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LESSON 2

Finance Fundamentals II

Note: Selected readings can be found under "Lesson 2" on your course website

Assigned Reading

1. Real Estate Division. 2014. CPD 151: *Real Estate Finance Basics*. Vancouver: UBC Real Estate Division.
Lesson 2: Finance Fundamentals II

Recommended Reading

1. Canadian vs US Mortgage Compounding
www.canadamortgage.com/articles/learning.cfm?DocID=29
2. Compound Interest and Mortgages
www.canadamortgage.com/articles/learning.cfm?DocID=2&CFID=5405295&CFTOKEN=56844346
3. Rule of 72 for Compounding
<http://beginnersinvest.about.com/cs/21jumpstreet/a/012501a.htm>

Learning Objectives

After completing this lesson, the student should be able to:

1. describe and differentiate various types of annuities;
2. explain the structure of Canadian mortgage loans;
3. calculate the loan amount, payment, interest rate, and amortization of a mortgage;
4. calculate the outstanding balance of a mortgage;
5. calculate the principal and interest portions of one or more mortgage payments;
6. discuss the present value of regular cash flows formula; and
7. calculate the future value of an interest only loan.

Instructor's Comments

This lesson picks up where Lesson 1 left off. In Lesson 1, we introduced financial analysis, with a focus on foundations and fluency. Interest rates were covered in extensive detail, as these are the basis for *all* financial analysis. However, the financial problems we covered were highly simplistic, in that they involved only a single deposit or repayment of principal (one-time cash flows). We will now proceed to remove this simplifying assumption and examine more realistic loans and investments, where cash flows are paid or received over time, either on a regular or irregular basis. This discussion introduces the use of the payment key on the calculator.

This lesson outlines the different types of annuities, with both regular and irregular payments. Most of the lesson focuses on situations where regular, uniform payments occur at regular intervals of time. This is most commonly seen in mortgage loans, where a large amount of principal is advanced and then repaid by set payments over a long time period; this is an *ordinary general annuity*. We will describe the features of the standard Canadian

mortgage loan and then calculate a variety of elements associated with it: loan amount, payments, amortization, interest rate, outstanding balances, and principal and interest portions of payments. The lesson concludes with a brief discussion of interest only loans.

Lesson 3 will examine financial situations involving the future value of regular and irregular pattern of payments (of interest and principal) occurring at regular and irregular intervals of time. This course concludes with net present value (NPV) and internal rates of return (IRR) calculations, tools that investors use to analyze irregular cash flows. NPV and IRR calculations are useful for investigation and comparison of real estate investments in a real world context.

Constant Payment Mortgages

The standard Canadian mortgage loan requires constant payments, uniform and regularly occurring. In the next several sections, we will illustrate the calculations necessary for constant payment loans, including loan amounts, periodic payments (monthly or otherwise), amortization periods, interest rates, and outstanding balances.

There are four basic financial components in all constant payment mortgage loans:

1. **The Loan Amount:** The loan amount (or face value of the mortgage) is the amount the borrower agrees to repay at the interest rate stated in the mortgage contract. In financial terms, the loan amount is the present value (PV) of the required payments.
2. **The Nominal Rate of Interest:** The frequency of compounding of the nominal interest rate must match the frequency of the payments. For example, if a loan called for interest at 5% per annum, compounded semi-annually with monthly payments, the equivalent nominal rate of interest with monthly compounding would need to be calculated as was shown at the end of the previous lesson.
3. **The Amortization Period:** The amortization period is used to calculate the size of the required payments. The amortization period must be specified in terms of the number of payment periods, so a loan calling for monthly payments over 25 years has 300 payment periods (25×12).
4. **The Payment:** The constant payment required to repay the loan amount over the amortization period is calculated such that, if payments are made regularly, the last payment will repay all remaining principal as well as interest due at the end of the final payment period.

The calculator also uses a fifth piece of information, the future value (FV). However, the future value is set to zero when doing basic calculations for constant payment mortgages because these mortgages are always completely paid off (have a future value of zero) at the end of the amortization period.

The examples in Lesson 1 illustrated the financial calculator's pre-programmed keys for calculating loan amounts (PV), future values (FV), amortization periods (N), interest rates (I/YR), and periods per year (P/YR). All of the simple problems in Lesson 1 assumed that there were no payments involved, i.e., $PMT = 0$. We now add payments into the mix by specifying an amount into the PMT key. By entering the known values of any five of the six variables (PV, FV, PMT, N, I/YR, and P/YR), the calculator can then determine the value of the unknown sixth variable.

Pre-programmed Mortgage Calculations with Excel

Similar to the calculator, Excel has pre-programmed financial functions to calculate the present value (PV), future value (FV), payment (PMT), amortization periods (NPER), and interest rate (RATE). The previous lesson involved single deposits or repayments (lump sums); therefore, the payment was always set to zero. In this lesson, we will build payments into the analyses, but still use the same pre-programmed functions introduced previously.

The following conditions must occur in order to use the calculator to analyze a constant payment mortgage:

1. The present value must occur at the *beginning* of the first payment/compounding period.
2. The payments must be equal in amount, occur at regular intervals, and be made at the *end* of each payment period.
3. The rate of interest must be stated as, or *converted to*, a nominal rate with compounding frequency matching the payment frequency.



Positive and Negative Cash Flows

Financial calculators and spreadsheets, such as Excel, have the financial formulas programmed in, so that users need only specify the amounts and the computer does the math. However, a problem with any calculator/spreadsheet is in specifying positive and negative cash flows. For example, in a typical mortgage loan, the borrower receives loan funds at the beginning of the loan term (cash in, so a positive amount) and makes periodic payments during the loan term and an outstanding balance payment at the end of the loan term (cash out, so negative amounts). The HP10BII/II+ requires opposite signs be used, so PV will be shown as positive, while PMT and FV will be shown as negatives. From the lender's perspective, this could be reversed with PV negative and PMT and FV positive; however, this difference has no mathematical impact. The need to specify negative cash flows or not will vary with every calculator and spreadsheet, depending on how it was programmed. You must be careful to ensure you are using your financial tool correctly, because one incorrect + or – and you will get incorrect answers.

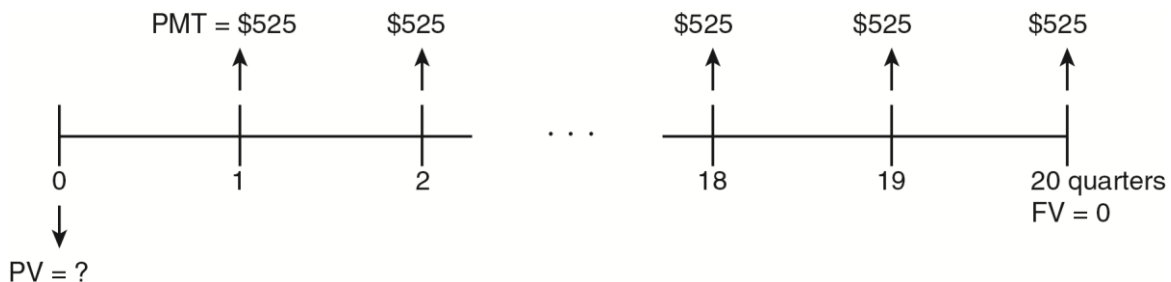
Annuities: Introducing Payments into Financial Calculations

Let's begin the discussion of annuities with a simple payment calculation in Illustration 2.1. In order to do calculations involving recurring payments, we can use the PMT key on the calculator. In order to use the PMT key, payments must be in the form known as an annuity. An annuity is a stream of equal payments that are spread evenly over time. An example of an annuity is the stream of payments on a constant payment mortgage, which is the most common application of the PMT key.

Illustration 2.1

John wants to purchase an investment that will give him payments of \$525 at the end of every quarter for the next five years. If John wants to earn an interest rate of 5% per annum, compounded quarterly, how much should he pay today for this investment?

Solution



In order to find out what John should pay, we can enter the information about the investment into the calculator and calculate the present value.

Press	Display	Comments
5 I/YR	5	Enter nominal interest rate
4 \blacksquare P/YR	4	Enter compounding frequency
5 \times 4 = 20 N	20	Enter number of payments
0 FV	0	No future value accumulated
525 PMT	525	Enter payment amount (cash received, positive)
PV	-9,239.64097	Compute present value (cash paid out, negative)

John should pay \$9,239.64 for this investment if he wants to earn $j_4 = 5\%$ interest.

Calculating Payments in Excel

We can easily add the payment element to the previous calculations that we calculated in Lesson 1. Recall that the pre-programmed functions that we used in Lesson 1 all had a PMT of 0. In this lesson, we expand our discussion to incorporate a value for the payment. Using the pre-programmed PV function, we enter the periodic rate, length of loan (NPER), payment, future value (0), and type (0). Note that we must match the payment frequency with the interest rate and the length of the loan. Similar to the calculator, since we have cash flows going in and out, either the payment or the present value will be negative. Since this illustration is from the investor's perspective, we enter the PMT as a positive; the resulting present value is a negative.

This is just one example of how payments or annuities might be used. The next section discusses annuities in more detail.



Frequency Alert!

It is vital that I/YR, N, and PMT keys all use the same frequency. In Illustration 2.1, the interest rate was compounded quarterly and the payments were every quarter, so these matched. When using the financial keys, and the PMT key in particular, if the payments were instead made semi-annually, the interest rate would have to be entered in the calculator as a j_2 rate – meaning an interest rate conversion would be required, to go from the j_4 rate in the problem to the j_2 rate needed to solve it.

Frequency Rule of Thumb

Mismatching frequency of I/YR, N, and PMT is one of the main mistakes students make in financial problems. The following tips should help avoid these problems:

1. If there is a PMT frequency stated in the financial problem, e.g. monthly mortgage payment, then this drives the frequency in the solution. For example, if you know that there are monthly mortgage payments, the interest rate entered into the calculator must be expressed as a nominal rate per annum, compounded monthly (j_{12}) and N must be entered in months.
2. If there is no PMT frequency stated in the financial problem, e.g., interest accrual loan which has no constant payments, then the compounding frequency of the interest rate given in the problem drives the question. For example, if the question stated there was an interest accrual loan with an interest rate of 10% per annum, compounded annually, you know that there are no payments; therefore, the payment frequency cannot drive the solution. The interest rate compounding frequency is the "key" – if you have a j_1 interest rate, then any data you enter into N must be expressed annually.

Note that the same rules also apply to Excel calculations.

Types of Annuities

Before we do more calculations of annuities, we need to define some terminology more explicitly. An *annuity* is defined as any investment with a sequence of periodic payments made at equal intervals of time over a specified term. The most common example is a mortgage loan that requires equal payments occurring at uniform intervals.

Consider a few basic definitions for annuities:

- The *term* of an annuity runs from the beginning of the first payment interval to the end of the last payment interval.
- The *compounding period* is the interval of time over which interest due on the principal is calculated. There are m compounding periods in one year, and n compounding periods in the term of the loan.
- The *payment period* is the period of time between payments on the annuity. Note that the payment periods and the compounding period are not necessarily of the same duration.

Several types of annuities exist, distinguished by the timing of the payments and the compounding of interest. Annuities that have payments at the beginning of payment periods are referred to as *annuities due*; those that have payments at the end of payment periods are referred to as *ordinary annuities*. Alternatively, annuities with equal frequencies of compounding and payment are known as *simple annuities*; those with different frequencies of compounding and payment are referred to as *general annuities*.

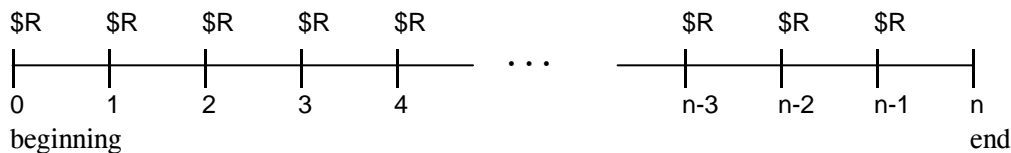
Therefore, there are four specific categories of annuities as illustrated by the following table:

COMPOUNDING PERIODS AND PAYMENT PERIODS		
PAYMENTS	Same	Different
Beginning	Simple Annuity Due	General Annuity Due
End	Ordinary Simple Annuity	Ordinary General Annuity

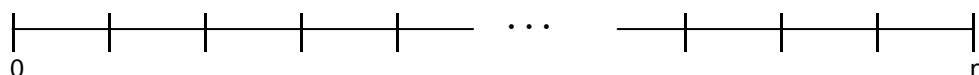
Simple Annuity Due

A *simple annuity due* has payments due at the beginning of the payment period and a payment interval identical to the interest compounding period. A stream of rent payments may be considered as an example of a simple annuity due.

Payment Periods



Compounding Periods



Note that no payment of \$R is required at the end of the last period.

Consider the following example of a simple annuity due.

Illustration 2.2

Chuck's Hockey Shop leases space along a busy street where Chuck sells his hockey sticks and accessories. The lease calls for monthly payments of \$1,850 to be made at the beginning of each month. Chuck wants to get out of his lease, but the landlord has informed him he will have to pay the present value of the three years of remaining lease payments, applying an interest rate of 6% per annum, compounded monthly. How much will Chuck have to pay today to cancel his lease?

Solution

In order to solve this problem, we need to recognize that the payments are made at the beginning (not the end) of the compounding period. On the calculator, we can easily account for this by setting the mode to beginning of period payments as shown in the calculator steps below. Once that is done, we can approach the calculation similar to Illustration 2.1: enter the information about the lease into the calculator and calculate the present value.

Press	Display	Comments
■ Beg/End	BEG	Payments due at beginning of period ¹
6 I/YR	6	Enter nominal interest rate
12 ■ P/YR	12	Enter compounding frequency
3 × 12 = 36 N	36	Enter number of payments
0 FV	0	No future value accumulated
1850 +/- PMT	-1,850	Enter payment amount
PV	61,115.436943	Compute present value

Chuck would have to pay \$61,115.44 to prepay the lease.²

Annuities Due with Excel

In Excel, we identify beginning of period payments by the TYPE in the pre-programmed functions. In all previous calculations, the TYPE is set to 0, which specifies end of period payments. If we change the TYPE to 1, Excel will automatically consider the payments at the beginning of the compounding period.

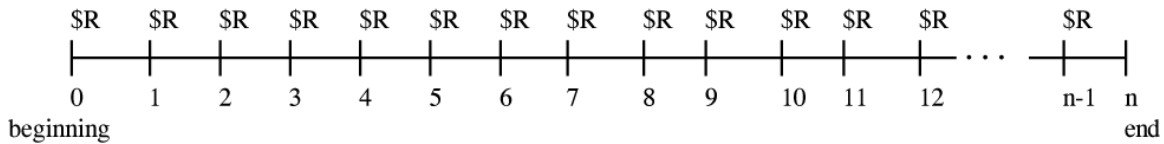
General Annuity Due

A *general annuity due* is one in which the payments are due at the beginning of the payment period and the payment period is not identical to the interest compounding period.

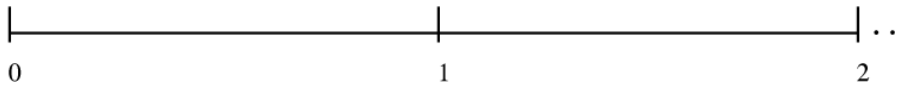
¹ Note that this button works as a toggle switch. If BEG is not showing on the display and you press ■ Beg/End, then BEG will appear and the calculator will assume the payments are paid or received at the beginning of the compounding period. If you press ■ Beg/End again, the BEG on the display will disappear and the calculator now assumes the payments are at the end of the compounding period. Ensure that BEG is only displayed for beginning of period payment calculations (not used commonly in this course or in financial analysis practice).

² An alternative method to calculate an annuity due is to leave the calculator in "End" mode, calculate the PV based one less N (35 in this case), and then add one payment to this PV. The PV of 35 payments is \$59,265.44, plus the \$1,850 payment = \$61,115.44. This explanation allows calculation of this type of problem on calculators that may not have a "Begin" setting.

Payment Periods



Compounding Periods



If we assume that in Illustration 2.2, the interest rate on the lease was 6% per annum, compounded semi-annually, then this illustration becomes an example of a general annuity due. The only difference between a simple annuity due and a general annuity due is that we require an interest rate conversion before we can calculate the present value of the lease. The calculator steps for the modified Illustration 2.2 are as follows:

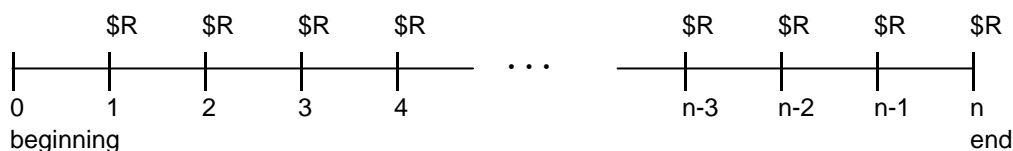
Press	Display	Comments
■ Beg/End	BEG	Payments due at beginning of period
6 ■ NOM%	6	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	6.09	Compute effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	5.926346	Compute nominal rate with monthly compounding
3 × 12 = 36 N	36	Enter number of payments
0 FV	0	No future value accumulated
1850 +/- PMT	-1,850	Enter payment amount
PV	61,178.794472	Compute present value

In this modified case, Chuck would have to pay \$61,178.79 to prepay the lease.

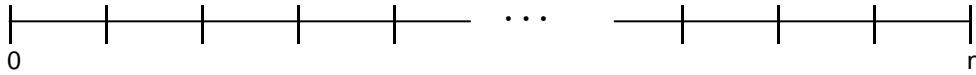
Ordinary Simple Annuity

An *ordinary simple annuity* is defined as an annuity in which the payments are due at the end of the payment period and the payment interval is identical to the interest compounding period.

Payment Periods



Compounding Periods



and where \$R represents the equal periodic payments.

The time diagram outlining the payment and compounding pattern on an ordinary simple annuity represents the repayment pattern applicable to some mortgage loans. Payments commence exactly one period, or time interval, after the loan is advanced and continue in a regular fashion until the loan is fully repaid. If the interest rate's compounding period matches the period for the payments, then this is an ordinary simple annuity.

Illustration 2.1 earlier in this lesson provided an example of an ordinary simple annuity. John receives payments of \$525 per quarter and wants to earn an interest rate of 5% per annum, compounded quarterly. The payments are the end of each quarter and the interest is compounded quarterly.

A mortgage loan where payments are made monthly and interest is compounded monthly is also an example of an ordinary simple annuity. Such repayment schedules are common in the United States where the great majority of mortgage loans are written with monthly payments and monthly compounding. However, the Interest Act prohibits mortgages in Canada from quoting monthly compounding (j_{12}) interest rates; only semi-annual compounding (j_2) and annual compounding (j_1) can be quoted.

Illustration 2.3 provides another example of an ordinary simple annuity.

Illustration 2.3

Joan and Andrew are planning to retire soon. They will invest \$100,000 today and then draw annual cash payments for 15 years. If their investment earns a rate of 4% per annum, compounded annually, how much can Joan and Andrew receive annually in cash payments?

Solution

In order to find out the annual payment that Joan and Andrew can receive, we enter the investment's information into the calculator and solve for the payment.

Press	Display ³	Comments
4 I/YR	4	Enter nominal interest rate
1 \blacksquare P/YR	1	Enter compounding frequency
15 N	15	Enter number of payments
0 FV	0	No future value accumulated
100000 +/- PV	-100,000	Enter investment amount
PMT	8,994.110037	Compute payment

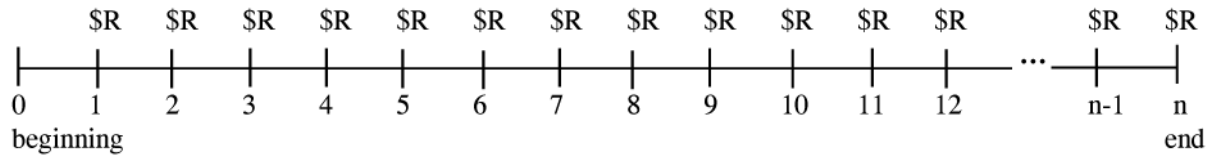
Joan and Andrew will receive annual payments of approximately \$8,994.

³ Note that since this calculation involves end of period payments, we should not have BEG on the display.

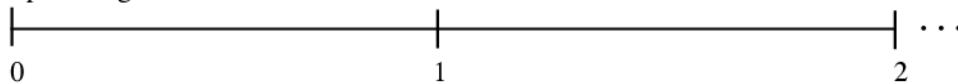
Ordinary General Annuity

An *ordinary general annuity* is one in which the payments are due at the end of the payment period and the payment period is not identical to the interest compounding period. Mortgage loans in Canada are nearly all ordinary general annuities, i.e., mortgage loans with monthly payments and interest rates compounded semi-annually.

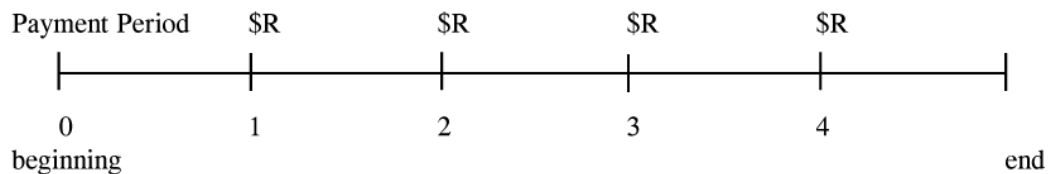
Payment Periods



Compounding Periods



It is also possible that the payments occur less frequently than the compounding of interest.



Compounding Periods

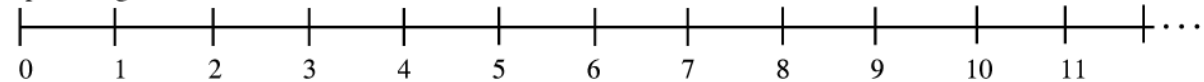


Illustration 2.4 provides an example of an ordinary general annuity.

Illustration 2.4

Tim and Chantal Jones have found a new home they want to purchase. However, in order to purchase the home, they must obtain financing. The Jones can only afford to pay \$2,500 per month. If loans with amortization periods and terms of 25 years have an interest rate of 5% per annum, compounded semi-annually, what is the maximum loan they can obtain?

Solution

The only difference between an ordinary simple annuity and an ordinary general annuity is that we require an interest rate conversion before we can calculate the desired answer. The calculator steps are as follows:

Press	Display	Comments
5 ■ NOM%	5	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	5.0625	Compute effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	4.948699	Compute nominal rate with monthly compounding
25 × 12 = 300 N	300	Enter number of payments
0 FV	0	No future value accumulated
2500 +/- PMT	-2,500	Enter payment amount
PV	429,845.008953	Compute present value

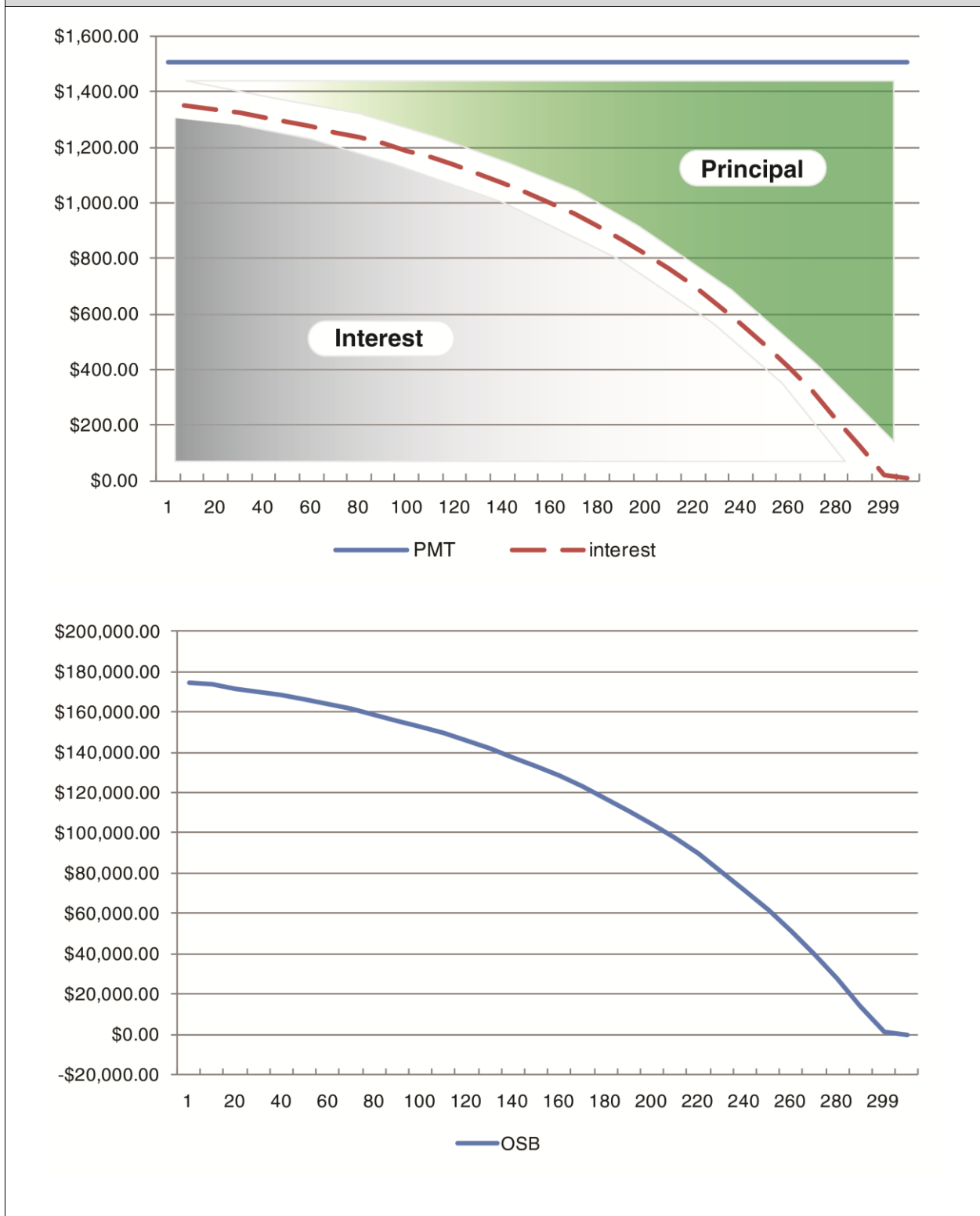
Tim and Chantal can obtain a maximum loan of \$429,845.01.

Annuities due are common in lease analysis problems. They are not commonly used in most financial problems because it is illegal to charge interest in advance. Therefore, nearly all loan calculations are based on ordinary annuities. The problems in the remainder of this lesson focus on ordinary general and ordinary simple annuities. As previously mentioned, most Canadian mortgages call for monthly payments and interest rates are expressed as annual rates with semi-annual compounding; therefore, ordinary general annuities are more common in practice. However, for simplicity, we will also consider examples of ordinary simple annuity problems to avoid the extra steps involved in manipulating interest rates.

Analysis of Constant Payment Mortgages

A constant payment mortgage loan has equal payments throughout the life of the loan. Payments are made on a specified day, at a specified frequency, for example, monthly, semi-monthly, biweekly, or weekly. Each payment includes all interest due for that period plus some repayment of principal. By the time a constant payment mortgage loan expires, the entire principal amount has been repaid by periodic payments. The early payments on a fully amortized constant payment loan are mainly interest with a little principal since the borrower must pay interest on a large outstanding principal amount. However, as each payment is made, an increasing portion of the principal is repaid thereby reducing the outstanding balance on which interest is charged during the next period. As a result of the decreasing balance on which interest is charged, the interest portion of the payment gradually decreases and the principal repayment portion increases. As the end of the loan approaches, little principal remains outstanding and relatively little interest is due. Thus, almost all of the payment is assigned to principal repayment. The top portion of Figure 2.1 illustrates how the principal and interest portions of the payment change over time and the bottom diagram illustrates how the outstanding balance changes over a 25-year amortization period on a \$175,000 mortgage loan with monthly payments of \$1,507.

Figure 2.1 – Principal/Interest Split and Outstanding Balance Amortization Curve for a Constant Payment Blended Loan



Constant payment loans can be classified as either *fully amortized* or *partially amortized*. Fully amortized means that the entire amount of principal is repaid by periodic payments and the final regular payment will repay the remaining principal balance and accrued interest. With a partially amortized loan, the regular payments of principal and interest are calculated to repay the debt over an amortization period which is longer than the loan term. At the end of the loan term, a large payment is required to repay the outstanding principal; in other words, the loan contract period is shorter than the amortization period. An example of a partially amortized loan is a mortgage amortized over 25 years with a 5-year contract term. The regular payments made by the borrower are based on the amortization period of 25 years; however, the borrower makes these payments for only 5 years. At the end of the 5 years, the principal balance that remains outstanding is repaid or refinanced by the borrower. Since payments made early in the life of a constant payment loan are largely comprised of interest, the outstanding balance due at the end of 5 years on a loan with a 25-year amortization is large.

Constant payment loans are also known as fixed rate loans. The interest rate is set and unchanged over the entire loan term. In contrast, variable rate loans allow changes in the interest rate during the loan term.

With the underlying structure of the standard Canadian mortgages now explained, we proceed to examine the loan's components and features: loan amount, payments, interest rate, amortization, outstanding balance, and principal and interest portions of payments.

fully amortized mortgage

a loan which is repaid completely by a series of payments over the full duration of the amortization period

partially amortized mortgage

a loan repayment scheme where the term is shorter than the amortization period; the payment is based on the longer amortization period

Calculation of Loan Amounts

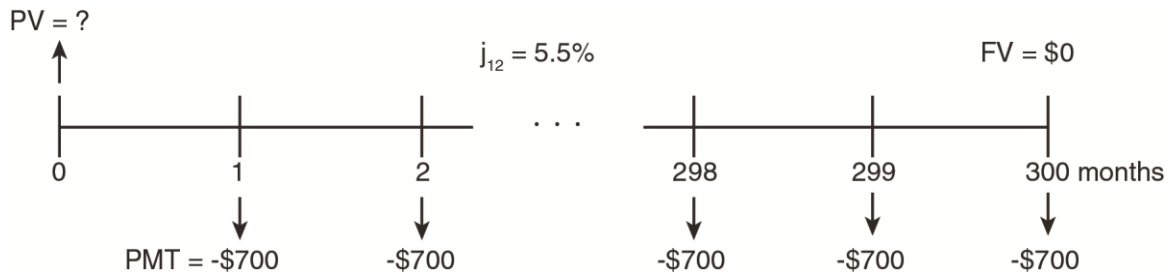
Illustration 2.5 (a)

An individual is considering the purchase of a residential condominium, but wants to limit mortgage payments to \$700 per month. If mortgage rates are 5.5% per annum, compounded monthly, and the lender will permit monthly payments to be made over a 25-year amortization period, determine the maximum allowable loan.

Solution

The financial terms of the proposed loan may be summarized as follows:

$$PV = ?; j_{12} = 5.5\%; PMT = \$700 \text{ per month}; N = 25 \times 12 = 300 \text{ months}$$



As the frequencies of payment and compounding correspond (both are monthly), the problem may be solved directly as follows and no interest rate conversion is necessary:

Press	Display	Comments
5.5 I/YR	5.5	Nominal rate with monthly compounding
12 ■ P/YR	12	Number of payments per year
700 +/- PMT	-700	Monthly payment
25 × 12 = N	300	Months in amortization period
0 FV	0	Zero future value at end of amortization ⁴
PV	113,990.271549	Present value (or loan amount)

The maximum loan based on the interest rate, payments, and amortization period specified is \$113,990.27.

Illustration 2.5 (b)

If the loan above called for interest at the rate of 7% per annum, compounded monthly, determine the maximum loan amount.

Solution

$$N = 300; j_{12} = 7\%; PMT = \$700; PV = ?$$

Because PMT, N, P/YR, and FV are already stored and do not require revision, the calculation is straightforward. We simply change the interest rate and then solve for the present value:

(continued)		
Press	Display	Comments
7 I/YR	7	Nominal rate with monthly compounding
PV	99,040.83237	Loan amount at $j_{12} = 7\%$

Therefore, increasing the interest rate from $j_{12} = 5.5\%$ to $j_{12} = 7\%$ has the effect of decreasing the maximum allowable loan by almost \$15,000 (from \$113,990.27 to \$99,040.83).

Dynamic Structure of Excel

Similar to the calculator, in order to change the interest rate, we only have to change the cell with the monthly interest rate from 0.45833333% to 0.58333333% and Excel automatically recalculates the present value.

Illustration 2.5 (c)

Recalculate the maximum loan, given the details from Illustration 2.5 (a) and assuming interest rates increased to $j_{12} = 10\%$.

(continued)		
Press	Display	Comments
10 I/YR	10	Nominal rate with monthly compounding
PV	77,033.061042	Loan amount at $j_{12} = 10\%$

⁴ A future value amount is not used in this problem because at the end of 300 months the entire principal amount (or outstanding balance) has been repaid, making the future value of the loan zero.

The maximum loan at an interest rate of $j_{12} = 10\%$ reduces to \$77,033.06.

From the preceding illustrations, it is clear that the rate of interest charged on a loan has a large impact on the size of the loan that a *fixed* series of payments will support. With constant payment mortgage loans, a large portion of each of the early payments is allocated to the payment of interest. Increased interest rates reduce the amount of each payment available for principal repayment, making a very large impact on an individual's ability to borrow a given amount.

A number of borrowers and lenders, particularly in the residential real estate market, repay loans using monthly payments. It is easier for borrowers to budget for 12 smaller payments, rather than 1 or 2 large payments during the year. However, as noted earlier, the Interest Act requires that all blended payment mortgages specify the rate of interest as compounded annually or semi-annually. To conform to this provision of the Interest Act, interest rates are typically quoted for semi-annual compounding, but most mortgage loans specify that payments are to be made monthly. This means that the techniques of mortgage analysis presented in Illustration 2.5 must include an extra step to convert the stated nominal rate with semi-annual compounding to an equivalent nominal rate with monthly compounding.

Illustration 2.6

A local trust company has been approached by a real estate investor desiring mortgage money. The investor will pay \$4,000 per month over a 15-year period. What amount will the trust company advance if it desires a yield (or interest rate) of $j_2 = 5\%$?

Solution

When a financial arrangement has a different frequency of compounding and payment, it is necessary to convert the given nominal rate of interest with the stated compounding frequency to an equivalent nominal interest rate, so that the compounding frequency matches the payment frequency. In this illustration, the lender demands a return on investment of 5% per annum, compounded semi-annually. On the other hand, the borrower makes payments on a monthly basis.

The first step to solve for the maximum allowable loan amount involves calculating the nominal rate of interest with monthly compounding that is equivalent to $j_2 = 5\%$.

Press	Display	Comments
5 ■ NOM%	5	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	5.0625	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	4.948699	Compute equivalent nominal rate with desired compounding frequency.



It is not necessary to “key” or type equivalent interest rates into the calculator. Instead, these interest rates should be calculated and used directly to avoid errors in re-entering the number and losing accuracy of your solutions.

The borrower will make 180 monthly payments (15 years × 12 payments per year) of \$4,000, and the rate of interest is 4.948699% per annum, compounded monthly. Since the full unrounded interest rate is already entered as the nominal interest rate with monthly compounding, it does not have to be entered again.

After determining the nominal rate, the maximum loan amount would be calculated as follows:

Press	Display	Comments
	4.948699	j_{12} rate displayed from previous calculation
4000 +/- PMT	-4,000	Payment per month
15 × 12 = N	180	Number of monthly payments
0 FV	0	
PV	507,534.472267	Present value or loan amount

The lender, desiring to earn 5% per annum, compounded semi-annually, would be willing to advance \$507,534.47 in exchange for the borrower's promise to pay \$4,000 per month for 180 months.

Exercise 1

A mortgage loan officer is reviewing a number of loan applications. The table below provides a summary of the loan terms under consideration. Calculate the maximum loan amount the borrowers should be offered for each loan:

Summary of Loan Terms

Loan	Size of Payment	Frequency of Payment	Amortization Period	Nominal Rate
1	\$500	monthly	20 years	$j_2 = 5\%$
2	\$1,712.15	monthly	25 years	$j_2 = 17\%$
3	\$6,000	annually	25 years	$j_1 = 4\%$
4	\$17,250	quarterly	15 years	$j_2 = 12\%$
5	\$623	monthly	25 years	$j_2 = 4.5\%$

Abbreviated Solution:

- $j_{12} = 4.948699\%$ $N = 240$ $PV = \$76,089.02$
- $j_{12} = 16.427423\%$ $N = 300$ $PV = \$122,953.40$
- $j_1 = 4\%$ $N = 25$ $PV = \$93,732.48$
- $j_4 = 11.825206\%$ $N = 60$ $PV = \$481,906.22$
- $j_{12} = 4.458383\%$ $N = 300$ $PV = \$112,561.96$

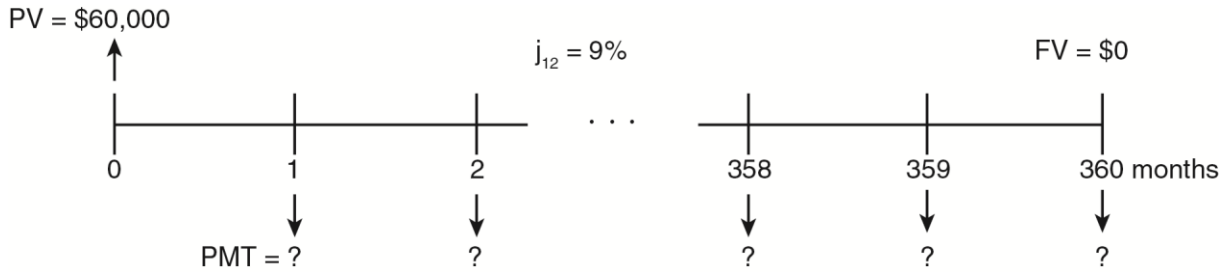
Calculation of Payments to Amortize a Loan

Illustration 2.7

A mortgage loan for \$60,000 is to be repaid by equal monthly payments over a 30-year period. The interest rate is 9% per annum, compounded monthly. Calculate the size of the required monthly payments.

Solution

$$j_{12} = 9\%; N = 12 \times 30 = 360; PV = \$60,000; PMT = ?$$



Press	Display	Comments
9 I/YR	9	Enter stated nominal rate
12 \blacksquare P/YR	12	Enter stated compounding frequency
60000 PV	60,000	Enter present value
30 \times 12 = N	360	Enter amortization period in months
0 FV	0	Loan is fully repaid at the end of 360 months
PMT	-482.77357	Size of required monthly payments

Since most of these mortgage loan illustrations are from the borrower's perspective, we enter the loan amount as a positive and the payment will result in a negative.

Unless stated otherwise, round repeated payments to the nearest cent.

The calculated monthly payments are \$482.77357. Since borrowers cannot make payments which involve fractions of cents, the payments must be rounded to at least the nearest cent. Regular rounding rules will apply unless the facts indicate otherwise. Therefore, the payments on this loan would be \$482.77 when rounded.

Rounding Payments in Excel

If payments are rounded to the nearest cent, you can use the pre-programmed ROUND function by indicating the values to be rounded and the number of digits to round. For dollar and cents rounding, choose 2 as the number of digits.

If payments are rounded up, you can use the pre-programmed ROUNDUP function by indicating the value to be rounded and the number of digits to round. To round up to the next higher dollar, enter 0 as the number of digits; to round up to the next higher ten dollars, choose -1 as the number of digits; and to round up to the next higher one hundred dollars, enter -2 as the number of digits.

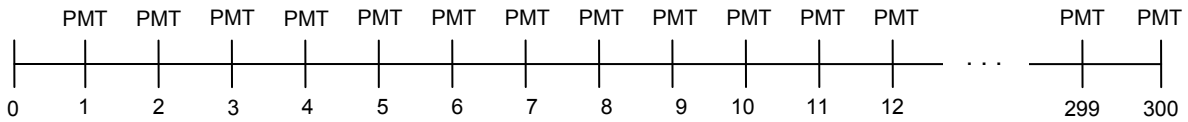
This illustration may be slightly simplistic, since interest rates for mortgage loans in Canada are not typically quoted using monthly compounding, but as annual rates with semi-annual compounding. Since borrowers prefer to make their payments monthly, interest rate conversions are often required when calculating the periodic payments of principal and interest to be made on a loan. If payments occur monthly, principal reductions occur monthly, and interest calculations must be made on a monthly basis. The nominal rate with semi-annual compounding must be converted to its equivalent rate with monthly compounding so the amount of interest charged each month can be calculated.

Illustration 2.8 calculates the equivalent nominal interest rates for a constant payment mortgage (ordinary general annuity) using the same method as outlined in the previous illustration.

Illustration 2.8

A constant payment mortgage of \$175,000 is to be repaid by monthly payments over a 25-year amortization period. The interest rate is 4% per annum, compounded semi-annually. What is the size of the required monthly payment to fully amortize the loan?

PV = \$175,000



Solution

First, find the nominal annual rate with monthly compounding and then find the amount of the payment.

Press	Display	Comments
4 ■ NOM%	4	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	4.04	Compute effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	3.967068	Compute nominal rate with monthly compounding
25 × 12 = 300 N	300	Enter number of payments
0 FV	0	No future value accumulated
175000 PV	175,000	Enter loan amount
PMT	-920.535359	Compute payment

The monthly payment is \$920.54.

Exercise 2

A prospective borrower has contacted five lenders and has collected the information summarized below. Calculate, for each loan alternative, the required payment rounded up to the next higher cent.

Summary of Loan Terms

Loan	Loan Amount	Nominal Rate	Amortization Period	Frequency of Payment
1	\$75,000	$j_2 = 4\%$	20 years	monthly
2	\$100,000	$j_1 = 7\%$	25 years	quarterly
3	\$51,125	$j_4 = 5\%$	240 months	monthly
4	\$60,000	$j_2 = 6.25\%$	25 years	monthly
5	\$260,000	$j_2 = 3.25\%$	25 years	biweekly

Abbreviated Solution:

1. $N = 240$ $j_{12} = 3.967068\%$ $PMT = \$453.18$
2. $N = 100$ $j_4 = 6.823410\%$ $PMT = \$2,091.14$
3. $N = 240$ $j_{12} = 4.979310\%$ $PMT = \$336.82$
4. $N = 300$ $j_{12} = 6.170140\%$ $PMT = \$392.85$
5. $N = 650$ $j_{26} = 3.225876\%$ $PMT = \$582.98$

Accelerated Biweekly Payments

Although payments are typically made on a monthly basis, most lenders offer other repayment options for constant payment mortgage loans. A number of borrowers are choosing to "accelerate" their payments by paying one-half of the regular monthly payment every two weeks. For example, consider Loan #5 in Exercise 2. If payments are biweekly, then the minimum payment is \$582.98 every 2 weeks. If the payments were monthly, the required payment would be \$1,264.04 per month. In both cases, this is the minimum payment to exactly pay off the loan in 25 years, with all interest and principal fully paid. In an accelerated biweekly loan, the borrower pays half the monthly payment, or $\$1,264.04 \div 2 = \632.02 , every 2 weeks. This effectively overpays \$49.04 every 2 weeks above the \$582.98 minimum. Over the loan's term, this \$49 every two weeks means nearly 3 years of payments are no longer required and saves over \$15,000 in interest.

Calculation of Interest Rates

Illustration 2.9

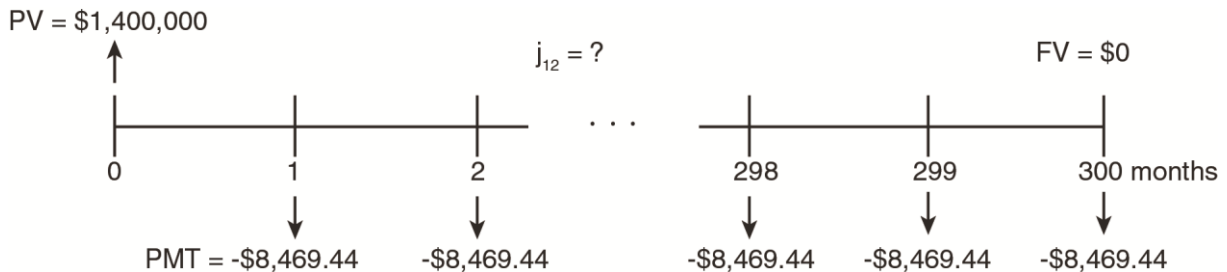
A mortgage calls for monthly payments of \$8,469.44 over 25 years. If the original loan advanced was \$1,400,000, calculate the annual rate of interest with semi-annual compounding (j_2).

Solution

Since the loan calls for monthly payments, one must first determine the nominal rate of interest with monthly compounding, and then convert this to an equivalent nominal rate with semi-annual compounding.

- (i) Calculate the nominal rate with monthly compounding:

$$PV = \$1,400,000; PMT = \$8,469.44; N = 300; j_{12} = ?$$



Press	Display	Comments
12 ■ P/YR	12	Enter payment frequency
1400000 PV	1,400,000	Enter present value
8469.44 +/- PMT	-8,469.44	Enter monthly payment
25 × 12 = N	300	Number of monthly payments
0 FV	0	Loan is fully paid off over 300 months
I/YR	5.346594	j_{12} rate

(ii) Calculate the equivalent nominal rate with semi-annual compounding:

$$j_{12} = 5.346594\% \rightarrow j_2 = ?$$

Press	Display	Comments
	5.346594	j_{12} rate displayed from previous calculation
■ EFF%	5.479579	Equivalent effective annual rate
2 ■ P/YR	2	Enter desired compounding frequency
■ NOM%	5.406503	Equivalent j_2 rate

The interest rate on this \$1,400,000 loan with monthly payments of \$8,469.44 over 25 years is $j_2 = 5.406503\%$.

Calculating Interest Rates in Excel

Use the pre-programmed RATE function and enter the length of loan (NPER), payment, present value, future value, type (0), and guess (can leave blank). Note that the interest rate calculated is a periodic rate (in this case monthly). Multiply by the original compounding frequency and then use the EFFECT and NOMINAL functions to obtain the equivalent j_2 rate.

Note also that either the present value or payment must be entered as a negative in order to obtain a solution. Since this loan is created from the borrower's perspective, we enter the payment as a negative and the loan amount as a positive.

Exercise 3

A private investor is considering three alternative mortgage investments, each of which involves a \$60,000 loan amount:

1. Under the first alternative, the investor will receive 60 monthly payments of \$1,104.93.
2. Under the second alternative, the investor will receive 66 monthly payments of \$1,025.05.
3. Under the third alternative, the investor will receive 54 monthly payments of \$1,207.48.

Assist the investor in choosing among these alternatives by calculating the yield, as a nominal rate with semi-annual compounding, in each case.

Abbreviated Solution:

- | | |
|--------------------------|--------------------|
| 1. $j_{12} = 3.997735\%$ | $j_2 = 4.031179\%$ |
| 2. $j_{12} = 4.395223\%$ | $j_2 = 4.435666\%$ |
| 3. $j_{12} = 3.684914\%$ | $j_2 = 3.713319\%$ |

The investor should choose the second alternative since it provides the highest return (rate of interest).

Calculation of Amortization Periods

The *amortization period* is the period of time that it takes to fully pay off a loan, given the required periodic payments. Most lenders will state the amortization period they require. The stated amortization period is then used to calculate the size of the required payments, i.e., to fully pay off the loan's principal plus all interest owing over this specified time period. The maximum amortization period for insured mortgages is 25 years. For uninsured mortgages, the maximum is typically 30-35 years, depending on the lender.

amortization period
the period of time that it takes to fully pay off a loan, given the required periodic payments

While mortgage payments may be based on a relatively long amortization period, it does not mean that the payments will continue throughout the entire period. Most mortgages written in Canada have a much shorter contractual term. In this case, the periodic payments, based on the amortization period, are made throughout the contractual term. At the end of the contractual term, the principal balance remaining (outstanding) at that time becomes due and payable.

Illustration 2.10

To facilitate the sale of his property, a vendor has agreed to provide partial financing to a purchaser. Under this mortgage, the vendor will loan the purchaser \$50,000, at a rate of 8% per annum, compounded semi-annually. The required payments are \$684.51 per month. What is the amortization period of this loan?

Solution

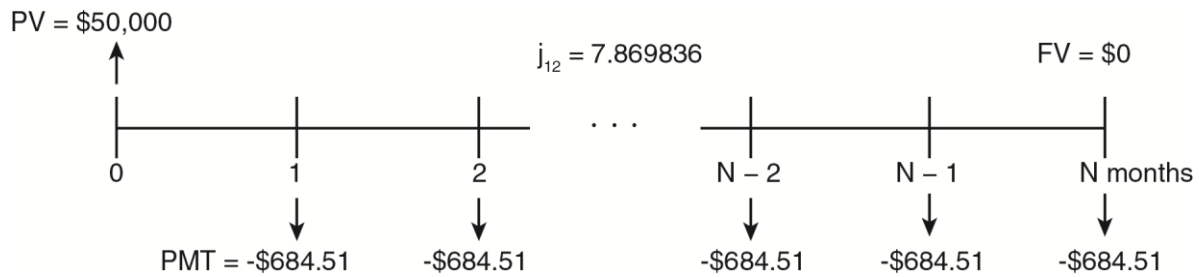
Since the contract rate is compounded semi-annually and the payments are made monthly, the interest rate given must be converted and expressed as an equivalent nominal rate with monthly compounding. Then we can calculate the amortization period.

(i) Calculate the equivalent nominal rate with monthly compounding:

$$j_2 = 8\% \rightarrow j_{12} = ?$$

Press	Display	Comments
8 ■ NOM%	8	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	8.16	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	7.869836	Compute equivalent nominal rate with monthly compounding

(ii) Calculate the number of months:



Press	Display	Comments
	7.869836	Displayed from previous calculation
50000 PV	50,000	Enter loan amount
684.51 +/- PMT	-684.51	Enter monthly payment
0 FV	0	Loan is fully paid off over N months
N	99.756695	Number of monthly payments

The amortization period is 99.756695, or approximately 100 months.

When computing amortization periods, it is usually the case that the amortization period will not be a round number, but will have several decimal places, as in the previous illustration. This indicates, in the case of the previous illustration, that the borrower must make 99 full monthly payments of \$684.51 and one final payment of an amount less than \$684.51.

Note that if the message, "no Solution" appears on the calculator's display when computing an amortization period, it indicates that the given payments are not large enough to ever repay the loan amount at this interest rate. In addition, this error is commonly experienced when the calculator user does not enter the correct sign for the payment (negative) or present value (positive).

Amortization Calculations in Excel

Use the pre-programmed function NPER for amortization calculations and enter the periodic rate, payment, present value, future value (0), and type (0). The solution will produce the number of periods to repay the loan (months in this case). To obtain the amortization in years, divide the answer by the compounding frequency.

As with previous calculations in Excel, either the present value or the payment must be entered as a negative in order to obtain a solution. If the payment is not large enough to repay the loan amount, an error message, #NUM, will appear in the cell. This error also occurs if the Excel user forgets to enter the correct sign for the PMT or PV.

Exercise 4

For each set of loan terms below, calculate the appropriate amortization period.

Summary of Loan Terms

Loan	Loan Amount	Nominal Rate	Payment
1	\$100,000	$j_{12} = 5\%$	\$659.96 per month
2	\$100,000	$j_2 = 6\%$	\$839.89 per month
3	\$50,000	$j_1 = 10\%$	\$6,000 per year
4	\$62,500	$j_2 = 11.5\%$	\$623.40 per month
5	\$451,000	$j_2 = 5.5\%$	\$1,100 per month

Abbreviated Solution:

- | | |
|---------------------------|--------------------------|
| 1. $j_{12} = 5\%$ | $N = 239.997341$ months |
| 2. $j_{12} = 5.926346\%$ | $N = 179.997514$ months |
| 3. $j_1 = 10\%$ | $N = 18.799246$ years |
| 4. $j_{12} = 11.233783\%$ | $N = 299.374382$ months |
| 5. $j_{12} = 5.438018\%$ | $N = \text{No Solution}$ |

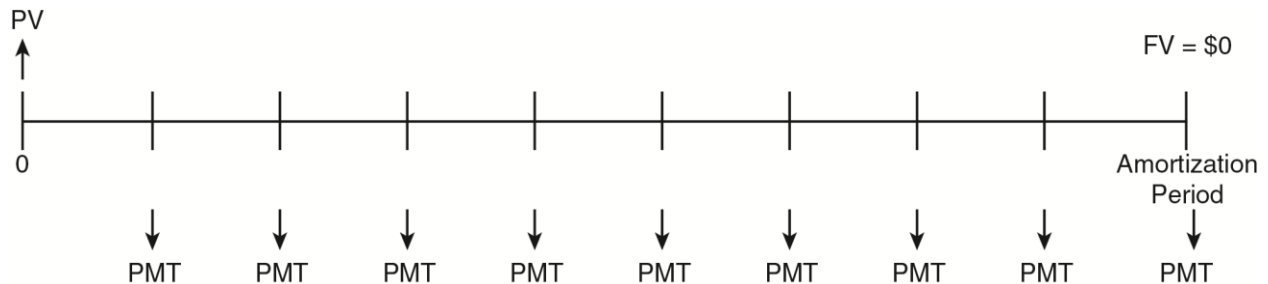
For loan 5, try increasing the payments to find the point at which they are large enough to amortize the loan.

Calculation of an Outstanding Balance (OSB)

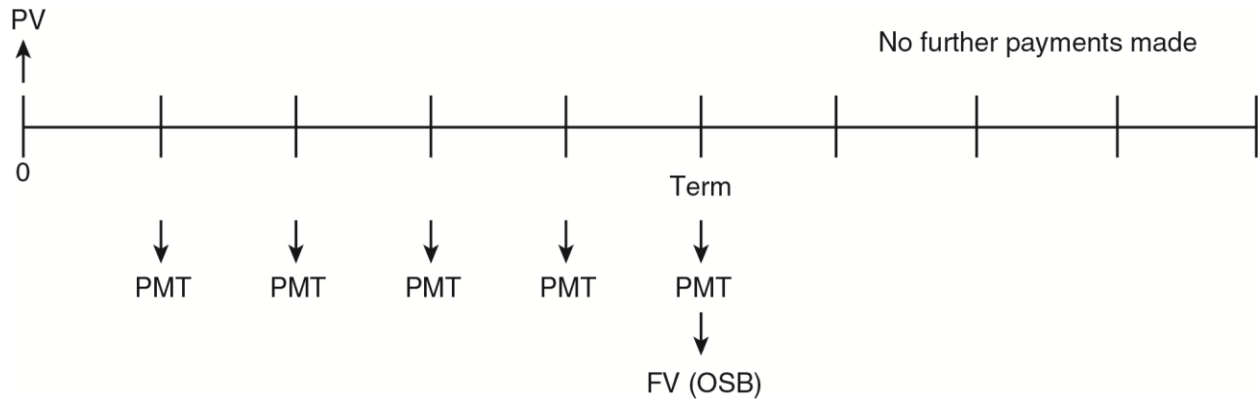
The loan's *outstanding balance (OSB)*, the amount of principal owing at a specific point in time, may need to be calculated for several reasons. Most vendors wish to know how much they will receive from the sale of their property after they have repaid the outstanding balance on their mortgage. While mortgage payments are calculated using the *amortization period*, the actual length of the mortgage contract may be different than the amortization period. The length of the mortgage contract is called the *term*. If the mortgage term and amortization period are the same length of time, the mortgage is said to be *fully amortized*. If the mortgage term is shorter than the amortization period, the mortgage is said to be *partially amortized*. Since mortgages are typically partially amortized with one to five year contractual terms, the amount of money which the borrower owes the lender when the contract expires must be calculated.

outstanding balance (OSB)
the amount of principal owing on a loan at a specific point in time

Payments are first calculated based on the full amortization period:



Outstanding balance is calculated at the end of the loan term:



As shown in the following illustration, the outstanding balance can be quickly calculated on your calculator.

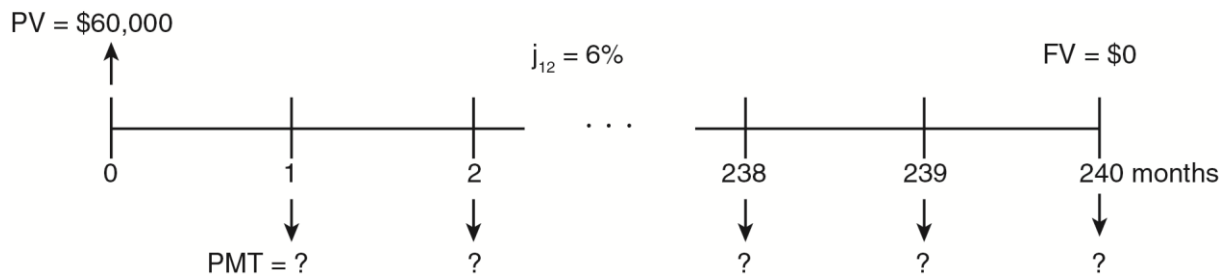
Illustration 2.11

A \$60,000 mortgage loan, written at $j_{12} = 6\%$, has a 20-year amortization period, a 3-year term and monthly payments rounded up to the next higher dollar. What is the outstanding balance of the mortgage at the end of its term? i.e., what is the outstanding balance just after the 36th payment (OSB_{36}) has been made?

Solution

- (i) Calculate the size of the required monthly payments:

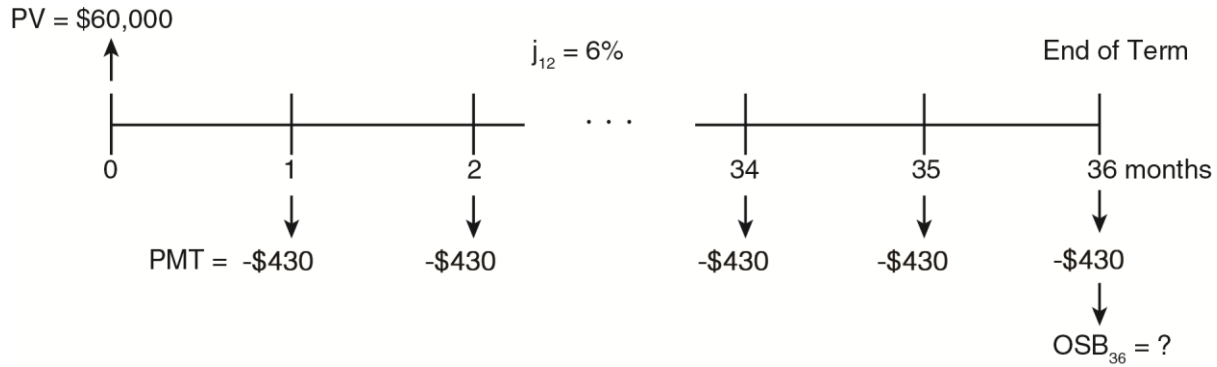
Amortization Period = $20 \times 12 = 240$ months
 PV = \$60,000; $j_{12} = 6\%$



Press	Display	Comments
6 I/YR	6	Enter nominal rate with monthly compounding
12 \blacksquare P/YR	12	Enter compounding frequency
60000 PV	60,000	Enter loan amount
$20 \times 12 = N$	240	Enter number of payments
0 FV	0	240 monthly payments fully amortize loan
PMT	-429.858635	Calculate payment

The monthly payment is \$430, rounded up to the next higher dollar.

- (ii) Calculate the outstanding balance due immediately after (with) the 36th monthly payment:



The HP 10BII/10BII+ calculator has a pre-programmed function that calculates outstanding balances. Before the outstanding balance can be calculated, the rounded payment must be entered into the calculator. The \$430 payment is slightly higher than the \$429.858635 payment required to repay the loan as calculated using the initial information. Since the loan amount and interest rate are set by contract, the increase in the size of the payment causes more principal to be repaid in each month than is required to amortize the loan. The rounding of the payment results in a faster repayment of the loan amount and, consequently, reduces the number of full payments needed to amortize the loan. When the new (larger) payment is entered, the pre-programmed function of your calculator revises the amortization period as part of the outstanding balance calculation:

Press	Display	Comments
	-429.858635	Payment from previous calculation
430 +/- PMT	-430	Enter rounded payment
N	239.84778	Recompute amortization period given higher payment (<i>see ALERT!</i>)
36 INPUT ■ AMORT ⁵	36 – 36	
===	54,886.306355	Outstanding balance after 36th monthly payment



ALERT!

Recalculating N

This step is included for illustrative purposes and is not required in outstanding balance calculations. Recomputing the amortization period will not affect further calculations such as the outstanding balance, as long as the rounded payment has been re-entered.

⁵ The HP10BII+ calculator generates a "running" error when calculating principal, interest, and outstanding balances (OSBs) using the AMORT and INPUT if you use floating decimal notation. You can solve this problem by calculating the outstanding balance using the following method: after re-entering the rounded up payment, enter the time period you wish to obtain the balance for, press N and then FV. For example, if you are working with monthly payments and want to determine the OSB after 36 payments, enter 36 N FV. Alternatively, instead of using floating decimal notation, you can fix the numbers of decimals to a specific number and then proceed with the calculation. For example, if you press ■ DISP →M, this will set the calculator display to 11 decimal places; if you press ■ DISP 9, this will set the calculator to 9 decimal places.

If this loan was fully amortized, it would take 239.84778 months to pay off the loan: 239 payments of \$430 plus a 240th payment which is less than \$430. However, this loan is partially amortized and the borrower will instead make 36 monthly payments of \$430 and pay the remaining balance of \$54,886.31 at the end of the 36th month.

Calculating Outstanding Balances using Excel

Use the pre-programmed FV function and enter the periodic rate, the term (NPER), payment, present value, and type (0). To determine the outstanding balance accurately, you must calculate it based on the rounded payment.

Readers may have noticed that while pressing the "=" sign three times, three different numbers appeared on the screen. The first number that appears is the principal paid in the 36th payment, the second number is the interest paid in the 36th payment, and the final number is the outstanding balance owing immediately after the 36th payment. These functions will be explained further in the next section.

Exercise 5

For each of the first four loans in Exercise 4, assume each loan has a two-year term and calculate the outstanding balance owing at the end of the term.

Abbreviated Solution:

1. OSB = \$93,872.43
2. OSB = \$91,206.14
3. OSB = \$47,900.00
4. OSB = \$61,474.51

Calculation of Principal and Interest Components of Payments

In addition to the outstanding balance, it is often necessary to calculate the principal and interest components of payments on constant payment mortgages. These calculations are important because interest on payments can sometimes be deducted as an expense for income tax purposes. As well, borrowers like to know how much principal they have paid off in a single payment or over a series of payments.

In a constant payment mortgage loan, the total payment is constant in total amount (comprised of principal plus interest), but the amount paid to principal and interest for each payment will change over time. As the outstanding balance on the loan declines over time, the proportion of interest and principal changes. Each payment includes all interest due plus some principal. In long-term amortized mortgages, the early payments on an amortized loan consist mostly of interest with little principal repayment. However, as more and more payments are made, increasingly more principal is repaid, reducing the outstanding balance on the loan as previously shown in Figure 1. As a result, the interest portion of the payment gradually decreases and the amount of the principal portion increases.

The principal and interest portions of individual payments as well as the principal and interest calculations paid over time can be illustrated with reference to an amortization schedule. Using the facts from Illustration 2.12 below, an amortization schedule is created that shows the principal and interest split for each payment and the outstanding balance over time. The calculations covered in this section can be determined with reference to outstanding balance values as shown below. However, the same calculations are easily performed on the calculator with pre-programmed functions as shown below.

Illustration 2.12

Three years ago, Tom and Nancy bought a house with a mortgage loan of \$175,000, written at $j_2 = 9.5\%$, with a 25-year amortization, monthly payments rounded up to the next higher dollar, and a 3-year term. Tom and Nancy are about to make their 36th monthly payment, the last one in the loan's term, and want to determine the following information:

- How much interest will they be paying with their 36th payment?
- How much principal will they be paying off with their 36th payment?
- What will the outstanding balance be immediately following the 36th payment, i.e., the amount they will have to refinance after the 36th payment?
- How much interest did they pay over the entire 3-year term?
- How much principal did they pay off during the 3-year term?
- What is the total amount of interest paid during the second year of the loan?

Solution

In order to answer any of the above questions, it is necessary to find the j_{12} interest rate and the monthly payments under the mortgage.

Press	Display	Comments
9.5 ■ NOM%	9.5	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	9.725625	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	9.31726	Compute nominal rate with monthly compounding
175000 PV	175,000	Enter present value
25 × 12 = N	300	Enter amortization period in months
0 FV	0	Payment is calculated to fully amortize loan
PMT	-1,506.798355	Compute the monthly payment
1507 +/- PMT	-1,507	Enter rounded payment

The monthly payment on the loan is \$1,507, rounded up to the next higher dollar.

The principal and interest portions of payments can be solved in more than one way. We can develop a spreadsheet that calculates this automatically (creating an amortization schedule), we can use the calculator's pre-programmed financial formulas to calculate it, or we can manually calculate it using a combination of the amortization schedule and mathematical formulas. We will demonstrate the manual method below – even if you do not use this method in practice, it is helpful to understand what the calculator or spreadsheet is doing mathematically in solving these problems.

Below is the amortization schedule calculated for this loan. To save space, we have shown only every 10th month and the 35th-36th months.

Amortization Schedule for Illustration 2.12

Month	Payment	Interest	Principal	OSB ⁿ
1	\$1,507.00	\$1,358.77	\$148.23	\$174,851.77
10	\$1,507.00	\$1,348.08	\$158.92	\$173,464.79
20	\$1,507.00	\$1,335.30	\$171.70	\$171,806.13
30	\$1,507.00	\$1,321.49	\$185.51	\$170,014.09
.
.
.
35	\$1,507.00	\$1,314.18	\$192.82	\$169,064.74
36	\$1,507.00	\$1,312.68	\$194.32	\$168,870.42
.
.
.
40	\$1,507.00	\$1,306.58	\$200.42	\$168,077.95
50	\$1,507.00	\$1,290.46	\$216.54	\$165,986.11
60	\$1,507.00	\$1,273.05	\$233.95	\$163,726.06
70	\$1,507.00	\$1,254.24	\$252.76	\$161,284.28
80	\$1,507.00	\$1,233.91	\$273.09	\$158,646.14
90	\$1,507.00	\$1,211.95	\$295.05	\$155,795.86
100	\$1,507.00	\$1,188.22	\$318.78	\$152,716.37
110	\$1,507.00	\$1,162.59	\$344.41	\$149,389.26
120	\$1,507.00	\$1,134.89	\$372.11	\$145,794.59
130	\$1,507.00	\$1,104.97	\$402.03	\$141,910.87
140	\$1,507.00	\$1,072.64	\$434.36	\$137,714.85
150	\$1,507.00	\$1,037.72	\$469.28	\$133,181.40
160	\$1,507.00	\$999.98	\$507.02	\$128,283.41
170	\$1,507.00	\$959.21	\$547.79	\$122,991.55
180	\$1,507.00	\$915.16	\$591.84	\$117,274.15
190	\$1,507.00	\$867.56	\$639.44	\$111,097.00
200	\$1,507.00	\$816.15	\$690.85	\$104,423.12
210	\$1,507.00	\$760.59	\$746.41	\$97,212.57
220	\$1,507.00	\$700.57	\$806.43	\$89,422.19
230	\$1,507.00	\$635.72	\$871.28	\$81,005.36
240	\$1,507.00	\$565.66	\$941.34	\$71,911.70
250	\$1,507.00	\$489.96	\$1,017.04	\$62,086.79
260	\$1,507.00	\$408.18	\$1,098.82	\$51,471.82
270	\$1,507.00	\$319.82	\$1,187.18	\$40,003.26
280	\$1,507.00	\$224.35	\$1,282.65	\$27,612.48
290	\$1,507.00	\$121.21	\$1,385.79	\$14,225.30
299	\$1,507.00	\$21.31	\$1,485.69	\$1,258.84
300	\$1,507.00	\$9.78	\$1,497.23	-\$238.38 ¹

¹ Note the final outstanding balance shows as a negative. In fact, the final payment will be for \$1,268.62 (less than the full regular payment) so that the outstanding balance is completely repaid.

A full amortization schedule is available under "Online Readings" on the Course Resources webpage.

In order to manually calculate the principal portion of any given payment, simply find the outstanding balance before and after the payment is made. The difference will be the principal repaid with that payment.

$$\text{Principal portion of payment} = \text{OSB}_{n-1} - \text{OSB}_n$$

The amortization schedule for Illustration 2.12 shows that the outstanding balance after the 35th payment is \$169,064.74 and the outstanding balance after the 36th payment is \$168,870.42. The difference of \$194.32 represents the principal portion of the 36th payment. Knowing that the total payment is \$1,507, the interest portion of the 36th payment is simply the difference between the total payment and the principal portion of the 36th payment – if the payment is \$1,507 and the principal repaid is \$194.32, then the interest portion of that payment must be \$1,312.68.

Alternatively, students could calculate the interest portion of the 36th payment by calculating the outstanding balance prior to the payment being made and multiplying it by the monthly interest rate to calculate interest owing for that month.

This formula is as follows:

$$I_n = OSB_{n-1} \times i_{mo}$$

Therefore, the interest portion of the 36th payment for Illustration 2.12 is calculated as follows:

$$I_{36} = OSB_{35} \times i_{mo}$$

where $i_{mo} = 0.776438317\%$ (the monthly equivalent to $j_2=9.5\%$)

$$I_{36} = \$169,064.74 \times 0.776438317\%$$

$$I_{36} = \$1,312.68$$

Following the same logic as above, the principal portion of the payment is the difference between the total payment and the calculated interest portion of the payment ($\$1,507 - \$1,312.68 = \$194.32$). We can see from the amortization schedule that the split of principal and interest will vary over the term. At the start of the loan term, most of the payment amount is interest. As principal is repaid, less interest accrues, and in later months, the principal portion increases until most of the payment is principal. For example, in the amortization schedule, we see that the principal and interest portions of the 36th payment are \$194.32 and \$1,312.68 and the principal and interest portions of the 280th payment are \$1,282.65 and \$224.35.

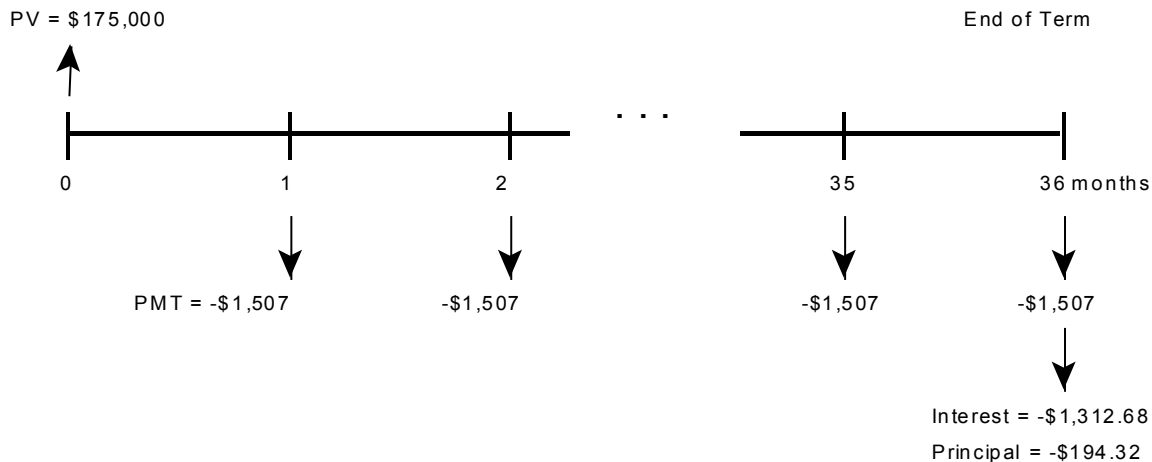
Referring back to Figure 2.1, these diagrams illustrate the principal and interest split of the payment as well as the outstanding balance over time for a constant payment blended mortgage loan. The line at the top shows the payment is constant at \$1,507, while the curving line below shows how much of each payment is interest paid (and the space above the curve shows how much of each payment is principal repayment). The curve illustrates that the proportion of interest and principal in payments changes over time, with early payments mostly interest and later payments mostly principal.

The answers to questions (a), (b), and (c) can all be found using the calculator's pre-programmed amortization function as shown in the following calculation.

Press	Display	Comments
	-1,507	Payment displayed from previous calculation
36 INPUT ■ AMORT	36 – 36	
=	-194.316609	Principal portion of 36 th payment
=	-1,312.683391	Interest portion of 36 th payment
=	168,870.419441	OSB_{36}

Calculating Interest and Principal Portions of a Payment using Excel

The pre-programmed functions, IPMT and PPMT, can be used for interest and principal calculations. To use these equations, enter the periodic rate, the period you want to know the principal and interest portions (PER), the amortization (NPER), present value, future value (0), and type (0). Note that these functions calculate PMT directly, but do not account for rounding – as a result, answers will differ slightly. If you want maximum accuracy, for NPER you need to enter the revised amortization period found by rounding off the payment and then re-computing N.



From the above calculations, questions (a), (b), and (c) can be answered. The amount of interest paid in the 36th payment is \$1,312.68. The amount of principal paid off in the 36th payment is \$194.32. The outstanding balance immediately following the 36th payment (OSB_{36}) is \$168,870.42. As expected, the principal paid and interest paid in the 36th payment total \$1,507, which is equal to the monthly payment.

If we want to calculate the principal and/or interest paid over a series of payments, we can once again solve for this using the outstanding balance amounts or use the pre-programmed function. Using outstanding balance amounts, to calculate the principal repaid over the term, take the initial loan amount minus the outstanding balance at the end of the term.

$$\text{Principal repaid over the loan term} = \text{Loan Amount} - OSB_n$$

For Illustration 2.12, the calculation is as follows:

$$\text{Principal repaid over term} = \text{Loan Amount} - OSB_{36}$$

$$\text{Principal repaid over term} = \$175,000 - \$168,870.42$$

$$\text{Principal repaid over term} = \$6,129.58$$

The interest paid over the term is simply the total of all payments made less the principal repaid in those payments. For Illustration 2.12, the calculation is as follows:

$$\text{Interest paid over term} = (36 \times \$1,507) - \$6,129.58$$

$$\text{Interest paid over term} = \$54,252 - \$6,129.58$$

$$\text{Interest paid over term} = \$48,122.42$$

In other words, over three years the borrower paid \$54,252 in total, of which \$6,129.58 represents the principal repaid. The difference means that \$48,122.42 of the total \$54,252 was interest paid. This result demonstrates

that most of the early payments cover interest and little is paid to principal. During the first three years of this loan, 89% of the payments are interest and 11% are principal repayment. Over time, this proportion will change and eventually reverse, with payments repaying more principal than interest until the loan is completely repaid.

The same pre-programmed amortization function can be used to find the amounts of principal and interest paid over a series of payments, as in questions (d) and (e). In this case, the series of payments will be the entire 36-month loan term, although the amortization function can be used over any series of payments.

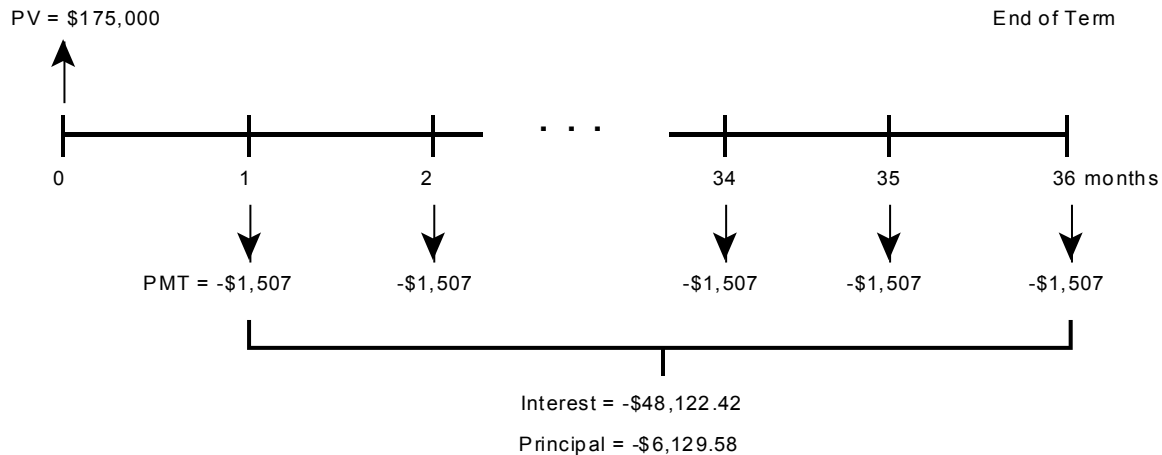
Questions (d) and (e) can be answered using the following calculation.

(continued)		
Press	Display	Comments
1 INPUT 36	36	Enter desired series of payments
■ AMORT	1 – 36	Series of payments being amortized
=	-6,129.580559	Total principal paid off in payments 1 – 36
=	-48,122.419441	Total interest paid in payments 1 – 36
=	168,870.419441	OSB ₃₆

The total amount of interest paid over the term of the loan was \$48,122.42. The total amount of principal paid off during the loan term was \$6,129.58.

Calculating Interest and Principal Paid over a Range of Payments using Excel

With the pre-programmed functions, CUMIPMT and CUMPRINC, enter the periodic rate, amortization (NPER), present value, starting period, ending period, and type (0). Again, note that answers will differ if you use a rounded versus unrounded payment. You must use the revised amortization (NPER) if you want to have full accuracy in determining the principal and interest over a range of payments.



To determine the total amount of interest paid during the second year using outstanding balances, take the difference between the total payments made during the second year and the principal paid during the second year. The total payments made during the second year total \$18,084 (\$1,507 x 12). The principal paid during the second year is \$2,037.34 (the outstanding balance at the end of year 1 of \$173,143.24 – the outstanding balance at the end of year 2 of \$171,105.90).

The interest paid during the second year is simply the total of all payments made less the principal repaid in those payments. For Illustration 2.12 (f), the calculation is as follows:

$$\text{Interest paid over term} = (12 \times \$1,507) - \$2,037.34$$

$$\text{Interest paid over term} = \$18,084 - \$2,037.34$$

$$\text{Interest paid over term} = \$16,046.66$$

Using the pre-programmed function, the calculation is as follows:

(continued)		
Press	Display	Comments
13 INPUT 24	24	Enter desired series of payments
■ AMORT	13 – 24	Series of payments being amortized
=	-2,037.339299	Total principal paid off during second year
=	-16,046.660701	Total interest paid during second year
=	171,105.90272	OSB24

The total amount of interest paid during the second year of the loan was \$16,046.66. It is important to note that the second year of the loan is comprised of monthly payments 13 through 24.

Exercise 6

A prospective borrower has contacted five lenders and has collected the information summarized below. Calculate, for each loan alternative, the amount of interest and principal paid during the 2nd and 24th payments, the interest and principal paid during the first and third years of the loan, and the total interest and principal paid during a 5-year term.

Loan	Loan Amount	Nominal Rate	Amortization Period	Frequency of Payment
1	\$175,000	$j_2 = 5\%$	25 years	monthly
2	\$200,000	$j_1 = 6\%$	20 years	quarterly
3	\$57,000	$j_4 = 12\%$	25 years	monthly
4	\$60,000	$j_{12} = 6.25\%$	20 years	monthly
5	\$260,000	$j_1 = 9\%$	25 years	biweekly

Abbreviated Solution:

	Loan 1	Loan 2	Loan 3	Loan 4	Loan 5
Desired interest rate	4.948699%	5.869538%	11.881961%	6.250000%	8.632067%
Rounded payment	\$1,017.81	\$4,264.44	\$595.37	\$438.56	\$976.44
Interest paid: 2nd pmt	\$720.46	\$2,915.26	\$564.09	\$311.84	\$862.83
Principal paid: 2nd pmt	\$297.35	\$1,349.18	\$31.28	\$126.72	\$113.61
Interest paid: 24th pmt	\$692.29	\$2,405.55	\$556.51	\$296.50	\$854.24
Principal paid: 24th pmt	\$325.52	\$1,858.89	\$38.86	\$142.06	\$122.20

(continued on the following page)

Interest paid: year 1	\$8,578.51	\$11,620.86	\$6,751.79	\$3,705.91	\$22,317.89
Principal paid: year 1	\$3,635.21	\$5,436.90	\$392.65	\$1,556.81	\$3,069.55
Interest paid: year 3	\$8,201.12	\$10,948.86	\$6,647.04	\$3,499.19	\$21,740.50
Principal paid: year 3	\$4,012.60	\$6,108.90	\$497.40	\$1,763.53	\$3,646.94
Interest paid: 5-year term	\$40,956.65	\$54,640.48	\$33,200.31	\$17,461.65	\$108,566.82

The Mathematics of Annuities

This lesson has illustrated that the annuity payments (and other components) can be calculated quite easily using the pre-programmed functions in the financial calculator or Excel. However, you may have noticed that we did not explain the underlying math formulas. Most practitioners who work with financial calculations do so with no need for the math – because the calculator or computer does the mathematics, they simply need to structure the problem to be solved. However, there are some instances where knowledge of the math formulas may come in handy, e.g., in resolving problems that do not fit how the calculator or computer software have been programmed. With this in mind, we will outline the underlying mathematics here.

In illustrating this math, we will follow this progression:

- First, we will outline the present value of a single lump sum cash flow, as discussed in Lesson 1.
- Second, we will expand the discussion to generalize to a formula for a series of regular (constant repeated) cash flows.
- Third, we will illustrate how the pre-programmed PMT calculation in the calculator and in Excel represents the same relationship.

Present Value of Cash Flows for Accrual Loans and Investments

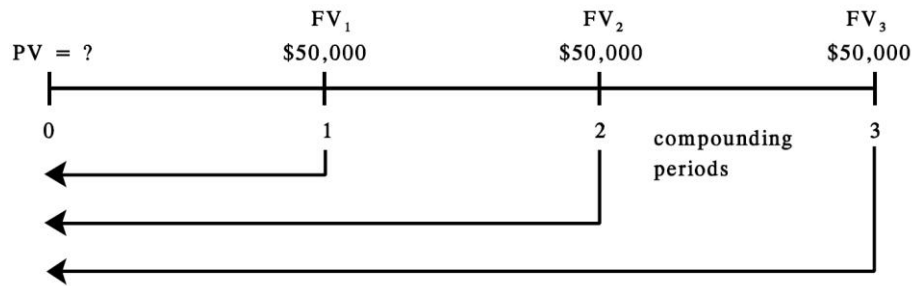
Illustration 2.13

A developer intends to repay a mortgage loan from the proceeds of lot sales from a proposed subdivision. Based on development forecasts, the developer plans to repay the loan in equal instalments of \$50,000 at the end of each of three consecutive months.

If the lender requires a lending rate of 6% per annum, compounded monthly (0.5% per month), what is the maximum the lender can advance to the developer at the beginning of the loan term?

In this case, it is necessary to calculate the maximum amount a lender should be willing to advance in exchange for a series of three regular payments of \$50,000 each. In order to illustrate the underlying math, we will first apply the present value of a lump sum formula discussed in Lesson 1. After this, we will examine more efficient techniques to solve this kind of problem, where an investment involves regular cash flows.

Time Diagram



Solution

To solve this problem, we will isolate each cash flow as a separate present value problem. We calculate the present value today of each \$50,000 cash flow received in future. We then sum these three present values to find the maximum loan today.

$$PV = FV_1 (1 + i_{mo})^{-1} + FV_2 (1 + i_{mo})^{-2} + FV_3 (1 + i_{mo})^{-3}$$

$$PV = \$50,000 \times (1 + .005)^{-1} + \$50,000 \times (1 + .005)^{-2} + \$50,000 \times (1 + .005)^{-3}$$

$$PV = \$148,512.41$$

Press	Display	Comments
6 I/YR	6	Enter stated nominal rate
12 \blacksquare P/YR	12	Enter stated compounding frequency
50000 +/- FV	-50,000	Enter expected future value
1 N	1	Number of compounding periods for FV_1
0 PMT	0	No payments
PV	49,751.243781	Computed present value
\rightarrow M	49,751.243781	Stores present value in M register
2 N	2	Number of compounding periods for FV_2
PV	49,503.725155	Computed present value
M+	49,503.725155	Adds new present value to stored value
3 N	3	Number of compounding periods for FV_3
PV	49,257.437966	Computed present value
M+	49,257.437966	Adds new present value to stored value
RM	148,512.406902	Present value of investment

Thus, if the lender requires a minimum yield of 0.5% per month, the lender would advance the developer no more than \$148,512.41.

Mathematically, we have solved for an annuity function using the relatively simple lump sum formula introduced in Lesson 1. This method requires separately discounting each amount in the problem. This is manageable here, with only 3 cash flows, but it clearly would be unmanageable for a loan requiring 360 monthly payments over 30 years. There are simply too many calculations – this would take too much time, plus there is more risk of error in so many repeated calculations.

A more expedient and efficient method of analysis is desirable. This is the basis for the annuity function described in the next section. However, the manual method described here is still applicable in practice, where cash flows are irregular in both timing and amount. The annuity formulas discussed in the next section cannot accommodate irregular cash flows.

Exercise 7

A developer is seeking financing for land subdivision project. She is seeking funds to purchase a 20-acre parcel and develop the necessary infrastructure to sell the building lots. Her marketing plan forecasts the following cash flows:

- May 30: \$3,000,000
- June 30: \$5,000,000
- Nov 30: \$6,000,000
- Dec 31: \$12,000,000

Based on these cash flows, what is the maximum a lender would loan out, if the lender requires an interest rate of $j_{12} = 9\%$? Assume this analysis is taking place on January 1st.

Abbreviated Solution:

Present Value of Cash Flows:

- May 30: 5 months, PV = \$2,889,987.61
- June 30: 6 months, PV = \$4,780,790.09
- Nov 30: 11 months, PV = \$5,526,569.65
- Dec 31: 12 months, PV = \$10,970,857.86
- Total PV = \$24,168,205.20

This manual method can be used to solve for irregular cash flows of this sort. However, once again, this kind of repetitive type of calculation is tedious and error-prone. Lesson 3 will illustrate the net present value (NPV) function, a powerful tool for problems with irregular cash flows.

Present Value of Regular Cash Flows

The problem in Illustration 2.13 required calculating the present value of three payments of \$50,000:

$$PV = \$50,000 \times (1 + .005)^{-1} + \$50,000 \times (1 + .005)^{-2} + \$50,000 \times (1 + .005)^{-3}$$

The present value of each cash flow was calculated separately and summed. However, this expression could be simplified by aggregating the terms that appear more than once in the equation. The \$50,000 payment appears three times, so this can be rearranged algebraically as follows:

$$PV = \$50,000 \times [(1 + .005)^{-1} + (1 + .005)^{-2} + (1 + .005)^{-3}]$$

This simplification works here only because every cash flow is of exactly the same size. The simplification here is not of great value, as it really only eliminates one step in the manual calculation required. However, this leads to another simplification that greatly reduces the number of calculations required, and ultimately permits the pre-programming functions of financial calculators and spreadsheet software. This second simplification results from the formula for what is known as a geometric progression: a series of numbers that change by a regular series of exponents [i.e. $(1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3}$] can be determined directly by using a single formula. Mathematically, the sum of the series is equal to a single value:

$$[(1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3}] = \left[\frac{1 - (1 + i)^{-3}}{i} \right]$$

Filling in i produces the following solution:

$$\left[\frac{1 - (1 + 0.005)^{-3}}{0.005} \right] = 2.970248$$

What this formula effectively calculates is that the promise of \$1 received at the end of each year for three years is worth \$2.97 today in present value terms. This function is known as the "present value of a \$1 annuity".⁶

Multiplying this present value factor by \$50,000 equals \$148,512.41. In other words, the promise of \$50,000 received at the end of each of three years is worth \$148,512.41 in present value terms today.

Summarizing this in equation form:

$$PV = \$50,000 \times (1 + .005)^{-1} + \$50,000 \times (1 + .005)^{-2} + \$50,000 \times (1 + .005)^{-3}$$

$$PV = \$50,000 \times \left[\frac{1 - (1 + 0.005)^{-3}}{0.005} \right]$$

As $(1.005)^{-3} = 0.985149$, then⁷

$$PV = \$50,000 \times \left[\frac{1 - 0.985149}{0.005} \right]$$

$$PV = \$50,000 \times \left[\frac{0.0148512}{0.005} \right]$$

$$PV = \$50,000 \times 2.970248$$

$$PV = \$148,512.41$$

⁶ Some readers may recognize this term as one of the "Six Functions of One Dollar". Before the introduction of financial calculators and desktop computers, there were published books of tables with these and other factors, for a variety of interest rates and periods. In a pre-computer world, this saved practitioners from needing to do the mathematics for present value and future value analysis; they simply looked up the appropriate factor. These books are now largely obsolete today.

⁷

Calculation

Press
1.005 ■ y^x 3 +/- =

Display
0.98514875931

Solving this problem on the calculator using the annuity formula is as follows:

Press	Display	Comments
6 ÷ 12 =	0.5	Monthly rate
%	0.005	Monthly rate expressed as a decimal
+ 1=	1.005	
■ y ^x 3 +/- =	0.985149	
+/- + 1 =	0.0148512	
÷ .005 =	2.970248	
× 50000 =	148,512.4069	Present value

Note this is exactly the same answer as was obtained using the more time-consuming method presented in the prior section. This formula makes it much easier to calculate the present value of a long-term payment stream, such as a 25-year mortgage with monthly payments or a 50-year land lease. All that would be required in these instances is to replace the -3 in the formula with -300 or -50. You can enter any number in this exponent as needed. If we substitute "n" for the number of periods, we can see the general formula of this relationship:

$$\text{PV of \$1 per period} = [(1 + i)^{-1} + (1 + i)^{-2} + \dots + (1 + i)^{-n}] = \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

This formula works in any problem where you have a cash flow at the *end* of every compounding period and the amounts of the cash flows are all *equal*.

Thus, the present value of an ordinary simple annuity of PMT per period for n periods at i% per period may be expressed as follows:

$$\text{PV} = \text{PMT} \times \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

The PV of \$1 per period formula requires equal cash flows received at the end of each period.

Shorthand notation has evolved to describe this function more quickly and easily. In this shorthand language, the term a[[n, j]] (pronounced A-N-J) is used as a replacement.

This formula eliminates the time-consuming steps in individually calculating the present value of an ordinary simple annuity, particularly where there are a large number of cash flows.

The mathematical expression is the "present value of \$1 per period", or more accurately "the present value of \$1 per period for n periods at i%". This can be expressed in shorthand format using the symbol a[[n, j]] (pronounced as A-N-J).

Therefore, the present value relationship⁸ for a stream of regular payments may be rewritten as:

$$\text{PV} = \text{PMT} \times a[[n, j]]$$

⁸ Note that this is the same relationship that we examined in the previous sections of this lesson.

where:

PV	=	Present value of the stream of regular cash flows
PMT	=	Dollar amount of the regular payment or cash flow
$a[[n, j]]$	=	Present value of \$1 per period for n periods, discounted at a nominal rate of $j\%$
n	=	Number of compounding periods
j	=	Interest rate per annum; requires compounding frequency ⁹

It is important to understand that $a[[n, j]]$ is not a mathematical formula; it is simply a shorthand or symbolic way of writing something that is otherwise very cumbersome. It represents the mathematical relationship to be used in solving problems where streams of regular payments are involved. The double square brackets " $[[,]]$ " are traditionally used to separate the values for n and j .

Returning to Illustration 2.13, in symbolic formulation, the problem is represented as follows:

$$PV = PMT \times a[[n, j_m]]$$

$$PV = \$50,000 \times a[[3, j_{12} = 6\%]]$$

In addition to the techniques demonstrated above, this present value relationship is also a pre-programmed operation on the financial calculator and the calculator's pre-programmed present value relationship may be used directly to solve the problem (as shown below). If the pre-programmed function is to be used, certain conditions must be met:

1. all payments occur at the end of each period;
2. the payment periods and compounding periods are identical;
3. payments are all for the same amount; and
4. the present value is to be determined at the beginning of the first compounding (and payment) period.

That is, it must be an ordinary simple annuity and the goal is to solve for the present value at the beginning of the first period. If these conditions do not prevail, one must either find equivalent values to create such a situation or use the detailed method of analysis presented in Illustration 2.13.

The solution using the pre-programmed functions on the calculator is as follows:

Press	Display	Comments
6 I/YR	6	Stated nominal rate
12 \blacksquare P/YR	12	Stated compounding frequency
50000 +/- PMT	-50,000	Monthly payment
3 N	3	Amortization period
0 FV	0	No balance owing at end of amortization period
PV	148,512.406902	Present value

⁹ Recall that the relationship between the nominal rate of interest, " j ," and the periodic rate of interest, " i ," is such that $i = j/m$ and $j = i \times m$. The formula uses " i " and the formulation uses " j ".

Excel Equivalent Calculation for Annuities

When calculating the present value of a series of payments, Excel can be used either to find the present value of individual payments (and add them together) OR to solve for the PV directly using the pre-programmed PMT function. Excel's PMT function operates much the same as the calculator (and is the same as we have already examined in previous sections of this lesson).

We have now illustrated several different ways to solve for problems where the present value must be calculated for a stream of regular, constant payments received in the future.

- This relationship may be stated as: $PV = PMT \times a[[n,j]]$
- We can solve for the present value of this relationship by determining the present value of the individual cash flows and summing them.

$$PV = PMT \times (1 + i)^{-1} + PMT \times (1 + i)^{-2} + \dots + PMT \times (1 + i)^{-n}$$

Or

$$PV = PMT \times [(1 + i)^{-1} + (1 + i)^{-2} + \dots + (1 + i)^{-n}]$$

- We can solve for the present value using the annuity formula.

$$PV = PMT \times \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

- We can rearrange this formula to solve for the PMT:

$$PMT = \frac{PV}{\left[\frac{1 - (1 + i)^{-n}}{i} \right]}$$

Or, simplifying this:

$$PMT = PV \left(i + \frac{i}{(1 + i)^n - 1} \right)$$

This formula has become more complicated ... fortunately, the payment can be easily solved using the financial calculator's PMT key or the PMT function in Excel.

All of these methods lead to the same answer – which you choose to apply is a matter of personal preference, depending on the tools at hand and your comfort with math. In most cases, practitioners will use the calculator or spreadsheet, as this is much easier and less prone to simple math errors. However, in complex situations, you may find it is helpful to be able to revert back to the underlying math, to work through a solution.

As well, keep in mind that you can also solve for any one of the variables: PV, PMT, i, or n, as long as you know the three other factors. Solving for i or n by rearranging the formula algebraically is a challenging mathematical exercise; this is much easier to do using the pre-programmed functions of the financial calculator or spreadsheet.

The following illustration provides an additional example of an annuity calculation, using the formula solution and the pre-programmed solution to determine the present value of a series of payments.

Illustration 2.14

An investor wishes to calculate the present value of a leasehold interest that is expected to result in cash payments of \$15,000 per annum for the next 40 years. The investor requires a yield of 5% per annum, compounded annually.

Find the present value of the leasehold interest at the rate of $j_1 = 5\%$.

Solution

This illustration involves calculating the present value of a cash flow of \$15,000 per annum for 40 different periods ($n = 1, 2, 3 \dots 40$), and adding the present values together. Given that the discount rate is $j_1 = 5\%$, this would require the following formula:

$$PV = \$15,000(1+.05)^{-1} + \$15,000(1+.05)^{-2} + \dots + \$15,000 \times (1+.05)^{-40}$$

$$PV = \$15,000 \times [(1.05)^{-1} + (1.05)^{-2} + (1.05)^{-3} + \dots + (1.05)^{-40}]$$

This problem can be expressed using the formula at the left. However, consider the same formula expressed using j_m instead of i :

$$PV = \$15,000 \times \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

$$PV = \$15,000 \times \left[\frac{1 - (1 + j_m / m)^{-n}}{j_m / m} \right]$$

$$PV = \$15,000 \times \left[\frac{1 - (1 + 0.05)^{-40}}{0.05} \right]$$

The mathematical solution outlined below would be the procedure to follow if a financial calculator was not available:

$$PV = \$15,000 \times \left[\frac{1 - 0.142046}{0.05} \right]$$

$$PV = \$15,000 \times \left[\frac{0.857954}{0.05} \right]$$

$$PV = \$15,000 \times 17.159086$$

$$PV = \$257,386.30$$

Press	Display	Comments
5	5	Stated nominal rate (expressed as j_1)
%	0.05	Monthly rate expressed as a decimal
+ 1=	1.05	
■ y^x 40 +/- =	0.142046	
+/- + 1 =	0.857954	
÷ .05 =	17.159086	
× 15000 =	257,386.29531	Present value

In symbolic formulation, this problem can be represented as follows:

$$PV = PMT \times a[[n, j_m]]$$

$$PV = \$15,000 \times a[[40, j_1 = 5\%]]$$

$$PV = \$257,386.30$$

Solving the same problem using the pre-programmed functions on the calculator, the steps are as follows:

Press	Display	Comments
5 I/YR	5	Stated nominal rate
1 ■ P/YR	1	Stated compounding frequency
15000 +/- PMT	-15,000	Payment
40 N	40	Length of loan
0 FV	0	No balance owing at end of amortization period
PV	257,386.29531	Present value

The present value of the leasehold interest is \$257,386.30.

The Six Functions of a Dollar

Financial analysis can be summarized into six mathematical functions:

1. Present value of \$1 received in future
2. Future value of \$1 received today
3. Present value of \$1 per period received in future
4. Future value of \$1 per period received in future
5. Payment per period to amortize (pay off) a \$1 present value (known as *amortizing factor*)
6. Payment per period to accumulate \$1 future value (known as *sinking fund factor*)

Lesson 1 illustrated (1) and (2). This lesson has focused on (3) and (5). Lesson 3 will examine (4) and (6) the future value functions, with particular emphasis on sinking funds.

See Appendix 1 for more details.

Interest Only Loans

For the final topic in this lesson, we will return to the basic foundations of interest calculations for loans. After we illustrated simple interest calculations, we stated that compound interest is the basis for nearly all financial calculations. However, interest will not compound or accrue if interest is paid as it is due. Recall that interest accrual means earning interest on the prior interest that has built-up or accrued but remains unpaid.

Interest accrual loans, where the interest remains unpaid until the end of the loan term, means much higher risk for lenders, as they are not only loaning the principal but also the accumulated interest. To avoid the income risk with interest accrual loans, most lenders require the interest earned be paid periodically rather than permitting it to accumulate as added debt. A lender may specify a loan as *interest only*, where the principal remains outstanding over the loan term but the borrower contracts to make regular payments of interest to the lender. Thus, the interest is paid when due, but the principal amount never declines.

The formula for calculating an interest only payment is as follows:

$$I = PV \times i$$

Where I = interest paid per period

PV = loan amount

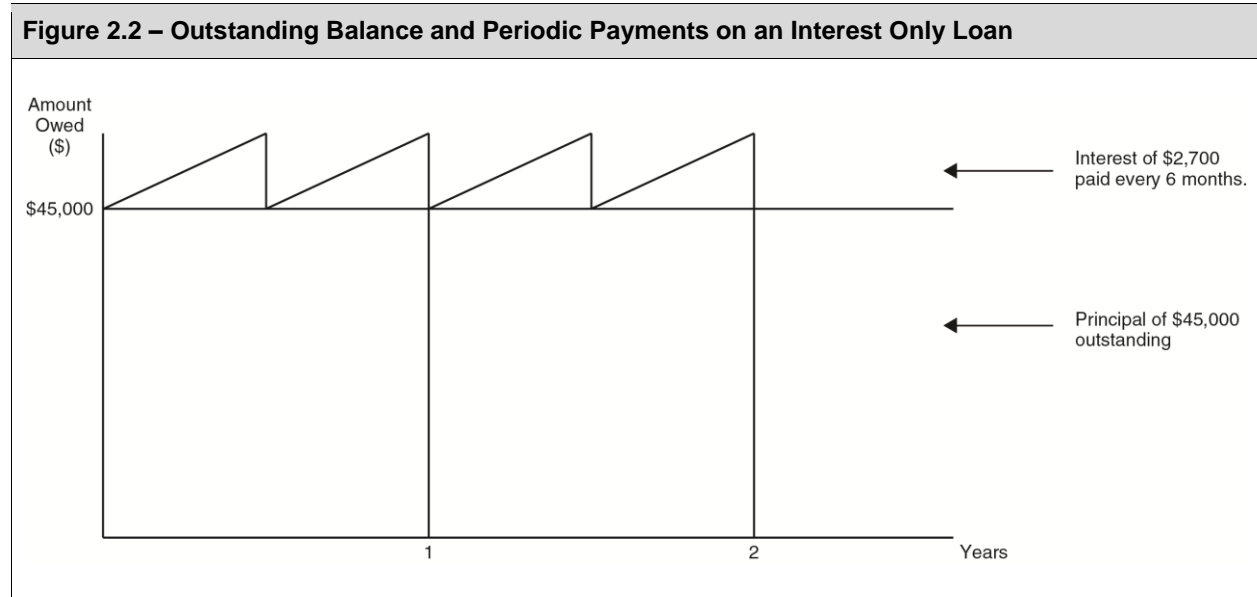
i = periodic interest rate

Illustration 2.15

A lender has agreed to a \$45,000 interest only loan. Interest will be charged at a rate of 6% at the end of every 6-month period over a two year term. The borrower must pay \$2,700 every 6 months during the term of the loan, and at the end of the term, \$45,000 plus the final interest payment. The loan's cash flows are summarized as follows:

\$45,000	Principal amount borrowed
\$ 2,700	Interest to be paid at end of six months ($45,000 \times .06$)
\$45,000	Amount owing during second six months
\$ 2,700	Interest to be paid at the end of the second six months
\$45,000	Amount owing during third six months
\$ 2,700	Interest to be paid at the end of the third six months
\$47,700	Total amount to be paid at the end of the final six-month period of the term

The financial situation during the term of the loan is shown in Figure 2.2.



Interest Only Loans in Practice

The interest only plan reduces the lender's income (interest) risk but still leaves the entire principal at risk. Interest only loans are often used in real estate development, where a developer needs funds for construction but wants to keep financing payments to a minimum during the construction period. Upon completion of construction, the developer will likely sell the property and pay out the lender from proceeds of the sale. In the past, lenders have not often approved these loans over long periods because the full amount of principal is outstanding and at risk over the life of the loan. However, it should be noted that interest only loans have been available for a long time in the United States residential lending market. In Canada, Canada Mortgage and Housing Corporation (CMHC) offers an interest only home owner mortgage insurance product. CMHC has aimed this product at borrowers who hold a strong track record in managing debt and can afford the mortgage principal following the initial interest period.

Illustration 2.16

Brad Hartfield has received a loan for \$250,000 to start a mini-golf course. The loan has an interest rate of 7% per annum, compounded monthly, and requires interest only payments every month. The loan must be repaid in 12 months. What is Brad's monthly interest only payment?

Solution

1. Mathematical/Formula

Calculate the periodic interest rate per monthly compounding period, press ÷ to convert this periodic rate to a decimal and multiply by the interest only loan amount. This will calculate the monthly interest only payment.

Press	Display	Comments
7 ÷ 12 =	0.583333	Convert the nominal rate to the periodic rate
%	0.00583333	Convert the periodic rate to a decimal
× 250000 =	1,458.333333	Calculate interest only payment

Brad will have to make monthly interest only payments of \$1,458.33.

2. Calculator

Since you never pay down the principal balance on an interest only loan, the present value and the future value will be the same. In an interest only loan, you can set N to equal any number because there is no principal portion of the payments paying down the loan, and each payment will be the interest portion of the loan.

Press	Display	Comments
7 I/YR	7	Enter nominal interest rate
12 ■ P/YR	12	Enter compounding frequency
250000 PV	250,000	Enter loan amount
250000 +/- FV	-250,000	Enter principal owing at end of loan
1 N	1	Set to any number
PMT	-1,458.333333	Compute interest only payment

Brad will have to make monthly interest only payments of \$1,458.33.

Interest Only Payments in Excel

Use the pre-programmed PMT function and enter the periodic rate, length of loan (NPER = any value), present value, future value, and type (0). Note that for an interest only loan, the present value and future value amounts must be the same amount and either the present value or future value amount must be entered as a negative. Alternatively, you can simply use the formula ($I = PV \times i$) to solve directly for an interest only payment.

Illustration 2.17

Sara Brown has received an interest only loan in the amount of \$850,000 which she will use to help build a new medical office. The loan has an interest rate of 5.5% per annum, compounded quarterly and requires interest only payments to be made quarterly. If the loan is for two years, how much are the interest only loan payments that Sara makes every quarter?

Solution

1. Mathematical/Formula

Calculate the periodic interest rate per quarterly compounding period, press % to convert this periodic rate to a decimal and multiply by the interest only loan amount. This will calculate the quarterly interest only payment.

Press	Display	Comments
5.5 ÷ 4 =	1.375	Convert the nominal rate to the period rate
%	0.01375	Convert the periodic rate to a decimal
× 850000 =	11,687.5	Calculate interest only payment

Sara will be required to make interest only payments of \$11,687.50 every quarter.

2. Calculator

Since you never pay down the principal balance on an interest only loan, the present value and the future value will be the same. In an interest only loan, set N to equal any number because there is no principal portion of the payments paying down the loan, and each payment will be the interest portion of the loan.

Press	Display	Comments
5.5 I/YR	5.5	Enter nominal interest rate
4 ■ P/YR	4	Enter compounding frequency
850000 PV	850,000	Enter loan amount
850000 +/- FV	-850,000	Enter principal owing at end of loan
1 N	1	Set to any number
PMT	-11,687.5	Compute interest only payment

Sara will be required to make interest only payments of \$11,687.50 every quarter.

Exercise 8

A prospective borrower has contacted four lenders and has collected the information summarized below. Calculate the required payment for each loan alternative.

Summary of Loan Terms

Loan	Loan Amount	Nominal Rate	Frequency of Payment
1	\$100,000	$j_2 = 8\%$	monthly
2	\$60,000	$j_1 = 4\%$	quarterly
3	\$125,000	$j_4 = 10\%$	monthly
4	\$90,000	$j_{12} = 6.25\%$	semi-annual
1	\$100,000	$j_2 = 8\%$	monthly

Abbreviated Solution:

1. $j_{12} = 7.869836\%$	$i_{mo} = 0.65582\%$	PMT = \$655.82
2. $j_4 = 3.941363\%$	$i_q = 0.985341\%$	PMT = \$591.20
3. $j_{12} = 9.917805\%$	$i_{mo} = 0.826484\%$	PMT = \$1,033.10
4. $j_2 = 6.331948\%$	$i_{sa} = 3.165974\%$	PMT = \$2,849.38

Summary

This lesson built on Lesson 1's foundations, where we examined single deposits or repayments of principal (one-time cash flows). In this lesson, we expanded the discussion to review the impact of receiving or paying regular cash flows over time. Adding payments to the simple present value situations in Lesson 1 adds considerable complexity to the financial problems. However, we showed that in situations where regular, uniform payments occur at regular intervals of time, shorthand methods have been developed to make the financial analysis relatively easy and straightforward. We illustrated these shorthand methods both mathematically and using the financial calculator and spreadsheet software.

Building payments into the calculations leads to a variety of practical financial applications, most notably amortized mortgage loans. This lesson detailed the various forms of annuities, focusing on the ordinary general annuity as the basis for the standard Canadian mortgage loan. We described the features of the standard Canadian mortgage loan and then calculated a variety of elements associated with it: loan amount, payments, amortization, interest rate, outstanding balances, and principal and interest portions of payments. The lesson concluded with a brief discussion of interest only loans.

Review and Discussion Questions

1. If you have a mortgage, calculate how much of your mortgage you actually paid off during the first year of your payments and how much you spent on interest payments.
2. The lesson discussed how the PV annuity formula can be rearranged mathematically to solve for PMT. Using algebra, carry out this rearranging, and restate the formula for PMT. (Hint: for those less mathematically inclined, you may wish to search the internet for assistance.)
3. The table below represents investment data collected by an investor for several properties he is considering buying. For each property, the investor has identified three important financial items. Assist the investor by calculating the missing piece of information for each property. Assume in each case that no payments are made (or costs incurred) during the investment horizon and ignore any possible income tax implications.

Property	Purchase Price (PV)	Estimated Future Value (FV)	Holding Period (N)	Interest Rate Earned (j_m)
(a)	\$100,000	\$185,000	6 years	? (j_1)
(b)	\$100,000	\$185,000	6 years	? (j_{12})
(c)	\$ 20,000	?	24 months	$j_{12} = 15\%$
(d)	?	\$187,921	5 years	$j_1 = 4\%$
(e)	\$ 55,000	\$125,000	? (years)	$j_4 = 10\%$
(f)	\$ 2,500	\$100,000	? (years)	$j_{12} = 8\%$
(g)	\$ 27,500	\$ 57,500	5 years	? (j_{52})
(h)	\$ 10,000	\$ 20,000	1 year	? (j_1)
(i)	\$ 10,000	\$ 20,000	2 years	? (j_1)
(j)	\$ 55,179	?	20 years	$j_{12} = 6\%$
(k)	\$ 55,179	?	20 years	$j_{12} = 12\%$

4. Calculate the missing piece of information for each of the following loans:

Loan	Purchase Price (PV)	Estimated Future Value (FV)	Payment (PMT)	Holding Period (N)	Interest Rate Paid (j_m)
(a)	\$ 3,576.24	?	\$ 0	6 years	$j_1 = 13\%$
(b)	\$ 4,775.00	\$ 9,061.57	\$ 0	5 years	? (j_1)
(c)	?	\$ 0.00	\$ 200/mo	18 months	$j_{12} = 6.5\%$
(d)	\$ 14,450.00	\$ 0.00	\$?/mo	24 months	$j_2 = 11\%$
(e)	\$ 895.75	\$ 5,000.00	\$ 0	300 months	? (j_{12})
(f)	\$ 25,000.00	\$ 0.00	\$?/bi-week	80 bi-weeks	$j_4 = 10.5\%$
(g)	\$ 650.23	?	\$ 0	3 days	$j_{365} = 6.5\%$
(h)	\$ 0.00	?	\$ 1,500/week	520 weeks	$j_{52} = 13.25\%$

5. (a) Your client is considering the purchase of a property at the listed price of \$179,000. She wishes to earn a minimum of 5.5% per annum, compounded annually. Presume that revenues will equal costs during the holding period. For what minimum price must this property be sold at the end of the five years in order to realize the 5.5% yield?
- (b) If you advise your client the selling price will more likely be \$200,000 at the end of five years, what is the maximum price she should pay today in order to still be able to earn the 5.5% yield?
- (c) If the client sells the property at the end of four years for \$200,000, what would be her yield if she paid the listed price?
- (d) If this same property could be sold in two parcels, one in two years' time at \$50,000 and in five years' time at \$100,000 - and the required yield was 7.25% per annum, compounded annually, what is the maximum price that the investor should pay for this property?
6. Colleen has borrowed \$10,000 from Lifetime Trust Company. The loan is interest accruing on which interest is to be charged at a rate of 6% per annum, compounded quarterly. How much will Colleen owe at the end of the 5-year term?
7. Consider an interest-only mortgage of \$100,000 with an interest rate of 10% per year, compounded monthly, a term of 10 years, and monthly payments.
- (a) How much principal is outstanding after the first year, i.e. after 12 monthly payments?
- (b) What is the 13th monthly payment?
- (c) What is the 17th monthly payment?
- (d) How much principal is owing after the 18th monthly payment?

-
8. Real estate developers often use interest only loans during construction. Why do you think this type of financing is favoured by developers.
 9. Mick Bunner wants to buy a boat and spend the rest of his days sailing the South Pacific with his new bride. He figures he can set aside \$500 per month towards this goal and that he will need to accumulate \$200,000.
 - (a) If he can earn interest of 6% per annum, compounded monthly, on his monthly deposits, will he have accumulated enough to go at the end of 5 years?
 - (b) If he decides he wants to go in 7 years, how much will he have to deposit per month?
 10. Terra Developers has been given a loan by its lender that is structured as follows: Stage 1 of the loan requires monthly interest only payments until construction of their mall is complete and 50% of tenant occupancy is reached. Stage 2 begins after this occupancy rate is reached, and Terra will then begin to pay down its principal with a standard mortgage loan. The loan will be for \$5.5 million, at an interest rate of 8% per annum, compounded monthly. Stage 2 of the loan has a 5-year term, a 25-year amortization, and monthly payments rounded up to the next higher dollar.
 - (a) Calculate the monthly interest only payment during Stage 1.
 - (b) Terra has forecasted that it will reach 50% occupancy in 4 years. What is the amount of outstanding principal at the end of year 4, i.e., end of Stage 1 of the loan?
 - (c) In Stage 2 of the loan, calculate the interest portion and the principal portion of the 48th payment. Also, determine the outstanding balance of the loan after the 48th payment has been made.
 - (d) In Stage 2, calculate the total interest paid over the loan term of 5 years.

Appendix 1

Six Functions of a Dollar

The *Compound Interest and Discount Factor Tables for Appraisers - The Six Functions of One Dollar* publication consists of three compounding functions and three discounting functions. These six functions form the basis for all financing mathematics. The functions are arranged into six columns:

- Amount of One Dollar (Column 1);
- Accumulation of One Dollar (Column 2);
- Sinking Fund Factor (Column 3);
- Reversion Factor (Column 4);
- Annuity Factor (Column 5); and
- Partial Payment Factor (Column 6).

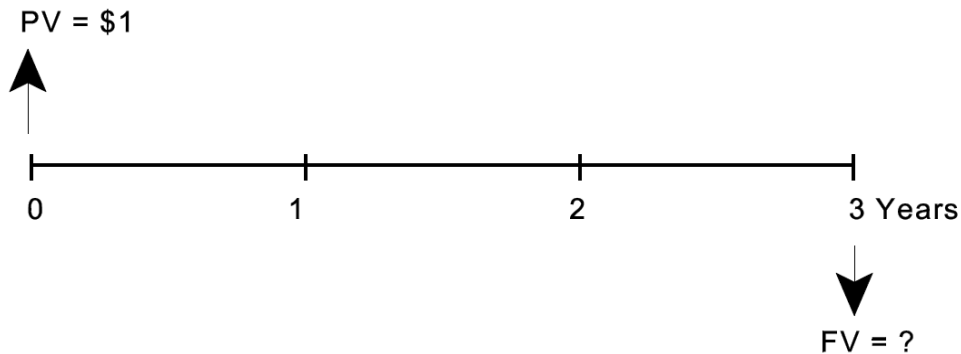
The factors in these columns are based on a value of one dollar. This creates a coefficient or multiplier which can then be applied to any dollar amount.

The following section will examine each of these six functions, explaining the purpose of each, illustrating it with a cash flow diagram, and listing the HP 10BII/II+ calculator steps that can be used to solve for the factor.

1. Amount of One Dollar

Column 1 shows the value at some future point in time of \$1 invested today at a given compound interest rate.

Example: Deposit \$1 for 3 years at 10% per annum, compounded annually.



Formula

$$\begin{aligned}
 S^n &= (1 + i)^n \\
 &= (1 + .10)^3 \\
 &= 1.331
 \end{aligned}$$

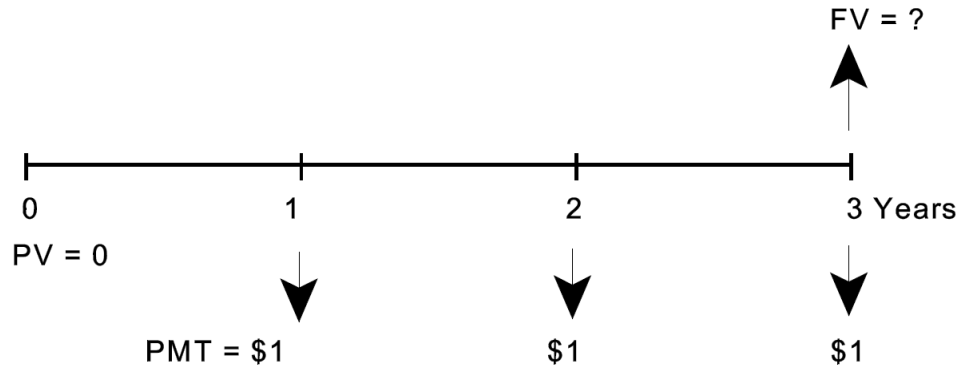
Press	Display	Comments
10 I/YR	10	Enter nominal interest rate
1 ■ P/YR	1	Enter compounding frequency
3 N	3	Enter the number of compounding periods
1 PV	1	Enter present value
0 PMT	0	No payments during term
FV	-1.331	Compute future value

The future value of \$1 deposited for 3 years at an interest rate of 10% per annum, compounded annually is \$1.33.

2. Accumulation of One Dollar

Column 2 shows the accumulated value at some future point in time of \$1 invested per period at a given compound interest rate.

Example: Deposit \$1 per year for 3 years at 10% per annum, compounded annually.



Formula

$$s[n, i\%] = \frac{S^n - 1}{i} \quad \text{where } S^n = (1+i)^n$$

$$= \frac{(1+i)^3 - 1}{.10}$$

$$= \frac{.331}{.10} = 3.31$$

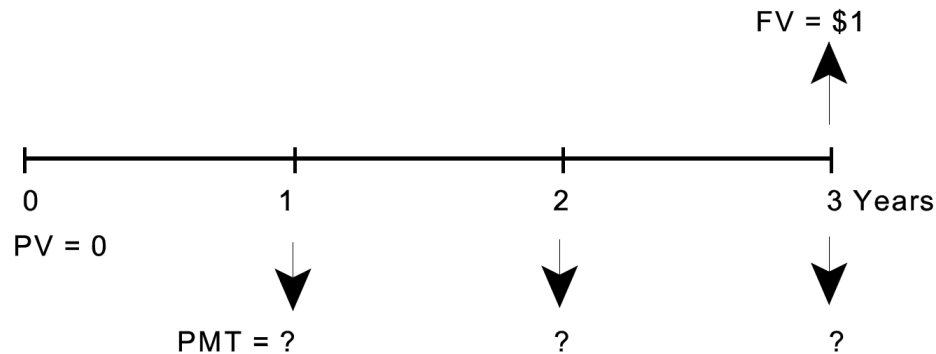
Press	Display	Comments
10 I/YR	10	Enter nominal interest rate
1 \blacksquare P/YR	1	Enter compounding frequency
3 N	3	Enter the number of payments
0 PV	0	Enter present value
1 +/- PMT	-1	Enter payments during term
FV	3.31	Computed future value

The future value at the end of 3 years of \$1 deposited each year into an account earning interest at 10% per annum, compounded annually, is \$3.31.

3. Sinking Fund Factor

Column 3 shows the amount which would have to be invested per period at a given compound interest rate in order to accumulate \$1 at some future point in time.

Example: Want to accumulate \$1 at the end of 3 years with an interest rate of 10% per annum, compounded annually.



Formula

$$\begin{aligned} \frac{1}{s[n, i\%]} &= \frac{i}{S^n - 1} \quad \text{where } S^n = (1 + i)^n \\ &= \frac{.10}{(1 + .10)^3 - 1} \\ &= \frac{.10}{1.331 - 1} = .302115 \end{aligned}$$

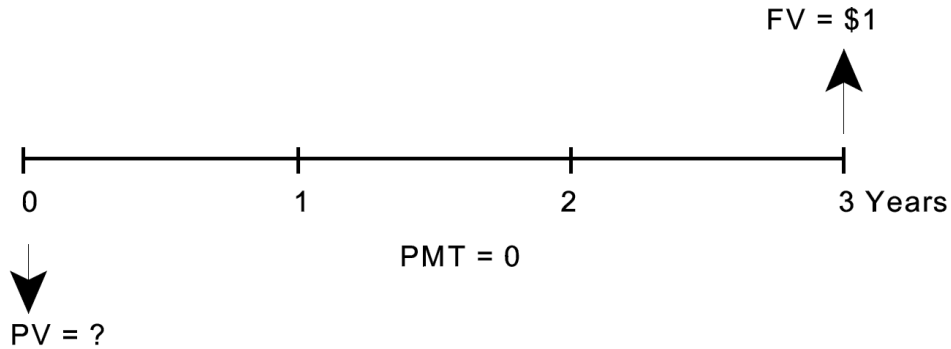
Press	Display	Comments
10 I/YR	10	Enter nominal interest rate
1 ■ P/YR	1	Enter compounding frequency
3 N	3	Enter the number of payments
0 PV	0	Enter present value
1 FV	1	Enter future value
PMT	-0.302115	Compute necessary payments per year

In order to accumulate \$1 at the end of 3 years in an account earning interest at 10% per annum, compounded annually, annual payments of \$0.30 would be required.

4. Reversion Factor

Column 4 shows the value today of an investment of \$1 at some future point in time discounted back to present value terms using a given compound interest rate.

Example: What is the value today of the promise of \$1 in three years time with an interest rate of 10% per annum, compounded annually.



Formula

$$\begin{aligned} \frac{1}{S^n} &= \frac{1}{(1+i)^n} \\ &= \frac{1}{(1+.10)^3} \\ &= \frac{1}{1.331} = .751315 \end{aligned}$$

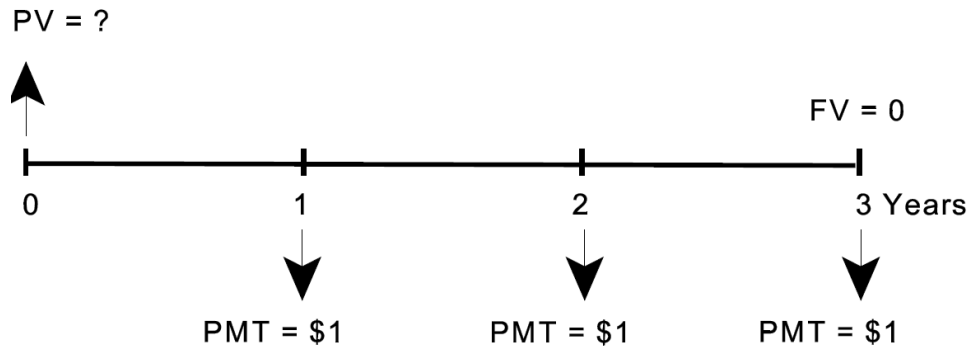
Press	Display	Comments
10 I/YR	10	Enter nominal interest rate
1 ■ P/YR	1	Enter compounding frequency
0 PMT	0	Enter payments during term
3 N	3	Enter the number of payments
1 FV	1	Enter future value
PV	-0.751314	Compute present value

The present value of \$1 to be received in 3 years is \$0.75 assuming a discount rate of 10% per annum, compounded annually.

5. Annuity Factor

Column 5 shows the value today of an investment of \$1 per period in the future discounted back to present value terms using a given compound interest rate.

Example: What is the value today of \$1 per year for 3 years discounted with an interest rate of 10% per annum, compounded annually.



Formula

$$\begin{aligned}
 a[n, i\%] &= \frac{1 - 1/S^n}{i}, \text{ where } 1/S^n = \frac{1}{(1+i)^n} \\
 &= \frac{1 - \frac{1}{(1+.10)^3}}{.10} \\
 &= \frac{1 - .7513}{.10} = 2.487
 \end{aligned}$$

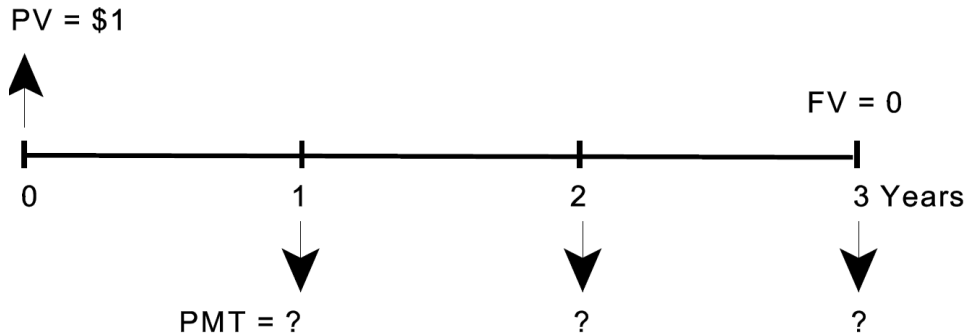
Press	Display	Comments
10 I/YR	10	Enter nominal interest rate
1 ■ P/YR	1	Enter compounding frequency
3 N	3	Enter the number of payments
1 +/- PMT	-1	Enter payments during term
0 FV	0	Enter future value
PV	2.486852	Compute present value

The present value of \$1 per year to be paid for 3 years is \$2.49, assuming a discount rate of 10% per annum, compounded annually.

6. Partial Payment Factor

Column 6 shows the amount which would have to be paid per period in order to amortize a loan of \$1 (i.e., pay off the loan amount plus all interest owing), given an amortization period and a compound interest rate.

Example: What would be required in order to repay all principal plus the interest owing on a loan of \$1 over 3 years at an interest rate of 10%, compounded annually?



Formula

$$\frac{1}{a[n, i\%]} = \frac{i}{1 - 1/S^n}$$

$$= \frac{.10}{1 - .7513}$$

$$= .402115$$

Press	Display	Comments
10 I/YR	10	Enter nominal interest rate
1 ■ P/YR	1	Enter compounding frequency
3 N	3	Enter the number of payments
1 PV	1	Enter present value
0 FV	0	Zero future value (loan fully repaid)
PMT	-0.402115	Compute required annual payment

In order to repay all principal and interest owing for a loan of \$1, assuming an interest rate of 10% per annum, compounded annually, the required payment would be \$0.40 per year.