Financial Math on Spreadsheet and Calculator

Version 4.0

© 2002 Kent L. Womack and Andrew Brownell

Tuck School of Business
Dartmouth College

Table of Contents

Introduction

This document will introduce you to basic financial concepts and help get you started solving finance-related problems. Concepts are introduced along with example problems. Our goal is to make you comfortable understanding the concepts *as well as* the calculations for basic financial problems. To that end, each example problem has a step-by-step solution for both calculator and spreadsheet. The underlying equation is also provided to help you understand the mathematics behind the calculator and spreadsheet functions.

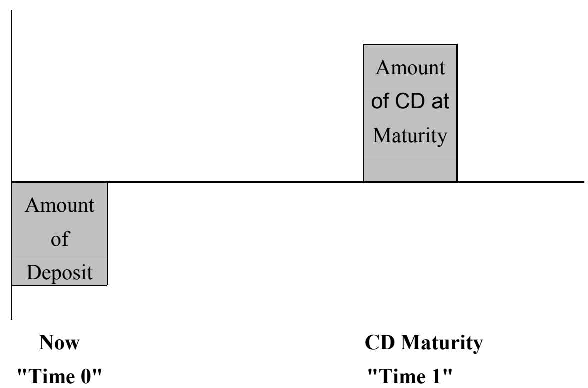
These problems are generally known as Time Value of Money problems, or TVM. As the name implies, TVM problems look at how the value of money changes over time. In your own experience, you know that inflation erodes the value of money over time and that if you invest money in a bank or CD today, it will be worth more in nominal terms in a year. But other factors, most notably risk, are also incorporated in the expected value of money over time. TVM concepts are omnipresent in the financial markets for pricing bonds as well as valuing companies.

TVM problems use a fairly standard nomenclature. But, as with many mathematical expressions, different sources (such as textbooks, calculator and spreadsheet manuals) may use slightly different notations to convey the same concept. We will use the following abbreviations throughout this document:

Notation	Concept	What it means
PV	Present Value	How much an investment is worth today
FV	Future Value	How much an investment is promised to be worth or pay at a specified future date
I%YR	Interest Rate or Discount Rate expressed as a percentage *per year*. Also called *Nominal interest rate*.	As an investor (or, debtor), the approximate rate you earn (or, pay) each year. *Example:* for a five year car loan with monthly payments bearing a nominal interest rate of 8.95%, I%YR = 8.95%
i	Interest rate or discount rate expressed as a percentage *per period*. Also called *Periodic interest rate*. Note that for instances where there is only one period per year, I%YR = *i*.	As an investor (or, debtor), the rate you earn (or, pay) each period. *Example:* for a five year car loan with monthly payments bearing a nominal interest rate of 8.95%, *i* = .746% (=.0895 / 12)
N or **Nper**	Total number of periods involved. HP calculators typically use **N**. *Excel* uses **Nper**.	*Example*: for a five year car loan with monthly payments, Nper = 60
P/YR	Periods per Year (we will assume that time is "sliced" into equal length periods)	*Example*: for a five year car loan with monthly payments, P/YR = 12
PMT	Amount of periodic payment	*Example*: for a $279.89 monthly car loan payment, PMT = 279.89
[ ]	Refers to a specific calculator button	*Example*: [PMT] means the PMT button

Performing TVM calculations—The General Framework

TVM problems calculate how the value of money changes over time. **The essential idea is much like foreign currency translation.** One can convert a certain number of dollars into yen or euros. TVM problems are similar except that the translation is from future time to current time values and visa versa, rather than from yen to dollars. Thus, it is helpful to think of TVM problems as a series of cash inflows and outflows that occur along a time line. A basic example is a bank certificate of deposit, or CD. A CD requires that you deposit a certain amount of money in exchange for a greater amount of money in return sometime in the future. That initial deposit has two important characteristics. First, it is an expense out of your pocket, and therefore can be viewed as a cash outflow. (TVM often (especially in Excel) uses a negative sign to represent a cash outflow.) Second, the deposit is made today, in the present, and thus the amount of the deposit is considered its Present Value, or PV. The amount you receive when the CD matures in the future, its Future Value, or FV, is a cash inflow. TVM regards cash inflows as positive amounts. To make the example simple, let's assume that there is only one period from the time of the deposit to the time the CD matures. A diagram of the CD cash flows would look like this:



As you know, what determines the value of the CD at maturity is the rate of interest earned on the initial deposit. So, for our example, the Future Value of the CD equals the Present Value of the deposit, appreciated by some Interest Rate. Expressed as a function, FV = *f*(PV, RATE).

In general, there are three steps in the process of correctly calculating a TVM problem. **It is important that these three steps become "second nature."**

1. a) <u>Identify</u> the present and future <u>cash flows</u>. We recommend that you think of a time line where the present is "time 0" and all future cash flows are laid out in future periods. It is even recommended that you draw the time line (as done in the examples) until visualizing the amount and timing of cash flows becomes second nature to you.

   b) <u>Determine</u> the appropriate <u>interest rate</u> (or, discount rate) to use, if necessary. In some problems (e.g. "What rate do you earn if . . . ?"), you will skip this step because you will solve for this interest rate once you are given the complete set of present and future cash flows.

   c) <u>Determine the compounding assumption</u> per year. Typically, interest rates (and hence, interest income received or expense paid) will either be compounded ***annually, semi-annually***, or ***monthly***. The default is "annual" unless you specifically know the convention for the asset you are pricing. Because most investors and banks do not consider interest earned until the end of the period, we will almost always use **END** mode, rather than BEGIN mode on calculator or spreadsheet.

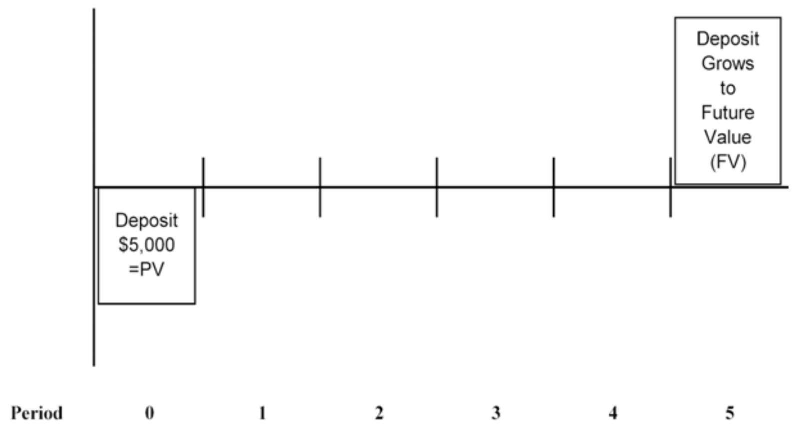
2. <u>Apply the appropriate equation</u> that converts future cash flows to present ones, or vice versa.

3. <u>Know</u> the appropriate <u>buttons</u> to push on the calculator or <u>formulae</u> to use in Excel.

Calculating Future Value

Future value (FV) is one of the simplest concepts in finance. The compound interest formula below tells us how much money invested today (its PV) will be worth at some future period (its FV). A good example of how future value works is illustrated by simple savings accounts. What would be the future value of $5,000 put into a savings account at the end of five years if it earns 5% interest yearly, compounded once per year?

Step One: Diagram the Cash Flows:



A) <u>Cash Flows</u>: Pay $5,000 today (Time 0) and receive and unknown amount in Time 5.
B) <u>Interest Rate</u>: is stated at 5%
C) <u>Compounding Assumption</u>: Once per year, at year end.

Step Two: Identify Formula Components

<u>Future Value Formula:</u> $\boxed{FV = PV(1+i)^{Nper}}$

<u>Note</u>: This formula is only appropriate when there is a <u>single</u> future cash flow. If the future payoff is a series of future cash flows, we use a different formula.

Present Value (i.e., the amount deposited today) =	**PV**	= $5,000
Interest Rate per period =	***i***	= 5.00%
Number of periods =	**Nper**	= 5

<u>Intuition</u>: The system of currency translation is what we call *compound interest* or "interest on interest." In this case, if we put $5,000 dollars to work earning 5% each year, at the end of one year, we will have $5,250. During the second year, we will earn 5% on $5,250, and so forth. So, using this standard convention of compound interest and the formula above, at the end of 5 years we would have $6,381.41.

Also note that the FV formula, like the other formulae presented here, <u>does not use</u> the TVM convention of signing cash outflows negative and cash inflows positive. However, financial calculators and Excel <u>do use</u> the inflow/outflow convention, so it is important to get used to thinking of cash flows being positive (you receive payments *in* = cash inflows) or negative (you make payments *out* = cash outflows).

Step Three: Solve for Future Value (FV)

Using a Calculator (HP17B)

5,000 [+/-] [PV]
1 [P/YR]
5 [I%YR]
5 [N]
➔ hit [FV] to solve

Answer = $6,381.41

Using a Calculator (HP12C)*

5,000 [CHS] [PV]
5 [i]
5 [n]
➔ [FV] to solve

Answer = $6,381.41

Using Excel

	A	B	C
1	PV	-5000	
2	I	5%	
3	Nper	5	
4	FV	=FV(B2,B3,,B1)	
		=$6381.41	

An Excel FV function can be inserted into cell B4 to perform the Future Value calculation. Cells B1, B2 and B3 are reference cells used by the Excel formula to calculate the Future Value.

* Note: [CHS] changes the sign of the number in the input register on the HP-12C. It is equivalent to [+/-] on other calculators.

Compounding Multiple Times Per Year

In the preceding problem, interest was collected/paid only once per year. However, as you can image, there can be more than one interest period per year. For example, most bonds assume semi-annual (twice per year) interest payments; and most bank loans and mortgages use a monthly interest convention. In each case, a nominal interest rate is quoted to you. That nominal rate is divided by the number of compounding periods per year to determine the periodic interest rate. The periodic rate is used to perform TVM calculations. Note that as the number of compounding periods increases, so does the total number of periods (Nper or N) such that for one year of monthly payments, N = 12. Thus, you must ask and know what the compounding method is for any nominal interest rate – and make the adjustment to the total number of periods – before you can perform a TVM calculation.

Example One: In the FV example above, the nominal rate (5%) equals the periodic rate because there is only one payment per year. If the bank had offered semi-annual compounding, the period rate would have been 2.5% (5% divided by 2 periods per year equals 2.5%) and the number of periods would double (5 years times 2 periods per year equals 10 periods).

Example Two: Assume another bank offers a savings rate of 4.95%, paid semi-annually. In which bank would you prefer to deposit your money for the next five years?

> Intuition: From our first example, you know how compound interest works. In this case, the compounding period is semi-annual, or every six months. Thus, the 4.95% nominal rate must be divided in two (*i=2.475% earned each six months)* because there are two periods per year. The total number of periods is likewise doubled (N=10). We know that in six months, we should have earned 2.475% on $5,000, or $123.75. Our bank account now shows a balance of $5,123.75 on which we will earn 2.475% interest over the next six months and so on.

Using the FV equation:		**Using a calculator (HP17B):**		**Using a calculator (HP12C)**	
PV	= 5,000	5,000	[+/-] [PV]	5,000	[CHS] [PV]
i	= 2.475%	4.95	[I%YR]	2.475	[i]
Nper	= 10	10	[N]	10	[n]
FV	➔ $6384.83	[FV] ➔ $6384.83		[FV] ➔ $6384.83	

Bank with 5% annual interest:
[FV] ➔ $6381.41

(Please refer to the previous page for calculation details.)

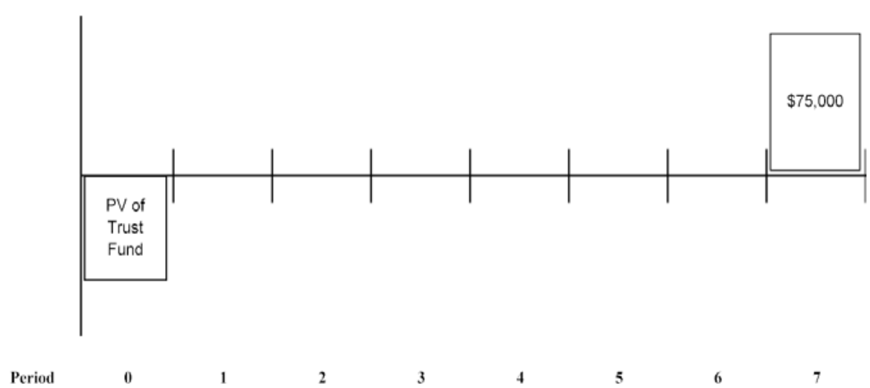
Thus, in this example, you are better off going with the new bank that offers a lower nominal rate, but more favorable compounding terms. At the end of five years, the 4.95% semi-annual interest account earns about $3.40 more interest than the 5% annually compounded account.

Calculating Present Value

Present Value (PV or NPV), which we address now, is probably the most important concept in finance. Essentially, Present Value tells us what value we should place today on cash flows that we will receive in the future.

Example: Assume that you just discovered that you are the beneficiary of a trust fund. In seven years you will receive $75,000 in cash. Assuming a 12% annual interest rate (or, synonymously, discount rate) is appropriate, what is the *present value* of your trust fund? The Present Value is approximately what you should be able to sell the promised future amount (FV) for today.

Step One: Identify Cash Flows



A) Cash Flows: The value today (Time 0) of the trust fund is unknown. The trust fund money, which you will receive in the future, is of course the Future Value, given to you at Time 7.
B) Interest Rate: is stated at 12%
C) Compounding assumption: the problem states annual compounding

Step Two: Identify Formula Components

Our original formula for Future Value was:

Future Value Formula:

$$\boxed{FV = PV(1+i)^{Nper}}$$

Present Value Formula:

$$\boxed{PV = \frac{FV}{(1+i)^{Nper}}}$$

You will note that in this case, we do not know the present value (PV), but we do know the future value (FV = $75,000). So, what do we do? We simply manipulate the Future Value Formula by dividing both sides by $(1 + i)^{Nper}$. Then we get what we call the **Present Value Formula**.

In our problem, the values we know are:

Future Value =	**FV**	= $75,000
Interest Rate per period =	***i***	= 12.00%
Number of periods =	**Nper**	= 7

Step Three: Solve for Present Value (PV)

Using a Calculator (HP17B)

75,000	[FV]
1	[P/YR], the "default"
12	[I%YR]
7	[N]
➔	hit [PV] to solve

Answer = -$33,926.19

Using a Calculator (HP12C)

75,000	[FV]
12	[i]
7	[n]
➔	[PV] to solve

Answer = -$33,926.19

Using Excel

	A	B	C
1	FV	75000	
2	I	12%	
3	Nper	7	
4	PV	PV(B2,B3,,B1)	
		=-$33926.19	

The Excel PV function can be inserted into cell B4 to perform the Present Value calculation (Insert | Function | Financial | PV; OR use the ***fx*** button). Cells B1, B2 and B3 are reference cells used by the Excel formula to calculate the Present Value.

Double-check and intuition: PV is displayed as negative because it represents a cash outflow. This is sometimes confusing because the equations we started with do not change the sign from positive to negative for Present Values and Future Values. The positive and negative values are a convention that most all calculators and spreadsheet functions use. Your inflow (what you will receive in seven years) was entered as a positive value. What you or someone else will pay out (in order to get the inflows) is "coded" as a negative. Notice that we did NOT enter a number for PMT. It should remain 0 for this problem. PMT is used when there is a *series* of future cash flows.

You should always ask at the end of a problem, "Does the answer I got, $33,926, make sense?" A quick check shows that after one year $33,926.19 earning 12% will become about $37,997. If we compounded 6 more times, would we get $75,000? Yes, a few calculator strokes would show that you would.

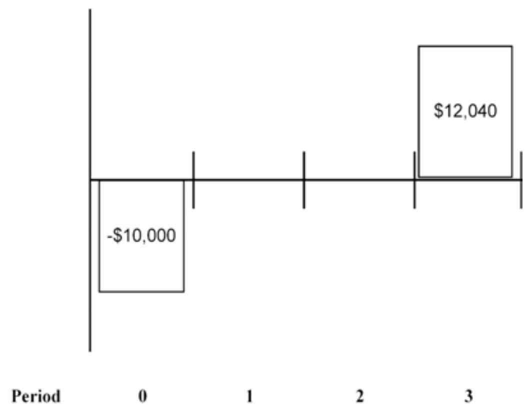
A very handy rule of thumb (remember this!) is that $100 earning 10% each year will double to $200 in about 7 years. In our case, we are earning 12% so $33,000 will more than double to $75,000 in seven years, so our answer is "in the ballpark".

Calculating the Interest Rate (or, the Discount Rate)

The next common problem is to calculate the rate of interest you are earning if you know what you must pay and what you will receive.

Example: A local bank is selling shares of what it calls an "investment pool." For a one-time investment of $10,000, the bank guarantees a return payment of $12,040 in three years. What is the effective annual interest rate on the investment pool?

Step One: Identify the Cash Flows



A) Cash Flows: You pay $10,000 now (Time 0) and receive $12,040 in three years (Time 3)

B) Interest Rate: Unknown

C) Compounding assumption: problem asks for the effective annual rate

Step Two: Identify Formula Components

$$\boxed{PV = \frac{FV}{(1+i)^{Nper}}} \qquad \text{OR} \qquad \boxed{FV = PV(1+i)^{Nper}}$$

Future Value	=	**FV**	= $12,040
Present Value	=	**PV**	= $10,000
Number of periods	=	**Nper**	= 3

Intuition: We know all the factors in the above equations except the interest rate, *i*. Since we have only one equation in one unknown variable, *i*, we know we can find *i*. There is an equation that isolates and solves for *i*, but you do not need to learn it. Your calculator or spreadsheet will do it for you.

Step Three: Solve for Nominal Interest Rate (I%YR)

Using a Calculator (HP17B)

12,040 [FV]
10,000 [+/-] [PV]
1 [P/YR]
3 [N]
➔ hit [I%YR] to solve

Answer = 6.38%

Using Calculator (HP12C)

12,040 [FV]
10,000 [CHS] [PV]
3 [n]
➔ [i] to solve

Answer = 6.38%

Using Excel

	A	B	C
1	FV	12,040	
2	PV	-10,000	
3	Nper	3	
4	I	=RATE(B3,,B2,B1)	
		= 6.38%	

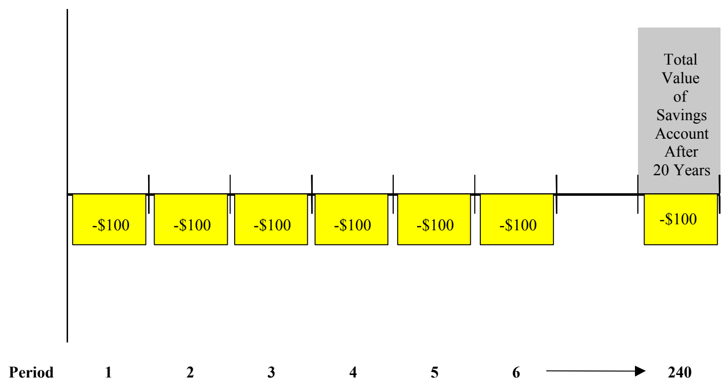
The RATE function can be inserted into cell B4 to perform the Interest Rate calculation. Cells B1, B2 and B3 are reference cells used by the Excel formula to calculate the Interest Rate.

Calculating the Future Value of an Annuity

An annuity is series of even payments made at fixed intervals.

Example: Again, with Future Value, it is sometimes easiest to think of the problem as a savings account. Imaging that you put $100 per month into a saving account that paid 0.40% monthly interest (*concept check: what nominal interest rate does 0.40% per month represent?*). If you kept on such a payment plan for 20 years, what would the value of the account grow to?

Step One: Identify the Cash Flows



A) Cash Flows: You pay $100 each month starting now

B) Interest Rate: Is stated as 0.40% monthly

C) Compounding assumption: monthly compounding

Step Two: Identify Formula Components

Future Value of Annuity Formula: $\boxed{FV = \frac{PMT}{i}\left[(1+i)^{Nper} - 1\right]}$

Monthly Deposits	=	**PMT**	=	$100	
Interest Rate per period	=	***i***	=	0.40%	
Number of periods	=	**Nper**	=	240	20 Yrs (N) x 12 Mos/yr (P/Yr)

Step Three: Solve for Future Value (FV)

Using a Calculator (HP17B)

100 [+/-] [PMT]
12 [P/YR]
4.80 [I%YR] = (0.40 x 12)
240 [N]
➔ hit [FV] to solve

Answer = $40,167.50

Using a Calculator (HP12C)

100 [CHS] [PMT]
0.40 [i]
240 [n]
➔ [FV] to solve

Answer = $40,167.50

Using Excel

	A	B	C
1	PMT	-100	
2	I	0.40%	
3	Nper	240	
4	FV	=FV(B2,B3,B1)	
		= $40167.50	

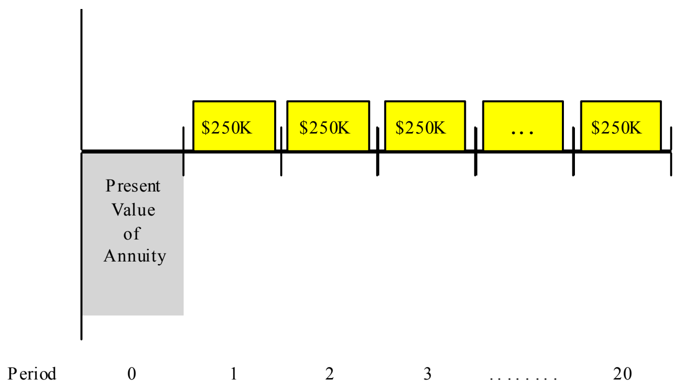
An Excel FV function can be inserted into cell B4 to perform the Future Value calculation. Cells B1, B2 and B3 are reference cells used by the Excel formula to calculate the Future Value.

Double-check: If we ignore the interest that we will earn over 20 years, the cash payments we will receive will be 240 x $100 = $24,000. However, with interest compounding over 20 years, we would expect considerably more. Our answer suggests that interest is about $16,000.

Calculating the Present Value of an Annuity

Example: What would you do if you won the lottery? Most lotteries give you the option to take a one-time cash payment or to receive annual payments for some period into the future? To answer this question, you need to calculate the present value of each of the lottery's annual payments to see if the sum total of them is higher or lower than the one-time cash payment. For example, consider a lottery that makes 20 annual payments of $250,000. What is the present value of the payments using a 7.5% annual discount rate? (Let's assume that 7.5% is a reasonable interest rate that we believe we could earn if we took the lump-sum payment.)

Step One: Identify the Cash Flows



A) Cash Flows: You receive 20 payments of $250K per year starting in Time 1. The value of the right to receive these payments in unknown at Time 0.

B) Interest Rate: stated at 7.5%

C) Compounding assumption: problem uses an annual rate

Step Two: Identify Formula Components

Present Value of Annuity Formula: $\boxed{PV = \frac{PMT}{i}[1-(\frac{1}{1+i})^{Nper}]}$

Intuition: It is not easy by simply looking at this formula to get an understanding of what it means. Most simply, it is just calculating the sum of the present values of each of the 20 payments if the interest rate used for all periods is *i*. You should just assume for our purposes that some mathematician found this solution, and so we use it. You should NOT memorize it, you should just understand that your calculator and spreadsheet use it when they calculate the PV of a series of future cash flows.

Monthly Deposits	=	**PMT**	=	$250,000
Interest Rate per period	=	***i***	=	7.5%
Number of periods	=	**Nper**	=	20

Step Three: Solve for Present Value (PV)

Using a Calculator (HP17B)

250,000 [PMT]
1 [P/YR]
7.5 [I%YR]
20 [N]
➔ [PV] to solve

Answer = -$2,548,622.84

Using a Calculator (HP12C)

250,000 [PMT]
7.5 [i]
20 [n]
➔ [PV] to solve

Answer = -$2,548,622.84

Using Excel

	A	B
1	PMT	250,000
2	I	7.50%
3	Nper	20
4	PV	= PV(B2,B3,B1)
		= -$2,548,622.84

An Excel PV function can be inserted into cell B4 to perform the Present Value calculation. Cells B1, B2 and B3 are reference cells used by the Excel formula to calculate the Present Value.

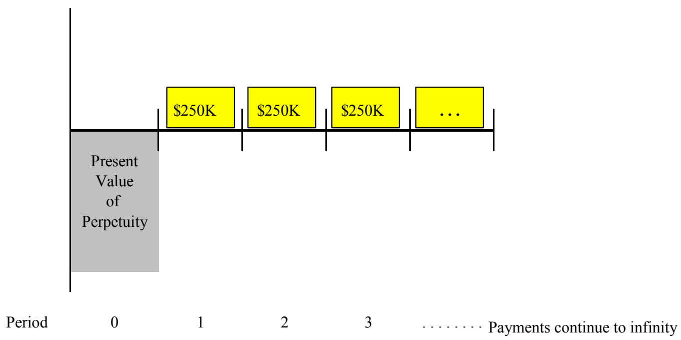
Therefore, take the lump sum if it is greater than $2,548,622.84. Otherwise take the series of 20 payments.

Calculating the Present Value of a Perpetuity

A perpetuity is an infinite series of payments made at fixed intervals.

Example: Consider a contract, perhaps sponsored by a government, that guarantees a regular fixed payment forever into the future. What if, instead of for 20 years, the lottery in the previous problem made annual payments in perpetuity?

Step One: Identify the Cash Flows



A) Cash Flows: You receive payments of $250K per year starting in Time 1 with an unknown value at Time 0

B) Interest Rate: stated at 7.5%

C) Compounding assumption: problem uses an annual rate

Step Two: Identify Formula Components

Present Value of Infinite Payment Stream (Perpetuity):

$$\boxed{PV = \frac{PMT}{i}}$$

Monthly Deposits	=	PMT	=	$250,000
Interest Rate per period	=	i	=	7.5%

Intuition: Again, it is not obvious why the above formula is equivalent to the sum of the infinite series of future cash flows. But, a mathematical proof will show that, using the same i for all future periods, it is.

Step Three: Solve for Present Value (PV)

Using the formula

250000/.075

Answer = -$3,333,333.33

Excel can be used to create a table to keep track of the payments through time. The payment one year from now is "worth" 232,558 (Row 7). You can see that each successive row is worth less. Excel cannot sum infinite payments, but a sum of the first 200 years will converge towards the correct answer since the present value of single payments approaches zero (as can be seen at Time 200 (Row 207), where the PV is close to zero)

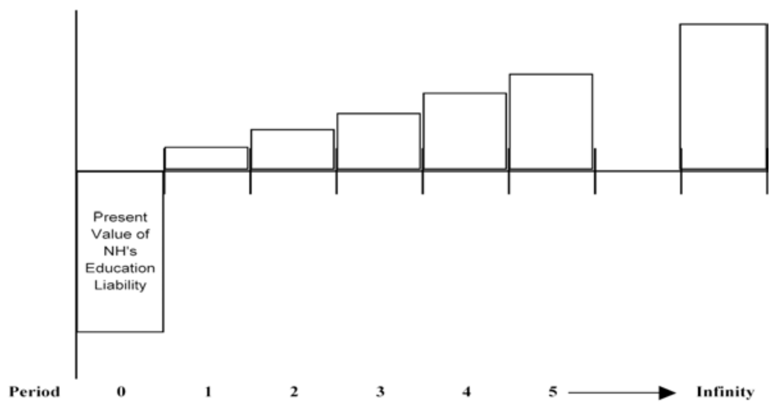
Using Excel

	A	B	C	D	E
1					
2					
3	i	7.5%			
4	"NET" PV	-3,333,333	=SUM(C7:C36)		
5					
6	TIME	PMT	PV		
7	1	250000.00	232,558	=B7/(1+$B$3)^A7	
8	2	250000.00	216,333	=B8/(1+$B$3)^A8	
9	3	250000.00	201,240	=B9/(1+$B$3)^A9	
10	4	250000.00	187,200	=B10/(1+$B$3)^A10	
11	5	250000.00	174,140	=B11/(1+$B$3)^A11	
12	6	250000.00	161,990	=B12/(1+$B$3)^A12	
13	7	250000.00	150,689	=B13/(1+$B$3)^A13	
14	8	250000.00	140,176	=B14/(1+$B$3)^A14	
15	9	250000.00	130,396	=B15/(1+$B$3)^A15	
16	10	250000.00	121,298	=B16/(1+$B$3)^A16	
17	11	250000.00	112,836	=B17/(1+$B$3)^A17	
...	...	...	...	...	
207	200	250000.00	0.13	=B207/(1+$B$3)^A207	

Calculating the Present Value of a <u>Growing</u> Perpetuity

Example: The Supreme Court of the State of New Hampshire recently ruled that the state government must provide funding so that every student in the state has access to "adequate" public education. While New Hampshire is still debating what the source of these funds will be, it is generally agreed that an adequate education for every public school child in the state will cost about $1 billion a year at the beginning of 2000. Unless the legislature opts to amend the state constitution, this obligation will continue into the future forever. Further, it is estimated that the cost of adequacy will increase each year at approximately the rate of inflation, about 3.00%. What is the present value of the state's obligation to fund public education if the appropriate discount rate is 5.50%? Assume that the first payment of this obligation will be at the end of year 2000.

Step One: Diagram the cash flows



A) <u>Cash Flows</u>: Unknown value at Time 0. Payments of increasing value are made in each year in perpetuity
B) <u>Interest Rate</u>: The problem states that 5.50% is appropriate
C) <u>Compounding assumption</u>: It is reasonable to assume annual, since payments are made only once per year

Step Two: Identify Formula

<u>Present Value of a Growing Perpetuity Formula</u>: $\boxed{PV = \dfrac{PMT}{i-g} \quad \text{for } i > g}$

PMT	=	$1.03 billion	=	$1 billion times (1+ *i*) = Payment in Period 1
i	=	5.50%		
g	=	3.00%		

<u>Intuition</u>: Just as in the level-payment perpetuity problem above, the equation above can be shown to correspond to the sum of the present values of an infinite series of growing cash flows. Note that in this equation, PMT represents the payment one period from present (i.e., Time 1, not Time 0).

Step Three: Solve for PV

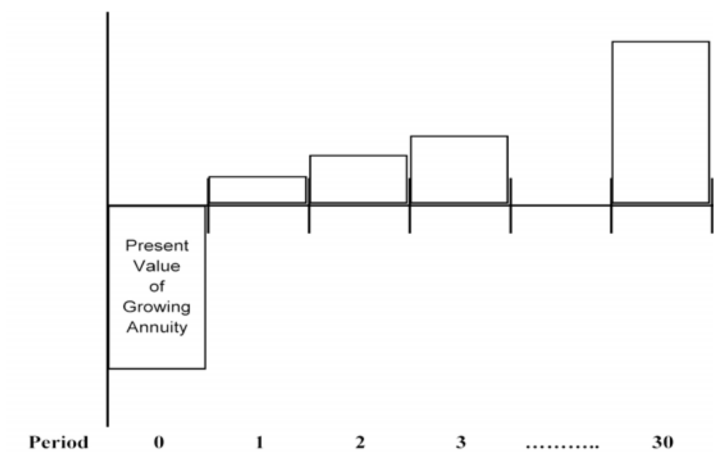
Use the above formula ➔ **Answer: $41.2 billion** = $1.03 billion divided by (5.50% - 3.00%)

We use $1.03 billion for the first payment because the $1 billion is at time zero, so the actual cost at the time of the first payment at time one is $1.03 billion. This result suggests that the state needs to set aside 41.2 billion now to completely fund the future liability.

Calculating the Present Value of a Growing Annuity

Example: Retirement or pension benefits that increase every year with a cost of living adjustment can be thought of as growing annuities. Consider a retired citizen receiving social security benefits. Next year, John Q. Public will begin receiving his social security benefit of $6,600. In the past, cost of living adjustments have averaged 3.5% while the U.S. Government's cost of capital has averaged 5% -- both are assumed to continue at those rates into the future. Suppose the benefits to the average citizen are assumed to last 30 years. What is the present value of the social security benefit John Q. Public will receive?

Step One: Diagram the Cash Flows:



A) Cash Flow: The value at Time 0 is unknown. Payments are received each year for the next 30 years.
B) Interest Rate: Given at 5.00%
C) Compounding assumption: Again, it is reasonable to assume annual, since payments are made once per year. (If the assumption is not annual, you will be given that information.)

Step Two: Identify Formula Components

Present Value of a Growing Annuity Formula:

$$\boxed{PV = \frac{PMT}{i-g}[1-(\frac{1+g}{1+i})^{Nper}] \quad \text{for } i > g}$$

Intuition: Again, it is difficult to see why this formula equals the sum of *Nper* future cash flows where the first cash flow is PMT and they grow each year after the first one at *g* percent. However, you can observe that, if you multiply out the two terms in the equation, the first term is identical to the PV of a growing perpetuity (see the previous formula). The second term subtracts the PV of the terms beyond *Nper*.

Payment	=	**PMT**	= $6,600
Interest Rate per period	=	***i***	= 5.00%
Number of periods	=	**Nper**	= 30
Growth rate of PMT	=	**g**	= 3.5%

Step Three: Solve for Present Value (PV)

Using Formula

This formula is not pre-programmed into most financial calculators. *When encountering this type of problem, you must use either the formula or a spreadsheet to derive a solution.*

Answer = $154,251

Using Excel

	A	B	C	D	E
1					
2	g	3.5%			
3	i	5.0%			
4	"NET" PV	154,251	=SUM(C7:C36)		
5					
6	TIME	PMT	PV		
7	1	6600.00	6,286	=B7/(1+$B$3)^A7	
8	2	=B7*(1+$B$2)	6,196	=B8/(1+$B$3)^A8	
9	3	=B8*(1+$B$2)	6,107	=B9/(1+$B$3)^A9	
10	4	=B9*(1+$B$2)	6,020	=B10/(1+$B$3)^A10	
11	5	=B10*(1+$B$2)	5,934	=B11/(1+$B$3)^A11	
12	6	=B11*(1+$B$2)	5,849	=B12/(1+$B$3)^A12	
13	7	=B12*(1+$B$2)	5,766	=B13/(1+$B$3)^A13	
14	8	=B13*(1+$B$2)	5,683	=B14/(1+$B$3)^A14	
15	9	=B14*(1+$B$2)	5,602	=B15/(1+$B$3)^A15	
16	10	=B15*(1+$B$2)	5,522	=B16/(1+$B$3)^A16	
17	11	=B16*(1+$B$2)	5,443	=B17/(1+$B$3)^A17	
...	...	...	...	...	
36	30	17,898.4	4141	=B7/(1+$B$3)^A36	
37					

A spreadsheet can be created to model the behavior of the cash flows over time. Each year, the social security payment (column B, "PMT") increases by 3.5% (cell B2). Column C calculates the present value of each PMT using the discount rate (i) of 5% (cell B3). Cell B4 sums column C to calculate a "net" present value. Note that column C contains the basic PV formula; the Excel PV function could have been used instead.

Advanced Application: Pricing Bonds

Note: These last bond examples are what we study near the end of the first MBA accounting course and in the first few days of the first MBA finance course. Our expectation is <u>not</u> that you have mastered this material, but that you are ready to understand the mechanics of the PV calculations of a bond, which has well-defined future cash flows.

You have already learned all of the financial math you need to price basic municipal, corporate and government bonds. An entity that issues a bond, often called the bond issuer, is essentially borrowing money from the bond purchaser (also called the bondholder). The issuer repays the loan by making a series of interest payments until the bond matures, at which time the principal, or stated amount, of the loan is repaid. Thus, a bond entitles the bondholder to two predetermined sets of cash flows:

(1) <u>Periodic interest payments</u>: The interest payments are made on a fixed schedule and calculated as the bond's stated interest rate (or coupon rate) times the bond's principal value or face value. For example, a \$100 bond with a 6.50% coupon would make semi-annual interest payments of \$100 x (6.500% / 2) = \$3.25.
(2) <u>Return of principal</u>: Principal is typically stated as the bond's ***par value*** and is typically denominated in units of \$1,000. But, we will calculate using denominations of \$100 FV and then scale to any size we need.

Because a bond sets forth a fixed schedule of cash flows, one can use TVM formulae to find the exact price of a bond (i.e., its present value) if the discount rate is known; OR one could find the discount rate if the present value (the price) of the bond is known.

<u>Semi-annual Cash Flows (a potential confusion in calculating PV)</u>

Calculations are relatively straightforward when the cash flows occur once a year and the interest rate charged is calculated and assessed at the end of the year. ("End of year" is standard convention; you pay or receive interest at the end of the period after the money has been "used".)

However, in many bond markets, a coupon is paid each half-year (semi-annually). In these markets, half of the yearly interest rate is charged each half year. In fact, the interest compounds twice a year rather than once a year. Hence, if you calculate out the future value of \$100 at the end of one year, you will notice that receiving 5% each half year (with compounding) creates <u>more</u> future value than 10% compounded once at year end.

Therefore, 10 % with semi-annual compounding is <u>not</u> equivalent to 10% with annual compounding. In many bond markets (including the US government bond market), prices (present values) and yields (the interest rate) are quoted using semi-annual compounding. Therefore, when you hear of the US government "long bond" (with 30 years to maturity) quoted at 6%, the technical "translation" of that quote is that you effectively receive 3% each half year, which is really slightly more than 6% per year.

One can either quote interest rates in terms of the Nominal Rate (stating or assuming that it is semi-annual compounding, or in terms of the Effective Rate (which is a calculation converting the semi-annual rate to an annual rate). Using "Effective Rate" quotes, even though they are correctly converting to the annual rate, is hardly ever done. The bond market, by convention, quotes nominal rates and *assumes* semi-annual terms. It does <u>not</u> make the conversion to an interest rate <u>as if</u> there were only one payment and interest rate per year. Simply, we get used to calculating in semi-annual mode, but when we quote the rate, we just quote it as two times the semi-annual rate.

<u>General Rules for bond calculations (Calculator HP 17B):</u>

- P/YR will equal 2, to represent semi-annual convention for bonds
- I%YR will equal the bond's nominal interest rate

- N will generally be 2 times the years remaining to maturity; again as a result of the semi-annual convention
- PMT will equal ($100 x Coupon Rate) / 2
- FV will equal 100. In general, use 100 as the future value and scale your answer up if necessary.

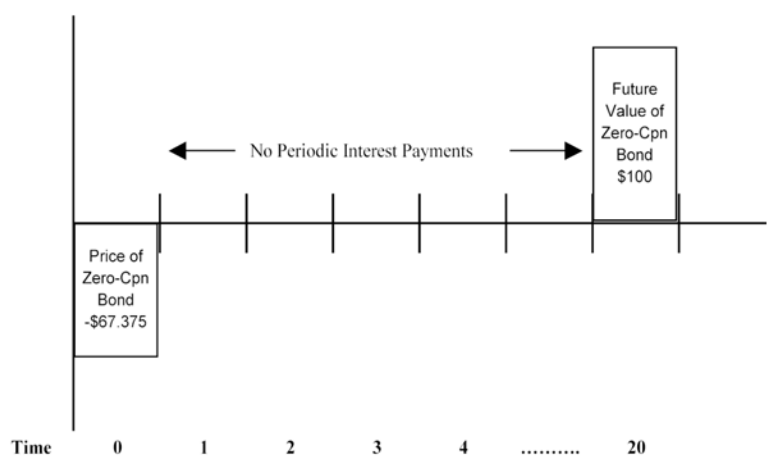
General Rules for bond calculations (Calculator HP 12C)
- I%YR will equal the bond's nominal interest rate
- i will equal the bond's nominal interest rate, I%YR, divided by 2 to represent the semi-annual convention for bonds
- n will generally be 2 times the years remaining to maturity; again as a result of the semi-annual convention
- PMT will equal ($100 x Coupon Rate) / 2
- FV will equal 100. In general, use 100 as the future value and scale your answer up if necessary.

Zero-Coupon Bonds

Zero-coupon bonds are perhaps the easiest to conceptualize. As the name implies, zero-coupon bonds do not pay periodic interest. Instead, the bonds are sold at deep discounts to par value. Thus, the interest rate is "built in" to the difference between the par value of the bond (paid at maturity) and its price today.

Example: Consider a $100 par value semi-annual zero-coupon (0.00% coupon) bond that matures in 10 years with a current price (its present value) of $67.375. What rate of interest is implicitly priced into the bond?

Step One: Diagram the cash flows



There are twenty periods (10 years, semi-annual) and no periodic interest payments are made.

Note that the future value of the bond equals its par value.

A) Cash Flows: The cost of the bond is known, $67.375 at Time 0. We also know the bond matures at $100 at Time 20
B) Discount Rate: Unknown, we will solve for this
C) Compounding assumption: Even though there are no periodic interest payments, the interest rate calculated for the zero-coupon bond must conform to the semi-annual convention used in the bond market. Therefore, you should assume two compounding periods per year.

Step Two: Determine Formula and Formula Components

In this case, we are solving for the Interest Rate (i) that will solve a basic FV or PV equation:

$$\boxed{FV = PV(1+i)^{Nper}} \quad \text{OR} \quad \boxed{PV = \frac{FV}{(1+i)^{Nper}}}$$

PV	=	-$67.375
FV	=	$100.00
Nper	=	20

Note: This problem is virtually identical to the example in *Calculating Present Value* above. The only difference is that we are compounding two times per year, which we can deal with several ways.

Step Three: Solve for Present Value

Using a Calculator (HP17B)

-67.375 [PV]
100 [FV]
0 [PMT] (= 0.00% x $100)
2 [P/YR]
10 [N]

➔ [I%YR] to solve

Answer: 3.99%

Note: An alternative method is to keep [P/YR] at 1 and put [N} = 20. Then, when you solve for [I%YR], you get 1.994%. per period. Thus, the annual interest rate (compounded semi-annually) = 2 x 0.1994 = 3.99%

Using a Calculator (HP12C)

67.375 [CHS] [PV]
100 [FV]
20 [n]

➔ [i] to solve

Answer: 1.994%

Nominal interest rate ➔

I%YR = 2 x 1.994% = 3.99%.

Using Excel

	A	B	C	D
1	PV	-67.375		
2	FV	100		
3	Nper	20		
4	I	0.019941		
5		=RATE(B3,,B1,B2)		
6				

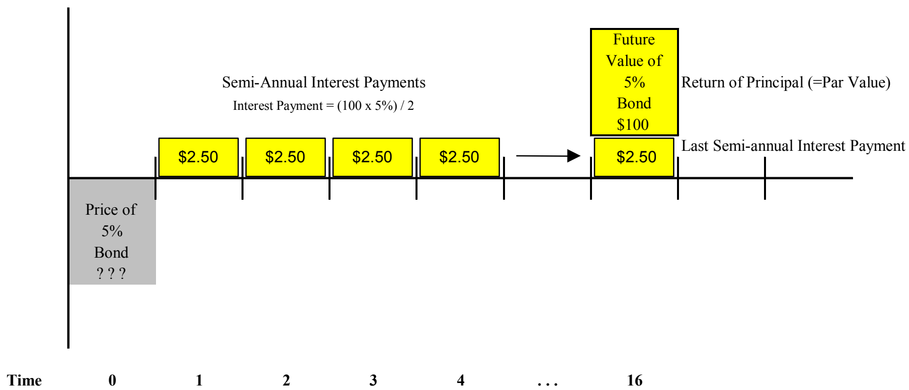
Note: RATE is calculated using 20 semi-annual periods. Therefore, to express RATE as a nominal rate, it must be doubled since there are two periods per year. Thus, the interest rate = 2 x 0.1994 = 3.99%

Coupon Bonds

Bonds that pay periodic interest in addition to the face value at maturity are called ***interest-bearing*** or ***coupon*** bonds. Interest payments typically are made in semi-annual installments up to and including the day of maturity. On the maturity date, a bond makes its final coupon payment in addition to returning the par value or remaining principal balance.

Example: Consider a $100 bond with a 5% annual coupon rate that matures in 8 years. What would be the price (PV) of the bond if the appropriate discount rate were 5.85%?

Step One: Diagram the Cash Flows



A) Cash Flows: Present Value at Time 0 is unknown. Time is in six-month-periods. Semi-annual coupons are paid beginning at Time 1. The bond matures at $100 at Time 16.
B) Discount Rate: Given at 5.85%
C) Compounding Assumption: Note that the quoted 5.85% is compounded semi-annually (since this is the standard convention for bonds) and should be translated to 2.925% per six month period.

Step Two: Determine Formula and Formula Components

The bond can be divided into two parts: principal and interest. The interest payments represent an annuity stretching 16 periods into the future. The principal represents a single payment received in period 16. Therefore, if one adds together the PV values from the Annuity Formula and the basic PV formula, one can find the price of the bond.

Interest Payments + *Principal*

$$\boxed{PV = \frac{PMT}{i}[1-(\frac{1}{1+i})^{Nper}]} \quad + \quad \boxed{PV = \frac{FV}{(1+i)^{Nper}}}$$

PMT	= $2.50	($100 x 5.00%) divided into semi-annual installments
Nper	= 16	8 years of semi-annual periods
FV	= $100	The amount of principal returned at maturity = Par Value
I%YR	= 5.85%	Remember i is the discount rate, not the coupon rate,
i	= 2.925%	and must be divided into semi-annual periods

Step Three: Solve for PV

Using a Calculator (HP 17B)

2.50	[PMT]
8	[N]
2	[P/YR]
5.85	[I%YR]
100	[FV]
➔	[PV] to solve

Answer: -$94.631

Using a Calculator (HP 12C)

2.50	[PMT]
16	[n]
2.925	[i]
100	[FV]
➔	[PV] to solve

Answer: -$94.631

Using Excel

	A	B	C
1	PMT	25	
2	Nper	16	
3	FV	100	
4	I	0.02925	
5	PV	-94.63079	
6		=PV(B4,B2,B1,B3)	

Note that both the calculator and Excel can incorporate the PMT feature directly into the PV calculation. The one difference is that Excel requires you to calculate a periodic interest rate, so that I = 0.02925 in the Excel model above. The HP calculator can makes this calculation for you automatically, as in the steps above. Alternatively, you may wish to keep track of the number of periods explicitly by changing [P/YR] to 1, [N] to 16, and [I%YR] to 2.925%.

A final note on pricing bonds using *Excel*: Excel has specialized bond pricing functions that can be accessed the same way as the PV and FV functions illustrated here. The bond pricing functions (detailed below) use dates, rather than number of periods, to make their calculations. You can use the built-in Help resource in Excel to practice using these functions if you like. However, the functions pre-suppose a sophisticated knowledge of bonds, so you may want to hold off using these functions until bonds are visited in the classroom.

PRICE	Calculates the price of $100 face value bond if given the date range, coupon and yield
YIELD	Calculates the yield of $100 face value bond if given the date range, coupon and price
ACCRINT	Accounts for partial first-period interest payments given the date range, coupon, and face value of the bond. Add to PRICE to get cash settlement price.

Practice Problems

Try to work through the following problems without looking at the answer key. The problems are provided to help you practice identifying the components of TVM problems as well as to get you used to working through the problems with your calculator.

1) The Bank of New Hampshire offers a two-year certificate of deposit (CD) with a stated annual percentage rate of 6.00%. A competing bank, Hanover Savings, offers its customers a money market account that pays 5.75% rate with monthly compounding. Assuming you would leave a deposit in the bank for the entire two years, in which bank would you rather deposit your money?

2) It is estimated that 7 years from now the total cost for one year of college will be $40,000.

    a) How much must be invested today in a CD paying 10.0 percent annual interest in order to accumulate the needed $40,000?

    b) If you only had $15,000 to invest, what rate of return would you need to reach the desired $40,000?

3) You need to purchase a computer on or before September 1 to begin the MBA program at Tuck. If the computer will cost $3,750, how much money would you need in the bank as of June 1, assuming the bank pays 3.0% interest compounded quarterly, to be able to purchase the computer on September 1?

4) Tuck students typically receive a signing bonus when they agree to full time employment upon completion of the MBA program. For the class of 2000, the signing bonus averaged about $20,000. Recently, the average signing bonus has gone up by about $2,000 per year, so that in two years, the average signing bonus is expected to reach $24,000. Using a discount rate of 10% and assuming the signing bonus is paid in exactly two years, what is the present value of the average bonus for the class of 2002?

5) Your poor, accident-prone friend Bob got into yet another car accident. Fortunately, Bob has good attorneys. As part of the settlement for his most recent injuries, Bob has a choice of taking $250,000 in cash now, or receiving 10 annual payments of $35,000. Bob, although a reckless driver, is a somewhat risk-averse investor who uses a 6.00% discount rate. Which settlement offer should Bob take?

6) Consider a stock that just paid its $2.00 annual dividend yesterday and assume that the dividend growth rate will be 5% a year. Assuming the dividend will continue into the future forever, what is the value of the stock TODAY if the appropriate discount rate is 14%?

7) What is the price of a bond with the following characteristics: 5.50% coupon, matures in three years, and has a yield (i.e., discount rate) of 6.00%?

8) What is the yield of a 6.75% four-year bond priced at 101.25?

Practice Problem Answers

1. The Bank of New Hampshire offers a two-year certificate of deposit (CD) with a stated annual percentage rate of 6.00%. A competing bank, Hanover Savings, offers its customers a money market account that pays 5.75% rate with monthly compounding. Assuming you would leave a deposit in the bank for the entire two years, in which bank would you rather deposit your money?

The easiest way to solve this problem is to think about what happens to $1.00 invested in each account over two years:

Bank of NH:		Hanover Savings:	
P/YR	= 1	P/YR	= 12
I%YR	= 6.00	I%YR	= 5.75
N	= 2	N	= 24
PV	= -1.00	PV	= -1.0
PMT	= 0.00	PMT	= 0.00
Solve FV = 1.1236		**Solve FV = 1.1216**	

You are better off with the Bank of NH CD.

2. It is estimated that 7 years from now the total cost for one year of college will be $40,000.

A) How much must be invested today in a CD paying 10.0 percent annual interest in order to accumulate the needed $40,000?

P/YR = 1
I%YR =10
N = 7
PMT = 0.00
FV = 40,000
Solve PV = -20,526.32

B) If you only had $15,000 to invest, what rate of return would you need to reach the desired $40,000?

P/YR = 1
N = 7
PMT = 0.00
PV = -15,000
FV = 40,000
Solve I%YR = 15.04%

3. You need to purchase a computer on or before September 1 to begin the MBA program at Tuck. If the computer will cost $3,750, how much money would you need in the bank as of June 1, assuming the bank pays 3.0% interest quarterly, to be able to purchase the computer on September 1?

I%YR = 3.0 / 4 (One-fourth of 3% each quarter)
N = 1
PMT = 0.00
FV = 3,750
Solve PV = -3,722.08

4. Tuck students typically receive a signing bonus when they agree to full time employment upon completion of the MBA program. For the class of 2000, the signing bonus averaged nearly $20,000. Recently, the average signing bonus has gone up by about $2,000 per year, so that in two years, the average signing bonus is expected to reach $24,000. Using a discount rate of 10.00% and assuming the signing bonus is paid in exactly two years, what is the present value of the average bonus for the class of 2002?

P/YR = 1
I%YR = 10
N = 2
PMT = 0.00
FV = 24,000
Solve PV = -19,834.71

5. Your unlucky, accident-prone friend Bob got into yet another car accident. Fortunately, Bob has good attorneys. As part of the settlement for his most recent injuries, Bob has a choice of taking $250,000 in cash now, or receiving 10 annual payments of $35,000. Bob, although a reckless driver, is a somewhat risk-averse investor who uses a 6.00% discount rate. Which settlement offer should Bob take?

P/YR = 1
I%YR = 6
N = 10
PMT = 35,000
Solve PV = -257,603.05 Bob should take the annuity since its PV is higher than $250,000

6. Consider a stock that just paid its $2.00 annual dividend yesterday and assume that the dividend growth rate will be 5% a year. Assuming the dividend will continue into the future forever, what is the value of the stock TODAY if the appropriate discount rate is 14%?

For this problem, you must use the Present Value of a Growing Perpetuity Formula

PMT = 2.10 (remember to always look one period from now, 2.10 = 2.00 x (1 + 5%))
i = 14%
g = 5%
Solve PV = 23.33 Yesterday's $2.00 dividend is irrelevant, you want to only pay for FUTURE CFs.

7. What is the price of a bond with the following characteristics: 5.50% coupon, matures in three years, and has a yield (i.e., discount rate) of 6.00%?

I%YR	= 6/2=3.0	Nominal discount rate
N	= 6	Three years of semi-annual payments
PMT	= 2.75	(100 x 5.50%)/2
FV	= 100	Bond convention assumes $100 FV

Solve PV = -98.64 so the fair price, given the discount rate, is $98.64 per $100 of face value of the bond.

8. What is the yield of a 6.75% four-year bond priced at 101.25?

N	= 8	four years of semi-annual payments
PMT	= 3.375	(100 x 6.75%)/2
PV	= -101.25	Given in problem
FV	= 100.00	Bond convention assumes $100 FV

Solve I%YR = 6.39% this means that if you pay $101.25 for every $100 of face value, you are earning 6.39 percent a year, compounded semi-annually