3.1 Simple Interest

- Definition: **I = Prt**
- **I = interest earned**
- **P = principal (amount invested)**
- **r = interest rate (as a decimal)**
- **t = time**

An example:

- Find the interest on a boat loan of $5,000 at 16% for 8 months.

- Solution: Use I = Prt
  - I = 5,000(0.16)(0.6667)
  - (8 months = 8/12 of one year = 0.6667 years)
    - I = $533.36

Total amount to be paid back

- The total amount to be paid back for the boat loan would be $5000 plus the interest of $533.36 for a total of $5,533.36.
- In general, the future value (amount) is given by the following equation:

$$A = P + Prt$$
$$= P(1 + rt)$$

Another example:

- Find the total amount due on a loan of \$600 at 16% interest at the end of 15 months.
- solution: A =P(1+rt)

A = 600(1+0.16(1.25))

A = \$720.00

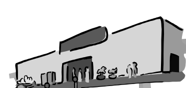
Interest rate earned on a note

- What is the annual interest rate earned by a 33-day T-bill with a maturity value of \$1,000 that sells for \$996.16?
- Solution: Use the equation A =P(1+rt)
- $1{,}000 = 996.16\left(1+r\left(\frac{33}{360}\right)\right)$
- 1000 = 996.16(1+r(0.09166))

- Solve for r:

$$1000 = 996.16(1+r(0.09166)) \rightarrow$$
$$1000=996.16+996.16(0.09166)r \rightarrow$$
$$\frac{1000-996.16}{996.16(0.09166)} = r \rightarrow$$
$$r = 0.042 = 4.2\%$$

Another application



- A department store charges 18.6% interest (annual) for overdue accounts. How much interest will be owed on a \$1080 account that is 3 months overdue?
- Solution:

A = P(1 + rt)

A = 1080(1+0.186(0.25))

A = 1080(1.0465)

A= 1130.22

I = 1130.22 – 1080 =50.22

3.2 Compound Interest

- Unlike simple interest, compound interest on an amount accumulates at a faster rate than simple interest. The basic idea is that after the first interest period, the amount of interest is added to the principal amount and then the interest is computed on this higher principal. The latest computed interest is then added to the increased principal and then interest is calculated again. This process is completed over a certain number of compounding periods. The result is a much faster growth of money than simple interest would yield.

An example

- As an example, suppose a principal of $1.00 was invested in an account paying 6% annual interest compounded monthly. How much would be in the account after one year?
- Solution:
- 1. amount after one month
- 2. amount after two months
- 3. amount after three months

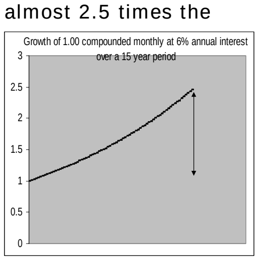
$$1+\frac{0.06}{12}(1)=1(1+0.005)=1.005$$

$$1.005(1+\frac{0.06}{12})=1.005(1.005)=1.005^2$$

$$1.005^2\left(1+\frac{0.06}{12}\right)=1.005^2(1.005)=1.005^3$$

Compound Interest

Growth of 1.00 compounded monthly at 6% annual interest over a 15 year period (Arrow indicates an increase in value of almost 2.5 times the original amount.)



General formula

- From the previous example, we arrive at a generalization: The amount to which 1.00 will grow after n months compounded monthly at 6% annual interest is :

$$\left(1+\frac{0.06}{12}\right)^n = (1.05)^n$$

- This formula can be generalized to
- where A is the future amount, P is the principal, r is the interest rate as a decimal, m is the number of compounding periods in one year and t is the total number of years. To simplify the formula, I
- $A = P\left(1+\frac{r}{m}\right)^{mt}$ = $A = P(1+i)^n$ where $i = \frac{r}{m}$ $n = mt$

Example

- Find the amount to which \$1500 will grow if compounded quarterly at 6.75% interest for 10 years.
- Solution: Use $A = P(1+i)^n$

$$A = \overbrace{1500}^{4}\left(\underbrace{1+}_{2}\overbrace{\frac{0.0675}{4}}^{1}\right)^{\overbrace{10(4)}^{3}}$$

$$A = 2929.50$$

- Helpful hint: Be sure to do the arithmetic using the rules for order of operations. See arrows in formula above

Same problem using simple interest

- Using the simple interest formula, the amount to which \$1500 will grow at an interest of 6.75% for 10 years is given by:
- A=P(1+rt)
- A=1500(1+0.0675(10))=2512.50, which is more than \$400 less than the amount earned using the compound interest formula.

Changing the number of compounding periods per year

To what amount will $1500 grow if compounded *daily at 6.75%* interest for 10 years?

Solution: $A = 1500\left(1 + \frac{0.0675}{365}\right)^{10(365)}$

$= 2945.87$

This is about $15.00 more than compounding $1500 quarterly at 6.75% interest.

Since there are 365 days in year (leap years excluded), the number of compounding periods is now 365. We divide the annual rate of interest by 365. Notice too that the number of compounding periods in 10 years is 10(365) = 3650.

Effect of increasing the number of compounding periods

- If the number of compounding periods per year is increased while the principal, annual rate of interest and total number of years remain the same, the future amount of money will increase slightly.

Computing the inflation rate

- Suppose a house that was worth $68,000 in 1987 is worth $104,000 in 2004. Assuming a constant rate of inflation from 1987 to 2004, what is the inflation rate?
- 1. Substitute in compound interest formula.
- 2. Divide both sides by 68,000
- 3. Take the 17th root of both sides of equation
- 4. Subtract 1 from both sides to solve for r.

- Solution:

$$104,000 = 68,000(1+r)^{17} \rightarrow$$

$$\frac{104,000}{68,000} = (1+r)^{17} \rightarrow$$

$$\sqrt[17]{\frac{104,000}{68,000}} = (1+r) \rightarrow$$

$$\sqrt[17]{\frac{104,000}{68,000}} - 1 = r = 0.0253$$

Inflation rate continued

- If the inflation rate remains the same for the next 10 years, what will the house be worth in the year 2014?
- Solution: From 1987 to 2014 is a period of 27 years. If the inflation rate stays the same over that period, r = 0.0253. Substituting into the compound interest formula, we have

$$A = 68,000(1+0.0253)^{27} = 133,501$$

Growth time of an investment

- How long will it take for \$5,000 to grow to \$15,000 if the money is invested at 8.5% compounded quarterly?
- 1. Substitute values in the compound interest formula.
- 2. divide both sides by 5,000
- 3. Take the natural logarithm of both sides.
- 4. Use the exponent property of logarithms
- 5. Solve for t.
- (*Note: you will most unlikely see this amount during your lifetime)*

- Solution:

$$15,000 = 5,000\left(1+\frac{0.085}{4}\right)^{4t} \rightarrow$$

$$3,000 = (1.02125)^{4t} \rightarrow$$

$$\ln(3,000) = \ln\left((1.02125)^{4t}\right) \rightarrow$$

$$\ln(3,000) = 4t\ln(1.02125) \rightarrow$$

$$\frac{\ln(3,000)}{4\ln(1.02125)} = t = 95.2$$

Annual percentage yield

- The simple interest rate that will produce the same amount in 1 year is called the **annual percentage yield (APY)**. To find the **APY,** proceed as follows: This is also referred to as the **effective rate.**

$$\begin{pmatrix} \text{amount at} \\ \text{simple interest} \\ \text{after 1 year} \end{pmatrix} = \begin{pmatrix} \text{amount at} \\ \text{compound interest} \\ \text{after 1 year} \end{pmatrix}$$

$$P(1+APY) = P\left(1+\frac{r}{m}\right)^{m} \rightarrow$$

$$1+APY = \left(1+\frac{r}{m}\right)^{m} \rightarrow$$

$$APY = \left(1+\frac{r}{m}\right)^{m} - 1$$

Effective Rate of interest

- What is the effective rate of interest for money that is invested at :
- A) 6% compounded monthly?
- General formula: $APY = \left(1+\frac{r}{m}\right)^m - 1$
- Substitute values: $APY = \underbrace{\left(1+\overset{1}{\frac{0.06}{12}}\right)^{\overset{3}{12}}}_{2} - 1 = 0.06168$
- Effective rate 0.06168
- Hint: Use the correct order of operations as indicated by the numbers

Computing the Annual nominal rate given the effective rate

- What is the annual nominal rate compounded monthly for a CD that has an annual percentage yield of 5.9%?
- 1. Use the general formula for APY.
- 2. Substitute value of APY and 12 for m (number of compounding periods per year).
- 3. Add one to both sides
- 4. Take the twelfth root of both sides of equation.
- 5. Isolate r (subtract 1 and then multiply both sides of equation by 12.

$$APY = \left(1+\frac{r}{m}\right)^m - 1$$

$$0.059 = \left(1+\frac{r}{12}\right)^{12} - 1$$

$$1.059 = \left(1+\frac{r}{12}\right)^{12}$$

$$\sqrt[12]{1.059} = \left(1+\frac{r}{12}\right)$$

$$\sqrt[12]{1.059} - 1 = \frac{r}{12}$$

$$12\left(\sqrt[12]{1.059} - 1\right) = r$$

$$0.057 = r$$

3.3 Future value of an Annuity;Sinking Funds

- **An annuity is any sequence of equal periodic payments.**
- **An ordinary annuity is one in which payments are made at the end of each time interval. If for example, $100 is deposited into an account every quarter (3 months) at an interest rate of 8% per year, the following sequence illustrates the growth of money in the account:**

$$100+100\left(1+\frac{0.08}{4}\right)+100(1.02)(1.02)+100(1.02)(1.02)(1.02)$$

$$100+\underbrace{100(1.02)}_{3^{rd}\text{ qtr}}+\underbrace{100(1.02)^2}_{2^{nd}\text{ qtr}}+\underbrace{100(1.02)^3}_{1^{st}\text{ qtr}}$$

General formula for future value of an annuity

- Here, R is the periodic payment, i is the interest rate per period and n is the total number of periods. S is the future value of the annuity:

$$S=R\frac{(1+i)^n-1}{i}$$

Notational changes

- FV= future value (amount)
- PMT=periodic payment
- i= interest rate per period
- n = total number of payments

$$FV=PMT\left(\frac{(1+i)^n-1}{i}\right)$$

Example

- Suppose a payment is made at the end of each quarter and the money in the account is compounded quarterly at 6.5% interest for 15 years. How much is in the account after the 15 year period?
- Solution: $$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

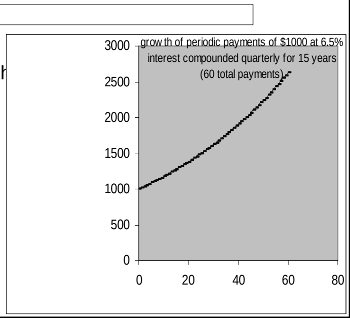
$$FV = 1000\left(\frac{\left(1+\frac{0.065}{4}\right)^{4(15)} - 1}{\frac{0.065}{4}}\right) = 100,336.68$$

Amount of interest earned

- How much interest was earned over the 15 year period?
- Solution:
- Each periodic payment was $1000. Over 15 years, 15(4)=60 payments made for a total of $60,000. Total amount in account after 15years is $100,336.68. Therefore, amount of accrued interest is $100,336.68-$60,000 = $40,336.68.

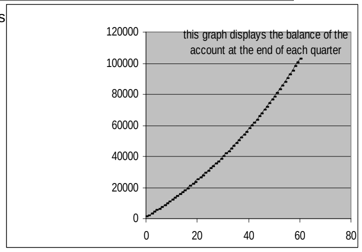
Graphical Display

- This graph
- displays the growth time



Balance in the Account at the end of each period

- This graph displays the balance in the account at the end of each quarter:



Sinking Fund

- Definition:Any account that is established for accumulating funds to meet future obligations or debts is called a sinking fund.
- The sinking fund payment is defined to be the amount that must be deposited into an account periodically to have a given future amount.

Sinking fund payment formula:

- To derive the sinking fund payment formula, we use algebraic techniques to rewrite the formula for the future value of an annuity and solve for the variable PMT:

$$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

$$FV\left(\frac{i}{(1+i)^n - 1}\right) = PMT$$

Sample problem

- How much must Harry save each month in order to buy a new car in three years if the interest rate is 6% compounded monthly?

$$FV\left(\frac{i}{(1+i)^n-1}\right)=PMT$$

$$12000\left(\frac{\frac{0.06}{12}}{\left(1+\frac{0.06}{12}\right)^{36}-1}\right)=pmt=305.06$$

Approximating interest rates

- Mr. Ray has deposited $150 per month into an ordinary annuity. After 14 years, the annuity is worth $85,000. What annual rate compounded monthly has this annuity earned during the 14 year period?
-
- Solution: Use the FV formula: Here FV = $85,000, PMT = $150 and n, number of payments is 14(12)=168. Substitute these values into the formula. Solution is approximated graphically.

Graphical solution

- Solution:

$$FV = PMT\left(\frac{(1+i)^n-1}{i}\right)$$

$$85,000=150\left(\frac{(1+i)^{14(12)}-1}{i}\right)$$

$$\frac{85,000}{150}=\left(\frac{(1+i)^{168}-1}{i}\right)$$

$$y=\left(\frac{(1+x)^{168}-1}{x}\right)=\frac{85,000}{150}=566.67$$

- By determining the point of intersection of the two graphs using a graphing calculator, we obtain an approximate solution of 0.013 or 1.3% rate of return.

Present Value of an Annuity; Amortization (3.4)

- In this section, we will address the problem of determining the amount that should be deposited into an account now at a given interest rate in order to be able to withdraw equal amounts from the account in the future until no money remains in the account. Here is an example: How much money must you deposit now at 6% interest compounded quarterly in order to be able to withdraw $3,000 at the end of each quarter year for two years?

Derivation of formula

- We begin by solving for P in the compound interest formula:

$$A = P\left(1+i\right)^{n} \rightarrow$$

$$P = A(1+i)^{-n}$$

Present value of the first four payments:

- Interest rate each period is 0.06/4=0.015

$$P_1 = 3000\left(1+\frac{0.06}{4}\right)^{-1}$$

$$P_2 = 3000\left(1.015\right)^{-2}$$

$$P_3 = 3000(1.015)^{-3}$$

$$P_4 = 3000(1.015)^{-4}$$

Derivation of short cut formula

- We could proceed to calculate the next four payments and then simply find the total of the 8 payments (there are 8 payments since there will be 8 total withdrawals – 2 years x four withdrawals per year = 8 total withdrawals). This method is tedious and time consuming so we seek a short cut method.

General formula

- In a manner similar to deriving previous formulas, the result is as follows: Here, R is the periodic payment, i is the interest rate per period, and n is the total number of periods. The present value of all payments is given by:

$$P = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

Back to our original problem:

- How much money must you deposit now at 6% interest compounded quarterly in order to be able to withdraw $3,000 at the end of each quarter year for two years?
- Solution: R = 3000, i=0.06/4=0.015, n = 8

$$P = R\left(\frac{1-(1+i)^{-n}}{i}\right) \rightarrow$$

$$P = 3000\left(\frac{1-(1.015)^{-8}}{0.015}\right) = 22,457.78$$

Interest earned

- The present value of all payments is \$22,457.78. The total amount of money withdrawn over two years is 3000(4)(2)=24,000. Thus, the accrued interest is the difference between the two amounts: 24,000-22,457.78 =1542.22.

Amortization

- Problem: A bank loans a customer \$50,000 to purchase a house at 4.5% interest per year. The customer agrees to make monthly payments for the next 15 years for a total of 180 payments. How much should the monthly payment be if the debt is to be retired in 15 years?
- Solution: The bank has bought an annuity from the customer. This annuity pays the bank a \$PMT per month at 4.5% interest compounded monthly for 180 months.

Amortization

- We use the previous formula for present value of an annuity and solve for PMT:

$$PV = PMT\left(\frac{1-(1+i)^{-n}}{i}\right) \to$$

$$PMT = PV\left(\frac{i}{1-(1+i)^{-n}}\right)$$

Solving the problem

- Care must be taken to perform the correct order of
- operations.
- 1. enter 0.045 divided by 12
- 2. 1 + step 1 result
- 3. Raise answer to -180 power.
- 4. 1 – step 3 result
- 5. Take reciprocal (1/x) of step 4 result. Multiply by 0.045 and divide by 12.
- 5. Finally, multiply that result by 50,000 to obtain 382.50

- Solution:

$$PMT = PV\left(\frac{i}{1-(1+i)^{-n}}\right) \rightarrow$$

$$PMT = 50,000\left(\frac{\frac{0.045}{12}}{1-\left(1+\frac{0.045}{12}\right)^{-180}}\right) = 382.50$$

Total amount of payments and interest paid

- If the customer makes a monthly payment of \$382.50 to the bank for 180 payments, then the total amount paid to the bank is the product of 382.50 and 180 = 68,850. Thus, the interest earned by the bank is the difference between 68,850 and 50,000 (original loan) = 18,850.