less likely to flip heads the next time, but this isn't true. Each flip of a coin is an independent event; that is, all previous flips do not influence the probability of flipping a head or a tail this time. In this lesson we are going to investigate this further.

Ask the students what the probability of flipping tails is on any given flip. (1/2 or 50%) Now ask your students what the probability of flipping tails is if the previous flip was tails. The answer is still 1/2, but the students may think it is less than one half. You could exaggerate the situation by asking them what the probability is of flipping tails after flipping twenty tails in a row. Again, the answer is still 1/2. Why is this?

There are a couple ways to explain this. One way is to test the hypothesis by looking at their list of heads and tails from when they actually flipped the coin. Now find all the times heads was flipped and write down what they flipped after flipping heads. They should see that, after flipping heads, they flipped another heads half the time and tails the other half. It is possible that their list of flips is not long enough to see that it all averages out; you could combine all the students actual flips into one big master list of all the coin flips.

Here are a couple of ways to explain why each time you

Introduction:

This lesson will show students that some events are related to others in a way that affects how the probabilities are calculated. That is, some events are dependent and some are independent.

This lesson follows on Keep on Rolling and Let's Go Fishing.

Objectives:

- To practice probabilities
- To understand the difference between independent and dependent events

Materials:

- Scratch paper and pencils
- Coins, at least one coin per student
- Standard 6-sided dice, at least one per student
- A standard 52-card deck of playing cards
- A calculator

flip a coin, it is an independent event. The first is to point out that the coin doesn't know that it's been flipped before. The coin itself only has two sides, and always has two sides no matter if it was flipped before or not. Another way to explain this is by, instead of flipping one coin ten times in a row, say, flip ten different coins each one time. Just because you flipped a head on one coin doesn't influence the probability of flipping a head on another coin. Finally, you could also pose the question to the students; What if someone flipped this coin five years ago? Would that influence how it flips today? The answer is no, and if they can understand that, then they may better be able to understand why it also doesn't matter that someone flipped the coin five minutes ago. The probability of flipping heads is still 1/2.

Ask the students if a die would remember what it has rolled before. Of course not, so rolling an even number 50 times in a row would be unlikely, but it would have no influence on whether the next roll is even or odd. Repeat the same sorts of experiments with dice and independence as we did with coins if the students need more convincing.

On the other hand, a "dependent" even is characterized by the fact that its outcome is influenced by previous events. To illustrate the difference between independent and dependent events, consider the following scenarios.

Drawing a red card from a full deck with replacement is an independent event. The probability of drawing a red card out of a full deck of cards is 1/2 since half the cards are red and the other half are black. If you then put that drawn card back into the deck (this is called replacement) and draw again, the probability of drawing a red card is still 1/2 because the deck still consists of half black and half red cards.

Drawing a red card from a deck of cards without replacement is a dependent event. Let's say that you draw a red card from a deck of cards and then you leave that card out of the deck (like we left the fish out of the pond in Let's Go Fishing). What's the probability now, with one red card removed, of drawing a red card? (25/51 -- is this the same as 26/52? Check! If 25/51 is the probability of pulling a red card after a red card has been removed, then 26/51 is the probability of pulling a black card after a red card has been removed. That is, the probability of pulling a black card increases after a red card has been removed.

To reinforce this idea, continue doing similar examples with suit, number, and face-cards versus non-face cards.

8. The Monty Hall Problem

Pose the following question to the students:

Suppose you're on a game show, and you're given the choice of three doors. Behind one of the doors is a new car; behind the other two are goats. You pick a door and the host (who knows what's behind all three of the doors) opens one of the doors you didn't choose and shows you a goat. Then the host asks you if you want to stick with the door you already chose, or if you want to switch to the other door which is still closed. What should you do? Should you keep the door you already chose, or should you switch?

Ask the students what they think about the question. Do they understand? Make sure everyone understands the question before proceeding. Ask them about their intuitions. Most people would say that it doesn't matter whether you switch or not; it's the same probability.

Now play the game 30 times with you as the host. Set up the three doors. Behind one put the car, and behind the other two put the goats. Have students take turns being the contestant, and tell them that you want to see what happens if they don't switch. As a group record how many of the 30 times the contestant got the car when the "Don't Switch" plan is used.

Repeat the game 30 more times, but now have the contestants always switch, and record what happens.

Was there a difference? (There should be more cars when you switch.) Have the students discuss what they see. If someone previously thought that changing or not makes no difference, has he changed his mind? Ask him to explain why or why not. Once everyone has a firm belief about whether changing doors matters or not, talk to them about how they should approach calculating the actual probabilities.

Here's one way of thinking about it: I either win or I lose. Suppose I go into the game with the decision to not switch. Then 1/3 of the time I'm going to pick the

Introduction:

This easily-stated but quite subtle question lead to quite a bit of discussions by mathematicians about how to approach the problem.

Don't rush discussion of this problem and its solution; it's fun, it's interesting, and there's a lot to learn.

Objectives:

- To have some fun with probability
- To learn a puzzle to share with friends and family

Materials:

- Scratch paper and pencils
- 3 "doors" (manila folders work well)
- 2 pictures of goats (different from each other) and one picture of a new car, hand-drawn or silly are encouraged

car, and since I'm not switching it doesn't matter what the game show host shows me. So I win with probability 1/3.

Now if I go into the game with the decision to switch no matter what, then when I originally choose the car, with probability 1/3, I actually lose because I'm going to switch to a goat. But if I originally choose a goat (with probability 2/3), then the host will show me the other goat, and offer me the door with the car behind it, so I'll win 2/3 of the time.

Thus I win twice as often if I decide to always switch (2/3 instead of 1/3).

Here are some good questions that can keep discussions going if your group catches on quickly:

1. Is it possible to win three times in a row if you never switch? (Yes) Why or why not?
2. If you win ten times in a row, are you less likely to win the next time? (No)
3. Why does playing the game more times (30 for each strategy) create more accurate data to understand what's happening?

Graph Theory:
It's just Dots and Lines

1. The Three Utilities Problem

Pose the following problem:

Alissa's family, Becky's family, and Christie's family live in three houses beside each other. Across the street are three utility companies, Gas, Water, and Electric. Each of the three houses need to be connected to each utility, but the utility companies can't let their lines cross.

Have the students draw the three houses and the three utilities, and have them [try to] connect each house to all three utilities without having the lines cross. "Lines" here do not have to be straight lines; they are simply a way to connect a house to a utility. (This can't be done, but let them struggle with it because they learn quite a bit in the struggle.)

If we draw the houses and utilities as dots, then we have a mathematical graph. A "graph" is a collection of vertices (dots), and edges (lines) connecting pairs of vertices. Any number of the vertices can be connected to other vertices, or maybe none at all.

The students will probably come up with some proposals to change the problem slightly to try to make it solvable. Let them explore with their ideas. Here are some questions to pose if they don't come up with any themselves:

1. Would it help to move the houses and utilities around? (Instead of on two parallel sides of the street, perhaps they're all in a circle, or the utilities are between some of the houses.) Try it!
2. What if there were only two houses? Or two utilities?
3. What if there were four houses and two utilities?
4. What if we had three dimensions (utilities lines could come out of the paper)?
5. What if the houses and utilities weren't on a flat piece of paper, but instead on a sphere (imagine the houses and utilities were on the Earth)?

The problem as originally stated is impossible because the graph we want to draw must contain edge crossings.

Introduction:

This introduction to the mathematical field of Graph Theory is through a classic puzzle that children have played for years.

Objectives:

- To learn what a "graph" is
- To view a word problem as a graph
- To learn what "planar" means

Materials:

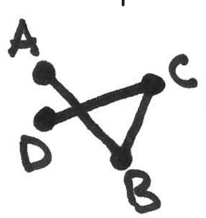
- Scratch paper and pencils
- Chalkboard and chalk, if available

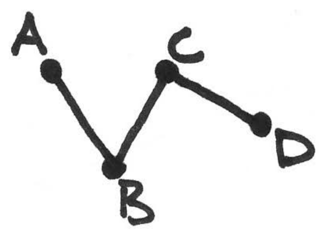
Taking it Further:

We say that a graph is "planar" if it can be drawn in the plane (that is, on a piece of paper), without any two edges crossing. The word "can" is important in that definition because if someone hands you a graph with edge crossings, it may be possible to draw the same graph without edge crossings. What's important about a graph is the collection of vertices and knowing which vertices are connected to which other vertices, not how or where the vertices and edges are drawn on the paper.

2. Euler's Magic Formula

Have your students review what they learned in the Three Utilities Problem by drawing some graphs on their paper. They should each draw three graphs; you might want to tell them to use 6 or fewer vertices to keep them from being too complicated. Have them color the vertices. (It doesn't matter if they use one or multiple colors; they should color just the vertices and not the edges. The coloring is just to be clear to everyone where the vertices are.) Then look at the collection of graphs. Which of the graphs they drew are planar and which are not planar? If there are graphs where edges cross ONLY at vertices, then those graphs are planar. Are the other graphs non-planar? Not necessarily. Recall that a graph is non-planar if it cannot be re-drawn to be a planar graph. For example, here is a graph that had edge-crossings, but has been redrawn to be planar:





Can all graphs be redrawn to be planar? (No, the Three Utilities Problem showed us that.)

Spend some time having your students re-draw the graphs with edge crossings to see if they seem to be planar are not.

Now we want to talk about just planar graphs, so remove all non-planar graphs from the work space. Have the students draw some more examples of planar graphs, and have them color the vertices.

Also, we want to only talk about connected graphs. A "connected graph" means that between any two vertices in the graph there is a series of edges that connects them. (What's not connected? One example is a graph that is simply two vertices and nothing else.) It's unlikely that the students drew any non-connected graphs, but if they did, those need to be removed from discussion now.

Introduction:

The equation discussed in this lesson, attributed to Leonard Euler (pronounced "OY ler"), shows an amazing relationship between the numbers of vertices, edges, and faces in a planar graph.

This lesson follows on the Three Utilities Problem.

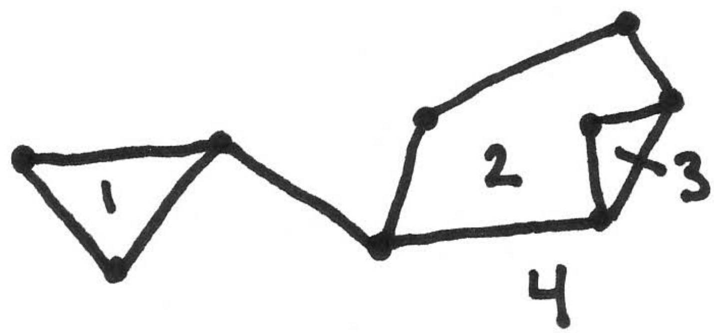
Objectives:

- To understand what Euler's Formula says
- To know how to count vertices, edges, and faces

Materials:

- Scratch paper and pencils
- Colored pencils or markers

For all the connected, planar graphs remaining, have the students count the number of vertices and the number of edges of each graph and write that information next to the graph. Also, have the students count the number of faces in each graph. A "face" is an area contained by the edges of the graph, and there is one infinite "face" that is all the area outside the graph. For example, the faces are numbered on this planar, connected graph:



Have the students count the number of faces on each of their graphs and write that number next to the graph as well.

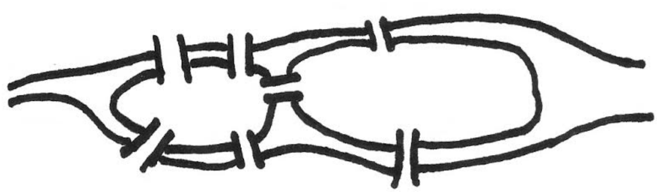
Now, for each graph, compute V-E+F (where V is the number of vertices, E is the number of edges, and F is the number of faces). It should be 2 in every case!

$$V - E + F = 2$$

That's Euler's Magic Formula!

3. The Seven Bridges Problem

In 1735 the people of Königsberg, Prussia liked to take walks around the city and across the seven bridges which connected the two sides of the Pregel River and the two islands within the river. The people wondered if it was possible to cross all seven bridges on an evening stroll without crossing any bridge more than once. The picture of what the bridges looked like is:



This can be modeled using graph theory by letting each side of the river be a vertex and each of the islands in the middle is another vertex, making four vertices in total. The bridges connecting the pieces of land are edges connecting vertices in our graph. A graph theory model of the bridges then looks like:

and we want to know whether we can cross all of the bridges. If you are outside, draw either of the above representations of the problem on the sidewalk large enough that the students can try walking the bridges and see if they can come up with a solution.

Give them some time to experiment with different starting places and different paths, then ask them to discuss what they learned. Is it possible? (It's not, but don't tell them that, let them conjecture it. They will learn the truth soon enough.) If you could change one thing to make it possible what would you change? (There are several different answers possible.) Why is this helpful? Are there other ways to change one thing and make a walk traversing all the bridges possible?

Introduction:

This puzzle about traversing the bridges of Königsberg (now Kaliningrad, Russia) caused the creation of the field of Graph Theory by Leonard Euler in 1735.

This lesson follows on the Three Utilities Problem.

Objectives:

- To think about paths on graphs
- To learn two classic graph theory problems
- To conjecture for themselves when a graph has an Eulerian path

Materials:

- Scratch paper and pencils
- Sidewalk chalk (enough for all) and permission to go outside and draw on the sidewalks, optional

The Highway Inspector Problem

A problem that's related to this is called the Highway Inspector Problem. Draw on the sidewalk (or paper) a big graph that has, say, 10 vertices, with many edges, say 20, between the vertices. Make sure that it's a connected graph (you can get from any vertex to any other vertex), but it doesn't have to be planar (that is, the graph can have edge-crossings). The vertices are representing cities, and the edges are representing highways between the cities. Make the graph physically large so that once again the students can walk around on the edges and vertices.

Ask the students the following question: Is it possible for a highway inspector to start at some city and drive over all the highways in the graph without driving on the same highway twice? Let them try this for a while, and when they are done, ask them to report on why it did or did not work. Again, if it didn't work, what would they like to change to make it work?

Give the students some chalk and ask them to draw a collection of cities and highways that will work for the highway inspector. What properties do they need for a graph to have so that it works for the highway inspector? While they are drawing graphs of their own, you draw some more examples, both that work and that don't work.

Let them draw examples, try each others' graphs, and discuss/argue about what they need.

After the students have had plenty of time to work with these two problems, bring them back together and ask them to talk about the similarities between the two problems. What are we looking for in each case? Let them say this in their own words, then say that we're looking for an **Eulerian path**, which is a path that starts at some vertex and allows us to walk through the graph, walking over each edge exactly once. (It can go through vertices more than once.)

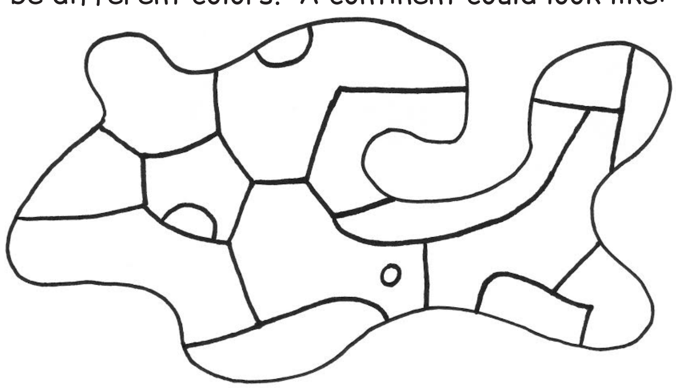
What do we need for an Eulerian path in a connected graph? The students have probably noticed that it's a problem when there are many vertices that have an odd number of edges coming out of them. The number of edges coming out of a vertex is called the "degree" of that vertex. It is possible to have an Eulerian path if the graph has exactly 0 or exactly 2 vertices with odd degree. (If it has 2 vertices of odd degree, then the path must begin on one of them and end on the other; if it has 0 vertices of odd degree, the path can begin and end on the same vertex.)

Make sure you all agree with this by drawing some graphs that have 0 vertices with odd degree and some graphs that have 2 vertices with odd degree, then finding the paths that are supposed to exist.

5. Coloring a Map

Have the students begin by looking through some atlases at pictures of colored maps. Have them describe the maps -- how many colors are used? Are there any rules that a cartographer follows about coloring a map? Lead them here into a discussion about countries (or states or boundaries of some sort) that touch each other need to be colored different colors. What do we mean by "touch"? If they share only a point in common (show The Four Corners states in the U.S.) then we won't consider them touching, but if they share part of a line segment, then we will.

For the time being, let's consider a continent consisting of countries where each country is only one piece. (For example, the U.S. is in several pieces because of Alaska and Hawaii.) Have the students draw such a continent containing 10 countries. The countries can be in crazy shapes, but each country must be contiguous. When they're done drawing, pass the map to someone else in the group, and have the students color each other's maps. The coloring should be correct; that is, countries that touch along a boundary (in more than a point) need to be different colors. A continent could look like:



Once they are done coloring the maps, ask them how many colors they used. Then ask them to think about what the minimum number of colors they needed was to color the map.

Have the students draw maps for each other again (again, with 10 countries; there's nothing special about 10 countries, but without direction, students tend to draw maps with hundreds of countries), but this time

Introduction:

First stated as a mathematical problem in 1852, the question arose originally by a cartographer wondering how many colors one must use to a color a map. The problem stumped mathematicians for over a one hundred years until a solution was proved in 1976.

This lesson follows on the Three Utilities Problem.

Objectives:

- To practice making conjectures
- To learn about map coloring
- To see connections with graph theory

Materials:

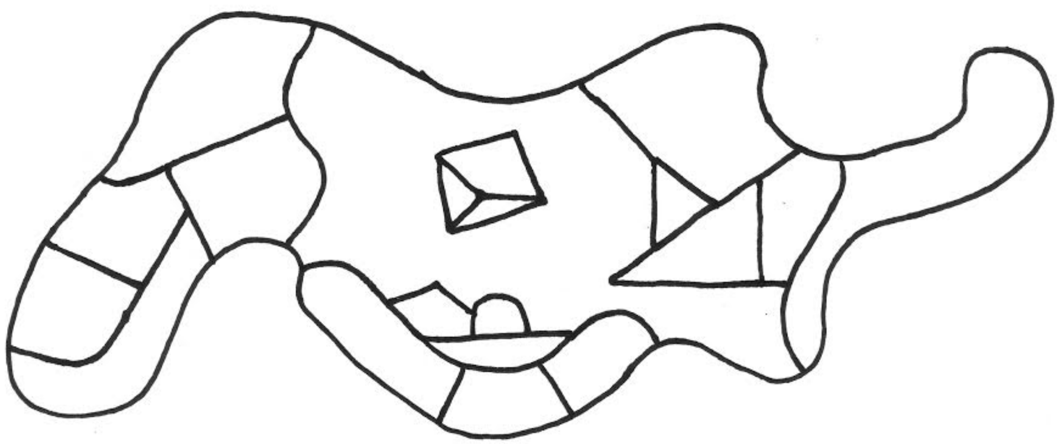
- Scratch paper and pencils
- Colored pencils or markers
- Access to some atlases containing colored maps

when they exchange with other students, ask the students to color the map using the smallest number of colors possible and still have a legal coloring. How many colors did each map take? (Don't tell the students this (so they get the joy of discovering it on their own), but it should never take more than four colors. If someone claims their map takes more than four colors, let the other students have a turn at coloring it until someone finds the error.)

Now ask the students to repeat this procedure of drawing a continent with ten countries, but tell them in advance this time that their partners are going to try to use the smallest number of colors possible, so they should make their maps as difficult to color as possible. How many colors did each map take?

Let the students continue this process as long as they'd like. Seeing other students' maps, they will probably come up with new map ideas of their own.

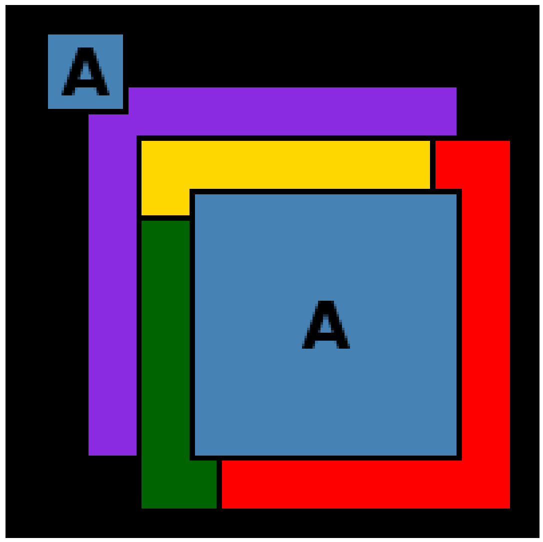
If no one has created a map that takes four colors (there are some), then introduce your own map into the mix, something like:



After the table is littered with colored maps, draw the students back together and ask them to make a conjecture from what they've seen. They should conjecture that any map (with contiguous countries) can be colored with just four colors. Don't worry about asking them to come up with justification for their answer; that took mathematicians over one hundred years and hundreds of pages of proof to do!

If the students are really into this idea and want to stretch it further, remind them that we required that the countries be contiguous regions. Can they come up with a map that requires more than four colors if we allow countries to be in two parts? (Both parts of the country must be the same color, of course.) This is a hard question, but give them as much time as they'd like to think about it and try things.

Here is a solution:



Iteration:
Again, and Again, and Again

★★★★

1. Spinning a Good Yarn

Discuss with the students what they think "iteration" means. (It means, simply, to repeat something.) Iteration in mathematics can be very interesting and lead to some quite surprising results.

Have a student cut a 1" (or you can do this all in cm if you like; the units are irrelevant) piece of yarn. Along the bottom (one of the short sides) of the long strip of graph paper, make a horizontal axis, and put 0 on the left. Write "number of iterations" under the horizontal axis, and "length" going up the left side. Now have a student plot a point that for the first iteration: we have a 1" piece of yarn, so the point is (1,1).

Now we could just repeat this same process, cutting another 1" piece of yard, but explain to the students that instead they are going to double it each time, and repeat that instruction. So in the second step, a student should cut a 2" piece of yarn, and plot (2,2) (for 2nd iteration and 2") on the graph. Take the pieces of yarn and lay them out on the floor, much like the graph paper graph, with the 1" piece on the left, then about an inch over put the 2" piece.

The next time, a student cuts a 4" piece of yarn and plots (3,4), and lays the 4" piece an inch to the right of the 2" piece. The next student cuts an 8" piece of yarn and plots (4,8). The next student cuts a 16" piece of yarn and plots (5,16). Next will be a 32" piece of yarn, and plot (6,32). When it's time to cut a 64" piece of yarn, you might point out that it's okay to measure off the 32" piece of yarn that's already been cut twice, to cut 64".

After plotting (7, 64), (8, 128), (9,256), you're probably running out of graph paper height. That's fine; let the students now connect those dots on their graph with a smooth curve, and the graph can be complete. The cutting is not, though! Let the students go on doubling as long as they are able. Finish by discussing with the students what they thought about the lesson and how quickly the numbers got huge. Explain that this is an example of exponential growth because in the n^{th} iteration you were plotting 2^n.

Introduction:

This is a fun and easy introduction to the idea of iteration -- doing the same thing over and over again. It also gives the students a chance to experience exponential growth is a very hands-on way.

This follows nicely on Graphing Equations, but that's not required to do this lesson.

Objectives:

- To learn about iteration
- To have a hands-on experience with exponential growth

Materials:

- Scratch paper and pencils
- Three or four sheets of graph paper, with their short sides taped together to make one long strip
- A jumbo skein of inexpensive yarn (remnants of several old skeins is fine, as well) OR a large ball of inexpensive string or twine
- Scissors
- Permission to use a long hallway, a gym, or go outdoors
- A yardstick or meter stick

2. Miss Katy's Party

The second-grade teacher Katy is going to have an end-of-the-term reception in the gym and she has 30 days to plan for it. The first day, she invites two of her friends, April and Karina, to come. The second day, Miss Katy invites two more friends, but unbeknownst to Miss Katy, so do April and Karina, The third day, the same thing happens: Miss Katy invites two more friends, but every person who knew about the party on day two also invited two more friends. This continues up until the 30th day, the day of the party.

Now Miss Katy has invited two people on each of 30 days, so she's invited 60 people. But other people were invited, too. If the gym holds 1000 people and everyone who's invited comes, is there enough room in the gym?

Let the students go on this one, figuring out for themselves how many people were invited. Give them time and space. If, after a while they seem stuck, ask them how many people in total know about the party at the end of the first day. How about at the end of the second day? The third day? (At the end of the first day, 3; at the end of the second day, 9; at the end of the third day, 27; at the end of the fourth day, 81; this continues and at the end of the n^{th} day, 3^n people are invited. Then by the end of the 30th day, 3^{30} people were invited, which is nearly 206 trillion people -- that's some party!)

If the students finish early, ask them how many people would have been invited if Miss Katy starts by inviting one person, and every day after that, all the people who knew about the party the day before each invited one more person. (Then the answer is 2^{30}, which is still too many for the gym!)

Another twist on this question is to ask the students how many days before the party Miss Katy can start inviting people if she follows her original invitation scheme and can only have 1000 people in the gym. (Only 6 days!)

Introduction:

Miss Katy is having a party in the gym, but with iterating the guest list, will there be enough room?

This could be tied in with How Big is the Gym?

Objectives:

- To learn about iteration
- To think about exponential growth

Materials:

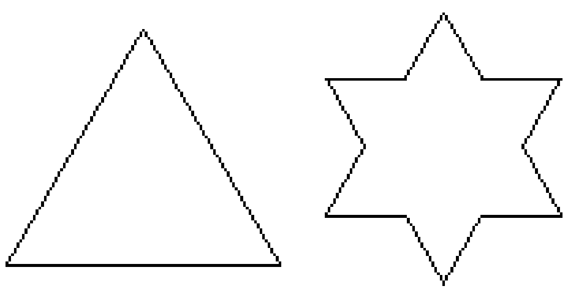
- Scratch paper and pencils
- Calculators for all

3. One Crazy Snowflake

Give the students each an equilateral triangle and ask them what's special about that triangle. (They should say things like "all the sides have the same length" and "all the angles at the corners are the same size".) Have them practice drawing equilateral triangles of various sizes on their paper.

When they get good at drawing equilateral triangles (perfection is not necessary, but neatness and precision count). Give them each a clean sheet of paper and tell them to trace their big equilateral triangle template onto the middle of their paper.

Each side of the triangle they should carefully mark into thirds. On the middle third of each side, they should build an equilateral triangle (whose third vertex is pointing out of the center triangle). The side lengths will all be 1/3 the size of the center triangle. Now the base side of the three new triangles (the one that was part of the center triangle), they should erase. Like this:



Now we're going to iterate this process. Now the shape has 12 sides, all the same length. Each side should be carefully marked into thirds, and an equilateral triangle pointing out should be built on each third. Then the base of each new triangle should be erased.

This process should continue as long as the students are able to still differentiate the sides from each other. At that point they should take a black marker and carefully outline the outside of the snowflake (ignoring all the erasure marks from the building process).

If we were able to let this process continue forever, the resulting snowflake (except we just have to imagine this, it would be impossible to draw) is called the Koch

Introduction:

This crazy snowflake, built by iterating a pattern, has a surprising property about its area and perimeter.

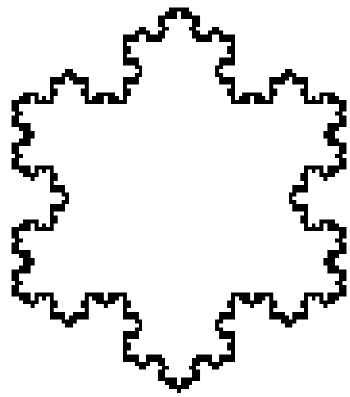
Objectives:

- To practice iteration
- To look for patterns
- To think about exponential growth

Materials:

- Scratch paper and pencils
- Calculators for all
- One equilateral triangle, about 4" on a side cut out from firm paper, for each of the students
- Black fine point markers, one for each student

Snowflake, and it would look something like this:



Let's talk about this snowflake. The outside curve of the snowflake has self-similarity. That is, if we focus our attention on one little cluster of the snowflake, it looks similar to other parts of the snowflake, only "zoomed in" or "zoomed out."

Is the area of the snowflake (if we were to color the inside) finite or infinite? (It's pretty easy to see that it's finite because if we draw a circle around the snowflake, its area is finite, and the area of the snowflake is smaller.)

What do we know about the perimeter of the snowflake? Suppose (just to make our lives easier), the original equilateral triangle we started with was 1 unit on each side. Let's compute the length of one side of the Koch Snowflake, but we'll do it by looking at how it changes with each iteration. Have the students draw a straight line segment like one side of the original equilateral triangle, and call that 1 unit of length. Then draw what happens in the next step of the iteration. The 1 interval gets replaced with 4 intervals, each 1/3 in length. So after the first step, we have 4/3 the original length.

Now have them draw the next step. We took each of those four segments and made them into four little segments, so sixteen segments in all, each 1/3 of the length they were the last time, or 1/9 of our unit length. So the total length is 16/9.

When they draw the next step, they will see that they will now have 64 segments each 1/27 of the unit length, so the total length is 64/27. Do they see a pattern yet? Each time we multiply the top by four and the bottom by three, so in the n^{th} stage, the length will $(4/3)^n$. As n gets really, really large (and goes toward infinity), what happens to this length? Investigate on a calculator. It goes to infinity!

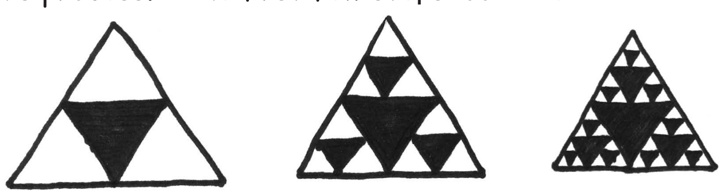
So the area of the Koch Snowflake is finite, but its perimeter is infinite!

4. Sierpinski's Triangle

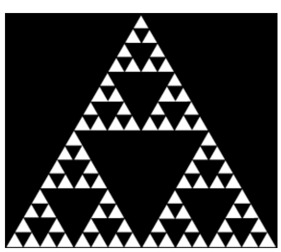
Start by having the students trace around the equilateral triangle on the middle of their papers.

We are going to iterate a process to create a beautiful triangle with self-similarity (that is, when we look at a small piece of the triangle, it will look like the whole triangle: we can "zoom in" or "zoom out" and we see the same design repeated).

Have the students "remove" an inverted triangle from the centers of each of their triangles by drawing the outline of an inverted triangle and coloring the inside. (See images below.) Now they should have three smaller triangles (still pointed up). From each of these three smaller triangles, remove an inverted triangle from the middle, leaving 9 smaller triangles. From each of these smaller triangles, remove an inverted triangle from the middle, leaving 27 smaller triangles. Continue this process. The first few steps look like:



If we were able to continue the process forever (which we cannot do; we can only draw an approximation of the result), we would get the Sierpinski Triangle, and it would look something like:



Notice the self-similarity in this picture. If you were to "zoom in" on any little upright triangle within the Sierpinski Triangle, you'd see exactly what you see at this stage.

You can try designing your own self-similar (or "fractal" image) by starting with **any** triangle, drawing the

Introduction:

Like One Crazy Snowflake, this shows the students another example of an iterated function and the interesting shape it creates.

Objectives:

- To iterate a function to create a beautiful shape

Materials:

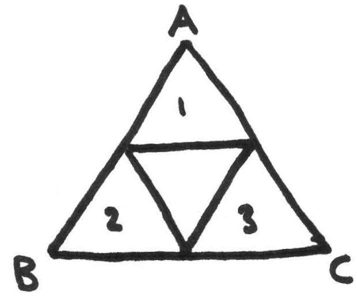
- Scratch paper and pencils
- One equilateral triangle, about 4" on a side cut out from firm paper, for each of the students
- One die
- Rulers

midpoints of the three sides, and connecting those midpoints to form an inverted triangle. Each of the three smaller triangles are the same as each other, and you can repeat the process on each of these. This iteration repeated indefinitely will form a Sierpinski-like triangle that has been "squished."

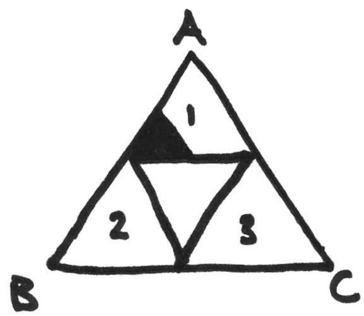
A remarkable result about these fractal images is that they can be generated by playing a game with a die. Start with an equilateral triangle with the vertices labeled A, B, and C. The rules of the game are easy: start with any point in the triangle, and roll a die. If the die says 1 or 2, move exactly halfway toward vertex A. If the die says 3 or 4, move exactly halfway toward B. If the die says 5 or 6, move exactly halfway toward vertex C. Then repeat this game over and over again. When we use this method to generate the Sierpinski Triangle, we throw out the first 100 or so points that we draw, then when you draw the next 10,000, you'll get an image that's imperceptibly different from the Sierpinski Triangle.

Why is this the case? Show the students with just a few rolls. Start with an empty equilateral triangle, and choose a point within it. Ask the student where that point goes if you have to move halfway toward A. Plot 10 or so points so they get the idea of what "halfway towards" means. (To plot a point halfway between a given point and a given vertex, let a ruler connect the two points, and measure the distance between the two points.)

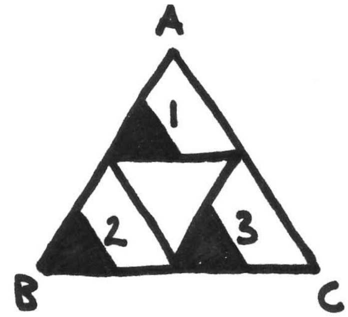
Then ask them to tell you where the points would end up if you took ANY point in the original triangle and moved halfway toward A. (You would end up in a smaller triangle whose vertices are A, the midpoint of AB and the midpoint of AC.) Similarly, what happens if you start with any point in the original triangle and you need to move halfway toward B? (You end up in a smaller triangle whose vertices are B, the midpoint of BA and the midpoint of BC.) A similar thing happens if you need to move halfway towards C. Let them plot points or think as a group or think on their own, however they want to approach this problem. After the first roll, you could end up in one of these regions:



Now starting with the image above which is the result of one roll, what if you are in one of the three shaded resions and the second roll told you to move halfway towards A? Plot some points and see if you can predict. You'd get an image like:



Similarly if you need to roll halfway toward B or C, so after two rolls, no matter where you start, you'd end up in one of these areas:



How about if you start in one of the shaded regions above and need to move halfway toward A? Let the students dive into this as much as they want.

You can see after each roll of the die you're getting closer and closer to lying on the Sierpinski Triangle; if you were to throw away the first 100 rolls, you couldn't tell the difference. And once you're on the Sierpinski Triangle and you play the die-rolling game, you stay on the Sierpinski Triangle.

5. Conway's Game of Life

John Conway's Game of Life is an artificial life simulation that uses iteration. To start, you need an infinite two-dimensional grid of squares. Obviously, we can't have an infinite grid, but we could start with a very large one, say 40x40, so that for our simple examples, it will seem infinite.

The only input we give the game is that we initially mark some of the cells on the grid as being "alive." The other cells are "dead." Notice that each cell has eight neighbors (the 8 cells surrounding it).

Start as a group by placing a few pennies (say 5) on the large grid to mark the alive cells; this grid is called Generation Zero. The next generation (and all after this) is generated by a set of rules:

1. A live cell with fewer than two live neighbors dies because it's lonely.
2. A live cell with two or three neighbors stays alive.
3. A live cell with more than three live neighbors dies because it's overcrowded.
4. A dead cell with exactly three neighbors comes to life.
5. A dead cell without exactly three live neighbors stays dead.

Using these rules to all cells in Generation Zero simultaneously gives us Generation 1. What does it mean to use them simultaneously? It means that for a given cell, to determine if it's alive in Generation One, we only look at its neighbors in Generation Zero.

For example, in the grid below the two live cells (the colored ones) marked with white circles each have exactly one live neighbor. Therefore by rule 1, each of those cells will die in the next generation. The live cell marked with the white square has exactly two live neighbors, and thus by rule 3 it will stay alive in the next generation. And the dead cell marked with the black square has exactly three live neighbors, so by rule 4 it will come to life in the next generation. All other dead cells remain dead by rule 5. Below we see generations zero and one.

Introduction:

This iterative process begins with a design drawn on a sheet of graph paper, and following a list of simple rules, it evolves over time. Created by John Conway and popularized by Martin Gardner, this example of a cellular automaton has received much attention and study.

Objectives:

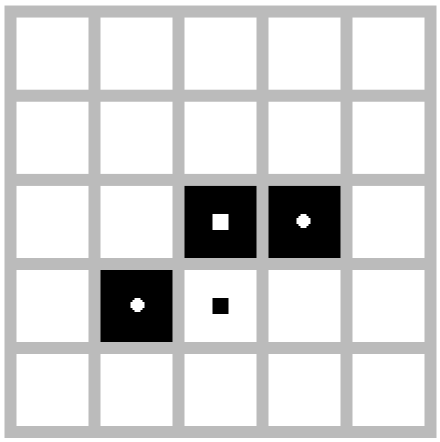
- To learn the rules of Conway's Game of Life
- To discover patterns

Materials:

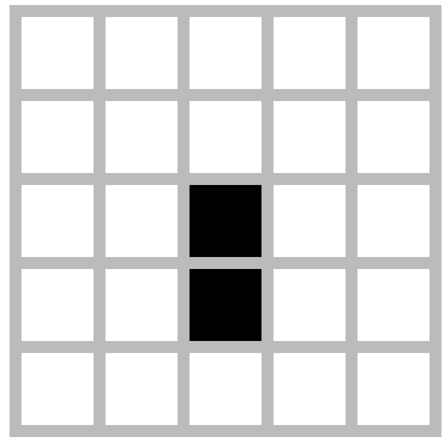
- Scratch paper and pencils
- Many sheets of graph paper
- One large sheet with a 40x40 grid large enough to hold pennies
- 100 pennies
- White board or black board on which to write the rules

Note to Leader:

This lesson is best if you have access to a computer lab where your students can use the internet. There is a nice article on Conway's Game of Life on Wikipedia, and that article contains links for programs that run the Game. It's easiest (and most fun) for students to see their creations evolve on a computer.

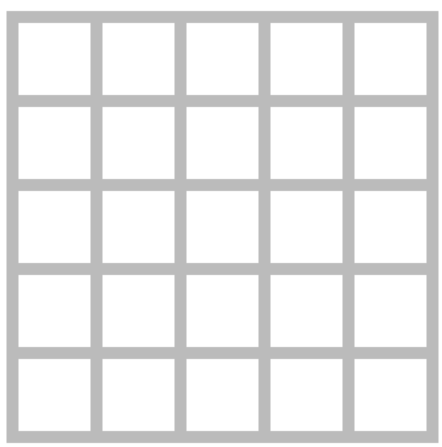


Generation Zero



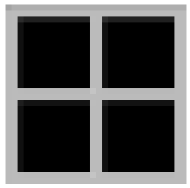
Generation One

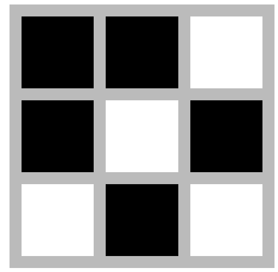
In Generation One, both live cells have exactly one live neighbor, so they both will die in the next generation. No dead cells have exactly three live neighbors so all dead cells remain dead. Thus Generation Two is the entirely dead grid, as seen below. No cells can come to life in an entirely dead grid, and thus will remain in this state for all future generations.



Generation Two

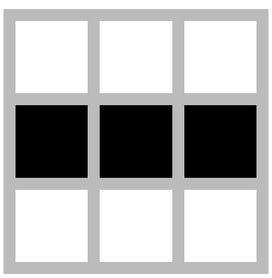
Now give it a try with your Generation Zero object. Practice on several different initial configurations so that the students understand how to apply the rules. Do they notice any neat patterns? Ask your students if they think it is possible for a configuration to have cells live forever. Let them play with this idea a while. Or ask if they think there is a configuration that remains unchanged forever. When a configuration remains unchanged forever, it is called a "still life." Can your students fine a still life? Here are two examples, called a Block and a Boat. See if your students can explain why they remain unchanged forever.

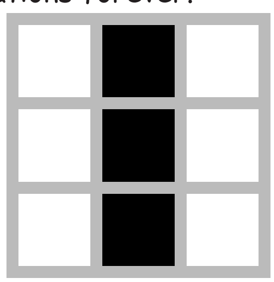


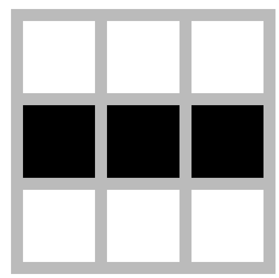


Block Boat

A second type of pattern occurs when a group of cells repeats itself. These patterns are called oscillators. The Blinker is a two-phase oscillator, meaning that it consists of two different configurations of cells. In Generation Zero, the middle cell in the middle row has two live neighbors, so it remains alive. The other two live cells each have one live neighbor, so they die. The middle cells in both the top and bottom rows each have exactly three live neighbors, so they come alive in the next generation. Then, when we apply the rules to Generation One, Generation Two is the same as Generation Zero, and thus this pattern oscillates between the two configurations forever.

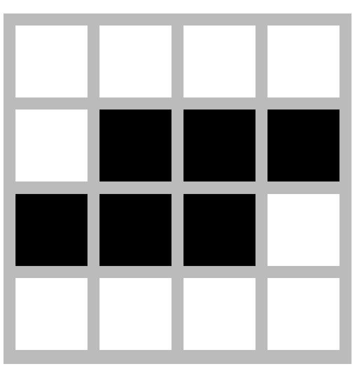


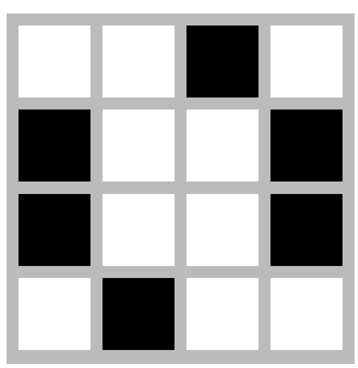


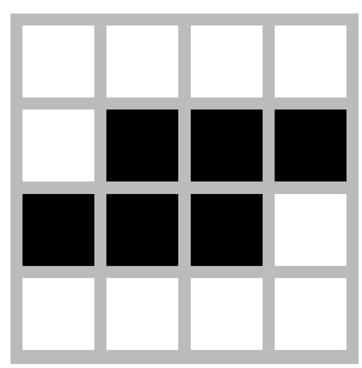


Blinker Generation Zero Blinker Generation One Blinker Generation Two

Another example of an oscillator is the Toad. The Toad is also a two-phase oscillator.

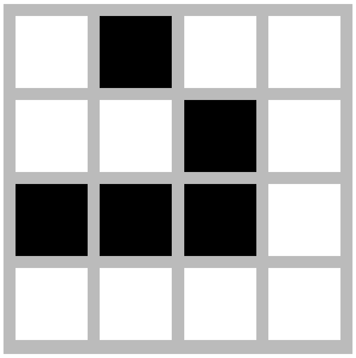




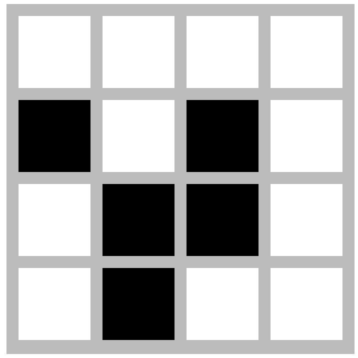


Toad Generation Zero Toad Generation One Toad Generation Two

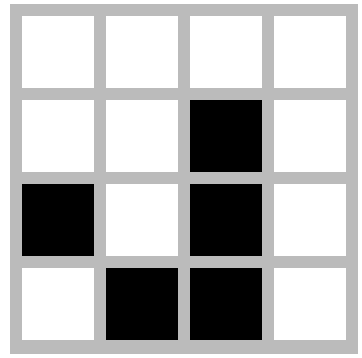
A third type of pattern occurs when a group of cells moves across the board forever. This is called a spaceship. The most famous example of a spaceship is the Glider.



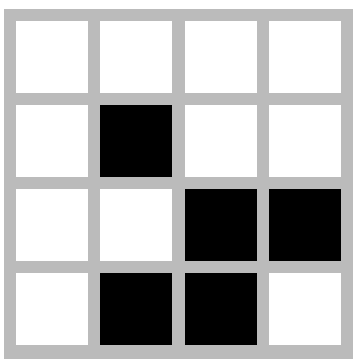
Glider Generation Zero



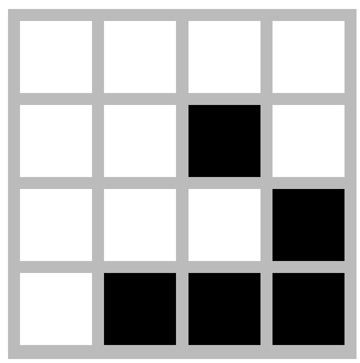
Glider Generation One



Glider Generation Two



Glider Generation Three



Glider Generation Four

After four transformations, the same initial configuration shifts down one cell and to the right one cell. In this way, the Glider moves across the plane forever.

Have the students play with their own versions of these patterns, and make sure they understand the rules for creating the next generation in each case. Have them start with their own simple configurations and see what happens after many generations.

If you are able to use a computer lab, let the students create their own more complicated initial conditions and let them run. They can even write out their names for an initial configuration and see how they evolve.

Taking it Further:
For a separate lesson, you can ask the students to create their own rules. What happens if they make it harder for a cell to die, or easier for a dead cell to come back to life? What if they want the future of a given cell to depend on more (or fewer) of the cell's neighbors? There are no right or wrong rules, and every set of rules will create interesting results.

Sets:
The Joy of Sets

1. The Basics of Sets

A set is simply a collection of zero or more objects. The objects can have traits in common or be totally unrelated. To write a set on paper, we use curly brackets, like this:

{x, y, z, dog, goat, -9.87}

Start by having the students write down some sets.

- Have them each write the set of their siblings. (For example, {Blaine, Blair, Jeff}; for an only child, this would be the empty set {}.)
- Have them write the set of the types of pets they have at home. (For example, this could look like {hamster, dog} even if they really have 7 hamsters and 3 dogs.)
- Give them a few minutes to write down the set of all states they've ever visited (you can choose as a group whether or not to count the one you live in).

The objects in the sets are called "elements." A set is just a collection of objects; there is no particular order to write the objects in the sets. That is, {1,2,3}={3,1,2}.

The two most basic ways of combining sets are union and intersection. The union of two sets A and B is the set containing all the elements that appear in either A or B. Union is denoted by ∪. For example,

{dog, cat, frog} ∪ {hamster, dog, goat}
= {dog, cat, frog, hamster, goat}.

Notice that even though dog is in both sets, it only appears once in the union.

Have the students practice unions by pairing up and taking unions of the sets they created before. After they've done that, ask them what they think the union of all their sets of pets would be. (It's just the collection of all different types of pets that appear in anyone's set.) Write the union of all siblings, pets, and states visited. Ask them to interpret what this union of all their sets means. (It means that, for example, all the states in the new list have been visited by at least one member of the group, or each type of pet listed is owned by at least one member of the group.)

Introduction:

Set Theory is a basic area of mathematics that has many intuitive ideas, and some quite surprising results. In this lesson the students will learn what sets are and how they combine.

Objectives:

- To learn what sets are
- To learn how sets combine

Materials:

- Scratch paper and pencils
- Yarn
- Chairs (one fewer than the number of students) and an open space to play a game

The other way of combining sets is by intersection. The intersection of sets A and B is the set containing all the elements that appear in both A and B. Intersection is denoted ∩. For example,
{dog, cat, frog} ∩ {hamster, dog, goat} = {dog}.

Again have the students practice by pairing up and taking intersections of sets that they formed before. Then ask them to write the intersection of all the students' siblings. Do they have any siblings in common? (Not unless they're all siblings themselves!) Ask them to write the intersection of all the students' pet lists and the intersection of all the students' states visited list. Are the intersections all empty? What would it mean if {cat} is the intersection of all the pet lists? (That everyone in the group has a cat.)

For a little more set practice, have them put the chairs (one fewer than the number of students in the group) in a circle. Have the students sit in the chairs, with one person standing in the middle. The student in the middle says something that applies to some or all of the students in the circle (for example, "is wearing jeans"), and any student to whom it applies has to get up and find a new chair. The student in the middle also tries to claim a chair so when all is said and done there will be someone new in the middle. The game continues for as long as you like. Meanwhile, you can be writing down the categories the students name and the sets of students to which is applies. When the students finish playing, you'll have a nice group of sets to use for examples.

Use two of those sets which have nothing in common, say "wearing a green shirt" and "wearing a blue shirt" and have the students wearing a green shirt stand with each other and the students wearing a blue shirt stand with each other. Wrap a piece of yarn around the students wearing a green shirt and a piece of yarn around the students wearing a blue shirt. Ask the students what the union of those two sets is. After they have answered, demonstrate by wrapping another piece of yarn around both sets.

Now use two sets which have one or more students in common (but which do not completely overlap!). Wrap each set individually in yarn, but there will be one or more students in both sets. Ask the students what the intersection of those two sets is. (They should see that any students wrapped in both groups is in the intersection.

Taking it Further:

Not all sets are finite. Can your students think of an example of an infinite set? Give them some time to work on that one. One example is the counting numbers. How do we write an infinite set? {1, 2, 3, 4, ...}, where the ellipses indicate that it goes on forever. Can the students think of another infinite set? (They could give many answers like the even numbers {2, 4, 6, 8, ...} or the prime numbers {2, 3, 5, 7, 11, ...} or the powers of 5 {1, 5, 25, 125, ...} or the set of all real numbers (which include the counting numbers, so they must be infinite!).

★★★★

2. How Many Subsets?

Given a finite set with, say, 10 elements in it, how many subsets of that set are there? A "subset" is a set all of whose elements are contained in the given set. We count the set with no elements in it, called the "empty set" {}, as a subset of every set since all the elements in the empty set are also in the given set. We also say that the given set is a subset of itself since it's a set all of whose elements are elements of the given set.

To answer this question, have students count all the subsets of a 2 element set, say {a,b}. (There should be four: {}, {a}, {b}, {a,b}.) Then count all the subsets of a 3-element set, say {a,b,c}. (There are 8: {}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}.) Then all the subsets of a 4-element set. (There are 16.) See if the students see a pattern. If they would like, they can write all the subsets of a 5-element set. (There are 32.) For an n-element set, there are 2^n subsets.

Why are there 2^n subsets? There are two different ways to think about this question. Consider listing a subset of the 4-element set {dog, cat, fish, goat}. For each of those four elements, we need to decide whether or not it's in the subset. Is "dog" in our subset? Yes or no? Is "cat" in our subset? Yes or no? Is "fish" in our subset? Yes or no? Finally, is "goat" in our subset? Yes or no? For each of those choices, we can make one of two decisions. There are four choices, so there are 2*2*2*2 different choices we could make, or 16 different subsets. Where's the empty set? It's when we choose "no" for all four questions. Where is the whole set? Remember, it, too, is a subset. It's when we choose "yes" for all four questions.

Another way to see how many subsets there are of, say, a 5-element set is to imagine that all the elements in a given 5-element set are fixed in an order, like (a, b, c, d, e). To make a subset, we could write down a 5-digit binary number, such as 10110, and wherever there's a 1, include that element of the set, and where there's a zero don't include the element. So 10110 would be the subset {a, c, d} and 01101 would be {b, c, e}. How many binary strings are there? 11111 in binary, which is 64!

Introduction:

Given a group of students, how many different subsets of students can you form?

It's great if you've covered How Many Locker Codes, and Binary: Talk Like a Computer.

Objectives:

- To count the number of subsets of a given finite set

Materials:

- Scratch paper and pencils

3. Are All Sets Created Equal?

How can you tell if two sets are the same size? Ask the students how they would tell. It's pretty easy if either or both of the sets are finite. If only one set is finite and the other is infinite, then they're not the same size. If they're both finite, you could just count the elements in the sets and see if they have the same number of elements. However, we're going to think about comparing the sizes of sets in a different way because the other way will make it easier when we talk about two infinite sets.

When children are little, before they can count, they can tell if two small sets are the same size by matching them up. Are there the same number of pots as lids in your kitchen cupboard? A little child would know the answer because he would just put lids on pots and find out if there are extra lids or pots. If they match up, the sets are the same size; if they don't match up, they're different sizes.

Are the set of boys and the set of girls the same size in your group? Have each girl hold hands with one boy (or stand across from, if that's more comfortable). Are there any girls or any boys not paired up? If everyone's paired up, then there are the same number of boys and girls. If anyone is unmatched, then the sets have different sizes.

Now find out amongst your students if the set of right hands is equal in size to the set of left hands. That is, is there a one-to-one relationship between right hands and left hands? (Each person could hold her own hand.) Is there a different one-to-one relationship? (Each person could match his right hand with someone else's left hand.) Is there another one-to-one relationship? (There are many different ones. How many would make a nice question for a different day!) As long as we find at least one one-to-one relationship, then the two sets have the same size.

Are the sets {cat, 7, goat} and (Al, Margaret, Bruno} the same size? Let the students discuss this and see if they can come up with more than one reason why. Yes, because they both have 3 elements, but we can also say

Introduction:

How can we compare the sizes of two different sets? It's pretty easy when they're finite, but much more interesting when they're infinite.

Objectives:

- To learn how mathematicians compare the sizes of sets
- To understand a one-to-one relationship
- To see that not all infinite sets are the same size

Materials:

- Scratch paper and pencils

yes because there's a one-to-one relationship, without needing to count the elements. The relationship

cat → Bruno
goat → Al
7 → Margaret

is a one-to-one relationship that matches of the elements of each set. There are other one-to-one relationships that we could form, but we only need one.

Now, what about infinite sets? This time we can't count the elements of the sets. (They're all infinite, but "infinity" is not a number, so we can't compare the numbers of elements in the sets.)

Consider the two infinite sets {1, 2, 3, 4, 5, ...} and {2, 4, 6, 8, ...}. Do these two sets have the same size? Let the students discuss this a while. We can't count the elements in the sets any more and say they're the same. Is there a way to make a one-to-one relationship between the two sets? We can't just create the relationship for the first few elements and stop, we need to say for any element in the first set what it matches up with in the second, and vice-versa. Is there a one-to-one relationship? (Yes, each number, say n, in the first set can get paired up with its double, 2n, in the second, and each number, say k, in the second is even, and it came from k/2 in the first.)

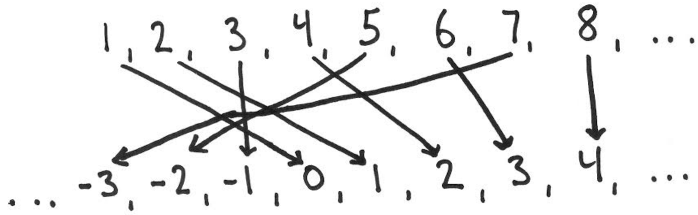
But wait! The second of those sets is a subset of the first! How can they be the same size? Well, we're talking about infinite sets, which have interesting properties! So an infinite set can be put in a one-to-one relationship with a subset of itself. Strange but true.

How about the sets {0, 1, 2, 3, 4, ...} and {1, 2, 3, 4, ...}? Are they the same size? Let the students wrestle with this a while. (Yes, a number n in the first set can be matched with n+1 in the second.)

How about the sets {1, 2, 3, ...} and {5, 25, 125, ...} where the second set is the powers of 5? (Yes, a number n in the first set can be matched with 5^n in the second.)

Taking it further:

What about the sets {1, 2, 3, ...} and {0, ±1, ±2, ±3, ...}, are they the same size? Let the students work on this for a while, then you can talk about the following one-to-one relationship:



4. Infinitely Much Mush

Infinity is a very odd concept; it's a property that sets can possess, but it doesn't act like a number, as the story below shows. Infinite sets have beautiful and interesting properties.

Read the story below together, and after each time the Lunch Lady Joanne is posed a question, stop reading and let the students try to figure out a solution. Give the students time to think and be creative -- mathematics requires creativity!

Joanne's Unbelievable Story

One day Joanne, the friendly lunch lady at Bridgeview Elementary, arrived for her hectic job. Bridgeview, where the motto is "Where the math is hard and the mush is free" has a policy of serving anyone lunch who requests it. Joanne had the job of lining up all the students at Bridgeview into a long line to get ready for lunch to be served. This was a hard job she did every day because there are infinitely many students at Bridgeview, and they all must stand quietly in line down Bridgeview's infinitely long hallway and wait their turns without fidgeting. Besides being friendly, Joanne was also good at her job and she had numbered all the positions in the hallway so that every child knew which position number he was in. She quickly had all the students lined up so that there was a first in line, a second, a third, a fourth, and so on, all the way down the hallway.

This day was extra challenging, though, because it was the day that the principal eats with the students, and she forgot to get in line when all the students were lining up. Where was she going to go? The students told her to go to the end of the line, but there IS no end of the line! How was the principal going to get her lunch? Joanne came to the rescue, though, and solved the problem.

How did she do it? Pause here and let the students work on finding a solution. Remember, there's a first place, and a second place, and so on, but there is no end of the line. So what place should the principal go? All the students need to eat!

Introduction:

This lesson presents an interesting example of how peculiar and fun infinite sets can be.

It's best if you've already done the lesson Prime Numbers: Building Blocks.

Objectives:

- To think about infinite sets and how they behave
- To realize that infinity is not a number

Materials:

- Scratch paper and pencils

Joanne got on the school microphone and she asked every student to take one step backwards in line, so that the first student became the second, and the second became the third, and so on all the way down the line. Then she let the principal get in the first spot in line.

Oh, but those weren't the end of her problems. It wasn't the day for just the principal to eat with the kids, but also all 27 teachers were supposed to eat, too, but they had forgotten to get in line. Now where should they go? She can't send them to the end of the line because there is no end of the line. She can't pull 27 students out of line and give those spots to the teachers because it's a state law that all children need to eat, so where would the 27 students go? But clever Joanne had a solution.

How did she do it?

Joanne grabbed the school microphone, and said, "Will everyone please move backward 27 places? That is, if you're standing on spot n, please move to spot n+27." All the children and the principal crabbed a little, but moved 27 spaces backwards in line. Now the principal was in spot 28 and the girl who used to be first in line before the principal came is now in spot 29, and so on, freeing up the first 27 spaces for the teachers.

Joanne was feeling a little tired, and the students hadn't yet even received their mush, but then she saw a bus pull up outside of the school. It seems the kitchen at Greenvalley Elementary across town had closed, and all the students at Greenvalley needed lunch. The problem was there were infinitely many of them, numbered, 1, 2, 3, ... How was she going to put them all in line?

How did she do it?

Joanne hadn't even had time to put down the microphone from the last announcement she made, when she again put it to her mouth and said, "Now that you're all in your new positions, I need you to move again." Infinitely many groans (or were those growls of hunger?) went up from the hallway. "Everyone, please, if you're in spot n, please move to spot 2n. So the person in spot 1 moves to 2, the person in spot 2 moves to 4, the person in spot 3 moves to 6, and so on all the way down the line. That puts the person in spot 137 into spot 274, so she has a ways to walk, but she's guaranteed a spot when she gets there. Please everyone move." Although they grumbled, everyone moved, and suddenly there were many spots open: all the odd numbered spots. So 1, 3, 5, 7, 9, ... were all open, and the first student from Greenvalley went into spot 1, the second went into spot 3, the third went into spot 5, and so on. They all were in line!

It had been a long day, and Joanne was thinking about going home, even though lunch had not yet been served. But she had one final hurdle to overcome. Outside, there suddenly appeared an entire caravan of buses, as far as the eye could see. There was a long line containing infinitely many buses. On each bus was a ball team from another town, all arriving for Bridgeview's free lunch. On each bus, the infinitude of ballplayers had numbers on their jerseys: 1, 2, 3,.... (The jersey with number 34,873,339,922 had very small font.) How was Joanne going to do it this time? How could she incorporate an infinity of infinities into the lunch line without telling anyone to go to the end of the line?

How did she do it?

She's amazing, that's how. She remembered that prime numbers are our friends. So with confidence and conviction, she got on the loudspeaker one final time and told all the Bridgeview students to move to stand on powers of 2. That is, the kid standing on spot n needed to move to spot 2^n. Then she got a blow horn and stood outside, assigning to each bus a different prime number as they pulled up. The first bus was assigned 3, the second 5, the third 7, the fourth 11, the fifth 13, the sixth 17, and so on. As the children exited the bus assigned the prime 3, the first child went to the spot 3^1=3 in line, the second went to spot 3^2=9 in line, the third went to spot 3^3=27 in line, and so on. As the children exited the bus assigned the prime 5, the first child went to the spot 5^1=5, the second went to spot 5^2=25, the third went to spot 5^3=125, and so on. Every child getting off every bus had a spot in line, and no one had to go hungry.

In fact, all the spots in line whose numbers were products of primes, like 10=2*5 and 12=3*2*2 and 14+2*7 and 15=3*5 (etc) had no one standing on them. There were infinitely many spaces left open in case someone else showed up! Exhausted, but with a smirk of satisfaction from a job well-done, Joanne removed her apron and hair net, and called it a day.

Geometry:
A Shapely Approach to
Mathematics

1. Snowflake Symmetry

Begin this lesson by giving the students an example or two things that have symmetry of reflection, then ask the students to identify things they see around them that have symmetry of reflection. For example, a wooden pencil with no writing on it might very well be symmetric about the line which goes the long way through its center. A perfectly square table top is symmetric about each line that divides the table in half (horizontal, vertical, and two diagonals). A child (who has symmetric hair) may very well look symmetric about a line vertically down the middle of her body. A hand, though, has no line of symmetry. Test out their ideas about symmetric objects by holding a mirror on the proposed line of symmetry and seeing if it appears that you see the whole object just by looking at half the object and its reflection.

Once the students are able to recognize symmetry, have them each fold a sheet of paper in half (horizontally or vertically) and show them how they can make cuts in the paper along the folded edge (without cutting off the entire folded edge), then unfold it. Do they see any symmetries of reflection? Depending on what shape they cut out, they may see one or more such lines. Have them present their findings to the others.

Now with a new sheet of paper, have them fold the paper in half as before, and half again. (Have them take time to be sure that their folds are neat and in halves.) Now they have two different folded sides. Have them cut a shape out of one of the folded sides and unfold it. What are their lines of symmetry? What if they cut shapes out of both sides? Now what are their lines of symmetry? Does it matter what shapes they cut out?

Repeat the folding with a new piece of paper, but this time, after two folds, have them fold a third time by bringing the two folded edges together to line up. Make sure there's a nice, crisp point in the middle of the folded paper so that the symmetries look nice. Again, have them make cuts on the sides, without completely cutting through one of the folded sides. Unfold and enjoy the symmetries!

Introduction:

An object has "symmetry of reflection" if there is a line across which it appears to have been reflected. That is, if you hold a mirror against the object at that line, the half of the object you can see along with its reflection in the mirror should look like the entire object.

Objectives:

- To understand what symmetry of reflection is
- To recognize when an object has such symmetry
- To create an object with symmetry of reflection

Materials:

- Blank white paper
- Scissors for each student
- Individual student mirrors or one larger mirror for your use

Taking it Further:

Which of the snowflakes you've created enjoy rotational symmetry? That is, imagine putting a pin in the very center of the snowflake, and spinning the snowflake around that point. Does the snowflake look like it did when it started before you get done turning it all the way around? Perhaps after 1/4 or 1/2 of the way around?

Other Ideas for Challenge Math

Geometry
2. An Introduction to Triangles
3. Angle Sums of a Triangle
4. Congruent Triangles
5. Regular Solids – create them with Polydrons or marshmallows and toothpicks; prove why there are only five platonic solids using the angle sum argument
6. Build mathematical geometric figures out of paper plates and staplers
7. Discuss Moebius Strips
8. Pythagoras – the theorem with proofs
9. Areas of Triangles
10. Pythagorean Triples
11. Circles and Pi
12. Flexagons

Math in Society—to help students see examples of math around them
1. Bowling Scores
2. Voting Theory – how to be fair in tallying elections
3. Fair Division – cake cutting and goods distribution
4. Guarding an Art Gallery

Math in Language and Literature –great books for children to get them thinking mathematically!
1. Talking Points – have a list of mathematical ideas that students have to describe to each other in words without writing anything down, like giving them a geometric figure to get their partner to draw just by using words, no hand gesturing and no writing
2. Martin Gardner's Word Ladders
3. Hooray for Diffindoofer Day
4. Math Curse
5. Franklin and his Magic Squares – maybe a separate lesson on magic squares
6. What's Smaller than a Pygmy Shrew?
7. Is a Blue Whale the Largest Thing there is?
8. On Beyond a Million
9. Ten Apples Up on Top! – and counting by tens
10. My Little Sister Ate One Hare – and triangular numbers
11. The King's Chessboard

Mathematical Diversions – fun games to play that keep young minds mathematically active

1. Dots and Lines
2. 24
3. Set
4. Mastermind
5. Rush Hour
6. Skunk
7. Polydrons
8. Fractal Paperfolding
9. Logic Puzzles – including ones using Venn Diagrams
10. YouTube videos by Vi Hart
11. YouTube videos by Numberphile