Unit 3

Congruence & Proofs

Module 1: Congruence, Proof and Constructions

Topic D – Congruence (G-CO.7, G-CO.8)

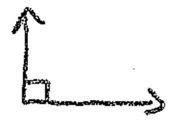
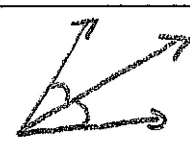
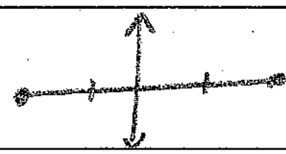
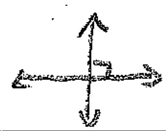
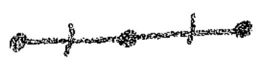
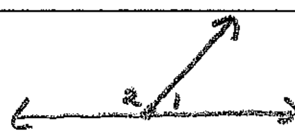
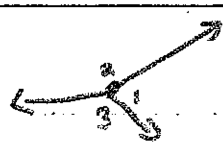
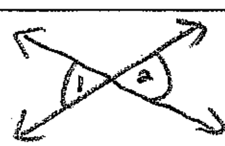
Topic E – Proving Properties of Geometric Figures (G-CO.9, G-CO.10, G-CO.11)

Unit 3
Congruence & Proofs

Lesson 1: Introduction to Triangle Proofs

Opening Exercise

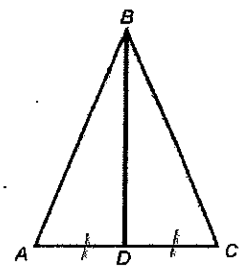
Using your knowledge of angle and segment relationships from Unit 1, fill in the following:

Definition/Property/Theorem	Diagram/Key Words	Statement
Definition of **Right Angle**		An angle that measures 90°
Definition of **Angle Bisector**		A ray that divides a angle into 2 ≅ angles.
Definition of **Segment Bisector**		A segment, line, or ray that divides a segment into two equal segments.
Definition of **Perpendicular**	 (⊥)	When two lines, segments or rays intersect forming 90° angle.
Definition of **Midpoint**		Divides a line segment into two equal segments
Angles on a Line		$\angle 1 + \angle 2 = 180^\circ$
Angles at a Point		$\angle 1 + \angle 2 + \angle 3 = 360^\circ$
Angles Sum of a Triangle		$\angle 1 + \angle 2 + \angle 3 = 180^\circ$
Vertical Angles		$\angle 1 \cong \angle 2$

Example 1

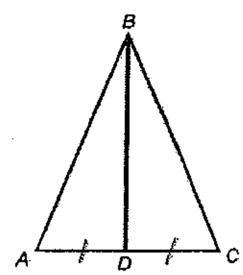
We are now going to take this knowledge and see how we can apply it to a proof. In each of the following you are given information. You must interpret what this means by first marking the diagram and then writing it in proof form.

a. Given: D is the midpoint of $\overline{AC}$



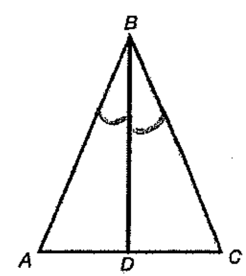
Statements	Reasons
1. D is the midpoint of $\overline{AC}$	1. Given
2. $\overline{AD} \cong \overline{DC}$	2. Def. of a midpoint

b. Given: $\overline{BD}$ bisects $\overline{AC}$



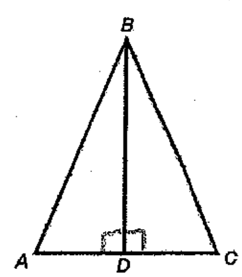
Statements	Reasons
1. $\overline{BD}$ bisects $\overline{AC}$	1. Given
2. $\overline{AD} \cong \overline{DC}$	2. Def. of a segment bisector.

c. Given: $\overline{BD}$ bisects $\angle ABC$



Statements	Reasons
1. $\overline{BD}$ bisects $\angle ABC$	1. Given
2. $\angle ABD \cong \angle DBC$	2. Def. of an angle bisector.

d. Given: $\overline{BD} \perp \overline{AC}$



Statements	Reasons
1. $\overline{BD} \perp \overline{AC}$	1. Given
2. $\angle ADB$ and $\angle CDB$ are right $\angle$'s	2. $\perp$ lines form right $\angle$'s
3. $\angle ADB \cong \angle COB$	3. All right $\angle$'s are $\cong$

Example 2

Listed below are other useful properties we've discussed that will be used in proofs.

Property / Postulate	In Words	Statement
Addition Postulate	Equals added to equals are equal.	$-2m - 24 = 3m + 6$ $+2m \quad +2m$
Subtraction Postulate	Equals subtracted from equals are equal.	$6d - 12 = 11d + 53$ $-6d \quad -6d$
Multiplication Postulate	Equals multiplied by equals are equal.	$\frac{3}{4}(\frac{4}{3}x) = (27)\frac{3}{4}$
Division Postulate	Equals divided by equals are equal.	$\frac{4x}{4} = \frac{20}{4}$
Partition Postulate	The whole is equal To the sum of its parts.	A B C $\overline{AB} + \overline{BC} = \overline{AC}$
Substitution	A quantity may be substituted for an equal quantity.	$d = 10$
Reflexive	Anything is equal to itself	$a = a$

<u>The two most important properties about parallel lines to remember</u>:

1. If lines are ∥, then Alt. Int ∠'s are ≅.

2. If lines are ∥, then Same side int ∠'s are supplementary

Lesson 2: Congruence Criteria for Triangles - SAS

Opening Exercise

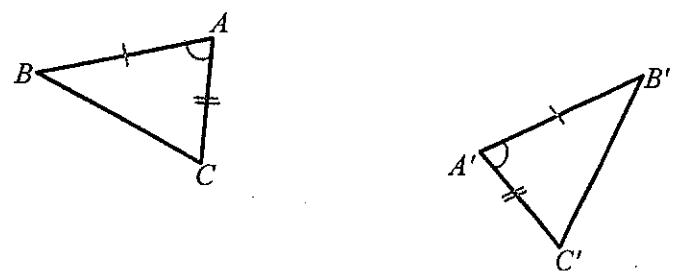
In Unit 2 we defined **congruent** to mean there exists a composition of basic rigid motions of the plane that maps one figure to the other.

In order to prove triangles are congruent, we do *not* need to prove all of their corresponding parts are congruent. Instead we will look at criteria that refer to fewer parts that will guarantee congruence.

We will start with:

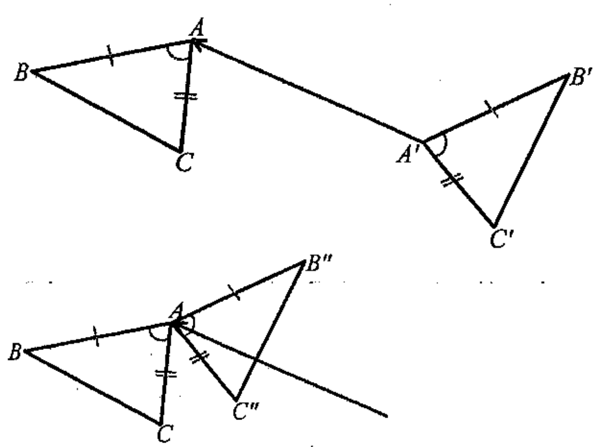
Side-Angle-Side Triangle Congruence Criteria (SAS)

- Two pairs of sides and the included angle are congruent

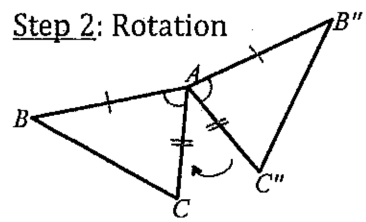


Using these distinct triangles, we can see there is a composition of rigid motions that will map $\Delta A'B'C'$ to ΔABC.

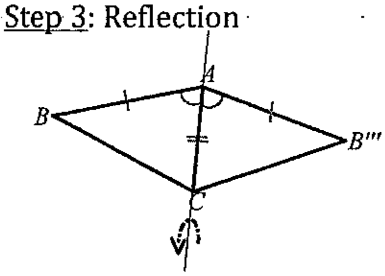
Step 1: Translation



Step 2: Rotation

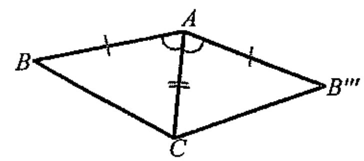
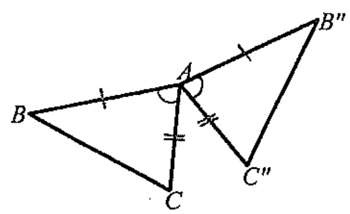


Step 3: Reflection



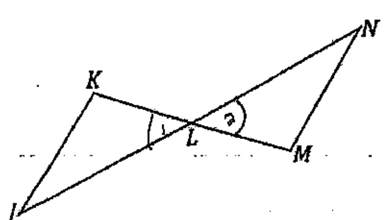
Example 1

What if we had SAS criteria for two triangles that were not distinct? Consider the following two cases and determine the rigid motion(s) that are needed to demonstrate congruence.

Case	Diagram	Rigid Motion(s) Needed
Shared Side		Reflection
Shared Vertex		Rotation

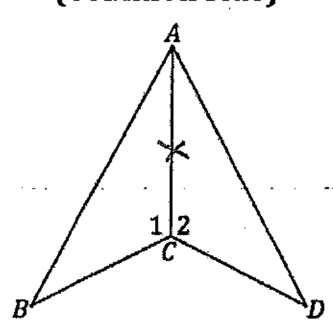
Two properties to look for when doing triangle proofs:

Vertical Angles



$\angle 1 \cong \angle 2$

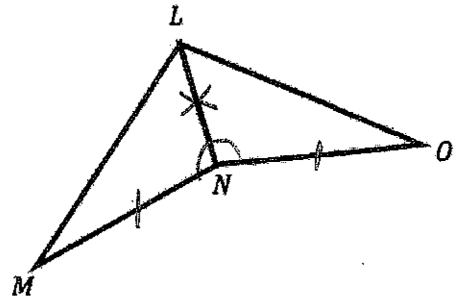
Reflexive Property (Common Side)



$\overline{AC} \cong \overline{AC}$

Examples

2. Given: $\angle LNM \cong \angle LNO$, $\overline{MN} \cong \overline{ON}$



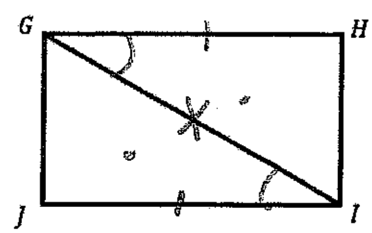
a. Prove: $\Delta LMN \cong \Delta LON$

Statements	Reasons
1) $\angle LNM \cong \angle LNO$, $\overline{MN} \cong \overline{ON}$	1) Given
2) $\overline{LN} \cong \overline{LN}$	2) Reflexive
3) $\Delta LMN \cong \Delta LON$	3) SAS

b. Describe the rigid motion(s) that would map ΔLON onto ΔLMN.

Reflection over $\overline{LN}$

3. Given: $\angle HGI \cong \angle JIG$, $HG \cong JI$



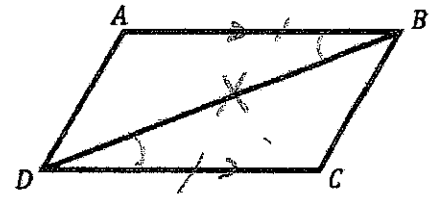
a. Prove: $\Delta HGI \cong \Delta JIG$

Statements	Reasons
1) $\angle HGI \cong \angle JIG$, $HG \cong JI$	1) Given
2) $\overline{GI} \cong \overline{GI}$	2) Reflexive
3) $\Delta HGI \cong \Delta JIG$	3) SAS

b. Describe the rigid motion(s) that would map ΔJIG onto ΔHGI.

Rotation 180° around the midpoint of $\overline{GI}$.

4. Given: $\overline{AB} \parallel \overline{CD}$, $\overline{AB} \cong \overline{CD}$



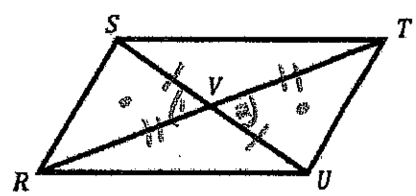
a. Prove: $\Delta ABD \cong \Delta CDB$

Statements	Reasons
1) $\overline{AB} // \overline{CD}$, $\overline{AB} \cong \overline{CD}$	1) Given
2) $\angle ABD \cong \angle CDB$	2) Alt. Int. $\angle$'s are $\cong$.
3) $\overline{DB} \cong \overline{DB}$	3) Reflexive
4) $\Delta ABD \cong \Delta CDB$	4) SAS

b. Describe the rigid motion(s) that would map ΔCDB onto ΔABD.

Rotation of 180° around Midpoint of $\overline{BD}$.

5. Given: $\overline{SU}$ and $\overline{RT}$ bisect each other



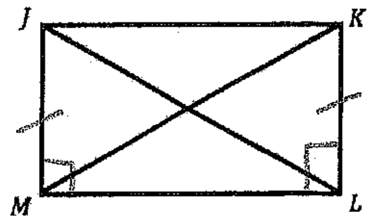
a. Prove: $\Delta SVR \cong \Delta UVT$

Statements	Reasons
1) $\overline{SU}$ and $\overline{RT}$ bisect each other	1) Given
2) $\overline{SV} \cong \overline{UV}$ and $\overline{RV} \cong \overline{TV}$	2) Def. of a segment bisector.
3) $\angle 1 \cong \angle 2$	3) Vertical $\angle$'s are $\cong$
4) $\Delta SVR \cong \Delta UVT$	4) SAS

b. Describe the rigid motion(s) that would map ΔUVT onto ΔSVR.

Rotation 180° ~~at~~ through V.

6. Given: $\overline{JM} \cong \overline{KL}, \overline{JM} \perp \overline{ML}, \overline{KL} \perp \overline{ML}$



a. Prove: $\Delta JML \cong \Delta KLM$

Statements	Reasons
1) $\overline{JM} \cong \overline{KL}$, $\overline{JM} \perp \overline{ML}$, $\overline{KL} \perp \overline{ML}$	1) Given
2) ∠JML and ∠KLM are right ∠'s	2) ⊥ lines form right ∠'s
3) ∠JML ≅ ∠KLM	3) All Right ∠'s are ≅.
4) $\overline{ML} \cong \overline{ML}$	4) Reflexive
5) △JML ≅ △KLM	5) SAS

b. Describe the rigid motion(s) that would map ΔJML onto ΔKLM.

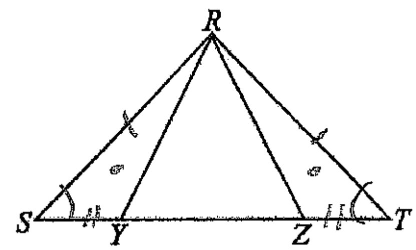
Reflection through the ⊥ Bisector of $\overline{LM}$

Lesson 3: Base Angles of Isosceles Triangles

Opening Exercise

<u>Given</u>: $\triangle RST$ is isosceles with $\angle R$ as the vertex,
$\overline{SY} \cong \overline{TZ}$

<u>Prove</u>: $\triangle RSY \cong \triangle RTZ$ (SAS)



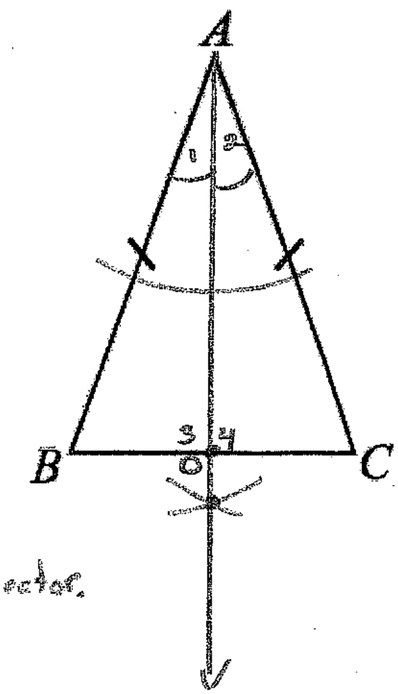
Statements	Reasons
1) $\triangle RST$ is isos. $\overline{SY} \cong \overline{TZ}$	1) Given
2) $\overline{RS} \cong \overline{RT}$	2) In an isos. $\triangle$, the sides are $\cong$.
3) $\angle RSY \cong \angle RTZ$	3) In an isos $\triangle$, the base angles are $\cong$
4) $\triangle RSY \cong \triangle RTZ$	4) SAS

Example 1

You will need a compass and a straightedge

We are going to prove that the base angles of an isosceles triangle are congruent!

Given: Isosceles ΔABC with $\overline{AB} \cong \overline{AC}$
Goal: To prove $\angle B \cong \angle C$



Step 1: Construct the angle bisector of the vertex $\angle$.

Step 2: ΔABC has now been split into two triangles. Prove the two triangles are $\cong$.

Statements	Reasons
1) Isos. ΔABC w/ $\overline{AB} \cong \overline{AC}$	1) Given
2) Construct angle bisector of A, label intersection w/ $\overline{BC}$ as D.	2) Auxilary Line
3) $\angle 1 \cong \angle 2$	3) Def. of an angle bisector.
4) $\overline{AD} \cong \overline{AD}$	4) Reflexive
5) $\Delta ABD \cong \Delta ACD$	5) SAS

Step 3: Identify the corresponding sides and angles.

$\angle 1 \to \angle 2$ $\quad$ $\overline{AB} \to \overline{AC}$

$\angle B \to \angle C$ $\quad$ $\overline{BD} \to \overline{CD}$

$\angle 3 \to \angle 4$ $\quad$ $\overline{AD} \to \overline{AD}$

Step 4: What is true about $\angle B$ and $\angle C$?

Congruent

Step 5: What types of angles were formed when the angle bisector intersected $\overline{BC}$? What does this mean about the angle bisector?

- Congruent → Also right $\angle$'s
- The angle bisector is the $\perp$ bisector.

Once we prove triangles are congruent, we know that their corresponding parts (angles and sides) are congruent. We can abbreviate this is in a proof by using the reasoning of:

CPCTC (Corresponding Parts of Congruent Triangles are Congruent).

To Prove Angles or Sides Congruent:

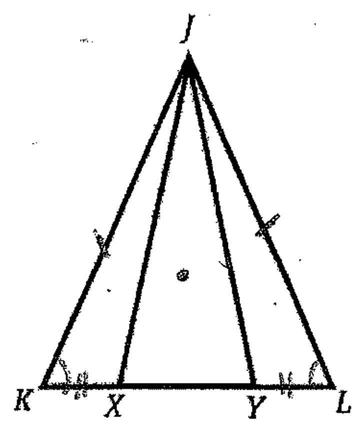
1. Prove the triangles are congruent (using one of the above criteria)
2. States that the angles/sides are congruent because of CPCTC.

To Prove Midpoint/Bisect/Perpendicular/Parallel:

1. Prove the triangles are congruent (using one of the above criteria)
2. State that the necessary angles/sides are congruent because of CPCTC.
3. State what you are trying to prove using def. of midpoint/def. of bisect/etc.

Example 2

Given: ΔJKL is isosceles, $\overline{KX} \cong \overline{LY}$
Prove: ΔJXY is isosceles

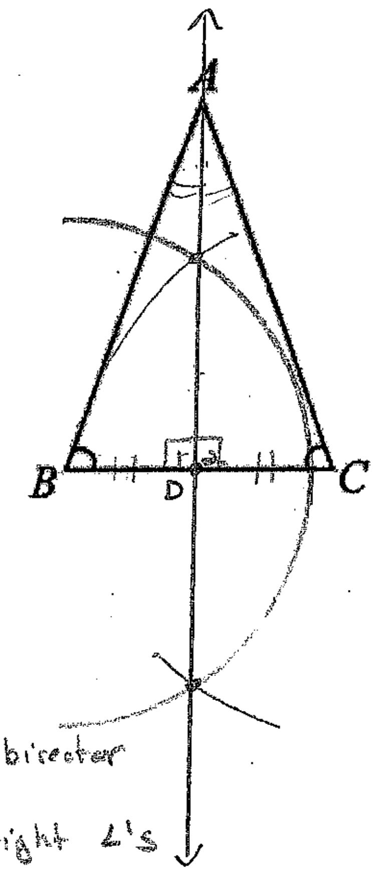


Statements	Reasons
1) ΔJKL is isos. $\overline{KX} \cong LY$	1) Given
2) $\angle JKX \cong \angle JLY$	2) In an Isos. Δ, the base $\angle$'s are $\cong$.
3) $\overline{JK} \cong \overline{JL}$	3) In an Isos. Δ, 2 sides are $\cong$.
4) $\Delta JKX \cong \Delta JLY$	4) SAS
5) $\overline{JX} \cong \overline{JY}$	5) CPCTC
6) ΔJXY is isosceles	6) A Δ with 2 $\cong$ sides is isosceles.

Lesson 4: Congruence Criteria for Triangles – ASA and SSS

Opening Exercise

You will need a compass and a straightedge



1. Given: ΔABC with $\angle B \cong \angle C$
   Goal: To prove $\overline{BA} \cong \overline{CA}$

   Step 1: Construct the perpendicular bisector to $\overline{BC}$.

   Step 2: ΔABC has now been split into two triangles.
   Prove $\overline{BA} \cong \overline{CA}$.

Statements	Reasons
1) ΔABC with $\angle B \cong \angle C$	1) Given
2) Construct ⊥ Bisector of $\overline{BC}$, label int. points D	2) Auxillary Line
3) $\overline{BD} \cong \overline{DC}$	3) Def. of a segment bisector
4) ∠1 and ∠2 are right ∠'s	4) ⊥ lines form right ∠'s
5) ∠1 ≅ ∠2	5) All right ∠'s are ≅.
6) $\overline{AD} \cong \overline{AD}$	6) Reflexive
7) $\Delta ABD \cong \Delta ACD$	7) SAS
8) $\overline{BA} \cong \overline{CA}$	8) CPCTC

2. In the diagram, ΔABC is isosceles with $\overline{AC} \cong \overline{AB}$. In your own words, describe how the properties of rigid motions can be used to show $\angle B \cong \angle C$.

   Reflection over line $\overline{AD}$

There are 5 ways to test for triangle congruence.

In lesson 1 we saw that we can prove triangles congruent using **SAS**. We proved this using rigid motions. Here's another way to look at it:

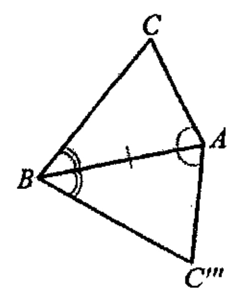
http://www.mathopenref.com/congruentsas.html

Today we are going to focus on two more types:

Angle-Side-Angle Triangle Congruence Criteria (ASA)

- Two pairs of angles and the included side are congruent

To prove this we could start with two distinct triangles. We could then translate and rotate one to bring the congruent sides together like we did in the SAS proof (see picture to the right).

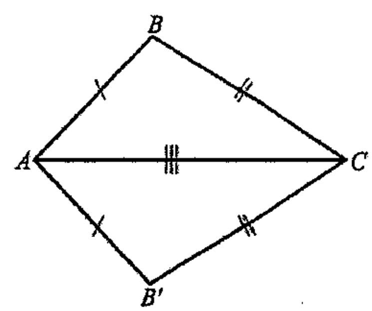


As we can see, a reflection over *AB* would result in the triangles being mapped onto one another, producing two congruent triangles.

http://www.mathopenref.com/congruentasa.html

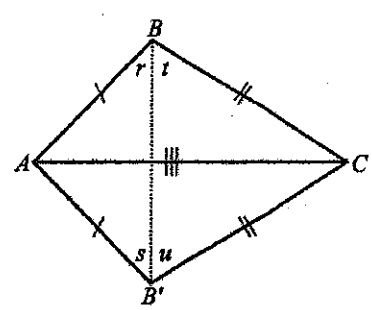
Side-Side-Side Triangle Congruence Criteria (SSS)

- All of the corresponding sides are congruent



Without any information about the angles, we cannot just perform a reflection as we did in the other two proofs. But by drawing an auxiliary line, we can see that two isosceles triangles are formed, creating congruent base angles and therefore, $\angle B \cong \angle B'$.

We can now perform a reflection, producing two congruent triangles.

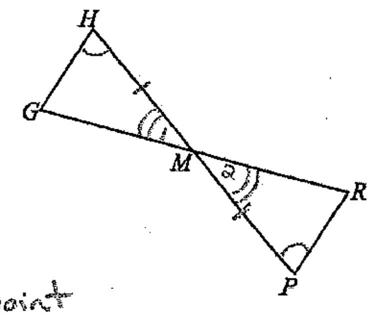


http://www.mathopenref.com/congruentsss.html

Exercises

Prove the following using any method of triangle congruence that we have discussed. Then identify the rigid motion(s) that would map one triangle onto the other.

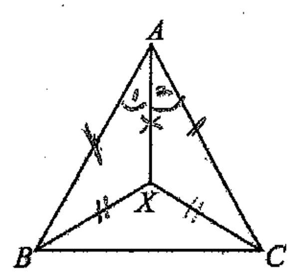
1. Given: M is the midpoint of $\overline{HP}$, $\angle H \cong \angle P$
Prove: $\Delta GHM \cong \Delta RPM$



Statements	Reasons
1) M is the midpoint of $\overline{HP}$, $\angle H \cong \angle P$	1) Given
2) $\overline{HM} \cong \overline{PM}$	2) Def. of a midpoint
3) $\angle 1 \cong \angle 2$	3) Vertical $\angle$'s are $\cong$.
4) $\Delta GHM \cong \Delta RPM$	4) ASA

Rotation 180° around M

2. Given: $\overline{AB} \cong \overline{AC}$, $\overline{XB} \cong \overline{XC}$
Prove: $\overline{AX}$ bisects $\angle BAC$

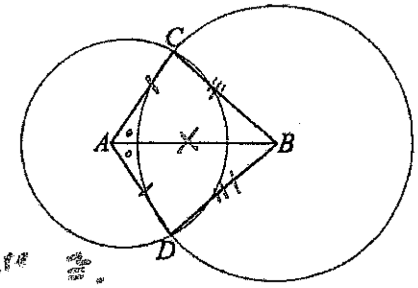


Statements	Reasons
1) $\overline{AB} \cong \overline{AC}$, $\overline{XB} \cong \overline{XC}$	1) Given
2) $\overline{AX} \cong \overline{AX}$	2) Reflexive
3) $\Delta ABX \cong \Delta ACX$	3) SSS
4) $\angle 1 \cong \angle 2$	4) CPCTC
5) $\overline{AX}$ bisects $\angle BAC$	5) Def. of an angle bisector

Reflection over $\overline{AX}$

Example 1

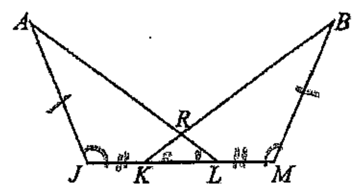
Given: Circles with centers A and B intersect at C and D.
Prove: $\angle CAB \cong \angle DAB$



Statements	Reasons
1) Circles center A and B intersect at C and D	1) Given
2) $\overline{CA} \cong \overline{AD}$, $\overline{BC} \cong \overline{BD}$	2) Radii in a circle are ≅.
3) $\overline{AB} \cong \overline{AB}$	3) Reflexive
4) $\triangle CAB \cong \triangle DAB$	4) SSS
5) $\angle CAB \cong \angle DAB$	5) CPCTC

Example 2

Given: $\angle J \cong \angle M$, $\overline{JA} \cong \overline{MB}$, $\overline{JK} \cong \overline{ML}$
Prove: $\overline{KR} \cong \overline{LR}$



Statements	Reasons
1) $\angle J \cong \angle M$, $\overline{JA} \cong \overline{MB}$, $\overline{JK} \cong \overline{ML}$	1) Given
2) $\overline{KL} \cong \overline{KL}$	2) Reflexive
3) $\overline{JK} + \overline{KL} \cong \overline{ML} + \overline{KL}$	3) Segment Addition Post.
4) $\overline{JL} \cong \overline{MK}$	3) Substitution
5) $\triangle AJL \cong \triangle BMK$	4) SAS
6) $\angle 1 \cong \angle 2$	6) CPCTC
7) $\overline{KR} \cong \overline{LR}$	7) In a △, sides opposite ≅ ∠'s are ≅.

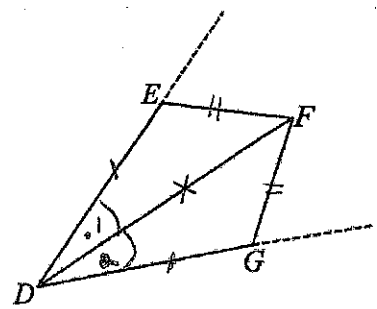
Lesson 5: Congruence Criteria for Triangles – SAA and HL

Opening Exercise

Write a proof for the following question. When finished, compare your proof with your partner's.

<u>Given</u>: $\overline{DE} \cong \overline{DG}$, $\overline{EF} \cong \overline{GF}$

<u>Prove</u>: $\overline{DF}$ is the angle bisector of $\angle EDG$



Statements	Reasons
1) $\overline{DE} \cong \overline{DG}$, $\overline{EF} \cong \overline{GF}$	1) Given
2) $\overline{DF} \cong \overline{DF}$	2) Reflexive
3) $\triangle DEF \cong \triangle DGF$	3) SSS
4) $\angle 1 \cong \angle 2$	4) CPCTC
5) $\overline{DF}$ is an angle bisector of $\angle EDG$	5) Def. of an angle bisector.

We have now identified 3 different ways of proving triangles congruent. What are they?

SAS, SSS, ASA

Does this mean any combination of 3 pairs of congruent sides and/or angles will guarantee congruence?

Sides → yes

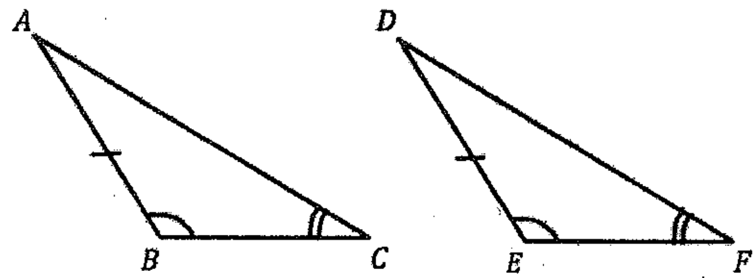
Angles → NO

Let's try another combination of sides and angles:

Side-Angle-Angle Triangle Congruence Criteria (SAA) (AAS)

- Two pairs of angles and a side that is not included are congruent

To prove this we could start with two distinct triangles.



If $\angle B \cong \angle E$ and $\angle C \cong \angle F$, what must be true about $\angle A$ and $\angle D$? Why?

$\angle A \cong \angle D$ because the sum of the angles of a Δ are always 180°. So if the two pairs of given $\angle$'s are $\cong$, the 3rd pair must be $\cong$.

Therefore, **SAA** is actually an extension of which triangle congruence criterion?

ASA

Let's take a look at two more types of criteria:

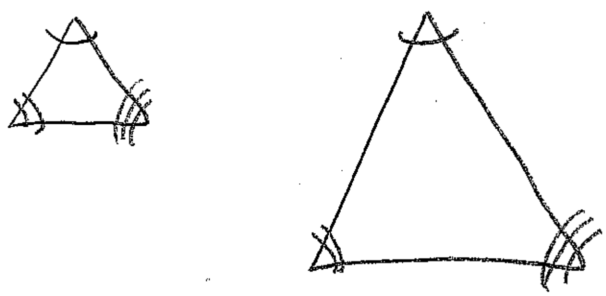
Angle-Angle-Angle (AAA)

- All three pairs of angles are congruent

http://www.mathopenref.com/congruentaaa.html

Does **AAA** guarantee triangle congruence? Draw a sketch demonstrating this.

No!



These are called Similar Triangles.

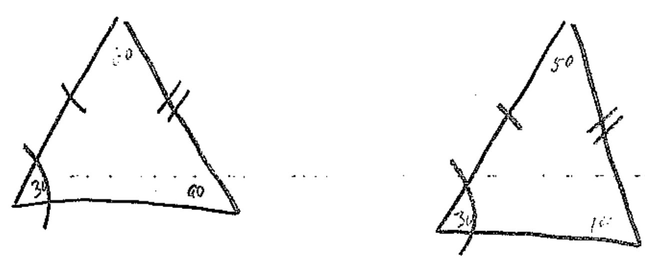
Side-Side-Angle (SSA)

- Two pairs of sides and a non-included angle are congruent

http://www.mathopenref.com/congruentssa.html

Does **SSA** guarantee triangle congruence? Draw a sketch demonstrating this.

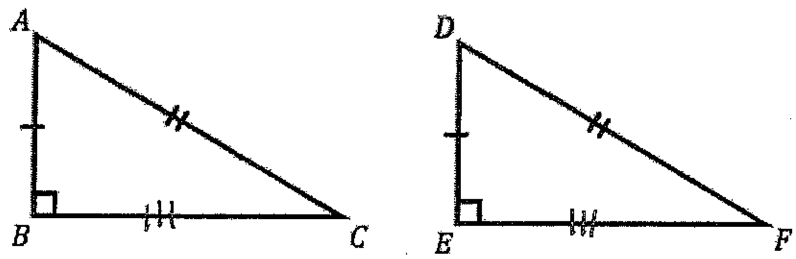
No!



There is a special case of **SSA** that does work, and that is when dealing with right triangles. We call this **Hypotenuse-Leg** triangle congruence.

Hypotenuse-Leg Triangle Congruence Criteria (HL)

- When two right triangles have congruent hypotenuses and a pair of congruent legs, then the triangles are congruent.



If we know two sides of a right triangle, how could we find the third side?

Pythagorean Theorem $(a^2+b^2=c^2)$

Therefore, **HL** is actually an extension of which triangle congruence criterion?

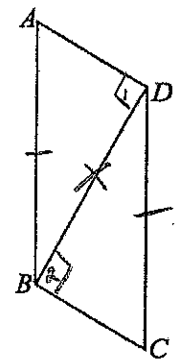
SSS and SAS

In order to use **HL** triangle congruence, you must first state that the triangles are right triangles!

Exercises

Prove the following using any method of triangle congruence that we have discussed. Then identify the rigid motion(s) that would map one triangle onto the other.

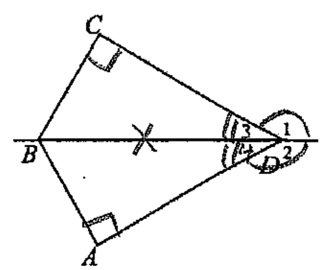
1. Given: $\overline{AD} \perp \overline{BD}$, $\overline{BD} \perp \overline{BC}$, $\overline{AB} \cong \overline{CD}$
   Prove: $\triangle ABD \cong \triangle CDB$



Statements	Reasons
1) $\overline{AD} \perp \overline{BD}$, $\overline{BD} \perp \overline{BC}$, $\overline{AB} \cong \overline{CD}$	1) Given
2) ∠1 and ∠2 are right ∠'s	2) ⊥ lines form right ∠'s
3) ∠1 ≅ ∠2	3) All right ∠'s are ≅.
4) △ADB and △CBD are right △'s	4) △'s containing a right ∠ are rt △'s.
5) $\overline{DB} \cong \overline{DB}$	5) Reflexive
6) △ABD ≅ △CDB	6) H-L

*Rotation 180° through the midpoint of $\overline{BD}$.

2. Given: $\overline{BC} \perp \overline{CD}$, $\overline{AB} \perp \overline{AD}$, $\angle 1 \cong \angle 2$
   Prove: $\triangle BCD \cong \triangle BAD$



Statements	Reasons
1) $\overline{BC} \perp \overline{CD}$, $\overline{AB} \perp \overline{AD}$, ∠1 ≅ ∠2	2) Given
2) ∠C and ∠A are right ∠'s	2) ⊥ lines form right ∠'s.
3) ∠C ≅ ∠A	3) All right ∠'s are ≅.
4) $\overline{BD} \cong \overline{BD}$	4) Reflexive
5) ∠1 and ∠3 are supplementary, ∠2 and ∠4 are supp.	5) ∠'s on a line.
6) ∠3 ≅ ∠4	6) Supplements of ≅ ∠'s are ≅.
7) △BCD ≅ △BAD	7) AAS

Lesson 6: Triangle Congruency Proofs

Opening Exercise

Triangle proofs summary. Let's see what you know!

List the 5 ways of proving triangles congruent:

1. SAS
2. SSS
3. ASA
4. AAS or SAA
5. HL

What two sets of criteria **CANNOT** be used to prove triangles congruent:

1. AAA
2. ASS (SSA)

In order to prove a pair of corresponding sides or angles are congruent, what must you do first?

Prove Δ's $\cong$.

What is the abbreviation used to state that corresponding parts (sides or angles) of congruent triangles are congruent?

CPCTC

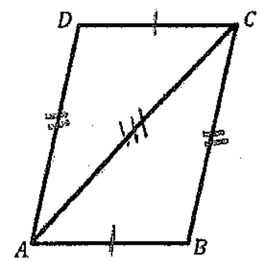
Exercises

Prove the following using any method of triangle congruence that we have discussed.

1. Given: $\overline{AB} \cong \overline{CD}$

   $\overline{BC} \cong \overline{DA}$

   Prove: $\Delta ADC \cong \Delta CBA$

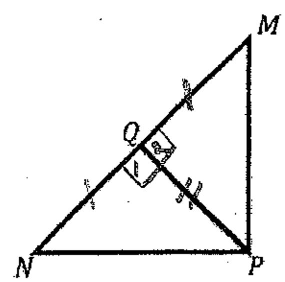


Statements	Reasons
1) $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{DA}$	1) Given
2) $\overline{AC} \cong \overline{AC}$	2) Reflexive
3) $\Delta ADC \cong \Delta CBA$	3) SSS

2. Given: $\overline{NQ} \cong \overline{MQ}$

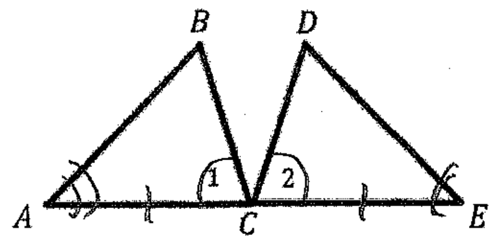
   $\overline{PQ} \perp \overline{NM}$

   Prove: $\Delta PQN \cong \Delta PQM$



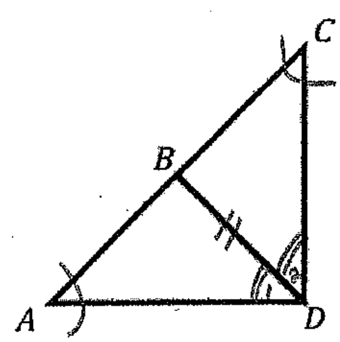
Statements	Reasons
1) $\overline{NQ} \cong \overline{mQ}$, $\overline{PQ} \perp \overline{Nm}$	1) Given
2) ∠1 and ∠2 are right ∠'s	2) ⊥ lines form right ∠'s
3) ∠1 ≅ ∠2	3) All right ∠'s are ≅.
4) $\overline{PQ} \cong \overline{PQ}$	4) Reflexive
5) ΔPQN ≅ ΔPQm	5) SAS

3. Given: $\angle 1 \cong \angle 2$, $\angle A \cong \angle E$,
C is the midpoint of $\overline{AE}$
Prove: $\overline{BC} \cong \overline{DC}$



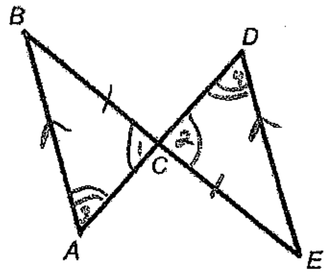
Statements	Reasons
1) $\angle 1 \cong \angle 2$, $\angle A \cong \angle E$ C is the midpoint of $\overline{AE}$	1) Given
2) $\overline{AC} \cong \overline{CE}$	2) Def. of a midpoint
3) $\triangle BAC \cong \triangle DEC$	3) ASA
4) $\overline{BC} \cong \overline{DC}$	4) CPCTC

4. Given: $\overline{BD}$ bisects $\angle ADC$
$\angle A \cong \angle C$
Prove: $\overline{AB} \cong \overline{CB}$



Statements	Reasons
1) $\angle A \cong \angle C$; $\overline{BD}$ bisects $\angle ADC$	1) Given
2) $\angle 1 \cong \angle 2$	2) Def. of an angle bisector.
3) $\overline{DB} \cong \overline{DB}$	3) Reflexive
4) $\triangle ADB \cong \triangle CDB$	4) AAS
5) $\overline{AB} \cong \overline{CB}$	5) CPCTC

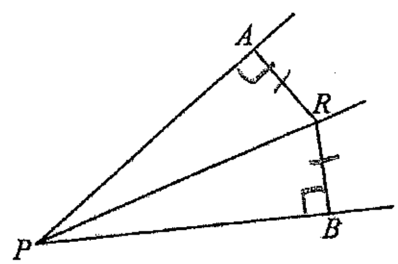
5. <u>Given</u>: $\overline{AD}$ bisects $\overline{BE}$
$AB \parallel DE$
<u>Prove</u>: $\Delta ABC \cong \Delta DEC$



Statements	Reasons
1) $\overline{AD}$ bisects $\overline{BE}$, $\overline{AB} \parallel \overline{DE}$	1) Given
2) $\overline{BC} \cong \overline{CE}$	2) Def. of a segment bisector.
3) $\angle 3 \cong \angle 4$	3) Alt. int. $\angle$'s are $\cong$ when the lines are //.
4) $\angle 1 \cong \angle 2$	4) Vertical $\angle$'s are $\cong$.
5) $\Delta ABC \cong \Delta DEC$	6) AAS

6. <u>Given</u>: $PA \perp AR$, $PB \perp BR$, $\overline{AR} \cong \overline{BR}$
<u>Prove</u>: PR bisects $\angle APB$

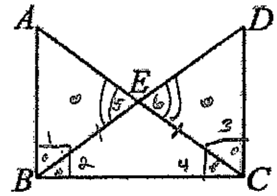
See HW Problem



Lesson 7: Triangle Congruency Proofs II

Prove the following using any method of triangle congruence that we have discussed.

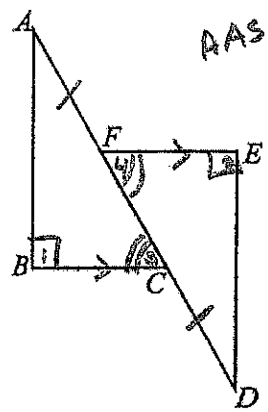
1. <u>Given</u>: $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{DC}$
   $\overline{DB}$ bisects $\angle ABC$
   $\overline{AC}$ bisects $\angle DCB$
   $\overline{EB} \cong \overline{EC}$
   <u>Prove</u>: $\Delta BEA \cong \Delta CED$



Statements	Reasons
1) $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{DC}$ $\overline{DB}$ bisects $\angle ABC$ $\overline{AC}$ bisects $\angle DCB$ $\overline{EB} \cong \overline{EC}$	1) Given
2) $\angle B = 90$; $\angle C = 90$	2) Def. of $\perp$
3) $\angle 1 = 45$; $\angle 3 = 45$	3) Def. of an angle bisector.
4) $\angle 1 \cong \angle 3$	4) Substitution
5) $\angle 5 \cong \angle 6$	5) Vertical $\angle$'s are $\cong$
6) $\Delta BEA \cong \Delta CED$	6) ASA

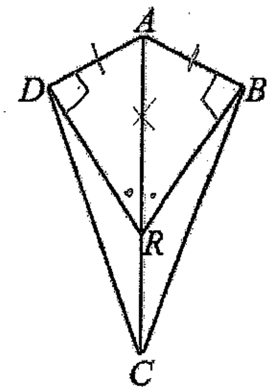
2. <u>Given</u>: $\overline{AB} \perp \overline{BC}$, $\overline{DE} \perp \overline{EF}$, $\overline{BC} \parallel \overline{EF}$, $\overline{AF} \cong \overline{DC}$
   <u>Prove</u>: $\Delta ABC \cong \Delta DEF$

AAS



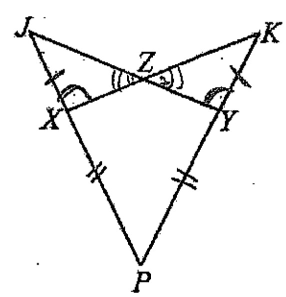
Statements	Reasons
1) $\overline{AB} \perp \overline{BC}$, $\overline{DE} \perp \overline{EF}$ $\overline{BC} \parallel \overline{EF}$; $\overline{AF} \cong \overline{DC}$	1) Given
2) $\angle 1$ and $\angle 2$ are right $\angle$'s	2) $\perp$ lines form right $\angle$'s
3) $\angle 1 \cong \angle 2$	3) All right $\angle$'s are $\cong$.
4) $\angle 3 \cong \angle 4$	4) If the lines are $\parallel$, alt. int $\angle$'s are $\cong$.
5) $\overline{FC} \cong \overline{FC}$	5) Reflexive
6) $\overline{AF} + \overline{FC} \cong \overline{DC} + \overline{FC}$	6) Segment Addition
7) $\overline{AC} \cong \overline{DF}$	7) Substitution
8) $\Delta ABC \cong \Delta DEF$	8) AAS

3. Given: $\overline{AD} \perp \overline{DR}$, $\overline{AB} \perp \overline{BR}$
$\overline{AD} \cong \overline{AB}$
Prove: $\angle ARD \cong \angle ARB$



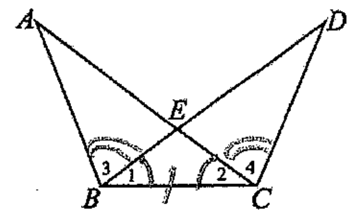
Statements	Reasons
1) $\overline{AD} \perp \overline{DR}$; $\overline{AB} \perp \overline{BR}$; $\overline{AD} \cong \overline{AB}$	1) Given
2) ∠D and ∠B are right ∠'s.	2) ⊥ lines form right ∠'s.
3) △ADR and △ABR are right △'s	3) In a △ if one angle is right, then the △ is right.
4) $\overline{AR} \cong \overline{AR}$	4) Reflexive
5) △ADR ≅ △ABR	5) HL
6) ∠ARD ≅ ∠ARB	6) CPCTC

4. Given: $\overline{XJ} \cong \overline{YK}$, $\overline{PX} \cong \overline{PY}$, $\angle ZXJ \cong \angle ZYK$
Prove: $\overline{JY} \cong \overline{KX}$



Statements	Reasons
1) $\overline{XJ} \cong \overline{YK}$, $\overline{PX} \cong \overline{PY}$, ∠ZXJ ≅ ∠ZYK	1) Given
2) ∠1 ≅ ∠2	2) Vertical ∠'s are ≅.
3) △JXZ ≅ △KYZ	3) AAS
4) $\overline{JZ} \cong \overline{KZ}$ and $\overline{XZ} \cong \overline{YZ}$	4) CPCTC
5) $\overline{JZ} + \overline{YZ} \cong \overline{KZ} + \overline{XZ}$	5) Segment Addition Post
6) $\overline{JY} \cong \overline{KX}$	6) Substitution

5. Given: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$
Prove: $\overline{AC} \cong \overline{BD}$

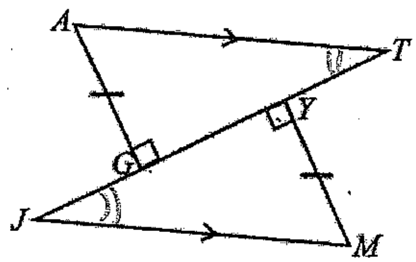


Statements	Reasons
1) ∠1 ≅ ∠2, ∠3 ≅ ∠4	1) Given
2) ∠1 + ∠3 ≅ ∠2 + ∠4	2) Angle Addition Post
3) $\overline{BC} \cong \overline{BC}$	3) Reflexive
4) △ABC ≅ △DCB	4) ASA
5) $\overline{AC} \cong \overline{BD}$	5) CPCTC

Lesson 8: Properties of Parallelograms

Opening Exercise

Based on the diagram pictured below, answer the following:



1. If the triangles are congruent, state the congruence.

   Yes, AAS $\triangle AGT \cong \triangle MYJ$

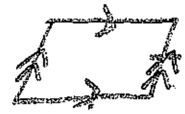
2. Which triangle congruence criterion guarantees they are congruent?

   AAS

3. Side TG corresponds with which side of ΔMYJ ?

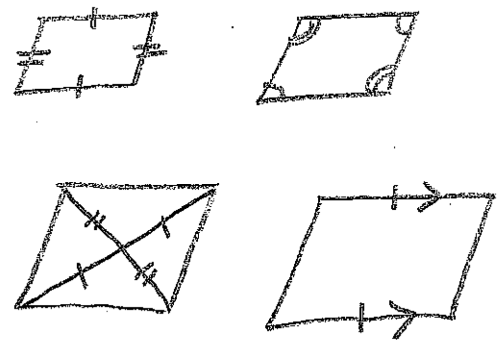
   $\overline{YJ}$

Vocabulary

Define	Diagram
Parallelogram – a quadrilateral in which both pairs of opp. sides are //.	

Using this definition of parallelograms and our knowledge of triangle congruence, we can prove the following properties of parallelograms:

- Opposite sides are congruent
- Opposite angles are congruent
- Diagonals bisect each other
- One pair of opposite sides are parallel and congruent



Example 1

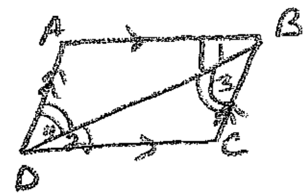
We are going to prove the following sentence:

If a quadrilateral is a parallelogram, then its opposite sides and angles are equal in measure.

Given: ABCD is a parallelogram.

Prove: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$
$\angle B \cong \angle D$; $\angle A \cong \angle C$

Diagram:



Proof:

Statements	Reasons
1) ABCD is a ▱	1) Given
2) $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \parallel \overline{BC}$	2) Def. of a ▱.
3) Construct diagonal $\overline{BD}$	3) Auxiliary Line
4) $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	4) Alt. int ∠'s are ≅.
5) $\overline{BD} \cong \overline{BD}$	5) Reflexive
6) $\triangle ABD \cong \triangle CDB$	6) ASA
7) $\overline{AB} \cong \overline{CD}$; $\overline{AD} \cong \overline{BC}$ $\angle A \cong \angle C$	7) CPCTC
8) $\angle 1 + \angle 3 \cong \angle 2 + \angle 4$	8) ≅ ∠'s added to ≅ ∠'s are ≅.

Example 2

Now that we have proven that opposite sides and angles of a parallelogram are congruent, we can use that it on our proofs!

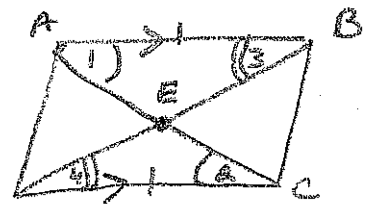
We are going to prove the following sentence:

If a quadrilateral is a parallelogram, then the diagonals bisect each other.

Given: ABCD is a ▱

Prove: $\overline{AE} \cong \overline{EC}$; $\overline{BE} \cong \overline{DE}$

Diagram:



Proof:

Statements	Reasons
1) ABCD is a ▱	1) Given
2) Construct diagonals $\overline{AC}$ and $\overline{BD}$, int. at E.	2) Auxilary Line
3) $\overline{AB} \cong \overline{DC}$	3) In a ▱, opposite sides are ≅.
4) $\overline{AB} \parallel \overline{DC}$	4) In a ▱, opposite sides are //.
5) $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	5) Alt. int ∠'s are ≅ when the lines are //.
6) $\triangle ABE \cong \triangle CDE$	6) ASA
7) $\overline{AE} \cong \overline{EC}$ $\overline{BE} \cong \overline{ED}$	7) CPCTC

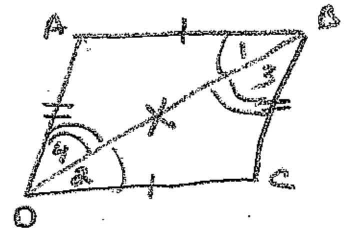
Example 3

We are going to prove the following sentence:

If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: Quad ABCD; $\overline{AB} \cong \overline{CD}$; $\overline{BC} \cong \overline{AD}$

Diagram:



Prove: ABCD is a ▱

Proof:

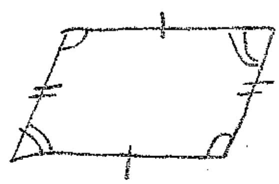
Statements	Reasons
1) Quad ABCD; $\overline{AB} \cong \overline{CD}$; $\overline{BC} \cong \overline{AD}$	1) Given
2) Construct Diagonal $\overline{DB}$	2) Auxilary Line
3) $\overline{BD} \cong \overline{BD}$	3) Reflexive
4) $\triangle ABD \cong \triangle CDB$	4) SSS
5) $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	5) CPCTC
6) $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \parallel \overline{BC}$	6) Alt. int ∠'s are ≅, so lines are ∥
7) ABCD is a ▱	7) Def. of ▱.

Lesson 9: Properties of Parallelograms II

Opening Exercise

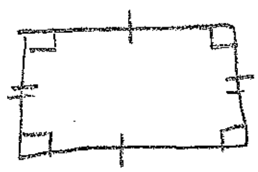
Draw a diagram for each of the quadrilaterals listed and draw in congruence markings where you believe they exist.

Parallelogram

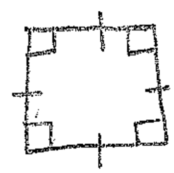


Rhombus

Rectangle

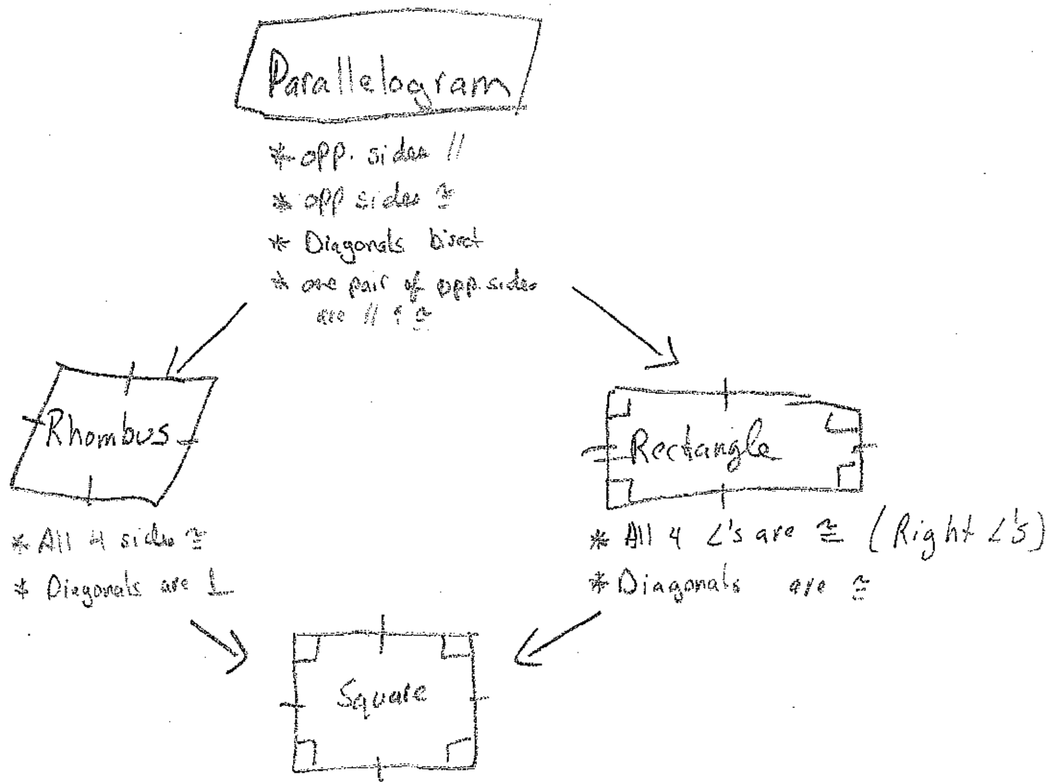


Square



Family of Quadrilaterals

Many of the quadrilaterals listed in the Opening Exercise share some of the same properties. We can look at this as a family:



The quadrilaterals at the bottom have all of the properties of the figures listed above it. Based on this, determine if the following are true or false. If it is false, explain why.

1. All rectangles are parallelograms.

   True

2. All parallelograms are rectangles.

   False, not all ▱ have 4 right ∠'s

3. All squares are rectangles.

   True

4. All rectangles are squares.

   False, not all rectangles have 4 ≅ sides

Example 1

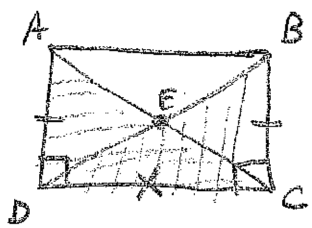
Prove the following sentence:

If a parallelogram is a rectangle, then the diagonals are equal in length.

Given: Rectangle ABCD with diagonals AC and BD intersecting at E.

Prove: $\overline{AC} \cong \overline{BD}$

Diagram:



Proof:

Statements	Reasons
1) Rectangle ABCD with diagonals $\overline{AC}$ and $\overline{BD}$ intersec. at E.	1) Given
2) $\angle D$ and $\angle C$ are right $\angle$'s. $\overline{AD} \cong \overline{BC}$	2) Property of a Rectangle.
3) $\angle D \cong \angle C$	3) All right $\angle$'s are $\cong$.
4) $\overline{DC} \cong \overline{DC}$	4) Reflexive
5) $\triangle ADC \cong \triangle BCD$	5) SAS
6) $\overline{AC} \cong \overline{BD}$	6) CPCTC

Example 2

Prove the following sentence:

If a parallelogram is a rhombus, the diagonals intersect perpendicularly.

Given:

Diagram:

Prove:

Proof:

Lesson 10: Mid-segment of a Triangle

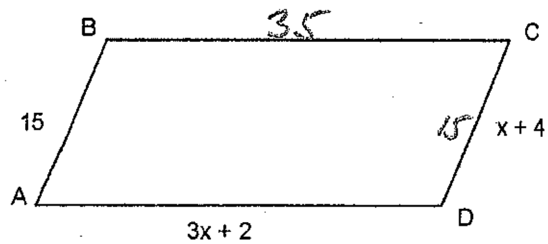
Opening Exercise

Using your knowledge of the properties of parallelograms, answer the following questions:

1. Find the perimeter of parallelogram *ABCD*. Justify your solution.

x+4 = 15
 -4 -4
x = 11

opposite sides of a ▱ are ≅.



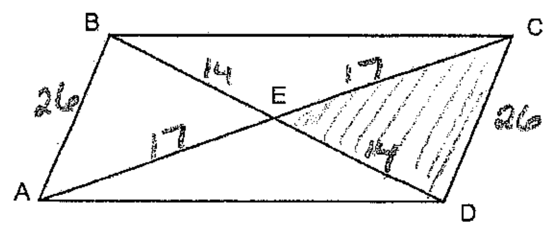
3(11)+2
= 35

P = 35(2) + 15(2)

P = 70 + 30

P = 100

2. If $AC = 34$, $AB = 26$ and $BD = 28$, find the perimeter of ΔCED. Justify your solution.



P = 17+14+26

P = 57

In a ▱, opposite sides are ≅ and diagonals bisect each other

Vocabulary

Define	Diagram
Mid-segment a segment connecting the midpoints of 2 sides of a Δ.	

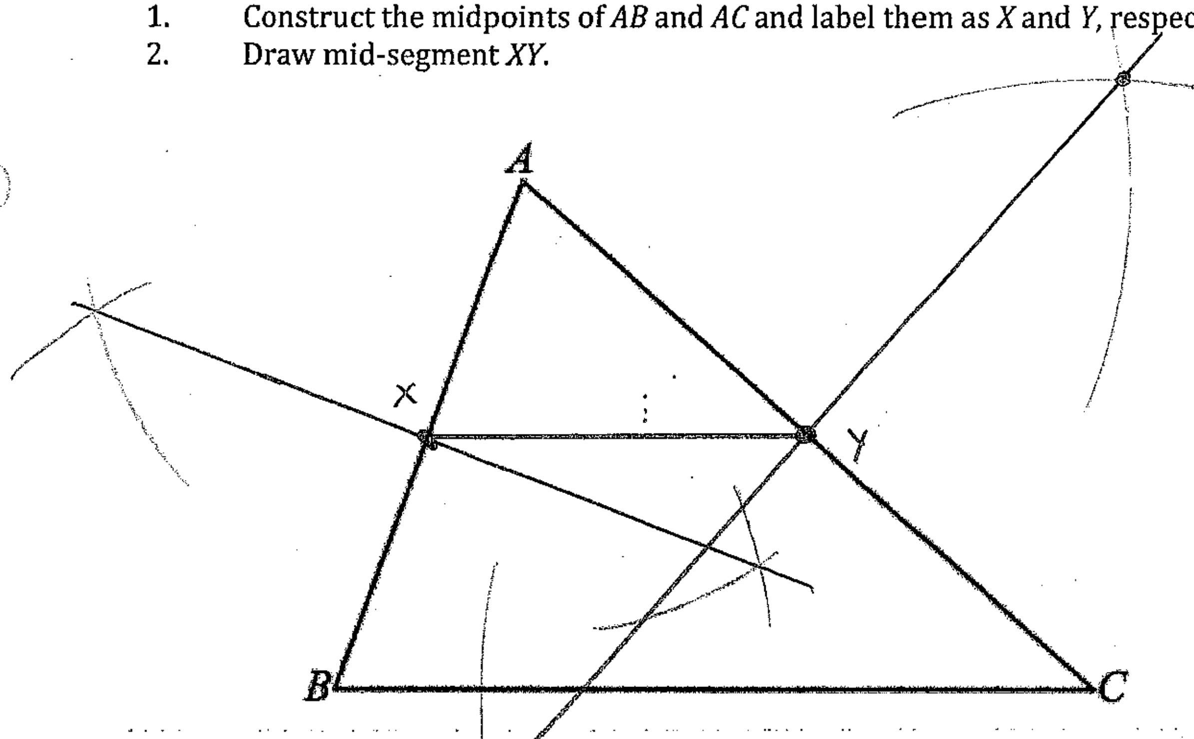
Example 1

You will need a compass and a straightedge

We are going to construct a mid-segment.

Steps:

1. Construct the midpoints of AB and AC and label them as X and Y, respectively.
2. Draw mid-segment XY.



Compare $\angle AXY$ to $\angle ABC$ and compare $\angle AYX$ to $\angle ACB$. Without using a protractor, what would you guess the relationship between these two pairs of angles is?

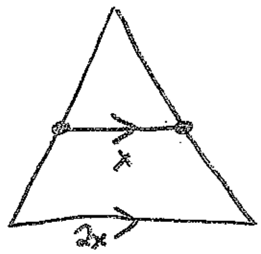
Congruent

What are the implications of this relationship?

$\overline{XY} \parallel BC$

Properties of Mid-segments

- The mid-segment of a triangle is parallel to the third side of the triangle.
- The mid-segment of a triangle is half the length of the third side of the triangle.



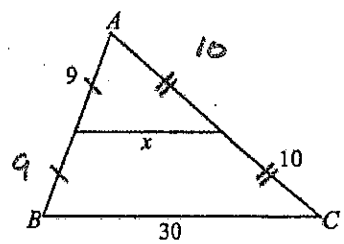
Exercises

Apply what your know about the properties of mid-segments to solve the following:

1. *a.* Find *x*.

$x = \frac{1}{2}(30)$

$x = 15$



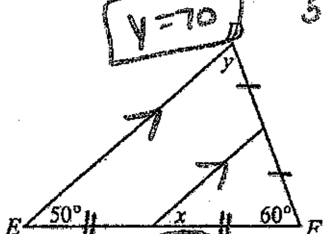
b. Find the perimeter of ΔABC

$P = 18 + 20 + 30$

$P = 68$

2. Find *x* and *y*.

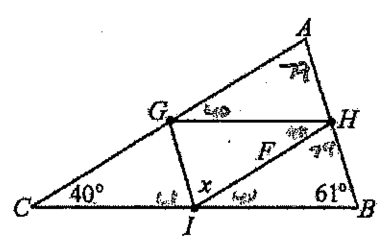
$y = 70$



$x = 50$

50+60
=110
180−110
=70

3. Find *x*.



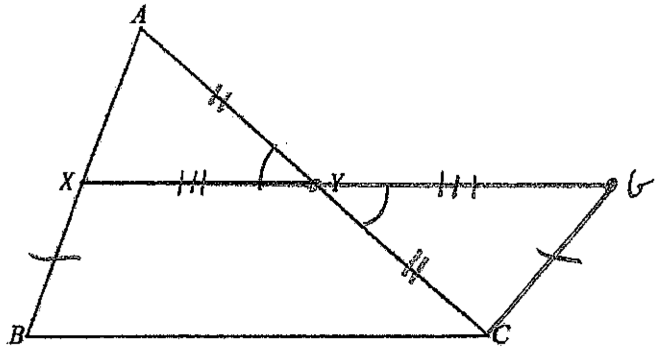
x = 79

Example 2

We are now going to prove the properties of mid-segments.

Given: XY is a mid-segment of ΔABC

Prove: $XY \parallel BC$ and $XY = \frac{1}{2}BC$

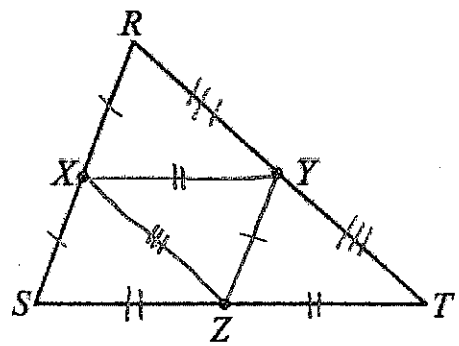


Statements	Reasons
1. XY is a mid-segment of ΔABC	1. Given
2. X is the midpoint of AB Y is the midpoint of AC	2. A mid-segment joins the midpoints
3. $AX \cong BX$ and $AY \cong CY$	3. Def. of Midpoint
4. Extend XY to point G so that $YG = XY$ Draw GC	4. Auxiliary Lines
5. $\angle AYX \cong \angle CYG$	5. Vertical ∠'s are ≅.
6. $\Delta AYX \cong \Delta CYG$	6. SAS
7. $\angle AXY \cong \angle CGY$, $AX \cong CG$	7. CPCTC
8. $BX \cong CG$	8. Substitution
9. $AB \parallel GC$	9. Alt. int ∠'s are ≅.
10. $BXGC$ is a parallelogram	10. One pair of opp. sides are ∥ and ≅
*11. $XY \parallel BC$	11. In a ▱, opposite sides are ∥
12. $XG \cong BC$	12. In a ▱, opposite sides are ≅
13. $XG = XY + YG$	13. A segment = sum of its parts
14. $XG = XY + XY$	14. Substitution
15. $BC = XY + XY$	15. Substitution
16. $BC = 2XY$	16. Substitution
*17. $XY = \frac{1}{2}BC$	17. Division Post

Lesson 11: Points of Concurrency

Opening Exercise

The midpoints of each side of ΔRST have been marked by points X, Y, and Z.



a. Mark the halves of each side divided by the midpoint with a congruency mark. Remember to distinguish congruency marks for each side.

b. Draw mid-segments XY, YZ, and XZ. Mark each mid-segment with the appropriate congruency mark from the sides of the triangle.

c. What conclusion can you draw about the four triangles within ΔRST? Explain why.

All ≅ b/c they all have the same congruency markings for SSS.

d. State the appropriate correspondences between the four triangles within ΔRST.

ΔRXY

ΔXSZ

ΔYZT

ΔZYX

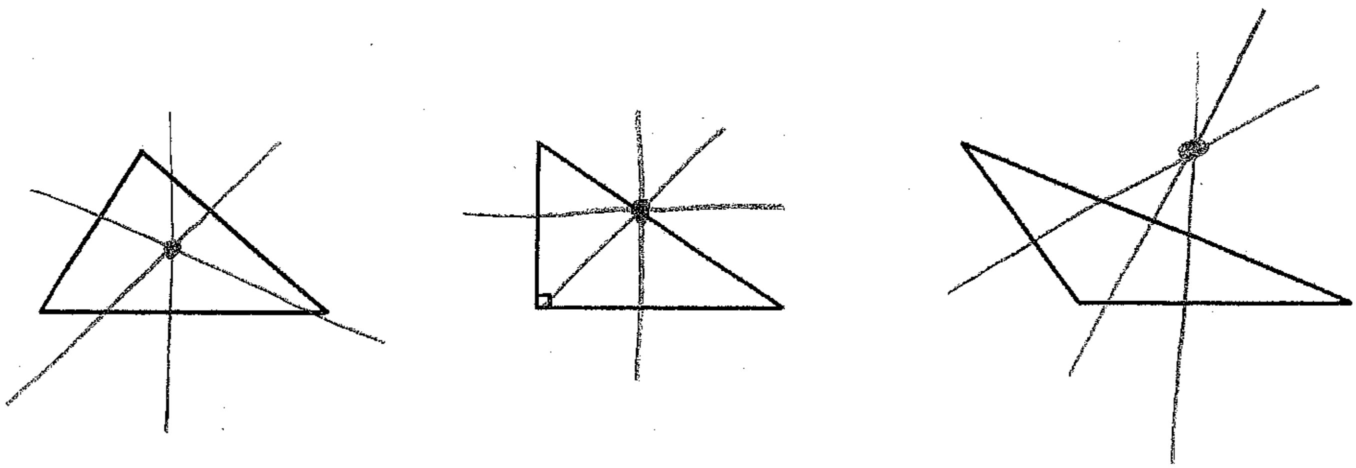
In Unit 1 we discussed two different points of concurrency (when 3 or more lines intersect in a single point).

Let's review what they are!

Circumcenter

- the point of concurrency of the 3 perpendicular bisectors of a triangle

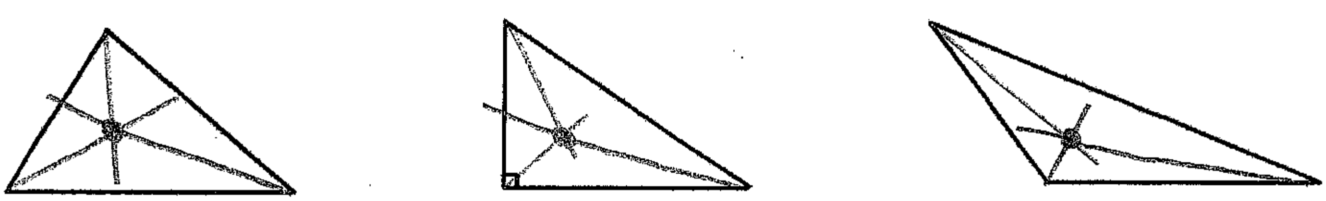
Sketch the location of the circumcenter on the triangles pictured below:



Incenter

- the point of concurrency of the 3 angle bisectors of a triangle

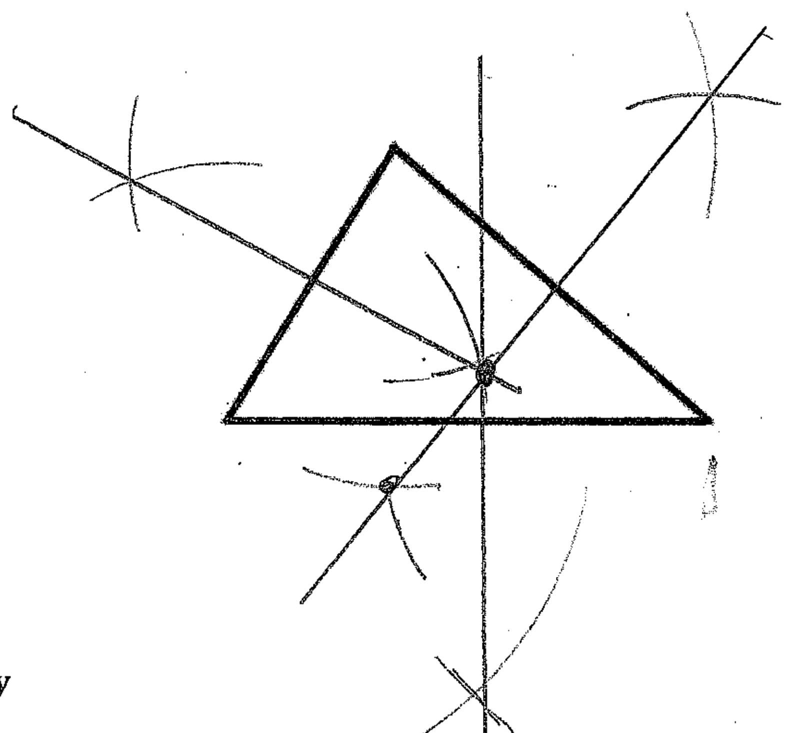
Sketch the location of the incenter on the triangles pictured below:



Example 1

You will need a compass and a straightedge

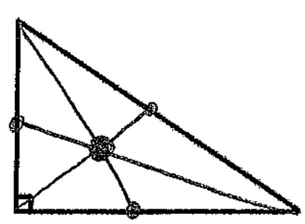
Construct the *medians* for each side of the triangle pictured below. A *median* is a segment connecting a vertex to the midpoint of the opposite side.

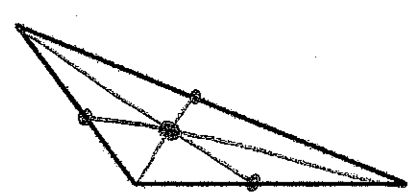


Vocabulary

- The point of intersection for 3 medians is called the centroid.
- This point is the *center of gravity* of the triangle.

We will use http://www.mathopenref.com/trianglecentroid.html to explore what happens when the triangle is right or obtuse. Sketch the location of the centroid on the triangles below:

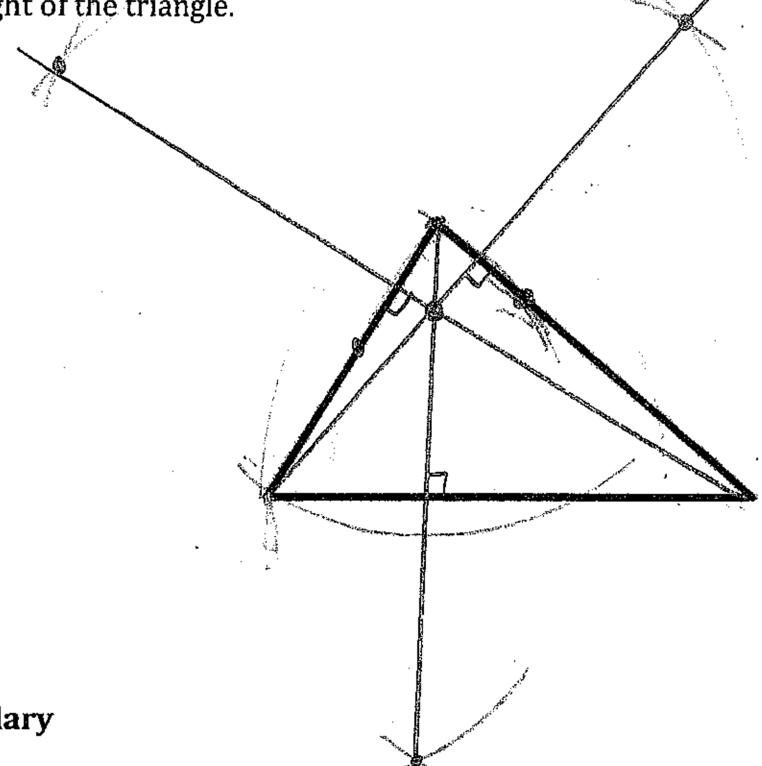




Example 2

You will need a compass and a straightedge

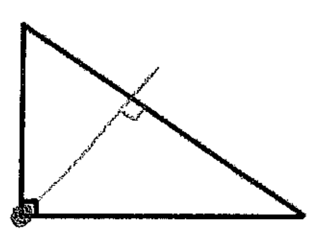
Construct the *altitudes* for each side of the triangle pictured below. An *altitude* is a segment connecting a vertex to the opposite side at a right angle. This can also be used to describe the height of the triangle.

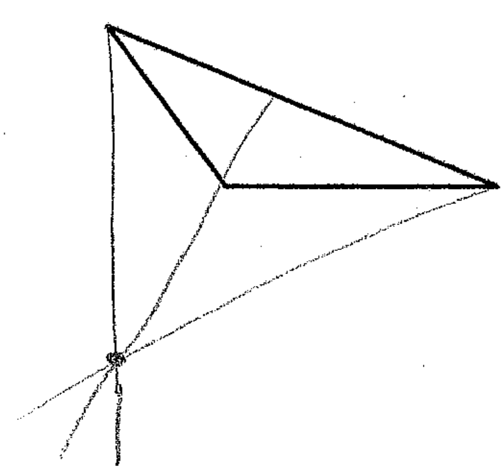


Vocabulary

- The point of intersection for 3 altitudes is called the ortho center.

We will use http://www.mathopenref.com/triangleorthocenter.html to explore what happens when the triangle is right or obtuse. Sketch the location of the orthocenter on the triangles below:





Lesson 12: Points of Concurrency II

Opening Exercise

Complete the table below to summarize what we did in Lesson 11. Circumcenter has been filled in for you.

Point of Concurrency	Types of Segments	What this type of line or segment does	Located Inside or Outside of the Triangle?
Circumcenter	*Perpendicular Bisectors*	*Forms a right angle and cuts a side in half*	*Both; depends on the type of triangle*
Incenter	Angle Bisectors	Cuts an angle in half.	Inside
Centroid	Medians	Connects vertex to opposite sides midpoints	Inside
Orthocenter	Altitudes	Connects vertex to opposite side at a right ∠	Both, Depends on the Δ.

Which two points of concurrency are located on the outside of an obtuse triangle?

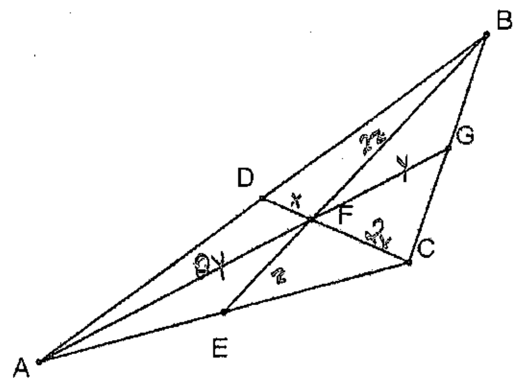
Circumcenter + Orthocenter.

What do these types have in common?

Both deal w/right ∠'s.

Example 1

A *centroid* splits the medians of a triangle into two smaller segments. These segments are always in a 2:1 ratio.



Label the lengths of segments *DF*, *GF* and *EF* as *x*, *y* and *z* respectively. Find the lengths of *CF*, *BF* and *AF*.

CF = 2x

BF = 2z

AF = 2y

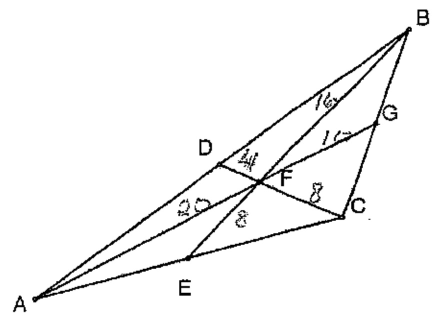
Exercises

1. In the figure pictured, $DF = 4$, $BF = 16$, and $GF = 10$. Find the lengths of:

   a. CF = 8

   b. EF = 8

   c. AF = 20



2. In the figure at the right, $EF = x + 3$ and $BF = 5x - 9$. Find the length of EF.

$2(x+3) = 5x-9$

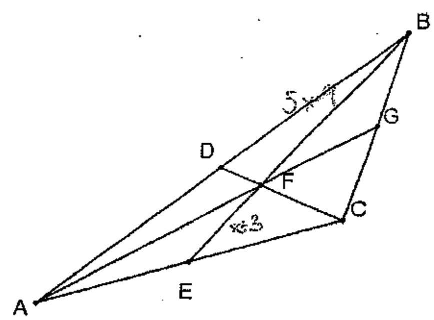
$2x+6 = 5x-9$

$-2x \quad -2x$

$6 = 3x-9$

$+9 \quad +9$

$\frac{15}{3} = \frac{3x}{3}$ $\boxed{x=5}$



EF = 5+3 = 8

3. In the figure at the right, $DC = 15$. Find DF and CF.

$x+2x=15$

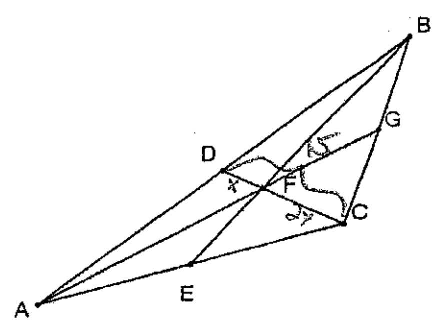
$3x=15$

$\frac{3x}{3} = \frac{15}{3}$

$x=5$

DF = 5

CF = 5(2) = 10

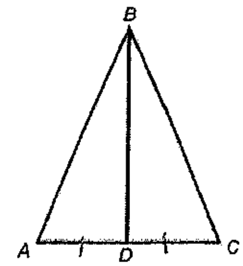


We can now use medians and altitudes in triangle proofs!

Here's how it looks:

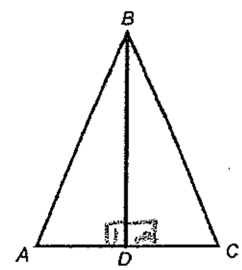
Given: $\overline{BD}$ is the median of $\triangle ABC$

Statements	Reasons
1. $\overline{BD}$ is the median of $\triangle ABC$	1. Given
2. D is the mp of $\overline{AC}$	2. A median goes to the midpoint
3. $\overline{AD} \cong \overline{DC}$	3. Def. of a midpoint



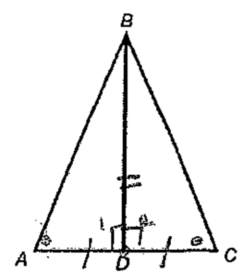
Given: $\overline{BD}$ is the altitude of $\triangle ABC$

Statements	Reasons
1. $\overline{BD}$ is the altitude of $\triangle ABC$	1. Given
2. $\overline{BD} \perp \overline{AC}$	2. Def. of altitude
3. ∠1 and ∠2 are right ∠'s	3. ⊥ lines form right ∠'s
4. ∠1 ≅ ∠2	4. All right ∠'s are ≅.



Example 2

Given: $\overline{BD}$ is the median of $\triangle ABC$, $\overline{BD} \perp \overline{AC}$
Prove: $\angle A \cong \angle C$



Statements	Reasons
1) $\overline{BD}$ is the median of $\triangle ABC$, $\overline{BD} \perp \overline{AC}$	1) Given
2) D is the midpoint of $\overline{AC}$	2) A median goes to the midpoint
3) $\overline{AD} \cong \overline{DC}$	3) Def. of a midpoint.
4) ∠1 and ∠2 are right ∠'s	4) ⊥ lines form right ∠'s
5) ∠1 ≅ ∠2	5) All right ∠'s are ≅.
6) $\overline{BD} \cong \overline{BD}$	6) Reflexive
7) $\triangle BAD \cong \triangle BCD$	7) SAS