Differential Equations
Equation of Motion
Converting the Form <u>A sin x + B cos x</u> to the Form <u>K sin(x + ϕ)</u>

Given the expression A sin x + B cos x, we are going to change it to the expression K sin(x + ϕ).

Find the value of K and ϕ.

$$K = \sqrt{A^2 + B^2}$$

$$A \sin x + B \cos x = \sqrt{A^2 + B^2}\left(\frac{A \sin x}{\sqrt{A^2 + B^2}} + \frac{B \cos x}{\sqrt{A^2 + B^2}}\right)$$

$$\sin \phi = \frac{B}{\sqrt{A^2 + B^2}} \qquad \cos \phi = \frac{A}{\sqrt{A^2 + B^2}}.$$

$$A \sin x + B \cos x = K(\sin x \cos \phi + \cos x \sin \phi) = K \sin(x + \phi).$$

<u>Example</u>: convert 5sin x - 7cos x

$$K = \sqrt{5^2 + (-7)^2} = \sqrt{25 + 49} = \sqrt{74} = 8.6$$

$$\sin \phi = \frac{-7}{8.6} \quad \phi = -54.5^\circ \qquad \cos \phi = \frac{5}{8.6} \quad \phi = 54.5^\circ$$

Since the sine is negative and the cosine is positive, ϕ must be in quadrant IV, and $\phi = -54.5^\circ$. The final equation is:

$$5\sin x - 7\cos x = 8.6\sin(x - 54.5)$$

When the equation of motion $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ is converted to $x(t) = A \sin(\omega t + \phi)$, where $A = \sqrt{c_1^{\ 2} + c_2^{\ 2}}$, A is the <u>amplitude</u> of free vibrations (ϕ is the <u>phase angle</u>).

<u>Example</u>: $2\sin x + 5\cos x$ is plotted. What is the amplitude of the graph?

$$K = \sqrt{2^2 + 5^2} = \sqrt{29} = 5.4 \qquad 2\sin x + 5\cos x = 5.4\sin(x + \phi)$$

Therefore the amplitude is 5.4.

Exercises: Convert:

1. $3 \sin x - \sqrt{3} \cos x$

2. $6 \sin x + 8 \cos x$

3. $2 \sin x + \cos x$

4. $\sin x + \cos x$

5. $\sqrt{3} \sin x - 3 \cos x$

Answers:

1. $2\sqrt{3} \sin(x - 30)$

2. $10 \sin(x + 53.1)$

3. $\sqrt{5} \sin(x + 26.6)$

4. $\sqrt{2} \sin(x + 45)$

5. $2\sqrt{3} \sin(x - 60)$