Logarithm Formulas

Expansion/Contraction Properties of Logarithms

These rules are used to write a single complicated logarithm as several simpler logarithms (called "expanding") or several simple logarithms as a single complicated logarithm (called "contracting"). Notice that these rules work for any base.

$\log_a(xy) = \log_a(x) + \log_a(y)$ (multiplication inside can be turned into addition outside, and vice versa.)

$\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$ (division inside can be turned into subtraction outside, and vice versa)

$\log_a(x^n) = n \cdot \log_a(x)$ (an exponent on everything inside can be moved out front, and vice versa)

Change of Base Formula

This formula is used to change a less helpful base to a more helpful one (generally base 10 or base e, since these appear on your calculator, but you can change to any base). In the formula below, a is the current base of your logarithm, and b is the base you would like to have instead.

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Cancellation Properties of Logarithms

These rules are used to solve for x when x is an exponent or is trapped inside a logarithm. Notice that these rules work for any base.

$\log_a(a^x) = x$ (this allows you to solve for x whenever it is in the exponent)

$a^{\log_a(x)} = x$ (this allows you to solve for x whenever it is inside a logarithm)

Logarithm Problems

1. Expand each expression (use expansion properties to expand as much as possible).

a. $\log_3(xy)$
Solution:

$$\log_3(xy) = \log_3(x) + \log_3(y) \qquad \text{(multiplication rule)}$$

b. $\log_7\left(\frac{x}{y}\right)$
Solution:

$$\log_7\left(\frac{x}{y}\right) = \log_7(x) - \log_7(y) \qquad \text{(division rule)}$$

c. $\ln(x^5)$
Solution:

$$\ln(x^5) = 5\ln(x) \qquad \text{(exponent rule)}$$

d. $\log\left(\frac{x^2 y^3}{z^4}\right)$

Solution: Whenever multiple rules seem to apply, you should always do them in *reverse* order of operations. Imagine you knew that $x = 5$, $y = 4$, and $z = 2$, and you wanted to evalaute $\frac{x^2 y^3}{z^4}$. Your work would be

$$\frac{5^2 \cdot 4^3}{2^4} = \frac{25 \cdot 64}{16} \qquad\qquad \text{(take care of exponents)}$$

$$= \frac{1600}{16} \qquad\qquad \text{(multiply numbers in numerator)}$$

$$= 100 \qquad\qquad \text{(divide)}$$

When expanding logarithms, you'll want to work in reverse. In this example, that means apply division rule, then the multiplication rule, then the exponent rule.

$$\log\left(\frac{x^2 y^3}{z^4}\right) = \log(x^2 y^3) - \log(z^4) \qquad\qquad \text{(division rule)}$$

$$= (\log(x^2) + \log(y^3)) - \log(z^4) \qquad\qquad \text{(multiplication rule)}$$

$$= \log(x^2) + \log(y^3) - \log(z^4) \qquad\qquad \text{(parentheses don't matter here)}$$

$$= 2\log(x) + 3\log(y) - 4\log(z) \qquad\qquad \text{(exponent rule)}$$

e. $\ln\left(\frac{x^2 - y}{wz^2}\right)$

Solution:

$$\ln\left(\frac{x^2 - y}{wz^2}\right) = \ln(x^2 - y) - \ln(wz^2) \qquad\qquad \text{(division rule)}$$

$$= \ln(x^2 - y) - (\ln(w) + \ln(z^2)) \qquad\qquad \text{(multiplication rule)}$$

$$= \ln(x^2 - y) - \ln(w) - \ln(z^2) \qquad\qquad \text{(distribute the minus sign)}$$

$$= \ln(x^2 - y) - \ln(w) - 2\ln(z) \qquad\qquad \text{(exponent rule)}$$

This problem has a couple tricky parts.

One is that you need to be careful about parentheses when you apply rules. When you have "$-\ln(wz^2)$", you're subtracting the *entire* quantity, so when you expand $\ln(wz^2)$ to $\ln(w) + \ln(z^2)$, you want to be subtracting this *entire* quantity. This is accomplished by writing "$-(\ln(w) + \ln(z^2))$" rather than "$-\ln(w) + \ln(z^2)$" (take a moment to convince yourself that the parentheses really make a difference here). Until you become comfortable with these kinds of problems, it's better to always write down parentheses and then decide later that you didn't actually need them. Having unnecessary parentheses doesn't hurt your answer, but not using parentheses where they are needed does.

The other thing to realize is that there is no rule to expand $\ln(x^2 - y)$. It is tempting to think that the division rule applies here, but read the rule very closely. It involves "log minus log", not "subtraction within a log". The same is true of addition; there is no rule to handle "addition within a log". Since the term cannot be expanded, we simply leave it as it is.

2. Contract each expression (use contraction properties to write as compactly as possible).

a. $\log_2(x) + \log_2(y)$

Solution:

$$\log_2(x) + \log_2(y) = \log_2(xy) \qquad\qquad \text{(multiplication rule)}$$

b. $\log_9(x) - \log_9(y)$

Solution:

$$\log_9(x) - \log_9(y) = \log_9\left(\frac{x}{y}\right) \qquad\qquad \text{(division rule)}$$

c. $5\log(x)$

Solution:

$$5\log(x) = \log(x^5) \qquad\qquad \text{(exponent rule)}$$

d. $\ln(x) + \ln(y) - 2\ln(z)$

Solution: When we expanded logarithms, we worked in reverse order of operations. Since contracting is the opposite of expanding, we should go back to the normal order of operations. For this example, that means taking care of the "times 2" in front of the "$\ln z$", then working from left to right after that.

$$\ln(x) + \ln(y) - 2\ln(z) = \ln(x) + \ln(y) - \ln(z^2) \qquad\qquad \text{(exponent rule)}$$
$$= \ln(xy) - \ln(z^2) \qquad\qquad \text{(multiplication rule)}$$
$$= \ln\left(\frac{xy}{z^2}\right) \qquad\qquad \text{(division rule)}$$

e. $2\log(2x) - 3(\log(y) + \log(z))$

Solution:

$$2\log(2x) - 3(\log(y) + \log(z)) = 2\log(2x) - 3\log(yz) \qquad\qquad \text{(multiplication rule)}$$
$$= \log((2x)^2) - \log((yz)^3) \qquad\qquad \text{(exponent rule)}$$
$$= \log(4x^2) - \log(y^3 z^3) \qquad\qquad \text{("distribute" the exponent, if desired)}$$
$$= \log\left(\frac{4x^2}{y^3 z^3}\right) \qquad\qquad \text{(division rule)}$$

Just like we saw earlier with the minus sign, you have to be careful when contracting using the power rule. When we move the number up that is to become an exponent, we want the *entire* contents of the logarithm to be affected by this new exponent. That's why we write, for example, "$\log((2x)^2)$" rather than "$\log(2x^2)$" (take a moment to convince yourself that the parentheses really make a difference here).

f. $\dfrac{\log_4(x) + 2\log_4(y)}{3\log(x) - \log(y)}$

Solution:

$$\frac{\log_4(x) + 2\log_4(y)}{3\log(x) - \log(y)} = \frac{\log_4(x) + \log_4(y^2)}{\log(x^3) - \log(y)} \qquad\qquad \text{(exponent rule)}$$
$$= \frac{\log_4(xy^2)}{\log(x^3) - \log(y)} \qquad\qquad \text{(multiplication rule)}$$
$$= \frac{\log_4(xy^2)}{\log\left(\frac{x^3}{y}\right)} \qquad\qquad \text{(division rule)}$$

It is tempting to try to contract the remaining two logarithms, but read the division rule very closely. It allows us to do something with "division within a log", not "log divided by log". There is no rule to handle this situation, so we simply leave it as it is. Similarly, there is no rule to handle "log times log", and so we cannot do anything to simplify problems where this appears.

3. Evaluate each expression (use change of base formula to convert to base 10 or base e).

a. $\log_2(5)$
Solution:

$$\begin{aligned}
\log_2(5) &= \frac{\log(5)}{\log(2)} && \text{(change of base formula)} \\
&\approx \frac{0.699}{0.301} && \text{(use calculator)} \\
&\approx 2.32 && \text{(divide)}
\end{aligned}$$

b. $\log_{1.05}(3.66)$
Solution:

$$\begin{aligned}
\log_{1.05}(3.66) &= \frac{\log(3.66)}{\log(1.05)} && \text{(change of base formula)} \\
&\approx \frac{0.563}{0.021} && \text{(use calculator)} \\
&\approx 26.8 && \text{(divide)}
\end{aligned}$$

4. Solve for x. Give approximate answers. (Hint: Simplify as much as possible, then use cancellation and expansion properties to isolate x.)

a. $10^x = 50$
Solution:

$$\begin{aligned}
10^x &= 50 \\
\log(10^x) &= \log(50) && \text{(take log of both sides)} \\
x &= \log(50) && \text{(cancellation property)} \\
x &\approx 1.70 && \text{(use calculator)}
\end{aligned}$$

b. $2 \cdot 5^{2x+1} = 2000$
Solution:

$$\begin{aligned}
2 \cdot 5^{2x+1} &= 2000 \\
5^{2x+1} &= 1000 && \text{(divide both sides by 2)} \\
\log_5(5^{2x+1}) &= \log_5(1000) && \text{(take } \log_5 \text{ of both sides)} \\
2x + 1 &\approx \log_5(1000) && \text{(cancellation property)}
\end{aligned}$$

Let's pause for a moment here. When we're done with this problem, we want some decimal number for x. Even though $\log_5(1000)$ looks complicated, it really is just some decimal number. Off to the side, let's figure out what decimal that is.

$$\begin{aligned}
\log_5(1000) &= \frac{\log(1000)}{\log(5)} && \text{(change of base formula)} \\
&\approx \frac{3}{0.699} && \text{(use calculator)} \\
&\approx 4.29 && \text{(divide)}
\end{aligned}$$

So, $\log_5(1000) \approx 4.29$. Let's return to our original problem with this new fact in hand.

$$\begin{aligned}
2x + 1 &= \log_5(1000) \\
2x + 1 &\approx 4.29 && \text{(substitute what we just found)} \\
2x &\approx 3.29 && \text{(subtract 1 from both sides)} \\
x &\approx 1.65 && \text{(divide both sides by 2)}
\end{aligned}$$

c. $4^{5x+3} = 4^{2x-9}$

Solution:

$$4^{5x+3} = 4^{2x-9}$$

$$\log_4(4^{5x+3}) = \log_4(4^{2x-9}) \qquad \text{(take } \log_4 \text{ of both sides)}$$

$$5x + 3 = 2x - 9 \qquad \text{(cancellation property)}$$

$$3x = -12 \qquad \text{(collect like terms)}$$

$$x = -4 \qquad \text{(divide both sides by 3)}$$

d. $4^{5x+3} = 10 \cdot 4^{2x-9}$

Solution:

$$4^{5x+3} = 10 \cdot 4^{2x-9}$$

$$\log_4(4^{5x+3}) = \log_4(10 \cdot 4^{2x-9}) \qquad \text{(take } \log_4 \text{ of both sides)}$$

$$\log_4(4^{5x+3}) = \log_4(10) + \log_4(4^{2x-9}) \qquad \text{(multiplication rule)}$$

$$5x + 3 = \log_4(10) + (2x - 9) \qquad \text{(cancellation property)}$$

$$5x + 3 = \log_4(10) + 2x - 9 \qquad \text{(parentheses don't matter here)}$$

$$3x = \log_4(10) - 12 \qquad \text{(collect like terms)}$$

$$3x \approx 1.66 - 12 \qquad \text{(use change of base formula)}$$

$$3x \approx -10.3 \qquad \text{(subtract)}$$

$$x \approx -3.43 \qquad \text{(divide both sides by 3)}$$

5. Solve for x. Give approximate answers. (Hint: Use contraction properties to write as a single logarithm, then use cancellation properties to isolate x.)

a. $\ln(x) = 3$

Solution: In solving problems of this kind (where x is trapped inside a logarithm), we make use of a technique that I refer to as $10^{\text{LHS}} = 10^{\text{RHS}}$ ("ten to the lefthand side equals ten to the righthand side"). The name is a little misleading, since we aren't always using 10, but the idea is that you introduce a new number on both sides of the equation somehow "underneath" the existing terms. It's easiest to see in an example.

$$\ln(x) = 3$$

$$e^{\ln(x)} = e^3 \qquad (e^{\text{LHS}} = e^{\text{RHS}})$$

$$x = e^3 \qquad \text{(cancellation property)}$$

$$x \approx 20.1 \qquad \text{(use calculator)}$$

In this example, we wanted to cancel the "ln", so we introduce an "e". We do *not* raise both sides to the power of e. Instead, e becomes the base and the LHS and RHS, respectively, go up in the exponent.

b. $\log_3(5x) - \log_3(3x) + \log_3(x) = 4$

Solution: The strategy with this problem is to use contraction rules to get our expression down to a single logarithm, then make use of the cancellation property.

$$\log_3(5x) - \log_3(3x) + \log_3(x) = 4$$

$$\log_3\left(\frac{5x}{3x}\right) + \log_3(x) = 4 \qquad \text{(division rule)}$$

$$\log_3\left(\frac{5}{3}\right) + \log_3(x) = 4 \qquad \text{(cancel the } x\text{)}$$

$$\log_3\left(\frac{5x}{3}\right) = 4 \qquad \text{(multiplication rule)}$$

$$3^{\log_3\left(\frac{5x}{3}\right)} = 3^4 \qquad (3^{\text{LHS}} = 3^{\text{RHS}})$$

$$\frac{5x}{3} = 81 \qquad \text{(cancellation property)}$$

$$5x = 243 \qquad \text{(multiply by sides by 3)}$$

$$x = 48.6 \qquad \text{(divide both sides by 5)}$$

c. $\ln(x) + \ln(x + 2) = \ln(8)$

Solution:

$$\ln(x) + \ln(x + 2) = \ln(8)$$

$$\ln(x(x + 2)) = \ln(8) \qquad \text{(multiplication rule)}$$

$$\ln(x^2 + 2x) = \ln(8) \qquad \text{(distribute the } x\text{)}$$

$$e^{\ln(x^2 + 2x)} = e^{\ln(8)} \qquad (e^{\text{LHS}} = e^{\text{RHS}})$$

$$x^2 + 2x = 8 \qquad \text{(cancellation property)}$$

$$x^2 + 2x - 8 = 0 \qquad \text{(move everything to one side)}$$

$$(x - 4)(x + 2) = 0 \qquad \text{(factor)}$$

So, $x = 4$ and $x = -2$ are *potential* solutions. There is a slight glitch, however. Graph $\ln(x)$ on your calculator, and you'll notice that it is only defined for $x > 0$. That is, expressions like $\ln(-4)$ or $\ln(0)$ produce an error, just like dividing by zero or taking the square root of a negative number. If we look at our original equation, $x = 4$ will go through fine, but $x = -2$ will cause a problem, since the terms $\ln(x)$ and $\ln(x+2)$ will both be undefined. So, we have to throw away $x = -2$, which leaves $x = 4$ as the unique solution. This problem of undefinedness is the same for *any* base. That is, $\log_a(x)$ is defined only for $x > 0$ no matter what base a you are using.

d. $\log_2(x + 1) - 2\log_2(x) = 1$

Solution:

$$\log_2(x + 1) - 2\log_2(x) = 1$$

$$\log_2(x + 1) - \log_2(x^2) = 1 \qquad \text{(exponent rule)}$$

$$\log_2\left(\frac{x + 1}{x^2}\right) = 1 \qquad \text{(division rule)}$$

$$2^{\log_2\left(\frac{x+1}{x^2}\right)} = 2^1 \qquad (2^{\text{LHS}} = 2^{\text{RHS}})$$

$$\frac{x + 1}{x^2} = 2 \qquad \text{(cancellation property)}$$

$$x + 1 = 2x^2 \qquad \text{(multiply both sides by } x^2\text{)}$$

$$0 = 2x^2 - x - 1 \qquad \text{(move everything to one side)}$$

$$0 = (2x + 1)(x - 1) \qquad \text{(factor)}$$

So, $x = -\frac{1}{2}$ and $x = 1$ are potential solutions. As before, we have to throw away the $x = -\frac{1}{2}$, since $\log_2(x)$ is undefined for this input.

6. A certain population of 1000 bacteria doubles in size every minute.

a. Model the growth of this population as a function of time (in minutes).

Solution: If you don't spot the function right away, think about how you might find out the number of bacteria you have during the first few minutes. If t dentoes the number of minutes that have passed, then we have 1000 bacteria when $t = 0$. What about $t = 1$? Our population is supposed to double, so we have $1000 \cdot 2$ (i.e. 2000) bacteria when $t = 1$. What about $t = 2$? We double the population again. Let's write it as $1000 \cdot 2 \cdot 2$ or $1000 \cdot 2^2$. Do you see the pattern? To get the number of bacteria at a certain minute, we multiply by 2 a certain number of times. The number of times we multiply is exactly the same as the number of minutes passed. If we call our function $f(t)$, then $f(t) = 1000 \cdot 2^t$.

b. How big is the population after three minutes and thirty seconds?

Solution: Using our notation from part a, the question says "What is $f(t)$ when $t = 3.5$?". To find out, we simply plug in this value for t.

$$
\begin{aligned}
f(3.5) &= 1000 \cdot 2^{3.5} && \text{(plug in } t = 3.5\text{)} \\
&\approx 1000 \cdot 11.3 && \text{(use calculator)} \\
&\approx 11,300 && \text{(multiply)}
\end{aligned}
$$

So, our population is approximately 11,300 bacteria after just 3.5 seconds.

c. When will the population reach 3000?

Solution: Using our notation from part a, the question says "What is t when $f(t) = 3000$?". Notice it does *not* say $t = 3000$. This is a very common mistake. We decided that t would be minutes, and the 3000 refers to bacteria, not minutes. Anyway, let's start to solve it.

$$
\begin{aligned}
3000 &= 1000 \cdot 2^t && \text{(plug in } f(t) = 3000\text{)} \\
3 &= 2^t && \text{(divide both sides by 1000)}
\end{aligned}
$$

We've solved problems like this before. We need t, but t is trapped up in an exponent. To get it down, we take the $\log_2$ of both sides.

$$
\begin{aligned}
3 &= 2^t \\
\log_2(3) &= \log_2(2^t) && \text{(take } \log_2 \text{ of both sides)} \\
\log_2(3) &= t && \text{(cancellation property)} \\
1.58 &\approx t && \text{(use change of base formula)}
\end{aligned}
$$

So, after approximately 1.58 seconds, our population will reach the desired size of 3,000 bacteria.