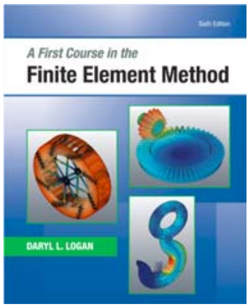
Chapter 3b – Development of Truss Equations



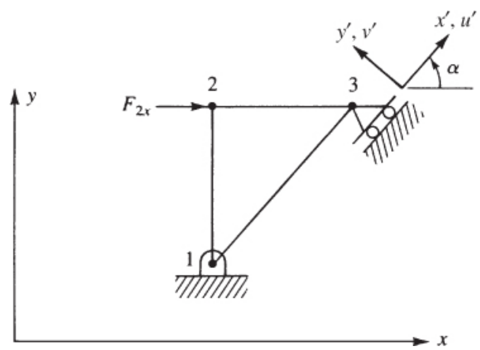
Learning Objectives

- To derive the stiffness matrix for a bar element.
- To illustrate how to solve a bar assemblage by the direct stiffness method.
- To introduce guidelines for selecting displacement functions.
- To describe the concept of transformation of vectors in two different coordinate systems in the plane.
- To derive the stiffness matrix for a bar arbitrarily oriented in the plane.
- To demonstrate how to compute stress for a bar in the plane.
- To show how to solve a plane truss problem.
- To develop the transformation matrix in three-dimensional space and show how to use it to derive the stiffness matrix for a bar arbitrarily oriented in space.
- To demonstrate the solution of space trusses.

Stiffness Matrix for a Bar Element

Inclined, or Skewed Supports

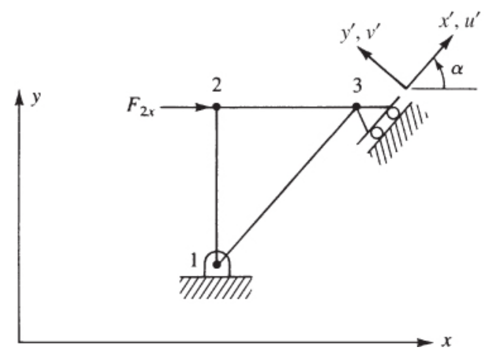
If a support is inclined, or skewed, at some angle α for the global *x* axis, as shown below, the boundary conditions on the displacements are not in the global *x*-*y* directions but in the *x'*-*y'* directions.



Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

We must transform the local boundary condition of $\boldsymbol{v'_3} = 0$ (in local coordinates) into the global *x-y* system.



Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

Therefore, the relationship between of the components of the displacement in the local and the global coordinate systems at node 3 is:

$$\begin{Bmatrix} u'_3 \\ v'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

We can rewrite the above expression as:

$$\{d'_3\} = [t_3]\{d_3\} \qquad [t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

We can apply this sort of transformation to the entire displacement vector as:

$$\{d'\} = [T_1]\{d\} \quad or \quad \{d\} = [T_1]^T\{d'\}$$

Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

Where the matrix $[T_1]^T$ is:

$$[T_1]^T = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

Both the identity matrix $[I]$ and the matrix $[t_3]$ are 2 x 2 matrices.

The force vector can be transformed by using the same transformation.

$$\{f'\} = [T_1]\{f\}$$

In global coordinates, the force-displacement equations are:

$$\{f\} = [K]\{d\}$$

Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

Applying the skewed support transformation to both sides of the equation gives:

$$\{f\} = [K]\{d\} \quad \Rightarrow \quad [T_1]\{f\} = [T_1][K]\{d\}$$

By using the relationship between the local and the global displacements, the force-displacement equations become:

$$\{d\} = [T_1]^T\{d'\} \qquad \Rightarrow \quad \{f'\} = [T_1][K][T_1]^T\{d'\}$$

Therefore the global equations become:

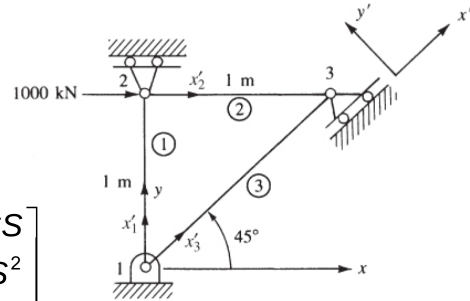
$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = [T_1][K][T_1]^T \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u'_3 \\ v'_3 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

Consider the plane truss shown below. Assume E = 210 GPa, A = 6 x 10^{-4} m^2 for element 1 and 2, and A = $\sqrt{2}$(6 x 10^{-4})m^2 for element 3.

Determine the stiffness matrix for each element.



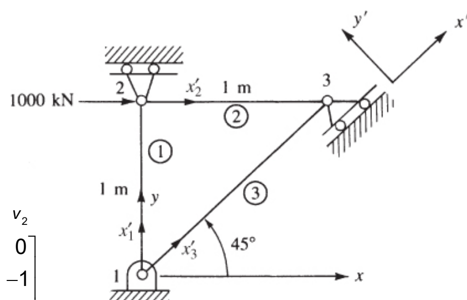
$$k = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The global elemental stiffness matrix for ***element 1*** is:

$$k = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$



$$\cos\theta^{(1)} = 0 \qquad \sin\theta^{(1)} = 1$$

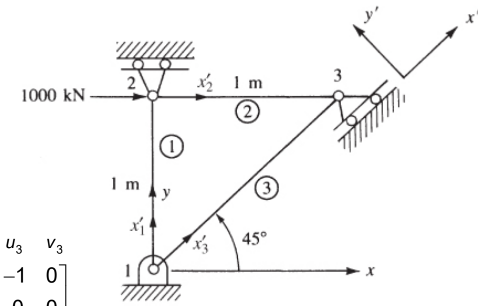
$$\mathbf{k}^{(1)} = \frac{(210\times 10^{6} kN/m^2)(6\times 10^{-4} m^{-2})}{1\,m} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The global elemental stiffness matrix for ***element 2*** is:

$$k = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$



$$\cos\theta^{(2)} = 1 \qquad \sin\theta^{(2)} = 0$$

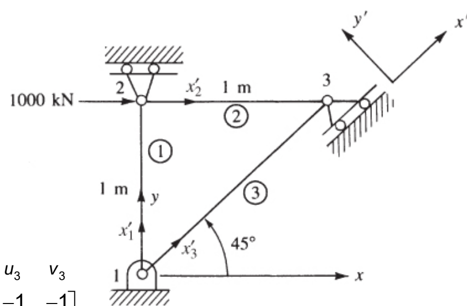
$$\mathbf{k}^{(2)} = \frac{(210\times 10^6 kN/m^2)(6\times 10^{-4} m^2)}{1\,m} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The global elemental stiffness matrix for ***element 3*** is:

$$k = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$



$$\cos\theta^{(3)} = \frac{\sqrt{2}}{2} \qquad \sin\theta^{(3)} = \frac{\sqrt{2}}{2}$$

$$\mathbf{k}^{(3)} = \frac{(210\times 10^6 kN/m^2)(6\sqrt{2}\times 10^{-4} m^2)}{2\sqrt{2}\,m} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

Using the direct stiffness method, the global stiffness matrix is:

$$\mathbf{K} = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

We must transform the global displacements into local coordinates. Therefore the transformation $[T_1]$ is:

$$[T_1] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{array}\right]$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The first step in the matrix transformation to find the product of $[T_1][K]$.

$$\begin{array}{ccc} & [T_1] & [K] \\ [T_1][K] = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} & \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{array}\right] & \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.414 & 0.707 \\ 0 & 0 & 0.707 & 0 & -0.707 & 0 \end{bmatrix} \end{array}$$

$$[T_1][K] = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.414 & 0.707 \\ 0 & 0 & 0.707 & 0 & -0.707 & 0 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The next step in the matrix transformation to find the product of $[T_1][K][T_1]^T$.

$$[T_1][K][T_1]^T = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \underbrace{\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.414 & 0.707 \\ 0 & 0 & 0.707 & 0 & -0.707 & 0 \end{bmatrix}}_{[T_1][K]} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{[T_1]^T}$$

$$[T_1][K][T_1]^T = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\ 0 & 0 & 0.707 & 0 & -0.5 & 0.5 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The displacement boundary conditions are: $u_1 = v_1 = v_2 = v'_3 = 0$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\ 0 & 0 & 0.707 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u'_3 \\ v'_3 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

By applying the boundary conditions the global force-displacement equations are:

$$1{,}260\times 10^{5}\, {}^{N}\!/_{m}\begin{bmatrix} 1 & -0.707 \\ -0.707 & 1.5 \end{bmatrix}\begin{Bmatrix} u_2 \\ u'_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} = 1{,}000\ kN \\ F'_{3x} = 0 \end{Bmatrix}$$

Solving the equation gives: $u_2 = 11.91\, mm \qquad u'_3 = 5.61\, mm$

Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

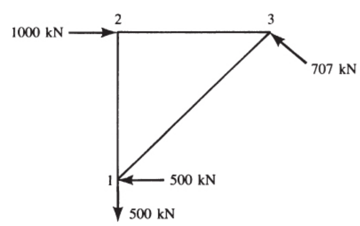
The global nodal forces are calculated as: $\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = [T_1][K][T_1]^T \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u'_3 \\ v'_3 \end{Bmatrix}$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = 1{,}260\times 10^{2}\, {}^{N}\!/_{mm}\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\ 0 & 0 & 0.707 & 0 & -0.5 & 0.5 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 11.91 \\ 0 \\ 5.61 \\ 0 \end{Bmatrix}$$

Therefore:

$F_{1x} = -500\ kN \qquad F_{1y} = -500\ kN$

$F_{2y} = 0 \qquad F'_{3y} = 707\ kN$



Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

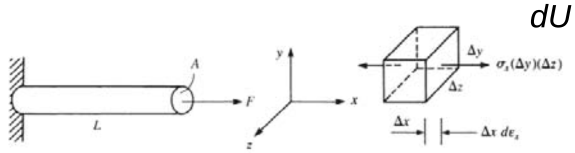
Let's derive the equations for a bar element using the principle of minimum potential energy.

The total potential energy, π_p, is defined as the sum of the internal strain energy U and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega$$

The differential internal work (strain energy) dU in a one-dimensional bar element is:

$$dU = \sigma_x(\Delta y)(\Delta z)(\Delta x)d\varepsilon_x$$



Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

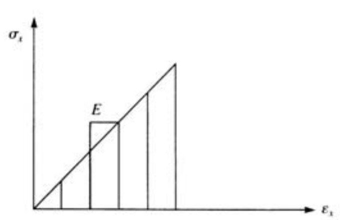
If we let the volume of the element approach zero, then:

$$dU = \sigma_x d\varepsilon_x dV$$

Summing the differential energy over the whole bar gives:

$$U = \int_V \left\{ \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \right\} dV = \int_V \left\{ \int_0^{\varepsilon_x} E\varepsilon_x d\varepsilon_x \right\} dV = \int_V \frac{1}{2} E\varepsilon_x^{\,2} dV$$

For a linear-elastic material (Hooke's law) as shown below:



$$\sigma_x = E\varepsilon_x \qquad U = \int_V \frac{1}{2}\sigma_x\varepsilon_x dV$$

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

The internal strain energy statement becomes

$$U = \frac{1}{2}\int_V \sigma_x \varepsilon_x dV$$

The potential energy of the external forces is:

$$\Omega = -\int_V X_{\mathbf{b}} u \, dV - \int_S T_{\mathbf{x}} u_s \, dS - \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{M}} f_{\mathbf{ix}} u_{\mathbf{i}}$$

where X_b is the body force (force per unit volume), T_x is the traction (force per unit area), and f_{ix} is the nodal concentrated force. All of these forces are considered to act in the local x direction.

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

1. Formulate an expression for the total potential energy.

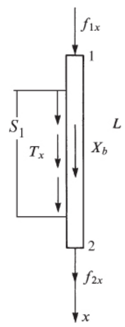
2. Assume a displacement pattern.

3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to the displacement parameters.

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Consider the following bar element, as shown below:



$$\pi_p = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, dx - f_{1x}u_1 - f_{2x}u_2 - \int_V X_{\mathbf{b}} u \, dV - \int_S T_{\mathbf{x}} u_s \, dS$$

We can approximate the axial displacement as:

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Using the stress-strain relationships, the axial strain is:

$$\varepsilon_x = \frac{du}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where ***N_1*** and ***N_2*** are the interpolation functions gives as:

$$\varepsilon_x = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \{\varepsilon_x\} = [B]\{d\}$$

$$B = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

The axial stress-strain relationship is: $\{\sigma_x\} = [D]\{\varepsilon_x\}$

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

For the one-dimensional stress-strain relationship [D] = [E] where E is the modulus of elasticity.

Therefore, stress can be related to nodal displacements as:

$$\{\sigma_x\} = [D][B]\{d\}$$

The total potential energy expressed in matrix form is:

$$\pi_p = \frac{A}{2}\int_0^L \{\sigma_x\}^T \{\varepsilon_x\}\, dx - \{d\}^T \{P\} - \int_V \{u\}^T \{X_{\mathbf{b}}\}\, dV - \int_S \{u\}^T \{T_{\mathbf{x}}\}\, dS$$

where {P} represented the concentrated nodal loads.

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

If we substitute the relationship between $\hat{u}$ and $\hat{d}$ into the energy equations we get:

$$\pi_p = \frac{A}{2}\int_0^L \overbrace{\{d\}^T [B]^T [D]^T}^{\sigma_x} \overbrace{[B]\{d\}}^{\varepsilon_x}\, dx - \{d\}^T \{P\}$$

$$- \int_V \{d\}^T [N]^T \{X_{\mathbf{b}}\}\, dV - \int_S \{d\}^T [N_s]^T \{T_{\mathbf{x}}\}\, dS$$

In the above expression for potential energy π_p is a function of the **d**, that is: $\pi_p = \pi_p(u_1,\ u_2)$.

However, [B] and [D] and the nodal displacements u are not a function of x.

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Integration the energy expression with respect to *x* gives:

$$\pi_p = \frac{AL}{2}\{d\}^T [B]^T [D]^T [B]\{d\} - \{d\}^T \{f\}$$

where

$$\{f\} = \{P\} + \int_V [N]^\mathbf{T} \{X_\mathbf{b}\}\, dV + \int_S [N]^\mathbf{T} \{X_\mathbf{b}\} dS$$

We can define the surface tractions and body-force matrices as:

$$\{f_\mathbf{s}\} = \int_S [N]^\mathbf{T} \{T_x\} dS \qquad \{f_\mathbf{b}\} = \int_V [N]^\mathbf{T} \{X_\mathbf{b}\}\, dV$$

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Minimization of π_p with respect to each nodal displacement requires that:

$$\frac{\partial \pi_p}{\partial u_1} = 0 \qquad \frac{\partial \pi_p}{\partial u_2} = 0$$

For convenience, let's define the following

$$\{U^*\} = \{d\}^T [B]^T [D]^T [B]\{d\}$$

$$\{U^*\} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Simplifying the above expression gives:

$$U^* = \frac{E}{L^2}\left(u_1^2 - 2u_1u_2 + u_2^2\right)$$

The loading on a bar element is given as:

$$\{d\}^T\{f\} = u_1 f_{1x} + u_2 f_{2x}$$

Therefore, the minimum potential energy is:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{AE}{2L}\left(2u_1 - 2u_2\right) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{AE}{2L}\left(-2u_1 + 2u_2\right) - f_{2x} = 0$$

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

The above equations can be written in matrix form as:

$$\frac{\partial \pi_p}{\partial (d)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} f_{1x} \\ f_{2x} \end{pmatrix} = 0$$

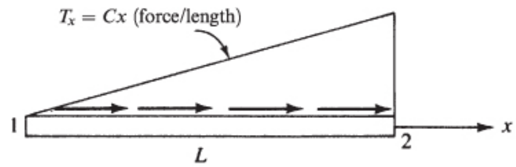
The stiffness matrix for a bar element is: $[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

This form of the stiffness matrix obtained from the principle of minimum potential energy is identical to the stiffness matrix derived from the equilibrium equations.

Stiffness Matrix for a Bar Element

Example 10 - Bar Problem

Consider the bar shown below:



The energy equivalent nodal forces due to the distributed load are:

$$\{f_0\} = \int_S [N]^{\mathrm{T}} \{T_x\} dS \qquad \{f_0\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \int_0^L \begin{Bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \{Cx\} dx$$

Stiffness Matrix for a Bar Element

Example 10 - Bar Problem

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \int_0^L \begin{Bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \{Cx\} dx = \begin{Bmatrix} \left| \dfrac{Cx^2}{2} - \dfrac{Cx^3}{3L} \right|_0^L \\ \left| \dfrac{Cx^3}{3L} \right|_0^L \end{Bmatrix} = \begin{Bmatrix} \dfrac{CL^2}{6} \\ \dfrac{CL^2}{3} \end{Bmatrix}$$

The total load is the area under the distributed load curve, or:

$$F = \frac{1}{2}(L)(CL) = \frac{CL^2}{2}$$

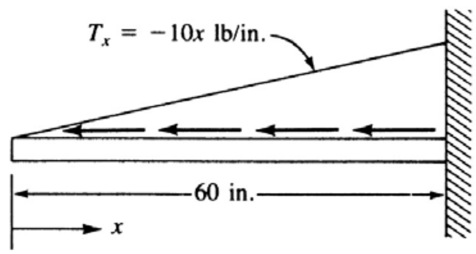
The equivalent nodal forces for a linearly varying load are:

$$f_{1x} = \frac{F}{3} = \frac{1}{3}\text{of the total load} \qquad f_{2x} = \frac{2F}{3} = \frac{2}{3}\text{of the total load}$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

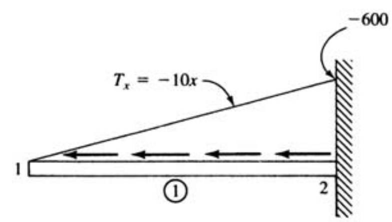
Consider the axially loaded bar shown below. Determine the axial displacement and axial stress. Let E = 30 x 10^6 *psi*, A = 2 in^2, and L = 60 *in*. Use (a) one and (b) two elements in the finite element solutions.



Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The one-element solution:



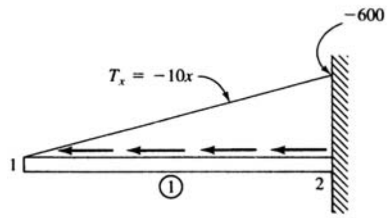
The distributed load can be converted into equivalent nodal forces using:

$$\{F_0\} = \int_S [N]^T \{T_x\} dS \qquad \{F_0\} = \begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \int_0^L \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \{-10x\} dx$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The one-element solution:



$$\begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \int_0^L \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \{-10x\}\, dx = \begin{Bmatrix} -\frac{10L^2}{2} + \frac{10L^2}{3} \\ -\frac{10L^2}{3} \end{Bmatrix} = \begin{Bmatrix} -\frac{10L^2}{6} \\ -\frac{10L^2}{3} \end{Bmatrix}$$

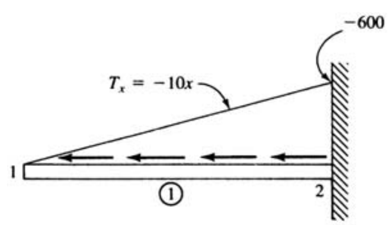
$$\begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \begin{Bmatrix} -6{,}000\ lb \\ -12{,}000\ lb \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The one-element solution:

$$\mathbf{k}^{(1)} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



The element equations are:

$$10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -6{,}000 \\ R_{2x} - 12{,}000 \end{Bmatrix} \qquad u_1 = -0.006\ in$$

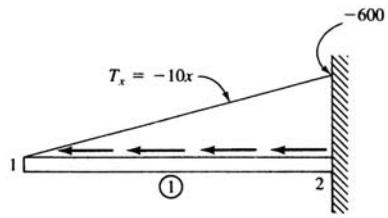
The second equation gives:

$$-10^6 (u_1) = R_{2x} - 12{,}000 \qquad \Rightarrow \qquad R_{2x} = 18{,}000\, lb$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The one-element solution:



The axial stress-strain relationship is: $\{\sigma_x\} = [D]\{\varepsilon_x\}$

$$\{\sigma_x\} = E[B]\{d\}$$

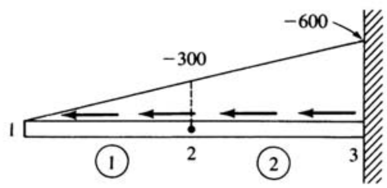
$$= E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = E\left(\frac{u_2 - u_1}{L}\right)$$

$$= 30 \times 10^6 \left(\frac{0 + 0.006}{60}\right) = 3{,}000 \ psi \ (T)$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



The distributed load can be converted into equivalent nodal forces.

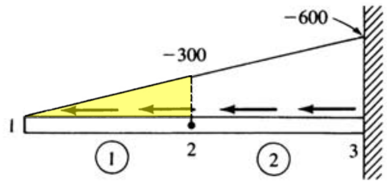
For element 1, the total force of the triangular-shaped distributed load is:

$$\frac{1}{2}(30 \ in.)(300 \ {}^{lb}\!/_{in}) = -4{,}500 \ lb$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



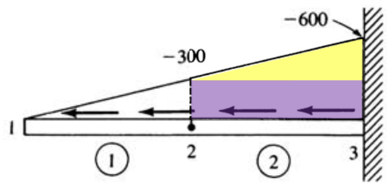
Based on equations developed for the equivalent nodal force of a triangular distributed load, develop in the one-element problem, the nodal forces are:

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{3}(4{,}500) \\ -\frac{2}{3}(4{,}500) \end{Bmatrix} = \begin{Bmatrix} -1{,}500\ lb \\ -3{,}000\ lb \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



For element 2, the applied force is in two parts: a triangular-shaped distributed load and a uniform load. The uniform load is: $(30\ in)(300\ lb / in) = -9{,}000\ lb$

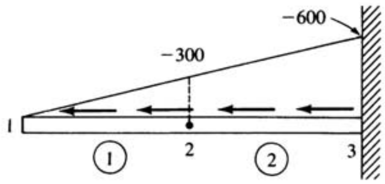
The nodal forces for element 2 are:

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -\left[\frac{1}{2}(9{,}000) + \frac{1}{3}(4{,}500)\right] \\ -\left[\frac{1}{2}(9{,}000) + \frac{2}{3}(4{,}500)\right] \end{Bmatrix} = \begin{Bmatrix} -6{,}000\ lb \\ -7{,}500\ lb \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



The final nodal force vector is:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} + f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1{,}500 \\ -9{,}000 \\ R_{3x} - 7{,}500 \end{Bmatrix}$$

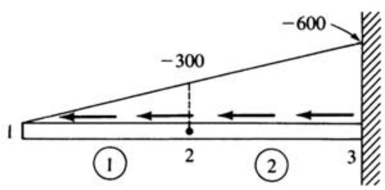
The element stiffness matrices are:

$$\mathbf{k}^{(1)} = \mathbf{k}^{(2)} = \frac{2AE}{L} \begin{array}{c} \begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



The assembled global stiffness matrix is:

$$\mathbf{K} = 2 \times 10^6 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

element 1

element 2

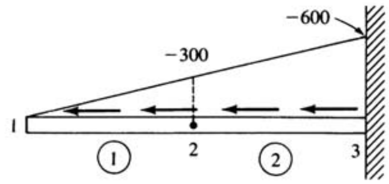
The assembled global force-displacement equations are:

$$2 \times 10^6 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1{,}500 \\ -9{,}000 \\ R_{3x} - 7{,}500 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



After the eliminating the row and column associated with u_{3x}, we get:

$$2\times10^6\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}-1{,}500\\ -9{,}000\end{Bmatrix}$$

Solving the equation gives: $u_1=-0.006\ in$

$u_2=-0.00525\ in$

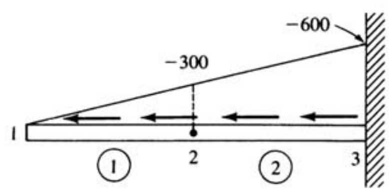
Solving the last equation gives:

$$-2\times10^6u_2=R_{3x}-7{,}500 \qquad \Rightarrow \qquad R_{3x}=18{,}000$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



The axial stress-strain relationship is:

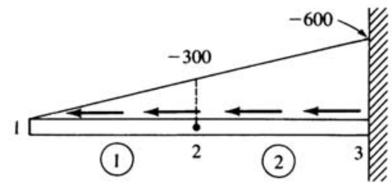
$$\sigma_x^{(1)}=E\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{2x}\end{Bmatrix}$$

$$=E\begin{bmatrix}-\frac{1}{30} & \frac{1}{30}\end{bmatrix}\begin{Bmatrix}-0.006\\ -0.00525\end{Bmatrix}=750\ psi\ \ (T)$$

Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:



The axial stress-strain relationship is:

$$\sigma_x^{(2)} = E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$= E\begin{bmatrix} -\frac{1}{30} & \frac{1}{30} \end{bmatrix}\begin{Bmatrix} -0.00525 \\ 0 \end{Bmatrix} = 5{,}250\ psi\ \ (T)$$

Stiffness Matrix for a Bar Element

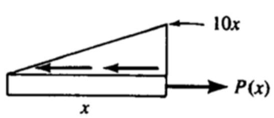
Comparison of Finite Element Solution to Exact Solution

In order to be able to judge the accuracy of our finite element models, we will develop an exact solution for the bar element problem.

The exact solution for the displacement may be obtained by:

$$u = \frac{1}{AE}\int_0^L P(x)dx$$

where the force ***P*** is shown on the following free-body diagram.



$$P(x) = \frac{1}{2}\left[x(10x)\right] = 5x^2$$

Stiffness Matrix for a Bar Element

Comparison of Finite Element Solution to Exact Solution

Therefore:

$$u = \frac{1}{AE}\int_0^L P(x)dx \qquad u = \frac{1}{AE}\int_o^x 5x^2 dx = \frac{5x^3}{3AE} + C_1$$

Applying the boundary conditions:

$$u(L) = 0 = \frac{5x^3}{3AE} + C_1 \quad \Rightarrow \quad C_1 = -\frac{5L^3}{3AE}$$

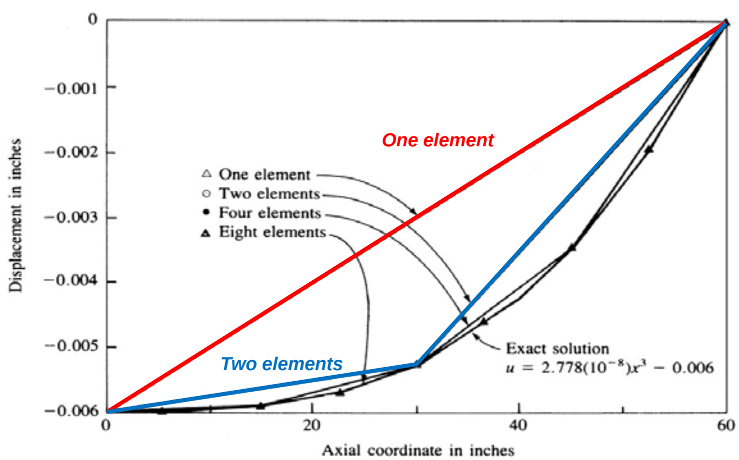
The exact solution for axial displacement is:

$$u(L) = \frac{5}{3AE}\left(x^3 - L^3\right) \qquad \sigma(x) = \frac{P(x)}{A} = \frac{5x^2}{A}$$

Stiffness Matrix for a Bar Element

Comparison of Finite Element Solution to Exact Solution

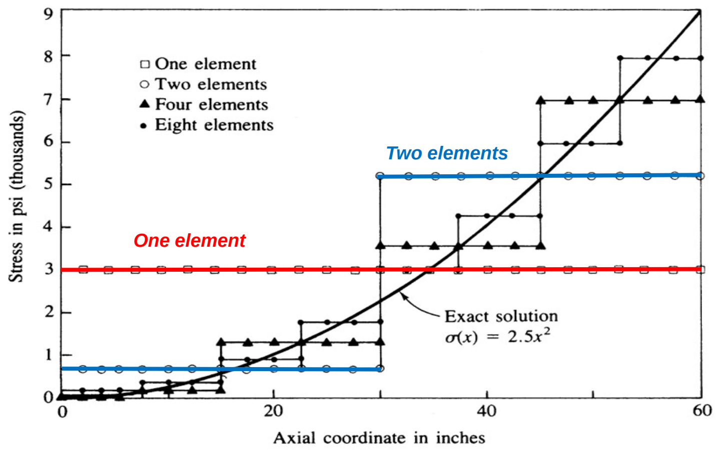
A plot of the exact solution for displacement as compared to several different finite element solutions is shown below.



Stiffness Matrix for a Bar Element

Comparison of Finite Element Solution to Exact Solution

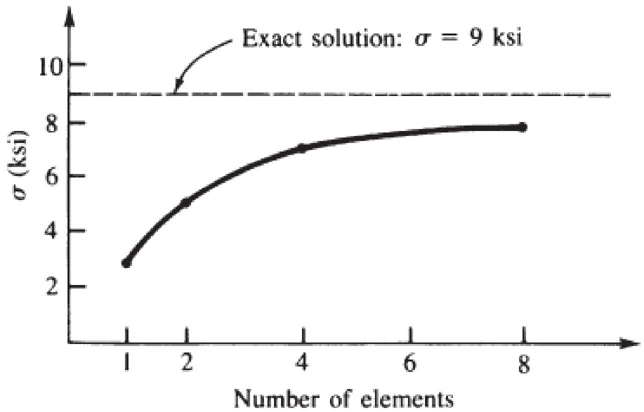
A plot of the exact solution for axial stress as compared to several different finite element solutions is shown below.



Stiffness Matrix for a Bar Element

Comparison of Finite Element Solution to Exact Solution

A plot of the exact solution for axial stress at the fixed end ($x = L$) as compared to several different finite element solutions is shown below.



Stiffness Matrix for a Bar Element

Galerkin's Residual Method and Its Application to a One-Dimensional Bar

There are a number of weighted residual methods.

However, the Galerkin's method is more well-known and will be the only weighted residual method discussed in this course.

In weighted residual methods, a trial or approximate function is chosen to approximate the independent variable (in our case, displacement) in a problem defined by a differential equation.

The trial function will not, in general, satisfy the governing differential equation.

Therefore, the substitution of the trial function in the differential equation will create a residual over the entire domain of the problem.

Stiffness Matrix for a Bar Element

Galerkin's Residual Method and Its Application to a One-Dimensional Bar

Therefore, the substitution of the trial function in the differential equation will create a residual over the entire domain of the problem.

$$\int_V R dV = \text{minimum}$$

In the residual methods, we require that a weighted value of the residual be a minimum over the entire domain of the problem.

The weighting function *W* allows the weighted integral of the residuals to go to zero.

$$\int_V RW \, dV = 0$$

Stiffness Matrix for a Bar Element

Galerkin's Residual Method and Its Application to a One-Dimensional Bar

Using Galerkin's weighted residual method, we require the weighting functions to be the interpolation functions N_i. Therefore:

$$\int_V RN_i \, dV = 0 \qquad i = 1, 2, \cdots, n$$

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Let's derive the bar element formulation using Galerkin's method. The governing differential equation is:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) = 0$$

Applying Galerkin's method we get:

$$\int_0^L \frac{d}{dx}\left(AE\frac{du}{dx}\right)N_i \, dx = 0 \qquad i = 1, 2, \cdots, n$$

We now apply integration by parts using the following general formula:

$$\int r ds = rs - \int s dr$$

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

If we assume the following:

$$\boxed{\begin{aligned} r &= N_i \\ ds &= \frac{d}{dx}\left(AE\frac{du}{dx}\right)dx \end{aligned}} \Rightarrow \begin{aligned} dr &= \frac{dN_i}{dx}dx \\ s &= AE\frac{du}{dx} \end{aligned}$$

then integration by parts gives:

$$\int r\,ds = rs - \int s\,dr$$

$$\int_0^L \frac{d}{dx}\left(AE\frac{du}{dx}\right)N_i\,dx = \left|N_i AE\frac{du}{dx}\right|_0^L - \int_0^L AE\frac{du}{dx}\frac{dN_i}{dx}dx = 0$$

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Recall that:

$$\frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 \qquad \frac{du}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Our original weighted residual expression, with the approximation for u becomes:

$$AE\int_0^L \frac{dN_i}{dx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}dx \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \left|N_i AE\frac{du}{dx}\right|_0^L$$

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Substituting $\boldsymbol{N_1}$ for the weighting function $\boldsymbol{N_i}$ gives:

$$AE\int_0^L \frac{dN_1}{dx}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}dx\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \left|N_1 AE\frac{du}{dx}\right|_0^L$$

$$AE\int_0^L \begin{bmatrix}-\frac{1}{L}\end{bmatrix}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}dx\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \frac{AEL}{L^2}(u_1 - u_2)$$

$$\left|N_1 AE\frac{du}{dx}\right|_0^L = AE\left.\frac{du}{dx}\right|_{x=0} = AE\varepsilon_x\big|_{x=0} = A\sigma_x\big|_{x=0} = f_{1x}$$

$$\Rightarrow \quad \frac{AE}{L}(u_1 - u_2) = f_{1x}$$

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Substituting $\boldsymbol{N_2}$ for the weighting function $\boldsymbol{N_i}$ gives:

$$AE\int_0^L \frac{dN_2}{dx}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}dx\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \left|N_2 AE\frac{du}{dx}\right|_0^L$$

$$AE\int_0^L \begin{bmatrix}\frac{1}{L}\end{bmatrix}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}dx\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \frac{AEL}{L^2}(-u_1 + u_2)$$

$$\left|N_2 AE\frac{du}{dx}\right|_0^L = AE\left.\frac{du}{dx}\right|_{x=L} = AE\varepsilon_x\big|_{x=L} = A\sigma_x\big|_{x=L} = f_{2x}$$

$$\Rightarrow \quad \frac{AE}{L}(-u_1 + u_2) = f_{2x}$$

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Writing the last two equations in matrix form gives:

$$\boxed{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}}$$

This element formulation is identical to that developed from equilibrium and the minimum potential energy approach.

Symmetry and Bandwidth

In this section, we will introduce the concepts of symmetry to reduce the size of a problem and of banded-symmetric matrices and bandwidth.

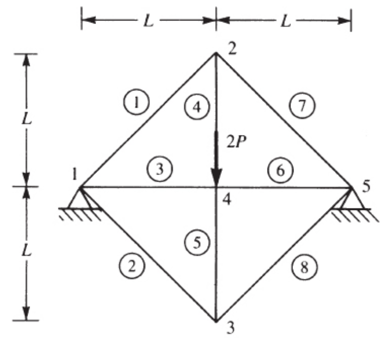
In many instances, we can use symmetry to facilitate the solution of a problem.

Symmetry means correspondence in size, shape, and position of loads; material properties; and boundary conditions that are mirrored about a dividing line or plane.

Use of symmetry allows us to consider a reduced problem instead of the actual problem. Thus, the order of the total stiffness matrix and total set of stiffness equations can be reduced.

Symmetry and Bandwidth - Example 1

Solve the plane truss problem shown below. The truss is composed of eight elements and five nodes.

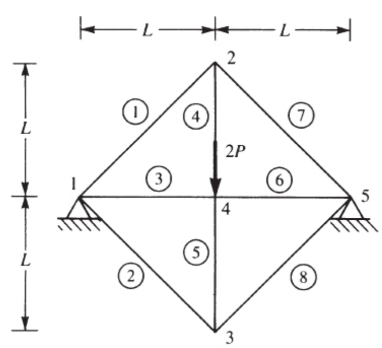


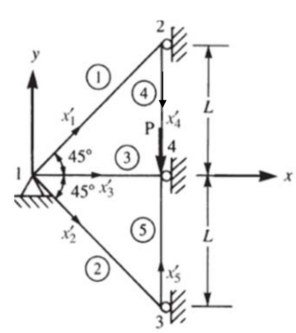
A vertical load of 2P is applied at node 4. Nodes 1 and 5 are pin supports. Bar elements 1, 2, 7, and 8 have an axial stiffness of ***AE*** and bars 3, 4, 5, and 6 have an axial stiffness of ***AE***.

Symmetry and Bandwidth - Example 1

In this problem, we will use a plane of symmetry.

The vertical plane perpendicular to the plane truss passing through nodes 2, 4, and 3 is the plane of symmetry because identical geometry, material, loading, and boundary conditions occur at the corresponding locations on opposite sides of this plane.

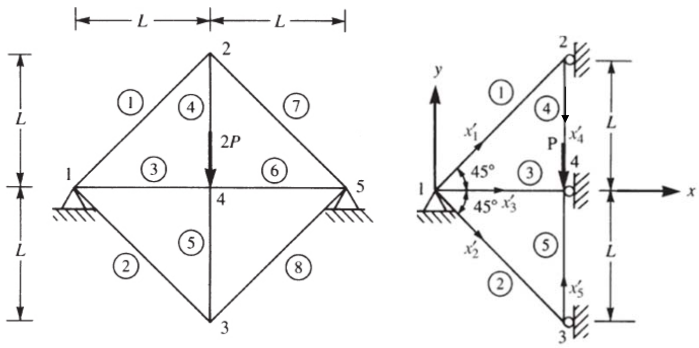




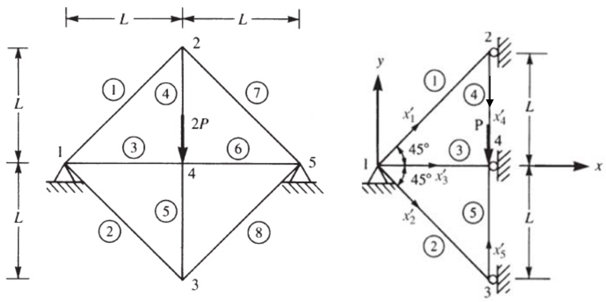
Symmetry and Bandwidth - Example 1

For loads such as ***2P***, occurring in the plane of symmetry, one-half of the total load must be applied to the reduced structure.

For elements occurring in the plane of symmetry, one-half of the cross-sectional area must be used in the reduced structure.



Symmetry and Bandwidth - Example 1



Element	θ	*C*	*S*	C^2	S^2	*CS*
1	45°	0.707	0.707	0.5	0.5	0.5
2	315°	0.707	-0.707	0.5	0.5	-0.5
3	0°	1	0	1	0	0
4	270°	0	-1	0	1	0
5	90°	0	1	0	1	0

Symmetry and Bandwidth - Example 1

$$\mathbf{k}^{(1)} = \frac{AE}{2L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{k}^{(2)} = \frac{AE}{2L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Element	θ	*C*	*S*	C^2	S^2	*CS*
1	45°	0.707	0.707	0.5	0.5	0.5
2	315°	0.707	-0.707	0.5	0.5	-0.5
3	0°	1	0	1	0	0
4	270°	0	-1	0	1	0
5	90°	0	1	0	1	0

Symmetry and Bandwidth - Example 1

$$\mathbf{k}^{(3)} = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \mathbf{k}^{(4)} = \frac{AE}{2L}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element	θ	*C*	*S*	C^2	S^2	*CS*
1	45°	0.707	0.707	0.5	0.5	0.5
2	315°	0.707	-0.707	0.5	0.5	-0.5
3	0°	1	0	1	0	0
4	270°	0	-1	0	1	0
5	90°	0	1	0	1	0

Symmetry and Bandwidth - Example 1

$$\mathbf{k}^{(5)} = \frac{AE}{2L} \begin{array}{c} \begin{array}{cccc} u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element	θ	C	S	C^2	S^2	CS
1	45°	0.707	0.707	0.5	0.5	0.5
2	315°	0.707	-0.707	0.5	0.5	-0.5
3	0°	1	0	1	0	0
4	270°	0	-1	0	1	0
5	90°	0	1	0	1	0

Symmetry and Bandwidth - Example 1

Since elements 4 and 5 lie in the plane of symmetry, one half of their original areas have been used in developing the stiffness matrices.

The displacement boundary conditions are:

$$u_1 = v_1 = u_2 = u_3 = u_4 = 0$$

By applying the boundary conditions the force-displacement equations reduce to:

$$\frac{AE}{2L} \left[\begin{array}{cc:c} 2 & 0 & -1 \\ 0 & 2 & -1 \\ \hdashline -1 & -1 & 2 \end{array} \right] \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}$$

Symmetry and Bandwidth - Example 1

We can solve the above equations by separating the matrices in submatrices (indicated by the dashed lines). Consider a general set of equations shown below:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix} \qquad \begin{aligned} K_{11}d_1 + K_{12}d_2 &= 0 \\ K_{21}d_1 + K_{22}d_2 &= F \end{aligned}$$

Solving the first equation for d_1 gives: $d_1 = -K_{11}^{-1}K_{12}d_2$

Substituting the above equation in the second matrix equation gives:

$$K_{21}\left(-K_{11}^{-1}K_{12}d_2\right) + K_{22}d_2 = F$$

Simplifying this expression gives:

$$\left(K_{22} - K_{21}K_{11}^{-1}K_{12}\right) d_2 = F$$

Symmetry and Bandwidth - Example 1

The previous equations can be written as: $k_c d_2 = F$

where: $k_c = K_{22} - K_{21}K_{11}^{-1}K_{12}$

Therefore, the displacements d_2 are: $d_2 = \left(k_c\right)^{-1} F$

If we apply this solution technique to our example global stiffness equations we get:

$$k_c = \frac{AE}{L}\left[[1] - \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{Bmatrix} \right]$$

$$\boxed{\frac{AE}{2L}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}}$$

Simplifying:

$$k_c = \frac{AE}{L}\left\{ [1] \quad - \left[\frac{1}{2}\right] \right\} = \frac{AE}{L}\left[\frac{1}{2}\right] \qquad \left(k_c\right)^{-1} = \frac{2L}{AE}$$

Symmetry and Bandwidth - Example 1

Therefore, the displacements d_2 are: $d_2 = v_4 = -\frac{2PL}{AE}$

The remaining displacements can be found by substituting the result for v_4 in the global force-displacement equations.

$$\begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{Bmatrix} \left[-\frac{2PL}{AE} \right]$$

Expanding the above equations gives the values for the displacements.

$$\begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{PL}{AE} \\ -\frac{PL}{AE} \end{Bmatrix}$$

Symmetry and Bandwidth

The coefficient matrix (stiffness matrix) for the linear equations that occur in structural analysis is always symmetric and banded.

Because a meaningful analysis generally requires the use of a large number of variables, the implementation of compressed storage of the stiffness matrix is desirable both from the viewpoint of fitting into memory (immediate access portion of the computer) and computational efficiency.

Symmetry and Bandwidth

Another method, based on the concept of the skyline of the stiffness matrix, is often used to improve the efficiency in solving the equations.

The ***skyline*** is an envelope that begins with the first nonzero coefficient in each column of the stiffness matrix (see the following figure).

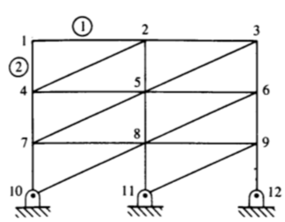
In skyline, only the coefficients between the main diagonal and the skyline are stored.

In general, this procedure takes even less storage space in the computer and is more efficient in terms of equation solving than the conventional banded format.

Symmetry and Bandwidth

A matrix is ***banded*** if the nonzero terms of the matrix are gathered about the main diagonal.

To illustrate this concept, consider the plane truss shown on below.



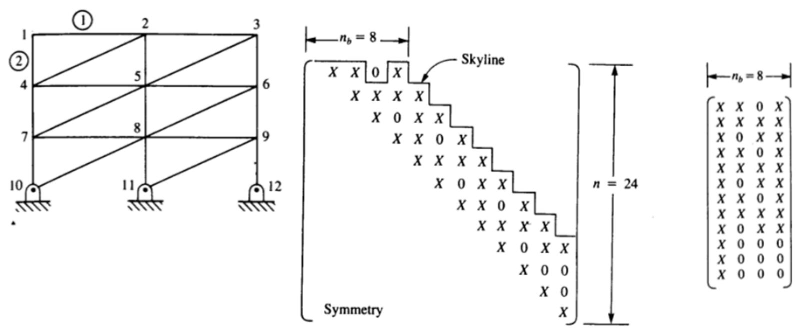
We can see that element 2 connects nodes 1 and 4.

Therefore, the 2 x 2 submatrices at positions 1-1, 1-4, 4-1, and 4-4 will have nonzero coefficients.

Symmetry and Bandwidth

The total stiffness matrix of the plane truss, shown in the figure below, denotes nonzero coefficients with *X*'s.

The nonzero terms are within the some band. Using a banded storage format, only the main diagonal and the nonzero upper codiagonals need be stored.



Symmetry and Bandwidth

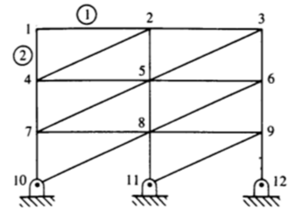
We now define the semibandwidth: n_b as $n_b = n_d(m + 1)$

where n_d is the number of degrees of freedom per node and m is the maximum difference in node numbers determined by calculating the difference in node numbers for each element of a finite element model.

In the example for the plane truss shown above,

$m = 4 - 1 = 3$ and $n_d = 2$;

$n_b = 2(3 + 1) = 8$



Symmetry and Bandwidth

Execution time (primarily, equation-solving time) is a function of the number of equations to be solved.

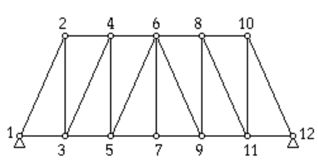
Without using banded storage of global stiffness matrix K, the execution time is proportional to **(1/3)*n*^{3}**, where *n* is the number of equations to be solved.

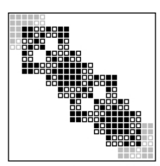
Using banded storage of K, the execution time is proportional to ***n*(*n*$_b$)^2**

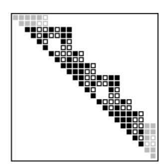
The ratio of time of execution without banded storage to that using banded storage is then **(1/3)(*n*/*n*$_b$)^2**

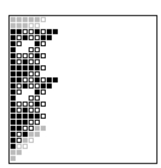
Symmetry and Bandwidth

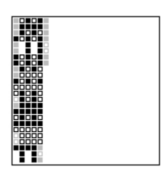
Execution time (primarily, equation-solving time) is a function of the number of equations to be solved.




square matrix


upper triangular matrix

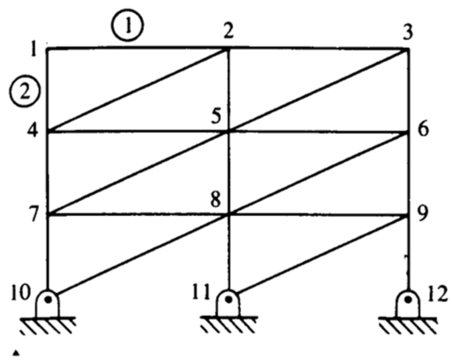

banded matrix


skyline matrix

Symmetry and Bandwidth

For the plane truss example, this ratio is $(1/3)(24/8)^2 = 3$

Therefore, it takes about three times as long to execute the solution of the example truss if banded storage is not used.



$$\overset{n_b = 8}{\longleftrightarrow}$$

$$\begin{bmatrix} x & x & 0 & x \\ x & x & x & x \\ x & 0 & x & x \\ x & x & 0 & x \\ x & x & x & x \\ x & 0 & x & x \\ x & x & 0 & x \\ x & x & x & x \\ x & 0 & x & x \\ x & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$$

Symmetry and Bandwidth

Several automatic node renumbering schemes have been computerized.

This option is available in most general-purpose computer programs. Alternatively, the ***wavefront*** or ***frontal*** method are popular for optimizing equation solution time.

In the ***wavefront method***, elements, instead of nodes, are automatically renumbered.

In the ***wavefront method*** the assembly of the equations alternates with their solution by Gauss elimination.

Symmetry and Bandwidth

The sequence in which the equations are processed is determined by element numbering rather than by node numbering.

The first equations eliminated are those associated with element 1 only.

Next the contributions to stiffness coefficients from the adjacent element, element 2, are eliminated.

If any additional degrees of freedom are contributed by elements 1 and 2 only these equations are eliminated (condensed) from the system of equations.

Symmetry and Bandwidth

As one or more additional elements make their contributions to the system of equations and additional degrees of freedom are contributed only by these elements, those degrees of freedom are eliminated from the solution.

This repetitive alternation between assembly and solution was initially seen as a ***wavefront*** that sweeps over the structure in a pattern determined by the element numbering.

The ***wavefront method***, although somewhat more difficult to understand and to program than the banded-symmetric method, is computationally more efficient.

Symmetry and Bandwidth

A banded solver stores and processes any blocks of zeros created in assembling the stiffness matrix.

These blocks of zero coefficients are not stored or processed using the wavefront method.

Many large-scale computer programs are now using the wavefront method to solve the system of equations.

Homework Problems

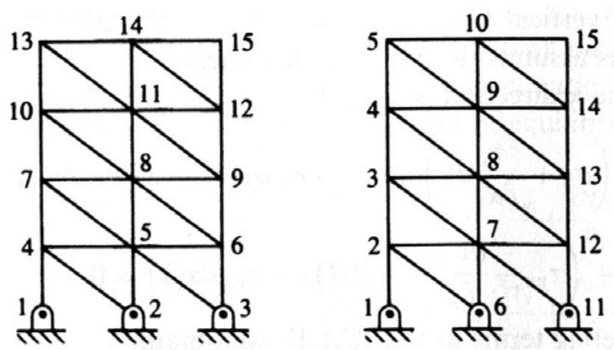
3b. Do problems ***3.50*** and ***3.55*** on pages 146 - 165 in your textbook “A First Course in the Finite Element Method” by D. Logan.

4. Use SAP2000 and solve problems ***3.63*** and ***3.67***.

Homework Problems

5. Do problem ***B.9*** in your textbook "A First Course in the Finite Element Method" by D. Logan.

   Determine the bandwidths of the plane trusses shown in the figure below. What conclusions can you draw regarding labeling of nodes?

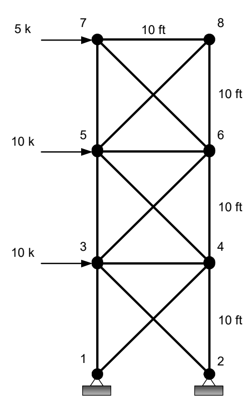


Homework Problems

6. Solve the following truss problems. You may use **SAP2000** to do truss analysis.

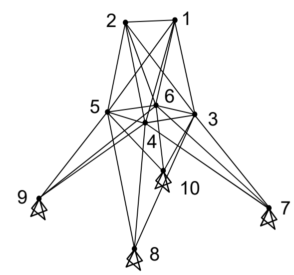
   a) For the plane truss shown below, determine the nodal displacements and element stresses.

   Nodes 1 and 2 are pin joints. Let $E = 10^7$ *psi* and the $A = 2.0$ in^2 for all elements.



Homework Problems

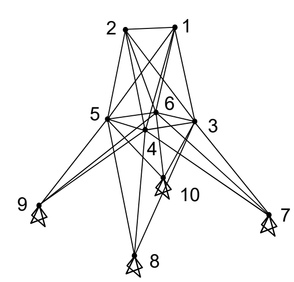
b) For the 25-bar truss shown below, determine the displacements and elemental stresses. Nodes 7, 8, 9, and 10 are pin connections.
Let $E = 10^7$ *psi* and the $A = 2.0$ in^2 for the first story and $A = 1.0$ in^2 for the top story. Table 1 lists the coordinates for each node. Table 2 lists the values and directions of the two loads cases applied to the 25-bar space truss.



Node	x (*in*)	y (*in*)	z (*in*)
1	-37.5	0.0	200.0
2	37.5	0.0	200.0
3	-37.5	37.5	100.0
4	37.5	37.5	100.0
5	37.5	-37.5	100.0
6	-37.5	-37.5	100.0
7	-100.0	100.0	0.0
8	100.0	100.0	0.0
9	100.0	-100.0	0.0
10	-100.0	-100.0	0.0
Note: 1 *in* = 2.54 *cm*			

Homework Problems

b) For the 25-bar truss shown below, determine the displacements and elemental stresses. Nodes 7, 8, 9, and 10 are pin connections.
Let $E = 10^7$ *psi* and the $A = 2.0$ in^2 for the first story and $A = 1.0$ in^2 for the top story. Table 1 lists the coordinates for each node. Table 2 lists the values and directions of the two loads cases applied to the 25-bar space truss.

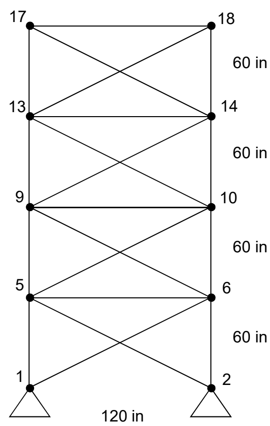


Case	Node	F_x (*kip*)	F_y (*kip*)	F_z (*kip*)
1	1	1.0	10.0	-5.0
	2	0.0	10.0	-5.0
	3	0.5	0.0	0.0
	6	0.5	0.0	0.0
2	1	0.0	20.0	-5.0
	2	0.0	-20.0	-5.0
Note: 1 *kip* = 4.45 *kN*				

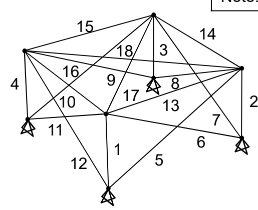
Homework Problems

c) For the 72-bar truss shown below, determine the displacements and elemental stresses. Nodes 1, 2, 3, and 4 are pin connections. Let $E = 10^7$ *psi* and the A = 1.0 in^2 for the first two stories and A = 0.5 in^2 for the top two stories. Table 3 lists the values and directions of the two loads cases applied to the 72-bar space truss.

Case	Node	F_x (*kip*)	F_y (*kip*)	F_z (*kip*)
1	17	0.0	0.0	-5.0
	18	0.0	0.0	-5.0
	19	0.0	0.0	-5.0
	20	0.0	0.0	-5.0
2	17	5.0	5.0	-5.0
Note: 1 *kip* = 4.45 *kN*				



(a)



(b)

End of Chapter 3b