Systems of First Order Linear Equations

(The 2×2 Case)

To Accompany "Elementary Differential Equations" by Boyce and DiPrima

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Preface

This manuscript is intended to accompany Chapter 7 of the book *Elementary Differential Equations* by Boyce and DiPrima. The elementary differential equations course at our university does not have linear algebra as a prerequisite, and so it is sometimes felt that the treatment given to first order linear systems in this text is somewhat too general for our audience.

The purpose of this manuscript is to treat the same topic in the same way, but to restrict our attention to the much simpler case of 2×2 matrices.

Section 1

Introduction

If n is a positive integer, then an $n \times n$ *first order system* is a list of n first order differential equations having the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1, \ldots, x_n, t) \\
\frac{dx_2}{dt} &= F_2(x_1, \ldots, x_n, t) \\
\frac{dx_3}{dt} &= F_3(x_1, \ldots, x_n, t) \\
&\vdots \\
\frac{dx_n}{dt} &= F_n(x_1, \ldots, x_n, t).
\end{aligned}$$

The differential system is called a *linear system* provided that $F_1, \ldots, F_n$ are linear functions of the variables $x_1, \ldots, x_n$.

Note 1.1. The variable t is the *independent* variable and $x_1, \ldots, x_n$ are *dependent* variables that are implicitly defined as functions of t.

Example 1.2. The 2×2 linear system

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -kx_1 - \gamma x_2 + \cos(t)
\end{aligned}$$

describes the motion of a 1 kg mass on a spring acted upon by an external force given by $F(t) = \cos(t)$, where t represents time, x_1 is the displacement of the mass from equilibrium at time t, x_2 is the velocity of the mass at time t, k is the spring constant, and γ is the damping constant.

Proposition 1.3. *An n^{th} order linear differential equation can be transformed into an $n \times n$ linear system.*

Rather than prove this, we will demonstrate how it can be done with some examples.

Example 1.4. Transform the given differential equation into a first order linear system:

$$y'' + ty' + 3y = \sin(t).$$

Solution. Let $x_1 = y$ and let $x_2 = y'$. Then $x_1' = x_2$ and the second order differential equation can be written

$$x_2' + tx_2 + 3x_1 = \sin(t).$$

Thus,

$$\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 - tx_2 + \sin(t). \end{cases}$$

Example 1.5. Transform the given differential equation into a first order linear system:

$$y''' - 2y'' + ty' - 5y = \cos(t).$$

Solution. Let $x_1 = y$, let $x_2 = y'$, and let $x_3 = y''$. Then $x_1' = x_2$, and $x_2' = x_3$, and the given third order differential equation can be written

$$x_3' - 2x_3 + tx_2 - 5x_1 = \cos(t).$$

consequently,

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 5x_1 - tx_2 + 2x_3 + \cos(t). \end{cases}$$

Motivation. While it can be very challenging to find a solution to a high order linear differential equation, computers may be used to find (with relative ease) solutions to first order linear systems.

We will primarily consider only 2×2 linear systems of the form:

$$\begin{cases} x_1' = p_1(t)x_1 + q_1(t)x_2 + g_1(t) \\ x_2' = p_2(t)x_1 + q_2(t)x_2 + g_2(t). \end{cases} \tag{1.1}$$

Theorem 1.6. *If the functions $p_1, p_2, q_1, q_2, g_1,$ and g_2 in (1.1) are continuous on an interval, then the linear system has a general solution on that interval.*

We note that the 2×2 linear system in (1.1) can be rewritten as the following matrix equation:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} p_1(t) & q_1(t) \\ p_2(t) & q_2(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

For this reason, it will be useful to review some properties of matrices. (See Sections 2 and 3.)

In our current context, an *initial value problem (IVP)* is a 2×2 linear system with two initial conditions:

$$x_1(t_0) = x_1^0 \quad \text{and} \quad x_2(t_0) = x_2^0.$$

We obtain a solution to the initial value problem by finding functions ϕ_1 and ϕ_2 so that $x_1 = \phi_1(t)$ and $x_2 = \phi_2(t)$ satisfy both the differential equations and the initial conditions.

Example 1.7. Solve the following initial value problem:

$$\begin{cases} x_1' = 2x_2 & x_1(0) = 3 \\ x_2' = -2x_1 & x_2(0) = 4. \end{cases}$$

Solution. Since we do not yet have any method for solving a 2×2 system of differential equations, we will transform this system of linear first order differential equations into a second order linear differential equation. Start with the first equation, and differentiate both sides:

$$x_1' = 2x_2 \implies x_1'' = 2x_2'.$$

However, the second equations says that $x_2' = -2x_1$, and so

$$x_1'' = 2x_2' = 2(-2x_1) = -4x_1,$$

which can also be written

$$x_1'' + 4x_1 = 0.$$

This second order linear differential equation has characteristic equation $r^2 + 4 = 0$, so that $r = 2i$ or $r = -2i$, and consequently the general solution for x_1 is

$$x_1 = c_1 \cos(2t) + c_2 \sin(2t).$$

Since $x_2 = \frac{1}{2}x_1'$, it follows that

$$x_2 = -c_1 \sin(2t) + c_2 \cos(2t).$$

At this point, we may use the initial conditions to solve for c_1 and c_2:

$$x_1(0) = c_1 = 3 \quad \text{and} \quad x_2(0) = c_2 = 4.$$

Therefore

$$\begin{cases} x_1 = 3\cos(2t) + 4\sin(2t) \\ x_2 = -3\sin(2t) + 4\cos(2t). \end{cases}$$

Remark. Note that $x_1^2 + x_2^2 = 25$, and so the solution to this initial value problem is a circle about the point $(0, 0)$ with radius 5 in the x_1x_2-plane.

Example 1.8. Solve the following initial value problem:

$$\begin{cases} x_1' = -\dfrac{1}{2}x_1 + 2x_2 & x_1(0) = -3 \\ x_2' = -2x_1 - \dfrac{1}{2}x_2 & x_2(0) = 3. \end{cases}$$

Solution. Once again, we will transform this linear system into a second order linear differential equation. Rewrite the first equation to solve for x_2 in terms of x_1 and its derivative:

$$x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1.$$

Differentiating both sides of this equation gives a formula for x_2' in terms of the first and second derivatives of x_1:

$$x_2' = \frac{1}{2}x_1'' + \frac{1}{4}x_1'.$$

Substitute these formulas for x_2 and x_2' into the second differential equation to get

$$\frac{1}{2}x_1'' + \frac{1}{4}x_1' = -2x_1' - \frac{1}{2}\Big(\frac{1}{2}x_1' + \frac{1}{4}x_1\Big).$$

Rewriting this equation, we have

$$\frac{1}{2}x_1'' + \frac{1}{2}x_1' + \frac{17}{8}x_1 = 0.$$

This is a second order linear differential equation with constant coefficients and can be solved in the usual way. Consequently, we have the following general solution for x_1:

$$x_1 = c_1e^{-t/2}\cos(2t) + c_2e^{-t/2}\sin(2t).$$

We have the formula $x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1$ that expresses x_2 in terms of x_1 and its first derivative. Thus, since

$$x_1' = \Big[-\frac{1}{2}c_1 + 2c_2\Big]e^{-t/2}\cos(2t) - \Big[2c_1 + \frac{1}{2}c_2\Big]e^{-t/2}\sin(2t),$$

it follows that

$$x_2 = c_2e^{-t/2}\cos(2t) - c_1e^{-t/2}\sin(2t).$$

Thus, we have the following general solution to the linear system:

$$\begin{cases} x_1 = c_1e^{-t/2}\cos(2t) + c_2e^{-t/2}\sin(2t) \\ x_2 = c_2e^{-t/2}\cos(2t) - c_1e^{-t/2}\sin(2t). \end{cases}$$

At this point, we may use the initial conditions to solve for c_1 and c_2:

$$x_1(0) = c_1 = -3 \quad \text{and} \quad x_2(0) = c_2 = 3.$$

Therefore, the initial value problem has solution

$$\begin{cases} x_1 = -3e^{-t/2}\cos(2t) + 3e^{-t/2}\sin(2t) \\ x_2 = 3e^{-t/2}\cos(2t) + 3e^{-t/2}\sin(2t). \end{cases}$$

Remark. It can be seen that $x_1^2 + x_2^2 = 18e^{-t}$, and so the solution parameterizes a spiral in the x_1x_2-plane with $(x_1, x_2) \to (0, 0)$ as $t \to \infty$.

Section 2

Review of Matrices

In the following discussion, we restrict our attention to 2×2 matrices. A 2×2 *matrix* is an array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{or} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Matrices are usually denoted by capital letters.

Notation 2.1. a_{ij} is the entry of the matrix A in the i^{th} row and the j^{th} column.

We now consider the various arithmetic operations that can be performed using matrices.

Matrix Addition: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{22} \\ b_{11} & b_{22} \end{pmatrix}$, then the matrix sum of A and B is the matrix

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}.$$

Scalar Multiplication: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and k is a real numbers, then A may be multiplied by k using the following rule:

$$kA = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}.$$

This is known as scalar multiplication because real numbers are also known as *scalars*. (Multiplying by a real number has the effect of making the "size" larger or smaller; that is, it changes the scale.)

Matrix Subtraction: We may now define matrix subtraction by means of the following rule:

$$A - B = A + (-1)B.$$

Definition 2.2. The 2×2 *zero matrix* is the 2×2 matrix

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The 2×2 zero matrix is also sometimes denoted $O_{2\times 2}$.

Remark 2.3. If A and B are two matrices, then $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j.

A 2×1 matrix is called a *column vector* and a 1×2 matrix is called a *row vector*. It is common to omit the word "column" or "row" and refer to each as a *vector*. Vectors are usually denoted by lower case letters either in bold (such as $\boldsymbol{x}$) or with an arrow drawn above (such as $\vec{x}$). In text, it is more common to use bold face, so we will write our vectors like this:

$$\text{Column vector:} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{Row vector:} \quad \boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}.$$

Vector Multiplication: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$A\boldsymbol{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Matrix Multiplication: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $B = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$. Then the product of A and B is given by the formula

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} ax + by & az + bw \\ cx + dy & cz + bw \end{pmatrix}.$$

Remark 2.4. Suppose that $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} z \\ w \end{pmatrix}$. That is, suppose that $\boldsymbol{v}$ and $\boldsymbol{u}$ are the columns of the matrix B. Then

$$AB = \begin{pmatrix} A\boldsymbol{v} & A\boldsymbol{u} \end{pmatrix}.$$

In other words, the columns of AB are given by the column vectors $A\boldsymbol{v}$ and $A\boldsymbol{u}$.

Example 2.5. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$. Compute AB and BA.

Solution. Computing directly,

$$AB = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 9 \\ 5 & 23 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & -9 \\ -1 & 26 \end{pmatrix}.$$

Note 2.6. Observe that $AB \neq BA$. Matrix multiplication is *noncommutative*, and so it is quite common for AB and BA to be unequal. Therefore, one must be very careful when multiplying matrices to ensure that the order is correct.

Definition 2.7. The 2×2 *identity matrix* is the 2×2 matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The 2×2 identity matrix is also sometimes denoted $I_{2\times 2}$.

Proposition 2.8. *The 2×2 identity matrix I is the unique 2×2 matrix for which $AI = A$ and $IA = A$ for all 2×2 matrices A.*

The proof of Proposition 2.8 is left to the interested reader.

Definition 2.9. The matrix A is called *invertible* if there is a matrix A^{-1} such that $AA^{-1} = I$ and $A^{-1}A = I$. If such a matrix A^{-1} exists, then it is called the *inverse* of A.

Proposition 2.10. *If a matrix has an inverse, then it is necessarily unique. (That is, a matrix can have only one inverse.)*

Like the previous proposition, we will leave the proof of Proposition 2.10 to the interested reader.

Example 2.11. If $A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$, then show that $A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$.

Solution. Computing directly:

$$\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} (1)(0)+(-2)(-\frac{1}{2}) & (1)(\frac{1}{2})+(-2)(\frac{1}{4}) \\ (2)(0)+(0)(-\frac{1}{2}) & (2)(\frac{1}{2})+(0)(\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} (0)(1)+(\frac{1}{2})(2) & (0)(-2)+(\frac{1}{2})(0) \\ (-\frac{1}{2})(1)+(\frac{1}{4})(2) & (-\frac{1}{2})(-2)+(\frac{1}{4})(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefor A^{-1} is the inverse of A, as required.

To understand why we are considering matrices and the idea of a matrix inverse, consider the following example of a system of linear equations.

Example 2.12. Solve the system

$$\begin{cases} x - 2y = 3 \\ \qquad 2x = 5. \end{cases}$$

Solution. The system can be written as a matrix equation as follows:

$$\begin{pmatrix} 1 & -2 \\ 2, & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Therefore, using the result of Example 2.11,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2, & 0 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -1/4 \end{pmatrix}.$$

It follows that the solution to the system of linear equations is $x = \frac{5}{2}$ and $y = -\frac{1}{4}$.

Example 2.12 is a specific example of a more general phenomenon. If A is a matrix and $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors such that $A\boldsymbol{x} = \boldsymbol{y}$, then

$$A^{-1}(A\boldsymbol{x}) = A^{-1}\boldsymbol{y} \quad \text{or} \quad \boldsymbol{x} = A^{-1}\boldsymbol{y}.$$

This works for any $n\times n$ matrix (not just 2×2 matrices), provided that A^{-1} exists. (That is, provided that A is invertible.)

Theorem 2.13. *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *then* $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

This theorem is easy to check and the proof is left to the reader.

Definition 2.14. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the *determinant* of A is the number $ad - bc$. The determinant of the matrix A is denoted by $\det(A)$ and is written

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Remark 2.15. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Theorem 2.16. *A is invertible if and only if* $\det(A) \neq 0$.

Again, this theorem is easy to check and is left to the reader.

Definition 2.17. If A is a matrix and $\det(A) = 0$, then A is called *singular*. If $\det(A) \neq 0$ (so that A is invertible), then A is called *nonsingular*.

Definition 2.18. The *zero vector* is the vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Theorem 2.19. *If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$, then* $\det(A) = 0$.

Proof. Suppose that A is invertible and $A\boldsymbol{v} = \mathbf{0}$. If we multiply on the left by the inverse of A, then we have

$$A^{-1}(A\boldsymbol{v}) = A^{-1}\mathbf{0} \quad \text{or} \quad (A^{-1}A)\boldsymbol{v} = \mathbf{0}.$$

From this, it follows that $\boldsymbol{v} = \mathbf{0}$, which is assumed to be false. We have arrived at a contradiction, and so we conclude that A is not invertible. We therefore conclude that $\det(A) = 0$. □

The matrix A has zero determinant if and only if one row is a multiple of the other. That is $\det(A) = 0$ if and only if

$$A = \begin{pmatrix} x & y \\ \lambda x & \lambda y \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \lambda x & \lambda y \\ x & y \end{pmatrix},$$

for some real number λ. (It is worth pointing out that we must consider two possible cases because λ could be zero in either matrix.)

Definition 2.20. Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called *linearly dependent* if there is a real number λ such that either $\boldsymbol{x} = \lambda\boldsymbol{y}$ or $\boldsymbol{y} = \lambda\boldsymbol{x}$. If two vectors are not linearly dependent, then they are called *linearly independent*.

Remark 2.21. We can now restate the fact preceding Definition 2.20 as follows: $\det(A) = 0$ if and only if the rows (or columns) are linearly dependent.

Remark 2.22. We consider two cases in Definition 2.20 for the same reason we consider two cases in the preceding fact: it is possible that either $\boldsymbol{x}$ or $\boldsymbol{y}$ is the zero vector. It is possible to rephrase the definitions of "linearly dependent" so that only one condition is required, which we will now do in the next theorem.

Theorem 2.23. *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are linearly dependent if and only if there exist nonzero real numbers* λ_1 *and* λ_2 *such that* $\lambda_1\boldsymbol{x} + \lambda_2\boldsymbol{y} = \mathbf{0}$.

It is straightforward to check that this theorem provides a condition that is equivalent to the one given in Definition 2.20. Indeed, Theorem 2.23 gives us a natural way to generalize the notion of linear dependence to more vectors.

Definition 2.24. The n vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are called *linearly dependent* if there exist real numbers $\lambda_1, \ldots, \lambda_n$, not all zero, such that $\lambda_1\boldsymbol{x}_1 + \cdots + \lambda_n\boldsymbol{x}_n = \mathbf{0}$. If a collection of vectors are not linearly dependent, they are said to be *linearly independent*.

Section 3

Eigenvectors and Eigenvalues

Definition 3.1. Let A be a square matrix. If $\boldsymbol{v}$ is a nonzero vector such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ or scalar λ, then $\boldsymbol{v}$ is called an *eigenvector* for A. The scalar λ is called the *eigenvalue* for $\boldsymbol{v}$.

Example 3.2. Show that $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

Solution. Computing directly:

$$A\boldsymbol{v} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5\boldsymbol{v}.$$

Therefore, $\boldsymbol{v}$ is an eigenvector for the matrix with eigenvalue $\lambda = 5$.

Proposition 3.3. *If $\boldsymbol{v}$ is an eigenvector for the matrix A, then so is $c\boldsymbol{v}$ for any nonzero scalar c.*

Proof. Suppose $\boldsymbol{v}$ is an eigenvector for A with eigenvalue λ, so that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Let c be any nonzero scalar. Then

$$A(c\boldsymbol{v}) = c(A\boldsymbol{v}) = c(\lambda\boldsymbol{v}) = \lambda(c\boldsymbol{v}).$$

Consequently, $c\boldsymbol{v}$ is also an eigenvector for A having the same eigenvalue λ. □

Example 3.4. Since $c\boldsymbol{v}$ is an eigenvector for the matrix A whenever $\boldsymbol{v}$ is an eigenvector for A, and with the same eigenvalue, we can choose c to be any nonzero scalar we wish so that our eigenvector has a convenient form.

For example, if A has eigenvector $\boldsymbol{v} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix}$ with eigenvalue λ, then we may choose $c = 6$, so that our eigenvector has the more convenient form $c\boldsymbol{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Since this eigenvector has the same eigenvalue λ, nothing has been lost by choosing a more convenient representative for our eigenvector.

This leads to a very natural question.

Question 3.5. How do we find eigenvectors and eigenvalues?

In order to find an answer to this question, suppose that A is a square matrix with eigenvector $\boldsymbol{v}$. (Note that $\boldsymbol{v} \neq \mathbf{0}$, by definition.) There exists some scalar λ (the eigenvalue) such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. This is equivalent to

$$A\boldsymbol{v} - \lambda\boldsymbol{v} = \mathbf{0} \quad \text{or} \quad \big(A - \lambda I\big)\boldsymbol{v} = \mathbf{0}.$$

(We note here that $(\lambda I)\boldsymbol{v} = \lambda\boldsymbol{v}$.) Since $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$, even though $\boldsymbol{v} \neq \mathbf{0}$, it follows that $A - \lambda I$ is a noninvertible matrix, and so $\det(A - \lambda I) = 0$, by Theorem 2.19.

Conclusion: In order to find eigenvectors, start by looking for eigenvalues, and look for eigenvalues by finding values of λ for which $\det(A - \lambda I) = 0$.

Example 3.6. Find the eigenvalues for the matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

Solution. We find the eigenvalues by solving the equation $\det(A - \lambda I) = 0$. We first compute the determinant:

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - (3)(1).$$

Simplifying this, we have

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5).$$

Consequently, if $\det(A - \lambda I) = 0$, then either $\lambda = 1$ or $\lambda = 5$. These are the two eigenvalues for the matrix A.

Definition 3.7. If A is a square matrix, then the polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial* of A.

Note that the eigenvalues of a matrix are the zeros of the characteristic polynomial for that matrix.

Once we know the eigenvalues for a matrix, we can find corresponding eigenvectors by solving the matrix equation $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$.

Example 3.8. Find the eigenvectors for the matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

Solution. We found the eigenvalues of A in Example 3.6 by solving the equation $\det(A - \lambda I) = 0$. The solutions to this equation were the eigenvalues $\lambda = 1$ or $\lambda = 5$. We therefore have two cases, one for each eigenvalue, and we find an eigenvector corresponding to each λ by solving the matrix equation $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$.

It will help to label the components of the vector $\boldsymbol{v}$, so we let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$ can be written:

$$\begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For convenience, we will distinguish the eigenvalues using subscripts, so that $\lambda_1 = 1$ and $\lambda_2 = 5$.

Case 1: $\lambda_1 = 1$. In this case, our matrix equation becomes

$$\begin{pmatrix} 2-1 & 3 \\ 1 & 4-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If we carry out the vector multiplication, this becomes

$$\begin{pmatrix} x+3y \\ x+3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} x+3y=0 \\ x+3y=0. \end{cases}$$

This system of two equations is really just one equation: $x + 3y = 0$. Consequently, if $y = r$ (where r is any number), then $x = -3r$. Therefore,

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3r \\ r \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} r.$$

This is an eigenvector corresponding to $\lambda = 1$ for *any* choice of r, so long as $\boldsymbol{v}$ is not the zero vector. Thus, we can choose r to be any nonzero number so that the eigenvector $\boldsymbol{v}$ has a convenient representation. In this case, we can simply pick $r = 1$ and the result is the eigenvector

$$\boldsymbol{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Remark 3.9. Notice that if we had instead parameterized our solutions by letting $x = s$, where s is any number, then we would have $y = -s/3$, and our conclusion would have been that $\boldsymbol{v}$ had the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ -s/3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} s.$$

It might seem at first that we have found a different solution, but we have just found a different way to represent the same family of eigenvectors. If we pick $s = -3$, then we have $\boldsymbol{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, which is the same vector we chose above. In fact, if we pick $s = -3r$, the we get exactly the same representation of the family we had before.

Case 2: $\lambda_2 = 5$. In this case, our matrix equation becomes

$$\begin{pmatrix} 2-5 & 3 \\ 1 & 4-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If we carry out the vector multiplication, this becomes

$$\begin{pmatrix} -3x+3y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} -3x+3y=0 \\ x-y=0. \end{cases}$$

This system may appear to be two equations, but once again we really only have one equation: $x - y = 0$. (The top equation is -3 times the bottom equation.) We see then that $x = y$, so if we pick $y = r$ (where r is any number), then $x = r$, as well. Therefore,

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} r.$$

Once again, we can choose r to be any nonzero number. As before, we pick $r = 1$, and so our eigenvector corresponding to $\lambda = 5$ is the vector $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

In both cases in Example 3.8, we started with a matrix equation, which we then converted into a system of two equations and two unknowns, but this system turned out to really only represent one equation. (In Case 1, both equations were identical and in Case 2, one equation was a scalar multiple of the other.) This is typical of what happens when solving these types of problems. Our basic assumption is that the eigenvalue λ satisfies the equation $\det(A - \lambda I) = 0$, which means that $A - \lambda I$ is a singular (and so noninvertible) 2×2 matrix. When a matrix is singular, the rows are necessarily linearly dependent. In the 2×2 case, this means that one row is a scalar multiple of the other, and so it will always be the case that there is only one equation to solve, even if at first it appears that there are two equations.

Notation 3.10. In Example 3.8, we found two eigenvectors for the matrix A, one for each eigenvalue:

$$\lambda_1 = 1: \ \boldsymbol{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 5: \ \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It will become confusing if we continue to call both eigenvectors $\boldsymbol{v}$, so we will usually distinguish them using a superscript, as we demonstrate here:

$$\lambda_1 = 1: \ \boldsymbol{v}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 5: \ \boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We will use superscripts instead of subscripts, because it is common to use subscripts when writing the components of the vector, so it is not unusual to see something like

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix}.$$

Example 3.11. Find the eigenvectors and eigenvalues for $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$.

Solution. We start by solving the quadratic equation $\det(A - \lambda I) = 0$ for λ in order to find the eigenvalues:

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 + 4 = 0.$$

Simplifying, this becomes

$$\lambda^2 + 2\lambda + 5 = 0.$$

This quadratic equation has two complex solutions:

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

To avoid ambiguity, let $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$.

Next, we find the eigenvectors corresponding to the eigenvalues.

Case 1: $\lambda_1 = -1 + 2i$. In this case, the matrix equation becomes

$$\begin{pmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which simplifies to

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -2ix - 4y \\ x - 2iy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Writing this as a system of equations, we have:

$$\begin{cases} -2ix - 4y = 0 \\ x - 2iy = 0. \end{cases}$$

Once again, this gives us only one equation (because we can multiply the second equation by $-2i$ to get the first equation). Thus, we have that $x - 2iy = 0$, or $x = 2iy$. We suppose that $y = r$, for some scalar r, and so in turn $x = 2ir$. Consequently,

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2ir \\ r \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i \right] r.$$

Once again, we can choose r to be any nonzero number, and so we pick $r = 1$. Therefore, the eigenvector corresponding to $\lambda_1 = -1 + 2i$ is the vector

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i.$$

Case 2: $\lambda_2 = -1 - 2i$. This case is essentially the same as the last, with appropriate sign changes. We can mimic the previous case to find the eigenvector $\boldsymbol{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$.

Notice that in Example 3.11, the eigenvalues satisfy the relationship $\lambda_2 = \overline{\lambda_1}$, and similarly the eigenvectors satisfy the relationship $\boldsymbol{v}^{(2)} = \overline{\boldsymbol{v}^{(1)}}$. This is typical when there are two complex eigenvalues. Indeed, we have the following proposition that ensures complex eigenvalues and complex eigenvectors will always occur in pairs that are related by complex conjugation.

Proposition 3.12. *Suppose A is a real matrix. If λ is an eigenvalue for A with eigenvector $\boldsymbol{v}$, then $\overline{\lambda}$ is an eigenvalue for A with eigenvector $\overline{\boldsymbol{v}}$.*

Proof. This proof is actually quite straightforward:

$$A\overline{\boldsymbol{v}} = \overline{A\boldsymbol{v}} = \overline{\lambda \boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}.$$

(Note that it is necessary for A to be a matrix with real entries.) □

Section 4

Basic Theory of Linear Systems

A 2×2 linear system has the form

$$\begin{cases} x_1' = p_1(t)x_1 + q_1(t)x_2 + g_1(t) \\ x_2' = p_2(t)x_1 + q_2(t)x_2 + g_2(t). \end{cases} \tag{4.1}$$

In matrix form, this linear system can be rewritten as follows:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} p_1(t) & q_1(t) \\ p_2(t) & q_2(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Let

$$A(t) = \begin{pmatrix} p_1(t) & q_1(t) \\ p_2(t) & q_2(t) \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

With this notation, the linear system in (4.1) can be written as

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t).$$

We are interested in finding solutions to this equation. We know from Theorem 1.6 when we can be sure that the system in (4.1) has a solution on an interval, and so we can rephrase that theorem in terms of our new notation.

Theorem 4.1 (Theorem 1.6 reformulated)**.** *If the component functions of $A(t)$ and $\boldsymbol{g}(t)$ are continuous an interval, then the matrix differential equation*

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$$

has a general solution on that interval.

Definition 4.2. A differential system $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$ is *homogeneous* if and only if $\boldsymbol{g}(t) = \boldsymbol{0}$ for every t.

Theorem 4.3. *If $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are linearly independent (on an interval) solutions to a homogeneous system, then the general solution is $\boldsymbol{x} = c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)}$ (on that interval).*

Remark 4.4. In a homogeneous linear system $\boldsymbol{x}' = A(t)\boldsymbol{x}$, the vector $\boldsymbol{x}$ is implicitly a function of t, so that $\boldsymbol{x} = \boldsymbol{x}(t)$. Thus, in Theorem 4.3, it is necessary for the vectors $\boldsymbol{x}^{(1)}(t)$ and $\boldsymbol{x}^{(2)}(t)$ to be linearly independent for each t in some interval.

Definition 4.5. If $\boldsymbol{x} = c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)}$ is the general solution of a homogeneous system, then we say that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a *fundamental set of solutions.*

We can check if two solutions form a fundamental set using something called the Wronskien.

Definition 4.6. The *Wronskien* of the vectors $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ is the determinant

$$W\big(\boldsymbol{x}^{(1)},\ \boldsymbol{x}^{(2)}\big) = \det\big(\boldsymbol{x}^{(1)}\ \boldsymbol{x}^{(2)}\big).$$

To be clear about the meaning of the notation used in Definition 4.6, when we write $\det\big(\boldsymbol{x}^{(1)}\ \boldsymbol{x}^{(2)}\big)$, we mean the determinant of the matrix having $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ as the columns. That is, if

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} \boldsymbol{x}_1^{(1)} \\ \boldsymbol{x}_2^{(1)} \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \begin{pmatrix} \boldsymbol{x}_1^{(2)} \\ \boldsymbol{x}_2^{(2)} \end{pmatrix},$$

then

$$W\Big(\boldsymbol{x}^{(1)},\ \boldsymbol{x}^{(2)}\Big) = \det\begin{pmatrix} \boldsymbol{x}_1^{(1)} & \boldsymbol{x}_1^{(2)} \\ \boldsymbol{x}_2^{(1)} & \boldsymbol{x}_2^{(2)} \end{pmatrix} = \begin{vmatrix} \boldsymbol{x}_1^{(1)} & \boldsymbol{x}_1^{(2)} \\ \boldsymbol{x}_2^{(1)} & \boldsymbol{x}_2^{(2)} \end{vmatrix}.$$

Theorem 4.7. *If $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions on an interval, then $W\big(\boldsymbol{x}^{(1)},\ \boldsymbol{x}^{(2)}\big) \neq 0$ on that interval.*

Proof. If $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions on an interval, then they are linearly independent (as vectors) at every point in that interval. If $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are linearly independent vectors, then one vector is not a scalar multiple of the other vector, and consequently the matrix they form is invertible, which implies that it has a nonzero determinant. □

Example 4.8. Consider the linear system

$$\begin{cases} x_1' = \frac{1}{t}x_1 + x_2 \\ x_2' = \frac{1}{t}x_2. \end{cases}$$

Show that the following are a fundamental set of solutions on some interval, and determine the largest intervals on which they form a fundamental set:

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} t^2 \\ t \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Solution. First, we show that $\boldsymbol{x}^{(1)} = \begin{pmatrix} t^2 \\ t \end{pmatrix}$ is a solution. In this case, we have $x_1 = t^2$ and $x_2 = t$. Then

$$\begin{cases} x_1' = 2t \\ x_2' = 1 \end{cases} \quad \text{and} \quad \begin{cases} \frac{1}{t}x_1 + x_2 = \frac{1}{t}(t^2) + (t) = 2t \\ \frac{1}{t}x_2 = \frac{1}{t}(t) = 1. \end{cases}$$

Since these are the same, $\boldsymbol{x}^{(1)}$ is a solution to the linear system.

Next, we show that $\boldsymbol{x}^{(2)} = \begin{pmatrix} t \\ 0 \end{pmatrix}$ is a solution. In this case, $x_1 = t$ and $x_2 = 0$. Then

$$\begin{cases} x_1' = 1 \\ x_2' = 0 \end{cases} \quad \text{and} \quad \begin{cases} \frac{1}{t}x_1 + x_2 = \frac{1}{t}(t) + (0) = 1 \\ \frac{1}{t}x_2 = \frac{1}{t}(0) = 0. \end{cases}$$

Again, these are the same, and so $\boldsymbol{x}^{(2)}$ is a solution to the linear system.

In order to show that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions, we compute the Wronskien:

$$W(t) = W\big(\boldsymbol{x}^{(1)},\ \boldsymbol{x}^{(2)}\big) = \det \begin{pmatrix} t^2 & t \\ t & 0 \end{pmatrix} = \begin{vmatrix} t^2 & t \\ t & 0 \end{vmatrix} = -t^2$$

Because $W \neq 0$ on the intervals $(-\infty, 0)$ and $(0, \infty)$, the vectors $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions on these intervals.

Notice that $W(0) = 0$, which means that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ do not form a fundamental set of solutions on any interval that contains $t = 0$. Since the component functions $p_1(t) = \frac{1}{t}$ and $q_2(t) = \frac{1}{t}$ are not continuous at $t = 0$, we do not expect the system to have a solution at that point.

Example 4.9. Suppose the 2×2 homogeneous linear system

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 \end{cases}$$

has the following two solutions:

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} t^2 \\ t \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \begin{pmatrix} 0 \\ 1/t \end{pmatrix}.$$

(a) On what intervals do these form a fundamental set of solutions?

(b) What conclusions can we draw about the continuity of the coefficient functions p_{ij}?

(c) Is it possible to compute p_{ij} for each i and j in $\{1, 2\}$?

Solution. (a) Let

$$X = \big(\boldsymbol{x}^{(1)}\ \boldsymbol{x}^{(2)}\big) = \begin{pmatrix} t^2 & 0 \\ t & 1/t \end{pmatrix}.$$

Then

$$W(t) = W\big(\boldsymbol{x}^{(1)},\ \boldsymbol{x}^{(2)}\big) = \det(X) = \begin{vmatrix} t^2 & 0 \\ t & 1/t \end{vmatrix} = t$$

The solutions $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions on an interval provided that $W(t) \neq 0$ for any t in that interval. Therefore, $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions on any interval that does not contain zero. In particular, the "largest" intervals over which $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions are the intervals $(0, \infty)$ and $(-\infty, 0)$.

(b) It must be the case that at least one of the coefficient functions p_{ij} is not continuous at zero. Otherwise, we would have a fundamental set of solutions valid over the entire interval $(-\infty, \infty)$, by Theorem 1.6. Without further analysis, however, we cannot determine which (or even how many) are not continuous at $t = 0$.

(c) Let

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}.$$

By assumption, both $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions to the matrix equation $\boldsymbol{x}' = P(t)\boldsymbol{x}$. Therefore,

$$P(t)\boldsymbol{x}^{(1)} = \frac{d}{dt}\boldsymbol{x}^{(1)} \quad \text{and} \quad P(t)\boldsymbol{x}^{(2)} = \frac{d}{dt}\boldsymbol{x}^{(2)},$$

or

$$P(t)\begin{pmatrix} t^2 \\ t \end{pmatrix} = \begin{pmatrix} 2t \\ 1 \end{pmatrix} \quad \text{and} \quad P(t)\begin{pmatrix} 0 \\ 1/t \end{pmatrix} = \begin{pmatrix} 0 \\ -1/t^2 \end{pmatrix}.$$

These two equations can be combined into one matrix equation in the following way:

$$P(t)\big(\boldsymbol{x}^{(1)} \;\; \boldsymbol{x}^{(2)}\big) = \Big(\frac{d}{dt}\boldsymbol{x}^{(1)} \;\; \frac{d}{dt}\boldsymbol{x}^{(2)}\Big),$$

which in this case is

$$P(t)\underbrace{\begin{pmatrix} t^2 & 0 \\ t & 1/t \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 2t & 0 \\ 1 & -1/t^2 \end{pmatrix}}_{X'}.$$

The domain of X is $(-\infty, 0) \cup (0, \infty)$, and on this set we have already seen that $\det(X) = t$. In particular, on this set $\det(X) \neq 0$, and so X is an invertible matrix. That means that X^{-1} exists, and so

$$P(t)X = X' \quad \text{implies that} \quad [P(t)X]X^{-1} = X'X^{-1}.$$

Matrix multiplication is associate, and thus we have

$$P(t)[XX^{-1}] = X'X^{-1} \quad \text{or} \quad P(t) = X'X^{-1}.$$

We can compute X^{-1} using the formula given in Theorem 2.13:

$$X^{-1} = \frac{1}{\det(X)}\begin{pmatrix} 1/t & 0 \\ -t & t^2 \end{pmatrix} = \begin{pmatrix} 1/t^2 & 0 \\ -1 & t \end{pmatrix}.$$

Therefore,

$$P(t) = X'X^{-1} = \begin{pmatrix} 2t & 0 \\ 1 & -1/t^2 \end{pmatrix}\begin{pmatrix} 1/t^2 & 0 \\ -1 & t \end{pmatrix} = \begin{pmatrix} 2/t & 0 \\ 2/t^2 & -1/t \end{pmatrix},$$

and so the original linear system can be identified as

$$\begin{cases} x_1' = \dfrac{2}{t}x_1 \\ x_2' = \dfrac{2}{t^2}x_1 - \dfrac{1}{t}x_2. \end{cases}$$

Note that p_{11}, p_{21}, and p_{22} are all discontinuous at $t = 0$ (but continuous everywhere else). The other coefficient function p_{12} is continuous everywhere (because it is the constant function $p_{12}(t) = 0$).

Special Case

We will focus now on the special case of real homogeneous linear systems with constant coefficients. That is, we will consider systems of the following type:

$$\begin{cases} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{cases} \qquad \text{or} \qquad \boldsymbol{x}' = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\boldsymbol{x},$$

where a, b, c, and d are real numbers.

Since the coefficients functions ate continuous everywhere (because they are constant functions), Theorem 1.6 (or Theorem 4.1) asserts that the system will have a general solution that is valid for all $t \in (-\infty, \infty)$.

Goal: *Find vectors* $\boldsymbol{x}$ *such that* $A\boldsymbol{x} = \boldsymbol{x}'$.

Based on our experience with solving second order linear differential equations, we speculate that the solution may be some type of exponential function. For this reason, we guess that the solution may be of the form

$$\boldsymbol{x} = \boldsymbol{v}e^{rt},$$

where r is a constant scalar and $\boldsymbol{v}$ is a constant vector. We will assume that the solution has this form for some r and $\boldsymbol{v}$ and determine the appropriate values for these constants later.

If $\boldsymbol{x} = \boldsymbol{v}e^{rt}$, then the derivative is $\boldsymbol{x}' = r\boldsymbol{v}e^{rt}$. If we substitute these values into the differential equation $A\boldsymbol{x} = \boldsymbol{x}'$, we have

$$A\big(\boldsymbol{v}e^{rt}\big) = r\boldsymbol{v}e^{rt},$$

which implies that $(A\boldsymbol{v})e^{rt} = (r\boldsymbol{v})e^{rt}$ for all values of t. Therefore,

$$A\boldsymbol{x} = \boldsymbol{x}' \quad \Longleftrightarrow \quad A\boldsymbol{v} = r\boldsymbol{v}.$$

Conclusion: $\boldsymbol{x} = \boldsymbol{v}e^{rt}$ is a solution to the equation $A\boldsymbol{x} = \boldsymbol{x}'$ provided that $A\boldsymbol{v} = r\boldsymbol{v}$. That is, $\boldsymbol{x} = \boldsymbol{v}e^{rt}$ is a solution provided that $\boldsymbol{v}$ is an eigenvector of the matrix A with eigenvalue r.

We recall that r is an eigenvalue of A if and only if it is a solution to the equation $\det(A - rI) = 0$. The equation $\det(A - rI) = 0$ is quadratic (in terms of the variable r), and so there are two roots to the equation, say r_1 and r_2. There are three cases to consider:

1. The eigenvalues are real and distinct ($r_1 \neq r_2$).
2. The eigenvalues are complex conjugates of each other ($r_1 = \overline{r_2}$).
3. There is exactly one real eigenvalue. ($r_1 = r_2$).

We will treat each of these three cases separately in the next few sections.

Definition 4.10. If $\boldsymbol{x}' = A\boldsymbol{x}$ is a homogeneous system with constant matrix A, then $\det(A - rI) = 0$ is called the *characteristic equation* of the system.

Why does all of this seem familiar?

If much of this section seems familiar, it is because this is not the first time we have encountered "homogeneous" equations or solved a "characteristic equation" that could be separated into the three cases enumerated above. These ideas appeared before when we considered solutions to a second order linear differential equation with constant coefficients. This is no coincidence.

Consider the second order linear homogeneous differential equation

$$ay'' + by' + cy = 0,$$

where a, b, and c are real numbers with $a \neq 0$. It is possible to transform this second order differential equation into a 2×2 linear system: Let $x_1 = y$ and $x_2 = y'$. Then we have the system

$$\begin{cases} x_1' = x_2 \\ ax_2' + bx_2 + cx_1 = 0. \end{cases}$$

Solving the second equation for x_2', this system can be written as

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2. \end{cases}$$

This 2×2 linear system can be rewritten as the matrix equation $\boldsymbol{x}' = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.$$

Let us now find the eigenvalues of the matrix A:

$$\det(A - rI) = \begin{vmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{vmatrix} = (-r)\Big(-\frac{b}{a} - r \Big) + \frac{c}{a} = r^2 + \frac{b}{a}r + \frac{c}{a}.$$

The eigenvalues of A are the values of r for which $\det(A - rI) = 0$, and so we must compute the solutions to the equation

$$r^2 + \frac{b}{a}r + \frac{c}{a} = 0.$$

First, multiply my the common denominator a:

$$ar^2 + br + c = 0.$$

Notice that this is the characteristic equation of the second order differential equation that we started with. Therefore, the eigenvalues of A are the same as the zeros of characteristic equation that defined A. It is for this reason that we called the equation $\det(A - rI) = 0$ the characteristic equation of the homogeneous linear system in Definition 4.10.

The next step is to determine the zeros of this equation (which gives the eigenvalues) and then compute the eigenvectors that correspond to the eigenvalues. However, exactly how we proceed from this point depends on the type of zeros (i.e., eigenvalues) of this characteristic equation. Are the zeros real or complex? Are the zeros repeated or distinct? It is precisely these questions that we address in the next few sections.

Section 5

Case 1: Distinct Real Eigenvalues

Consider a homogeneous 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where A is a 2×2 real matrix with constant entries. In this section we will assume that A has two distinct real eigenvalues.

Theorem 5.1. *Suppose A is a 2×2 real matrix with constant entries that has two distinct real eigenvalues r_1 and r_2. If r_1 has eigenvector $\boldsymbol{v}^{(1)}$ and r_2 has eigenvector $\boldsymbol{v}^{(2)}$, then*

$$\boldsymbol{x}^{(1)} = \boldsymbol{v}^{(1)} e^{r_1 t} \quad \textit{and} \quad \boldsymbol{x}^{(2)} = \boldsymbol{v}^{(2)} e^{r_2 t}$$

form a fundamental set of solutions to the homogeneous linear system $\boldsymbol{x}' = A\boldsymbol{x}$.

Proof. We saw in Section 4 that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions to this system. (See the "Conclusion" of the "Special Case.") It remains to show that they form a fundamental set of solutions. To see this, compute the Wronskien:

$$W(t) = \det\left(\boldsymbol{x}^{(1)}\ \boldsymbol{x}^{(2)}\right) = \det\left(\boldsymbol{v}^{(1)} e^{r_1 t}\ \boldsymbol{v}^{(2)} e^{r_2 t}\right) = e^{(r_1+r_2)t} \det\left(\boldsymbol{v}^{(1)}\ \boldsymbol{v}^{(2)}\right).$$

An exponential is never zero. The determinant is also nonzero, because $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$ are linearly independent vectors. (Otherwise, one would be a multiple of the other, and so they would have the same eigenvalue.) Since the Wronskien is never zero, the two solutions form a fundamental set of solutions over $(-\infty, \infty)$. □

Example 5.2. Find the general solution for the homogeneous differential system $\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \boldsymbol{x}$.

Solution. From Example 3.6 we know that this matrix has two distinct real eigenvalues $r_1 = 1$ and $r_2 = 5$. From Example 3.6, we know that the eigenvectors are

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the general solution is

$$\boldsymbol{x} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}.$$

Example 5.3. Find the general solution for the homogeneous differential system $\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{x}$.

Solution. Let A be the matrix. To find the eigenvalues of A, we solve the characteristic equation. To that end, we compute

$$\det(A - rI) = \begin{vmatrix} -2-r & 1 \\ -5 & 4-r \end{vmatrix} = (-2-r)(4-r) + 5 = r^2 - 2r - 3.$$

This quadratic factors to $(r-3)(r+1)$, and so the two eigenvalues are $r_1 = 3$ and $r_2 = -1$, which are distinct and real.

Case 1: $r_1 = 3$.

Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$(A - 3I)\boldsymbol{v} = \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + y \\ -5x + y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently, we have that $-5x + y = 0$, or $y = 5x$. Therefore,

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 5x \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} x.$$

Pick $x = 1$ and let the eigenvector corresponding to $r_1 = 3$ be the vector $\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Case 2: $r_2 = -1$.

Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$(A + I)\boldsymbol{v} = \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + y \\ -5x + 5y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently, we have that $-x + y = 0$, or $y = x$. Therefore,

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x.$$

Again, pick $x = 1$ and let the eigenvector corresponding to $r_2 = -1$ be the vector $\boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Conclusion: By Theorem 5.1, the general solution is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Geometric Representation

Suppose that a 2×2 homogeneous linear system of the form $\boldsymbol{x}' = A\boldsymbol{x}$ has a solution $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The solution $\boldsymbol{x}$ is a function of t, which means that solving the linear system gives x_1 and x_2 as functions of t. Consequently, the solution to the linear system gives a parameterization $x_1 = x_1(t)$ and $x_2 = x_2(t)$. This parameterization describes a curve in the x_1x_2-plane. In this context, we call the x_1x_2-plane *phase plane* and a collection of graphs of curves parameterized by solutions in the phase plane is called a *phase portrait*.

Example 5.4. Graph several solutions to the linear system $\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \boldsymbol{x}$.

Solution. In Example 5.2, we saw that the general solution to this system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}.$$

Different choices for c_1 and c_2 provide different solutions. We wish to plot the graphs for several different choices for c_1 and c_2. (See Figure 5.A for the actual plots.)

1. Let $c_1 = 0$ and $c_2 = 1$. Then
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} = \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}.$$
It follows that $x_1 = e^{5t}$ and $x_2 = e^{5t}$. Observe that this implies that $x_2 = x_1$ for all values of t, and so the graph of this solution lies on the line through the origin with slope 1. Since $x_1 > 0$ and $x_2 > 0$ for each value of t, the graph of this solution is the portion of the line that appears in the first quadrant.

2. Let $c_1 = 0$ and $c_2 = -1$. This case is exactly like the previous case, except that here $x_1 = -e^{5t} < 0$ and $x_2 = -e^{5t} < 0$ for each value of t, and so the graph of this solution is the portion of the line $x_2 = x_1$ occurring in the third quadrant.

3. Let $c_1 = 1$ and $c_2 = 0$. Then
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} = \begin{pmatrix} -3e^{t} \\ e^{t} \end{pmatrix}.$$
It follows that $x_1 = -3e^t$ and $x_2 = e^t$. Consequently, the graph of this solution lies on the line
$$x_1 = -3x_2 \quad \text{or} \quad x_2 = -\frac{1}{3}x_1.$$
This is a line through the origin with slope $-1/3$. Since $x_1 < 0$ and $x_2 > 0$ for each value of t, the graph of this solution is the portion of the line that is in the second quadrant.

4. Let $c_1 = -1$ and $c_2 = 0$. This case is similar to the previous case, except that here $x_1 = 3e^t > 0$ and $x_2 = -e^t < 0$ for each value of t, and so the graph of this solution is the portion of the line $x_2 = -\frac{1}{3}x_1$ occurring in the fourth quadrant.

5. Let $c_1 = 1$ and $c_2 = 1$. Then
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} = \begin{pmatrix} -3e^{t} + e^{5t} \\ e^{t} + e^{5t} \end{pmatrix}.$$
It follows that $x_1 = -3e^t + e^{5t}$ and $x_2 = e^t + e^{5t}$. This graph is more difficult to describe that the previous curves. It is plotted with the other curves in Figure 5.A. Observe that $x_2 > 0$ for all values of t, so that the graph of this solution always lies above the x_1-axis.

6. Let $c_1 = -1$ and $c_2 = -1$. This case is like the previous one. We have
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} = \begin{pmatrix} 3e^{t} - e^{5t} \\ -e^{t} - e^{5t} \end{pmatrix}.$$
It follows that $x_1 = 3e^t - e^{5t}$ and $x_2 = -e^t - e^{5t}$. As before, this graph is difficult to describe. In this case, notice that $x_2 < 0$ for all values of t, and so the graph of the solution in this case always lies below the x_1-axis. (Again, see Figure 5.A.)

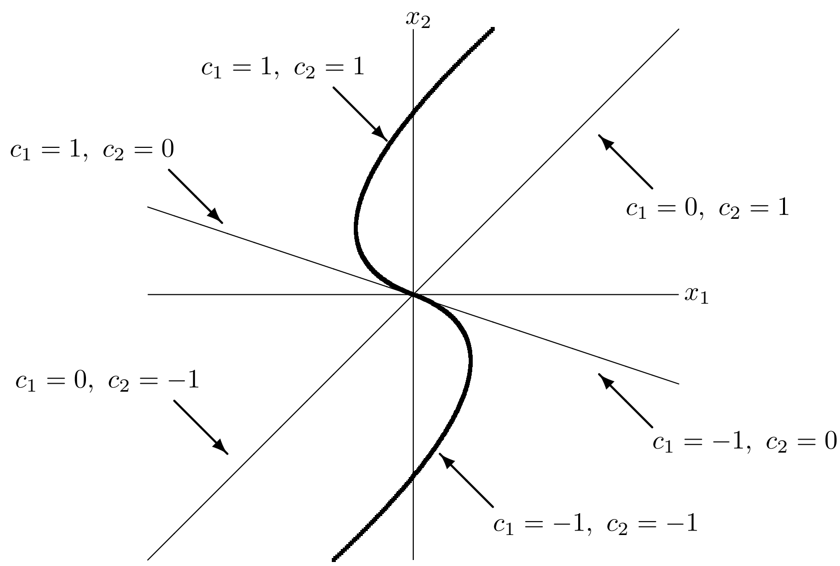


Figure 5.A: Incomplete phase portrait for Example 5.4.

The graphs in Figure 5.A give an incomplete picture of what is actually happening with the solutions. It might look as though there are thee solutions represented, but there are in fact six. The three curves that we see in Figure 5.A are not the solutions themselves, but rather represent the paths that the solutions take as $t \to \infty$ in the x_1x_2-plane. Keep in mind that x_1 and x_2 are functions of t, so if we think of t as time, then the point (x_1, x_2) moves in the plane as time passes (that is, as $t \to \infty$). The curves that we see in Figure 5.A are the paths made by (x_1, x_2) as it moves around in the plane as time passes. Exactly which path (x_1, x_2) takes in the plane depends on where it starts (at $t = 0$, for example).

In order to give a more complete phase portrait for Example 5.4, we should indicate the direction of motion. In this particular example, the point (x_1, x_2) moves away from the origin as $t \to \infty$. Consequently, in order to represent this motion away from the origin, it is customary to draw an arrow indicating the direction of motion. The phase portrait in Figure 5.B was created using Mathematica and includes the graphs of several solutions as well as their direction of motion away from the origin.

Note that none of the solutions represented in Figure 5.A or Figure 5.B actually pass through the origin, although they approach it as $t \to -\infty$.

The way to interpret a phase portrait, such as the one in Figure 5.B, is to imagine placing a point-sized particle into the x_1x_2-plane and allowing it to move in the direction indicated by the vector at whichever point the particle is currently located.

In this phase portrait, the particle moves away from the origin as $t \to \infty$, regardless of where the particle's initial position is on the phase plane (as long as it does not actually start at the origin). In a case like this, the origin is known as an *unstable node.* The term *unstable* is used to mean that motion is in almost all cases away from the node.

Example 5.5. Find the general solution and sketch a phase portrait for the linear system

$$\boldsymbol{x}' = \begin{pmatrix} -4 & 0 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}.$$

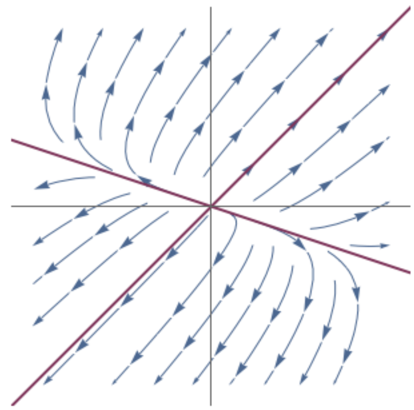


Figure 5.B: A more complete phase portrait for Example 5.4.

Solution. Denote the coefficient matrix by A. The eigenvalues of the matrix A are the solutions to the quadratic equation $\det(A - rI) = 0$. Since

$$\det(A - rI) = \begin{vmatrix} -4-r & 0 \\ 1 & -2-r \end{vmatrix} = (-4-r)(-2-r) - 0,$$

the quadratic equation we must solve is

$$r^2 + 6r + 8 = 0.$$

The solutions to this equation are $r_1 = -4$ and $r_2 = -2$, and so these are our eigenvalues.

To find eigenvectors for the matrix A, we need to solve the matrix equation $(A - rI)\boldsymbol{v} = \mathbf{0}$ when $r = r_1$ and when $r = r_2$.

Case 1: $r_1 = -4$.

Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$(A - r_1 I)\boldsymbol{v} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $x + 2y = 0$, and so $x = -2y$. Therefore, an eigenvector corresponding to the eigenvalue $r_1 = -4$ is any nonzero vector of the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} y.$$

We may choose any vector such that $y \neq 0$. If we pick $y = 1$, then the eigenvector corresponding to $r_1 = -4$ is $\boldsymbol{v}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Case 2: $r_2 = -2$.

Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$(A - r_2 I)\boldsymbol{v} = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $x = 0$, and so an eigenvector corresponding to the eigenvalue $r_2 = -2$ is any nonzero vector of the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} y.$$

Again, we may choose any vector such that $y \neq 0$. If we pick $y = 1$, then the eigenvector corresponding to $r_2 = -2$ is $\boldsymbol{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now that we have the eigenvalues and eigenvectors, we can conclude that the general solution to this linear system is

$$\boldsymbol{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}.$$

It remains to sketch the phase portrait. In Figure 5.C, we display a phase portrait generated using Mathematica. The dark lines represent the solutions where either $c_1 = 0$ or $c_2 = 0$.

Notice that in this phase portrait, motion is *towards* the origin as $t \to \infty$. In a case like this, the origin is known as a *asymptotically stable node.*

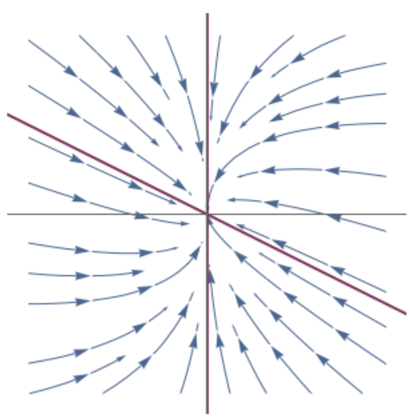


Figure 5.C: Phase portrait for Example 5.5.

Example 5.6. Find the general solution and sketch a phase portrait for the linear system

$$\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{x}.$$

Solution. As usual, denote the coefficient matrix by A. The eigenvalues of the matrix A are the solutions to the quadratic equation $\det(A - rI) = 0$, and so we start by computing the determinant:

$$\det(A - rI) = \begin{vmatrix} -2-r & 1 \\ -5 & 4-r \end{vmatrix} = (-2-r)(4-r) + 5.$$

Thus, we must solve the quadratic equation

$$r^2 - 2r - 3 = 0.$$

The solutions to this quadratic equation are $r_1 = -1$ and $r_2 = 3$, and so these are our eigenvalues.

To find eigenvectors for the matrix A, we need to solve the matrix equation $(A - rI)\boldsymbol{v} = \mathbf{0}$ when $r = r_1$ and when $r = r_2$.

Case 1: $r_1 = -1$.

Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$(A - r_1 I)\boldsymbol{v} = \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + y \\ -5x + 5y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $-x + y = 0$, and so $x = y$. (Notice that both equations give the same relationship between x and y.) We conclude that an eigenvector corresponding to the eigenvalue $r_1 = -1$ is any nonzero vector of the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x.$$

We pick $x = 1$, and so choose the eigenvector corresponding to $r_1 = -1$ to be $\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Case 2: $r_2 = 3$.

Once again, let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$(A - r_2 I)\boldsymbol{v} = \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + y \\ -5x + y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $y = 5x$, and so an eigenvector corresponding to the eigenvalue $r_2 = 3$ is any nonzero vector of the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 5x \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} x.$$

Again, we pick $x = 1$, and so the eigenvector we choose to correspond to $r_2 = 3$ is $\boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Now that we have the eigenvalues and eigenvectors, we can conclude that the general solution to this linear system is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

In Figure 5.D, we display a phase portrait generated using Mathematica. Again, the dark lines represent the solutions where either $c_1 = 0$ or $c_2 = 0$.

In this phase portrait, the motion is neither predominantly towards nor away from the origin. In this case, the origin is called a *saddle point* (and not a node). Saddle points are always unstable, because the motion tends away from the point as $t \to \infty$ for almost all paths.

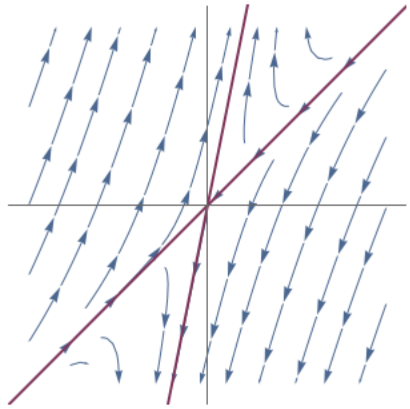


Figure 5.D: Phase portrait for Example 5.6.

In Figures 5.B, 5.C, and 5.D, we saw three different types of phase portraits. These three types are typical of linear systems where the corresponding matrix has two eigenvalues that are real and distinct. Indeed, the qualitative behavior of the phase portrait is determined by the eigenvalues. The origin will be an unstable node when both eigenvalues are positive, an asymptotically stable node when both eigenvalues are negative, and a saddle point when the eigenvalues have opposite signs.

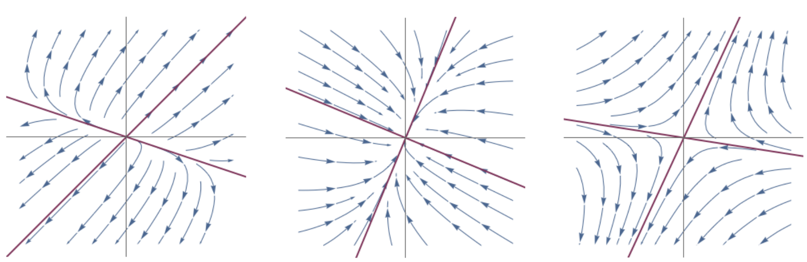
At this point it is natural to ask "What happens if one of the eigenvalues is zero?" We encourage the interested reader to consider this question and try to answer it by constructing an example and finding solutions. Start by asking yourself what the matrix A would have to look like if it had a zero eigenvalue.

Example 5.7. Solve the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Solution. In Example 5.6, we found that the general solution to the linear system is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$



$r_1 > 0$ and $r_2 > 0$ $\quad$ $r_1 < 0$ and $r_2 < 0$ $\quad$ $r_1 > 0$ and $r_2 < 0$

Figure 5.E: Phase portraits if eigenvalues are real, distinct, and nonzero.

The initial condition tells us that when $t = 0$,

$$\boldsymbol{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Consequently, we have the following system of equations with unknowns c_1 and c_2:

$$\begin{cases} c_1 + c_2 = -3 \\ c_1 + 5c_2 = 1. \end{cases}$$

The solution to this system is $c_1 = -4$ and $c_2 = 1$. Therefore, the solution to the initial value problem is

$$\boldsymbol{x} = -4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

In Example 5.7, we found the solution to an initial value problem arising from a linear system we studied in Example 5.6. A phase portrait corresponding to this linear system appears in Figure 5.D. We can interpret the solution to the initial value problem geometrically using this phase portrait. Solutions to the linear system are represented by directed curves in the phase portrait and initial conditions are points in the x_1x_2-plane that lie on one of these curves. In our current example (that is, in Example 5.7), the initial condition corresponds to the point $(-3, 1)$. Plotting this point on the phase plane, we see there is precisely one solution passing through this point. In Figure 5.F, we plot this point and highlight the curve passing through it. This highlighted curve represents the solution we found in Example 5.7.

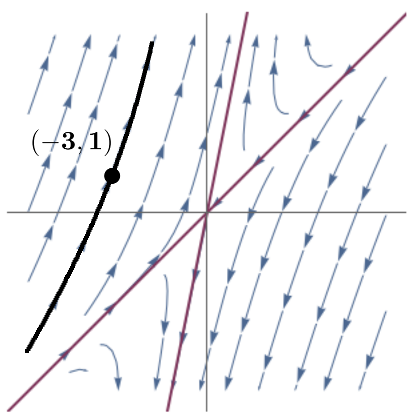


Figure 5.F: Phase portrait for Example 5.7.

Section 6

Case 2: Complex Eigenvalues

Consider a homogeneous 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where A is a 2×2 real matrix with constant entries. In this section we will assume that A has two complex (non-real) eigenvalues. (To be completely precise, we assume that each eigenvector has a nonzero imaginary part.)

If A has two complex eigenvalues, then it is necessarily the case that they are complex conjugates, by Proposition 3.12, which says that if r is an eigenvalue with eigenvector $\boldsymbol{v}$, then $\overline{r}$ is an eigenvalue with eigenvector $\overline{\boldsymbol{v}}$.

We know from Section 4 that if r is an eigenvalue for A with eigenvector $\boldsymbol{v}$, then $\boldsymbol{x} = \boldsymbol{v}e^{rt}$ is a solution to the linear system $\boldsymbol{x}' = A\boldsymbol{x}$. Consequently, if A is a matrix with real entries, and if r is an eigenvalue for A with eigenvector $\boldsymbol{v}$, then two solutions to the linear system are

$$\boldsymbol{y}^{(1)} = \boldsymbol{v}e^{rt} \quad \text{and} \quad \boldsymbol{y}^{(2)} = \overline{\boldsymbol{v}}e^{\overline{r}t}.$$

Furthermore, since r is not a real number, it follows that r and $\overline{r}$ are distinct. Also, since $\boldsymbol{v}$ is an eigenvector, both it and $\overline{\boldsymbol{v}}$ are (by definition) nonzero vectors. It follows from the same proof used in Theorem 5.1 that $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ form a fundamental set of solutions.

If we were just looking for any fundamental set of solutions, we could stop here. And we have never said that we were looking for any specific fundamental set of solutions, so why would we need anything other than what we have? In some sense, we could just say that $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ form a fundamental set of solutions and be done with this case. However, there is one drawback with these solutions, albeit not exactly of a mathematical nature. The original problem was to find solutions to the *real* linear system $\boldsymbol{x}' = A\boldsymbol{x}$. We have found two solutions, but both of them are *complex*. Since we started with a *real* system, we would prefer to have *real* solutions.

Recall that if $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ form a fundamental set of solutions, then the general solution is given by $c_1\boldsymbol{y}^{(1)} + c_2\boldsymbol{y}^{(2)}$. In particular, this is a solution for any choice of scalars c_1 and c_2. Thus, any linear combination of solutions is also a solution. (This is known as the *Law of Superposition.*) It will suffice, therefore, to find two real-valued linearly independent linear combinations of $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$. This might sound difficult at first, but it is actually quite easy.

Theorem 6.1. *Suppose A is a 2×2 real matrix with constant entries that has complex eigenvalues r and corresponding eigenvector $\boldsymbol{v}$, then*

$$\boldsymbol{x}^{(1)} = \Re(\boldsymbol{v}e^{rt}) \quad \textit{and} \quad \boldsymbol{x}^{(2)} = \Im(\boldsymbol{v}e^{rt})$$

form a fundamental set of solutions to the linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where $\Re(\boldsymbol{v}e^{rt})$ and $\Im(\boldsymbol{v}e^{rt})$ denote the real and imaginary parts of $\boldsymbol{v}e^{rt}$, respectively.

Proof. This basically follows from the fact that

$$\Re(\boldsymbol{v}e^{rt}) = \frac{\boldsymbol{v}e^{rt} + \overline{\boldsymbol{v}e^{rt}}}{2} = \frac{\boldsymbol{v}e^{rt} + \overline{\boldsymbol{v}}e^{\overline{r}t}}{2} = \frac{1}{2}\boldsymbol{y}^{(1)} + \frac{1}{2}\boldsymbol{y}^{(2)}$$

and

$$\Im(\boldsymbol{v}e^{rt}) = \frac{\boldsymbol{v}e^{rt} - \overline{\boldsymbol{v}e^{rt}}}{2i} = \frac{\boldsymbol{v}e^{rt} - \overline{\boldsymbol{v}}e^{\overline{r}t}}{2i} = \frac{1}{2i}\boldsymbol{y}^{(1)} + \frac{1}{2i}\boldsymbol{y}^{(2)}.$$

We still need to show that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions, but that can be shown by computing the Wronskien. $\square$

Example 6.2. Solve the linear system $\boldsymbol{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}$.

Solution. We leave it to the reader to show that the matrix A defining this homogeneous linear system has the eigenvalue $r = -1 + 2i$. In order to find a corresponding eigenvector, let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and solve the matrix equation $(A - rI)\boldsymbol{v} = \mathbf{0}$, or

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Consequently (using the bottom equation),

$$x - 2iy = 0 \quad \text{or} \quad x = 2iy.$$

Thus,

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2iy \\ y \end{pmatrix} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} y = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] y.$$

As usual, we can pick y to be any scalar (real or complex), so long as it is nonzero. If we pick $y = 1$, then our eigenvector is

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Observe that

$$\boldsymbol{v}e^{rt} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] e^{(-1+2i)t} = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] \left[\cos(2t) + i\sin(2t)\right]$$

$$= e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin(2t)\right] + ie^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos(2t)\right].$$

Since $\boldsymbol{x}^{(1)}$ is the real part of this expression, and $\boldsymbol{x}^{(2)}$ is the imaginary part, we conclude that

$$\boldsymbol{x}^{(1)} = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin(2t)\right]$$

and

$$\boldsymbol{x}^{(2)} = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos(2t)\right].$$

By Theorem 6.1, these form a fundamental set of solutions for this linear system.

It is also possible to sketch a phase portrait for the system in Example 6.2. Observe that for any real scalars c_1 and c_2, we have the solution

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} -2c_1 \sin(2t) + 2c_2 \cos(2t) \\ c_1 \cos(2t) + c_2 \sin(2t) \end{pmatrix}.$$

Then

$$\begin{aligned}
x_1^2 + 4x_2^2 &= e^{-2t}\Big[\Big(-2c_1\sin(2t) + 2c_2\cos(2t)\Big)^2 + 4\Big(c_1\cos(2t) + c_2\sin(2t)\Big)^2\Big] \\
&= e^{-2t}\Big[4c_1^2\sin^2(2t) - 4c_1c_2\sin(2t)\cos(2t) + 4c_2^2\cos(2t)^2 \\
&\qquad\qquad + 4c_1^2\cos^2(2t) + 4c_1c_2\sin(2t)\cos(2t) + 4c_2^2\sin^2(2t)\Big] \\
&= 4e^{-2t}(c_1^2 + c_2^2).
\end{aligned}$$

Therefore,

$$\frac{x_1^2}{2^2} + \frac{x_2^2}{1^2} = \frac{c_1^2 + c_2^2}{e^{2t}}.$$

For fixed constants c_1 and c_2, and if t is constant, then this is the equation of an ellipse. Notably, however, t is not fixed, and as t increases to ∞, the right side of this equation diminishes to zero. Consequently, the path traced out in the phase plane by these solutions are elliptical spirals with trajectories towards the origin. A phase portrait for Example 6.2 can be found in Figure 6.A.

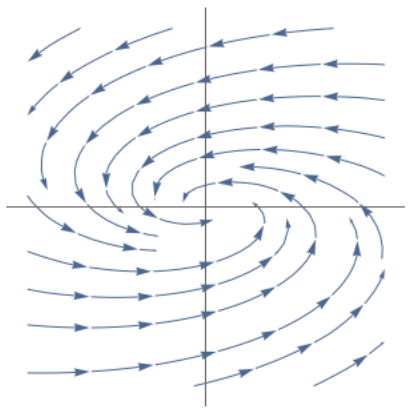


Figure 6.A: Phase portrait for Example 6.2.

When a phase portrait contains curves that spiral in towards the origin, like the one in Figure 6.A, the origin is an *asymptotically stable spiral point*. The decaying behavior, that leads to the spiraling motion towards the origin (and hence the asymptotically stable nature of the point), is a result of the term e^{-t} appearing in the general solution. In particular, it is a result of the real part of the eigenvalue being negative.

When a real 2×2 linear system has eigenvalues with positive real part, the trajectories are reversed, and so the motion is spiraling away from the origin. In this case, the origin will be an *unstable spiral point*.

Example 6.3. Solve the linear system $\boldsymbol{x}' = \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} \boldsymbol{x}$.

Solution. Direct computation shows that the matrix defining this homogeneous linear system has an eigenvalue $r = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. In order to find the corresponding eigenvector, let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and solve the matrix equation

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -r & 2 \\ -2 & 1-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 2 \\ -2 & \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Consequently,

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)x+2y=0.$$

If x and y are chosen so that this equation is true (so long as x and y are not both zero), then $\boldsymbol{v}$ is an eigenvector. Thus, in order to ease computation, choose $x=4$. Then $y=1+\sqrt{3}i$. Therefore, our choice of eigenvector is

$$\boldsymbol{v}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\1+\sqrt{3}i\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}+i\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}.$$

We begin by computing

$$\boldsymbol{v}e^{rt}=\begin{pmatrix}4\\1+\sqrt{3}i\end{pmatrix}e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)t}.$$

Our fundamental set of solutions will be given by the real and imaginary parts of this expression. In order to identify the real and imaginary parts, we use Euler's Formula:

$$\boldsymbol{v}e^{rt}=e^{t/2}\left[\begin{pmatrix}4\\1\end{pmatrix}+i\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\right]\left[\cos\left(\tfrac{\sqrt{3}}{2}t\right)+i\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right].$$

Expanding this, we find that

$$\begin{aligned}\boldsymbol{v}e^{rt}=e^{t/2}&\left[\begin{pmatrix}4\\1\end{pmatrix}\cos\left(\tfrac{\sqrt{3}}{2}t\right)-\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right]\\&+ie^{t/2}\left[\begin{pmatrix}4\\1\end{pmatrix}\sin\left(\tfrac{\sqrt{3}}{2}t\right)+\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\cos\left(\tfrac{\sqrt{3}}{2}t\right)\right].\end{aligned}$$

Therefore, since $\boldsymbol{x}^{(1)}$ is the real part of this expression, and $\boldsymbol{x}^{(2)}$ is the imaginary part, we conclude that

$$\boldsymbol{x}^{(1)}=e^{t/2}\left[\begin{pmatrix}4\\1\end{pmatrix}\cos\left(\tfrac{\sqrt{3}}{2}t\right)-\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right]$$

and

$$\boldsymbol{x}^{(2)}=e^{t/2}\left[\begin{pmatrix}4\\1\end{pmatrix}\sin\left(\tfrac{\sqrt{3}}{2}t\right)+\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\cos\left(\tfrac{\sqrt{3}}{2}t\right)\right].$$

These form a fundamental set of solutions for the linear system, by Theorem 6.1.

The linear system in Example 6.3 is defined by a real 2×2 matrix that has complex eigenvalues. In this example (as opposed to Example 6.2), the real part of the eigenvalue is positive. Consequently, the motion of the solution is an elliptical spiral away from the origin. Because the motion is away from the origin, the origin is called an *unstable spiral point*. A phase portrait for Example 6.3 is given in Figure 6.B.

Examples 6.2 and 6.3 discuss solutions to 2×2 real linear systems when the eigenvalues have negative or positive (respectively) real part. It is also possible for a 2×2 real linear system to have purely imaginary eigenvalues. (That is, it is possible for the eigenvalues to have real part equal to zero.)

Example 6.4. Solve the linear system $\boldsymbol{x}'=\begin{pmatrix}0&2\\-2&0\end{pmatrix}\boldsymbol{x}$.

Solution. In this case, the matrix A defining the homogeneous linear system has eigenvalues $r=2i$ and $\bar{r}=-2i$. We will identify an eigenvector $\boldsymbol{v}$ corresponding to $r=2i$. (The eigenvector corresponding

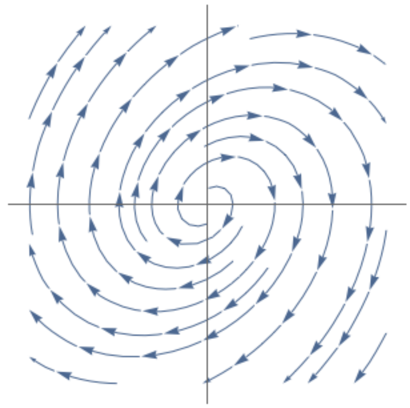


Figure 6.B: Phase portrait for Example 6.3.

to $\overline{r}$ will be $\overline{\boldsymbol{v}}$.) Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and solve the matrix equation $(A - rI)\boldsymbol{v} = \boldsymbol{0}$, or

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -r & 2 \\ -2 & -r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2ix + 2y \\ -2x - 2iy \end{pmatrix}.$$

This provides the following scalar equations:

$$\begin{cases} -2ix + 2y = 0 \\ -2x - 2iy = 0. \end{cases}$$

It may appear at first that these are two different equations, but multiplying the first equation by $-i$ will produce the second equation. Solving the second equation for x in terms of y, we see that $x = -iy$, and so $\boldsymbol{v}$ will be an eigenvector for A if

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -iy \\ y \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} y = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] y,$$

provided that $y \neq 0$. We will, as usual, choose $y = 1$.

In order to determine the fundamental set of solutions, we proceed as before, making use of Euler's Theorem:

$$\boldsymbol{v}e^{rt} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{2it} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \Big[\cos(2t) + i \sin(2t) \Big].$$

Our fundamental set of solutions will be given by the real and imaginary parts of this expression, and so we expand:

$$\boldsymbol{v}e^{rt} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin(2t) \right] + i \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(2t) \right].$$

Therefore, since $\boldsymbol{x}^{(1)}$ is the real part of this expression, and $\boldsymbol{x}^{(2)}$ is the imaginary part, we conclude that

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin(2t)$$

and

$$\boldsymbol{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(2t).$$

These form a fundamental set of solutions for the linear system, by Theorem 6.1.

To understand what happens to the phase portrait when the eigenvalues are purely imaginary (that is, when the eigenvalues have no real part), let us consider the general solution to the linear system given in Example 6.4:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \sin(2t) - c_2 \cos(2t) \\ c_1 \cos(2t) + c_2 \sin(2t) \end{pmatrix}.$$

Then

$$\begin{aligned} x_1^2 + x_2^2 &= \Big(c_1 \sin(2t) - c_2 \cos(2t)\Big)^2 + \Big(c_1 \cos(2t) + c_2 \sin(2t)\Big)^2 \\ &= c_1^2 \sin^2(2t) - 2c_1c_2 \sin(2t)\cos(2t) + c_2^2 \cos(2t)^2 + c_1^2 \cos^2(2t) + 2c_1c_2 \sin(2t)\cos(2t) + c_2^2 \sin^2(2t) \\ &= c_1^2 + c_2^2. \end{aligned}$$

Therefore, the graph of this solution lies on the curve $x_1^2 + x_2^2 = c_1^2 + c_2^2$. In other words, a solution's path lies on a circle and the radius of that circle is determined by the values of c_1 and c_2 (which in turn are determined by the problem's initial conditions). A phase portrait for Example 6.4 is given in Figure 6.C.

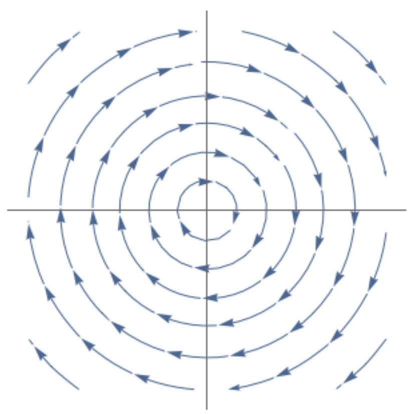


Figure 6.C: Phase portrait for Example 6.4.

Solutions to the linear system in Example 6.4 appear as orbits around the origin. In this case, the origin is called a *center*. Because the motion around a center does not tend away from the point, a center is always stable.

In general, when the eigenvalues are purely imaginary, the origin will be a center with elliptical orbits. The orbits in Example 6.4 are circular, but a circle is a special type of ellipse.

In order to understand why the nature of the phase portraits is determined by the real part of the eigenvalue, suppose that $\boldsymbol{x}' = A\boldsymbol{x}$ is a 2×2 linear system with A a real matrix, and suppose A has eigenvalue $r = \lambda + i\mu$ with corresponding eigenvector $\boldsymbol{v}$. Then a fundamental set of solutions is given by the real and imaginary parts of

$$\boldsymbol{v}e^{rt} = \boldsymbol{v}e^{(\lambda+i\mu)t} = \boldsymbol{v}e^{\lambda t}e^{i\mu t} = e^{\lambda t}\left[\boldsymbol{v}e^{i\mu t}\right].$$

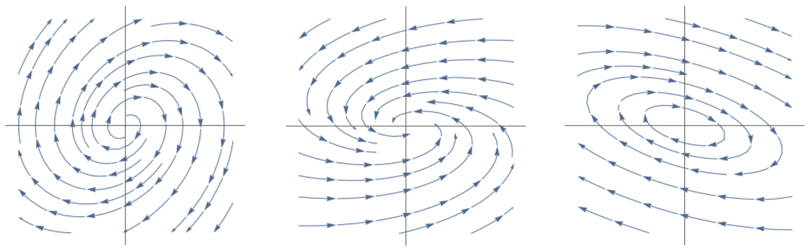


Figure 6.D: Phase portraits if eigenvalues are complex (and nonzero).
Eigenvalue: $r = \lambda + i\mu$

Therefore, a fundamental set of solutions is given by

$$\boldsymbol{x}^{(1)} = e^{\lambda t}\,\Re\left(\boldsymbol{v}e^{i\mu t}\right) \quad \text{and} \quad \boldsymbol{x}^{(2)} = e^{\lambda t}\,\Im\left(\boldsymbol{v}e^{i\mu t}\right),$$

where we note that $e^{\lambda t}$ is a real number (which is why we can factor it out). It follows that the general solution has the form

$$c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)} = e^{\lambda t}\left[c_1\Re\left(\boldsymbol{v}e^{i\mu t}\right) + c_2\Im\left(\boldsymbol{v}e^{i\mu t}\right)\right].$$

The numbers c_1 and c_2 are constants (and depend on the initial conditions for the problem). The vector $\boldsymbol{v}$ is also constant. Thus, as t changes, these constants do not change. The only terms that change with t are the exponential terms $e^{\lambda t}$ and $e^{i\mu t}$. While the term $e^{i\mu t}$ does depend on t, Euler's Formula tells us that $|e^{i\mu t}| = 1$ for all t. This means that the term $e^{i\mu t}$ does not contribute much (if anything) to the size of the solution as t tends to ∞, but $e^{i\mu t}$ is a periodic function (again referring to Euler's Formula). This is what gives the paths the elliptical (or rotational) motion. The size of the solution, and hence the long-term behavior of the solution, is determined by the term $e^{\lambda t}$. If $\lambda > 0$, then $e^{\lambda t} \to \infty$ as $t \to \infty$, and so the paths spiral away from the origin. If $\lambda < 0$, then $e^{\lambda t} \to 0$ as $t \to \infty$, and so the paths decay towards the origin. If $\lambda = 0$, then $e^{\lambda t} = 1$ for all t, and so the path of the solution is effected only by the rotational movement of the $e^{i\mu t}$ term, which is why this case leads to elliptical (or circular) orbits.

The different types of phase portraits for the complex eigenvalue case are demonstrated in Figure 6.D. The type of phase portrait is determined by the real part of the eigenvalue. In the figure, we assume the eigenvalue is $r = \lambda + i\mu$, so that the qualitative behavior of the solutions is determined by the nature of λ—whether it is zero, positive, or negative.

What has not been discussed here is how to determine if the rotational motion we see in the phase portraits is clockwise or counterclockwise. We encourage the reader to experiment with various linear systems (using some form of technology) and draw some conclusions.

Example 6.5. Solve the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 5 \\ -2 & -1 \end{pmatrix}\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

Solution. Let A be the matrix that defines the homogeneous linear system. The characteristic polynomial for this matrix is

$$\det(A - rI) = \begin{vmatrix} 1-r & 5 \\ -2 & -1-r \end{vmatrix} = (1-r)(-1-r) + 10 = r^2 + 9.$$

Thus, the characteristic equation is $r^2 + 9 = 0$, and so the eigenvalues for this matrix are $r = 3i$ and $\overline{r} = -3i$.

It is sufficient to identify an eigenvector $\boldsymbol{v}$ corresponding to $r = 3i$ (because the eigenvector corresponding to $\overline{r} = -3i$ will be $\overline{\boldsymbol{v}}$). We wish solve the matrix equation $(A - rI)\boldsymbol{v} = \mathbf{0}$. Thus, we compute

$$(A - rI)\boldsymbol{v} = \begin{pmatrix} 1-r & 5 \\ -2 & -1-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-3i & 5 \\ -2 & -1-3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, as usual. Consequently, we wish to find x and y (not both zero) so that

$$\begin{pmatrix} (1-3i)x + 5y \\ -2x + (-1-3i)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This leads to the scalar equations

$$\begin{cases} (1-3i)x + 5y = 0 \\ -2x + (-1-3i)y = 0. \end{cases}$$

Once again, it may appear at first that these are two different equations; however, observe that

$$\left(\frac{1-3i}{-2}\right)\Big[-2x + (-1-3i)y\Big] = (1-3i)x + 5y,$$

and so the two equations have the same solutions.

Solving the second equation for x in terms of y, we see that $x = \left(\frac{1+3i}{-2}\right)y$, and so $\boldsymbol{v}$ will be an eigenvector for A if

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1+3i)/(-2) \\ 1 \end{pmatrix} y,$$

provided that $y \neq 0$. For ease of computation we will choose $y = -2$. Then

$$\boldsymbol{v} = \begin{pmatrix} 1+3i \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + i\begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

In order to determine the fundamental set of solutions, we proceed as before, making use of Euler's Theorem:

$$\boldsymbol{v}e^{rt} = \begin{pmatrix} 1+3i \\ -2 \end{pmatrix} e^{3it} = \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} + i\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right] \Big[\cos(3t) + i\sin(3t)\Big].$$

Our fundamental set of solutions will be given by the real and imaginary parts of this expression, and so we expand, as before, to get:

$$\boldsymbol{v}e^{rt} = \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}\cos(3t) - \begin{pmatrix} 3 \\ 0 \end{pmatrix}\sin(3t)\right] + i\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}\sin(3t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix}\cos(3t)\right].$$

We will let $\boldsymbol{x}^{(1)}$ be the real part of this expression and $\boldsymbol{x}^{(2)}$ be the imaginary part, and so

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\cos(3t) - \begin{pmatrix} 3 \\ 0 \end{pmatrix}\sin(3t)$$

and

$$\boldsymbol{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\sin(3t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix}\cos(3t).$$

These form a fundamental set of solutions for the linear system, by Theorem 6.1, and so the general solution to this system is

$$c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)} = \begin{pmatrix} c_1\cos(3t) - 3c_1\sin(3t) + c_2\sin(3t) + 3c_2\cos(3t) \\ -2c_1\cos(3t) - 2c_2\sin(3t) \end{pmatrix}.$$

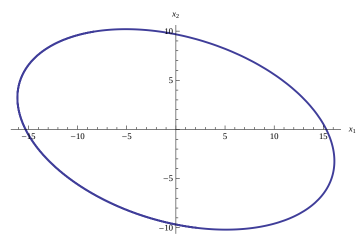
To solve the initial value problem, we must find values for the constants c_1 and c_2 so that the solution $\boldsymbol{x} = c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)}$ satisfies the given initial condition:

$$\boldsymbol{x}(0) = \begin{pmatrix} c_1 + 3c_2 \\ -2c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Therefore, $c_1 = 5$ and $c_2 = -1$, and so the solution to the initial value problem is

$$\boldsymbol{x}(t) = \begin{pmatrix} 2\cos(3t) - 16\sin(3t) \\ -10\cos(3t) + 2\sin(3t) \end{pmatrix}.$$

This solution parameterizes an ellipse in the x_1x_2-plane.



A phase portrait for this linear system appears in Figure 6.E. The ellipse graphed here is the curve belonging to one of the solutions represented in Figure 6.E.

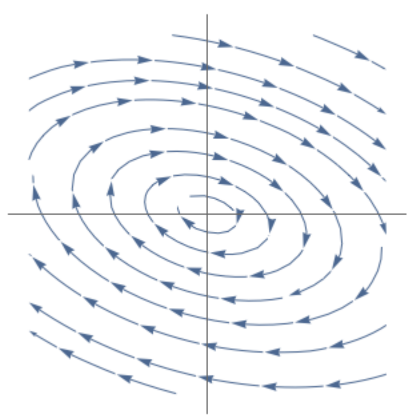


Figure 6.E: Phase portrait for Example 6.5.

Section 7

Fundamental Matrices

Suppose that $P(t)$ is a 2×2 matrix having entries that are real valued functions of t. Suppose also that the linear system

$$\boldsymbol{x}' = P(t)\boldsymbol{x}$$

has a fundamental set of solutions

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \begin{pmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{pmatrix}.$$

The *fundamental matrix* of the differential system is the matrix having columns $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$:

$$X = (\boldsymbol{x}^{(1)} \ \boldsymbol{x}^{(2)}) = \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) \end{pmatrix}.$$

The matrix X is nonsingular for all values of t for which $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ form a fundamental set of solutions, because $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are linearly independent on any interval for which they form a fundamental set of solutions.

Remark 7.1. The results in this section of the textbook are intended to be used when solving a general $n \times n$ linear differential system, where n is any natural number such that $n \geq 2$. We are only considering the case $n = 2$, however, which is a much simpler context, and so the results of this section are not needed in our work.

Section 8

Case 3: Repeated Eigenvalues

Once again, we consider a homogeneous 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where A is a 2×2 real matrix with constant entries. In this section we will assume that A has one real eigenvalue. That is to say, we assume that the characteristic polynomial of A has the form $\det(A - rI) = (r - r_0)^2$, so that $r = r_0$ is a root of the characteristic equation having multiplicity 2.

When a 2×2 real matrix has one real eigenvalue (or, in other words, a repeated eigenvalue), it is possible for there to be two linearly independent eigenvectors, both corresponding to the one eigenvalue. This can be seen in Example 8.1, below.

Example 8.1. Show that the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has one eigenvalue and two linearly independent eigenvectors.

Solution. Observe that

$$\det(A - rI) = \begin{vmatrix} 2-r & 0 \\ 0 & 2-r \end{vmatrix} = (2-r)^2.$$

The only zero of this polynomial is $r = 2$, and so A has only one eigenvalue.

To find eigenvectors corresponding to the eigenvalue $r = 2$, we solve the matrix equation $(A - rI)\boldsymbol{v} = \boldsymbol{0}$. Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, as usual. Then

$$(A - rI)\boldsymbol{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is satisfied by all choices of x and y, which means that $\boldsymbol{v}$ is an eigenvector for any x and y, provided they are not both zero. Let

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\boldsymbol{v}^{(1)}$ is the eigenvector obtained from choosing $x = 1$ and $y = 0$, and $\boldsymbol{v}^{(2)}$ is the eigenvector obtained from choosing $x = 0$ and $y = 1$. These two eigenvectors are linearly independent.

In Example 8.1, we found that an eigenvector was any nonzero vector having the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It can be said that this is the "general form" for an eigenvector. The fact that there are two completely free parameters ("free" in the sense that there is no equation relating one to the other), one in each component of the vector, indicates that there can be found two linearly independent eigenvectors.

We already know that whenever r is an eigenvalue of A with eigenvector $\boldsymbol{v}$, the vector $\boldsymbol{x} = \boldsymbol{v}e^{rt}$ is a solution to the linear system $\boldsymbol{x}' = A\boldsymbol{x}$. Consequently, if r is an eigenvalue with linearly independent eigenvectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$, then both $\boldsymbol{v}^{(1)}e^{rt}$ and $\boldsymbol{v}^{(2)}e^{rt}$ are solutions to the linear system. Furthermore, since $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$ are linearly independent vectors, the Wronskien of $\boldsymbol{v}^{(1)}e^{rt}$ and $\boldsymbol{v}^{(2)}e^{rt}$ is always nonzero. This leads to the next theorem.

Theorem 8.2. *Suppose A is a 2×2 real matrix with constant entries that has exactly one real eigenvalue r. If r has two linearly independent eigenvectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$, then*

$$\boldsymbol{x}^{(1)} = \boldsymbol{v}^{(1)}e^{rt} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \boldsymbol{v}^{(2)}e^{rt}$$

form a fundamental set of solutions to the homogeneous linear system $\boldsymbol{x}' = A\boldsymbol{x}$.

Proof. This follows from the comments before the statement of the theorem, and so a full proof does not need to be given here. □

Example 8.3. Find the general solution to the system $\boldsymbol{x}' = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solution. From Example 8.1, we know that A has the one eigenvalue $r = 2$ that corresponds to the two linearly independent eigenvectors

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, by Theorem 8.2, the general solution to this system is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}.$$

A phase portrait for this example can be found in Figure 8.A.

Example 8.1 contained a 2×2 matrix A with one eigenvalue that had two linearly independent eigenvectors. In this case, it was easy to find a fundamental set of solutions to the associated differential system $\boldsymbol{x}' = A\boldsymbol{x}$.

It is possible, however, to find a 2×2 matrix with one real eigenvalue for which their does not exist two linearly independent eigenvectors.

Example 8.4. Show that the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ has one eigenvalue, and that there are not two linearly independent eigenvectors corresponding to this eigenvalue.

Solution. Observe that

$$\det(A - rI) = \begin{vmatrix} 2-r & -1 \\ 1 & -r \end{vmatrix} = (2-r)(-r) + 1 = r^2 - 2r + 1.$$

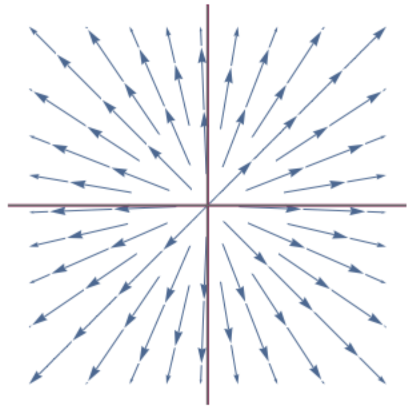


Figure 8.A: Phase portrait for Example 8.3.

The only zero of this polynomial is $r = 1$, and so A has only one eigenvalue.

To find eigenvectors corresponding to the eigenvalue $r = 1$, we solve the equation $(A - rI)\boldsymbol{v} = \mathbf{0}$. Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, as usual. Then

$$(A - rI)\boldsymbol{v} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This matrix equation is satisfied if $x = y$, and so the vector $\boldsymbol{v} = \begin{pmatrix} x \\ x \end{pmatrix}$ is an eigenvector for any choice of x, provided that it is nonzero. We pick $x = 1$, and consequently choose $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as our eigenvector corresponding to the eigenvalue $r = 1$. Since all eigenvectors are scalar multiples of $\boldsymbol{v}$, no two eigenvectors are linearly independent.

In Example 8.4, we found a matrix having one eigenvalue and no two linearly independent eigenvectors. We see that the matrix A has eigenvalue $r = 1$ with eigenvector $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Consequently, we know that the differential system $\boldsymbol{x}' = A\boldsymbol{x}$ has solution

$$\boldsymbol{x}^{(1)} = \boldsymbol{v}e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}.$$

However, in order to have a fundamental set of solutions, which is needed to find the general solution to the differential system, it is necessary to find a *second* solution to this system (and one that is not a scalar multiple of the solution we already have). We are left with the natural question: *"How do we find a second solution?"*

When solving a second order differential equation of the form

$$\alpha y'' + \beta y' + \gamma y = 0,$$

where α and β and γ are real numbers such that $\alpha \neq 0$, the first step is to solve the characteristic equation $\alpha r^2 + \beta r + \gamma = 0$. As we saw in Section 4, a 2×2 linear differential system with real constant coefficients can be reinterpreted as a second order linear differential equation with real constant coefficients (and *vice versa*).

When the characteristic equation for a second order linear differential equation with constant real coefficients had only one (repeated) real root r, we found that the general solution had the form

$$c_1 e^{rt} + c_2 t e^{rt}.$$

In other words, we multiplied the first solution by t to get the second solution. Perhaps the same strategy will work in our current context.

Statement of Problem: Suppose $\boldsymbol{x}' = A\boldsymbol{x}$ is a linear differential system where A is a 2×2 matrix with real (and constant) entires. Suppose also that A has one real eigenvalue r with eigenvector $\boldsymbol{v}$ and that there is no eigenvector which is not a scalar multiple of $\boldsymbol{v}$.

First Guess: Suppose the system has solutions $\boldsymbol{x}^{(1)} = \boldsymbol{v}e^{rt}$ and $\boldsymbol{x}^{(2)} = \boldsymbol{v}te^{rt}$.

We already know that $\boldsymbol{x}^{(1)}$ is a solution. We need to determine if $\boldsymbol{x}^{(2)}$ is also a solution. If it is, then it must be true that $\frac{d}{dt}\boldsymbol{x}^{(2)} = A\boldsymbol{x}^{(2)}$. Let's compute both sides of this equation and see if they are indeed equal:

$$\frac{d}{dt}\boldsymbol{x}^{(2)} = \frac{d}{dt}\left[\boldsymbol{v}te^{rt}\right] = \boldsymbol{v}e^{rt} + \boldsymbol{v}rte^{rt},$$

and

$$A\boldsymbol{x}^{(2)} = A(\boldsymbol{v}te^{rt}) = (A\boldsymbol{v})te^{rt} = (r\boldsymbol{v})te^{rt} = \boldsymbol{v}rte^{rt}.$$

Thus, in order for these two quantities to be the same (for all values of t), it must be true that

$$\boldsymbol{v}e^{rt} + \boldsymbol{v}rte^{rt} = \boldsymbol{v}rte^{rt} \quad \text{or} \quad \boldsymbol{v}e^{rt} = \boldsymbol{0}.$$

The exponential e^{rt} is never zero, and $\boldsymbol{v} \neq \boldsymbol{0}$ because it is an eigenvector, and eigenvectors are nonzero (by assumption).

We see, then, that our first guess was not quite correct. However, we can perhaps learn something from our mistake. If we set $\boldsymbol{x}$ equal to a term that looks like a vector times te^{rt}, then that terms appears on both sides of the equation $\boldsymbol{x}' = A\boldsymbol{x}$. The problem is that on one side we also get a term that looks like a vector times e^{rt}, so perhaps we should include an e^{rt} term in our guess.

Second Guess: Suppose the solutions are $\boldsymbol{x}^{(1)} = \boldsymbol{v}e^{rt}$ and $\boldsymbol{x}^{(2)} = \boldsymbol{v}te^{rt} + \boldsymbol{w}e^{rt}$.

In this guess, we suppose there is some vector $\boldsymbol{w}$ so that $\boldsymbol{x}^{(2)} = \boldsymbol{v}te^{rt} + \boldsymbol{w}e^{rt}$, but at this point, we do not make any assumptions on what $\boldsymbol{w}$ looks like. The vector $\boldsymbol{w}$ is *undetermined* at this point. We assume that there is some $\boldsymbol{w}$ so that this choice of $\boldsymbol{x}^{(2)}$ is a solution and then determine what $\boldsymbol{w}$ must look like in order for that assumption to be true.

Once again, we compute both sides of the equation $\frac{d}{dt}\boldsymbol{x}^{(2)} = A\boldsymbol{x}^{(2)}$, but this time we have some flexability, because we can choose $\boldsymbol{w}$ to be whatever is needed to make the both sides equal. First, we compute both sides:

$$\frac{d}{dt}\boldsymbol{x}^{(2)} = \frac{d}{dt}\left[\boldsymbol{v}te^{rt} + \boldsymbol{w}e^{rt}\right] = \boldsymbol{v}e^{rt} + \boldsymbol{v}rte^{rt} + \boldsymbol{w}re^{rt},$$

and

$$A\boldsymbol{x}^{(2)} = A(\boldsymbol{v}te^{rt} + \boldsymbol{w}e^{rt}) = (A\boldsymbol{v})te^{rt} + (A\boldsymbol{w})e^{rt} = \boldsymbol{v}rte^{rt} + (A\boldsymbol{w})e^{rt}.$$

Thus, in order for these two quantities to be the same (for all values of t), it must be true that

$$\boldsymbol{v}e^{rt} + \boldsymbol{v}rte^{rt} + \boldsymbol{w}re^{rt} = \boldsymbol{v}rte^{rt} + (A\boldsymbol{w})e^{rt},$$

which simplifies to

$$(\boldsymbol{v} + r\boldsymbol{w})e^{rt} = (A\boldsymbol{w})e^{rt}.$$

As before, the exponential term e^{rt} is never zero, and so we conclude that

$$\boldsymbol{v} + r\boldsymbol{w} = A\boldsymbol{w} \quad \text{or} \quad A\boldsymbol{w} - r\boldsymbol{w} = \boldsymbol{v}.$$

Therefore, in order for this choice of $\boldsymbol{x}^{(2)}$ to be a solution, we must find a vector $\boldsymbol{w}$ so that

$$(A - rI)\boldsymbol{w} = \boldsymbol{v}.$$

Definition 8.5. Suppose A is a square matrix and r is an eigenvalue of A with corresponding eigenvector $\boldsymbol{v}$. A nonzero vector $\boldsymbol{w}$ satisfying the equation $(A - rI)\boldsymbol{w} = \boldsymbol{v}$ is called a *generalized eigenvector*.

Remark 8.6. If A is a square matrix having eigenvalue r with eigenvector $\boldsymbol{v}$, then

$$(A - rI)\boldsymbol{v} = \boldsymbol{0}.$$

A generalized eigenvector is a vector $\boldsymbol{w}$ that satisfies the equation

$$(A - rI)\boldsymbol{w} = \boldsymbol{v}.$$

Consequently, if we multiply $\boldsymbol{w}$ by the matrix $(A - rI)$ twice, we have

$$(A - rI)^2\boldsymbol{w} = (A - rI)\boldsymbol{v} = \boldsymbol{0}.$$

Our generalized eigenvector $\boldsymbol{w}$ is, in a sense, one "step" removed from an eigenvector. We have to multiply it by $(A - rI)$ twice to get $\boldsymbol{0}$ instead of just once.

Based on our work so far, we can conclude that our "Second Guess" provides a fundamental set of solutions, provided that the undetermined vector $\boldsymbol{w}$ is a generalized eigenvector. We can, therefore, state the following theorem, which answers the question we asked after Example 8.4.

Theorem 8.7. *Suppose A is a 2×2 real matrix with constant entries that has exactly one real eigenvalue r. If r does not have two linearly independent eigenvectors, then*

$$\boldsymbol{x}^{(1)} = \boldsymbol{v}e^{rt} \quad \textit{and} \quad \boldsymbol{x}^{(2)} = \boldsymbol{v}te^{rt} + \boldsymbol{w}e^{rt}$$

form a fundamental set of solutions to the homogeneous linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where $\boldsymbol{v}$ is an eigenvector and $\boldsymbol{w}$ is a generalized eigenvector.

Proof. This follows from the comments before the statement of the theorem, and so a full proof does not need to be given here. □

Example 8.8. Find the general solution to the system $\boldsymbol{x}' = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution. From Example 8.4, we know that A has only one eigenvalue, and that eigenvalue is $r = 1$. We also found that an eigenvector for the eigenvalue $r = 1$ is

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It remains to find a generalized eigenvector $\boldsymbol{w}$. Let $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and solve the equation $(A - rI)\boldsymbol{w} = \boldsymbol{v}$:

$$\begin{pmatrix} 2-r & -1 \\ 1 & -r \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This matrix equation reduces to the scalar equation $w_1 - w_2 = 1$, or equivalently $w_1 = 1 + w_2$. Therefore,

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 + w_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_2.$$

We may choose any value we wish for w_2 so long as $\boldsymbol{w}$ is not the zero vector. There is, in fact, no value of w_2 for which $\boldsymbol{w} = \boldsymbol{0}$, and so we may choose w_1 to be any convenient value. Therefore, we choose $w_2 = 0$, and so

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In conclusion, by Theorem 8.7, the general solution to this system is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right].$$

A phase portrait for this example can be found in Figure 8.B.

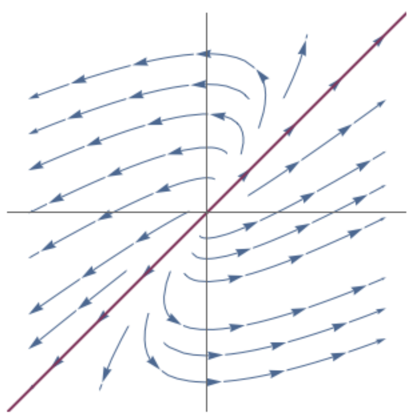


Figure 8.B: Phase portrait for Example 8.8.

Remark 8.9. In Example 8.8, we found that a generalized eigenvector had the form

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where k is any scalar. Note that the eigenvector $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ appears in this expression, so that in fact

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\boldsymbol{v}.$$

Thus, by Theorem 8.7, the general solution to this system is

$$\begin{aligned} \boldsymbol{x} = c_1\boldsymbol{v}e^t + c_2(\boldsymbol{v}te^t + \boldsymbol{w}e^t) &= c_1\boldsymbol{v}e^t + c_2\left(\boldsymbol{v}te^t + \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\boldsymbol{v}\right]e^t\right) \\ &= (c_1 + kc_2)\boldsymbol{v}e^t + c_2\left[\boldsymbol{v}te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t\right]. \end{aligned}$$

The constants c_1 and c_2 and k appearing in the above general solution are all completely arbitrary. Therefore, we can rename $c_1 + kc_2$ as c_1, and we are left with

$$\boldsymbol{x} = c_1\boldsymbol{v}e^t + c_2\left[\boldsymbol{v}te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t\right],$$

which is the form of the general solution we obtained in Example 8.8.

In Example 8.8, we made the choice $k = 0$. We are allowed to choose $k = 0$ because we are looking for a set of linearly independent vectors $\{\boldsymbol{v}, \boldsymbol{w}\}$. The vector $k\boldsymbol{v}$ is a scalar multiple of $\boldsymbol{v}$, and so that part of $\boldsymbol{w}$ is redundant. That is, we already have a vector that takes care of that, and so we don't need k to be nonzero.

Example 8.10. Find the solution to the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Solution. Let A be the matrix defining the linear differential system. We first find the eigenvalues of A by determining the zeros of the characteristic polynomial:

$$\det(A - rI) = \begin{vmatrix} 2-r & 3 \\ 0 & 2-r \end{vmatrix} = (2-r)^2.$$

This polynomial has a double root of $r = 2$, and thus there is only one real eigenvalue, which is $r = 2$.

The next step is to compute all eigenvectors corresponding to the eigenvalue $r = 2$ by solving the equation $(A - 2I)\boldsymbol{v} = \boldsymbol{0}$. Let $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, as usual, and compute:

$$(A - 2I)\boldsymbol{v} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 0 \end{pmatrix}.$$

This vector is the zero vector when x is any real number and $y = 0$. Thus, all eigenvectors have the form

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}x,$$

where $x \neq 0$. We choose $x = 1$ and let $\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

There are not two linearly independent eigenvectors corresponding to the eigenvalue $r = 2$, and so we must compute a generalized eigenvector by finding a vector $\boldsymbol{w}$ such that $(A-2I)\boldsymbol{w} = \boldsymbol{v}$. Let $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and compute:

$$(A-2I)\boldsymbol{w} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 3w_2 \\ 0 \end{pmatrix}.$$

We require that this vector be equal to $\boldsymbol{v}$, and so we want to choose w_1 and w_2 so that

$$\begin{pmatrix} 3w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consequently, we must choose $w_2 = \frac{1}{3}$. We may choose w_1 to be any real number, and thus we pick $w_1 = 0$. Therefore,

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix},$$

and hence the general solution to this linear differential system is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} e^t \right].$$

It remains to determine which constants c_1 and c_2 are needed for this solution to satisfy the initial condition. Observe that

$$\boldsymbol{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2/3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Consequently, we must have $c_1 = 2$ and $c_2 = 6$, and so the solution to the initial value problem is

$$\boldsymbol{x} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 6 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} e^t \right],$$

which we may simplify as

$$\boldsymbol{x} = \begin{pmatrix} 2e^t + 6te^t \\ 2e^t \end{pmatrix}.$$

A phase portrait for this example can be found in Figure 8.C. The solution to the initial value problem is indicated by the dark curve.

Remark 8.11. If $\boldsymbol{v}$ is an eigenvector for the matrix A corresponding to the eigenvalue r, then so is any nonzero scalar multiple $k\boldsymbol{v}$. For this reason, we may choose k to be any convenient nonzero value. When finding a generalized eigenvector $\boldsymbol{w}$, however, we have less choice. Given the eigenvector $\boldsymbol{v}$, we must choose $\boldsymbol{w}$ so that is satisfies the equation $(A - rI)\boldsymbol{w} = \boldsymbol{v}$. In other words, the choice of $\boldsymbol{w}$ is determined by the choice of $\boldsymbol{v}$. This is why in Example 8.10, there is a $1/3$ in the generalized eigenvector $\boldsymbol{w}$. If we want our $\boldsymbol{w}$ to have integer entries, then it is necessary to make a different choice for $\boldsymbol{v}$.

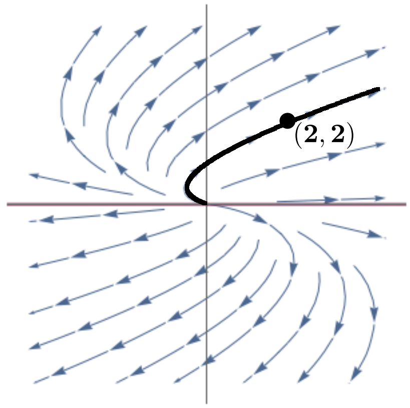


Figure 8.C: Phase portrait for Example 8.10.

The computation of c_1 and c_2 in Example 8.10 was made much easier by the fact that the initial conditions were given at $t = 0$. It is possible, however, for a different value of t to be used when specifying initial conditions.

Example 8.12. Find the solution to the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(2) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Solution. Again, let A be the matrix defining the linear differential system. This is the same linear differential system from Example 8.10, and thus the general solution is

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} e^t \right].$$

The difference between this example and Example 8.10 is that we now have a different initial condition. We wish to determine that constants c_1 and c_2 that are needed for the general solution to satisfy the given initial condition. Observe that

$$\boldsymbol{x}(1) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^2 + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} 2e^2 + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} e^2 \right] = \begin{pmatrix} c_1 + 2c_2 \\ c_2/3 \end{pmatrix} e^2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Consequently, we must have $c_1 = 4e^{-2}$ and $c_2 = -3e^{-2}$, and so the solution to the initial value problem is

$$\boldsymbol{x} = 4e^{-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t - 3e^{-2} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} e^t \right],$$

which we may simplify as

$$\boldsymbol{x} = \begin{pmatrix} 4 - 3t \\ -1 \end{pmatrix} e^{t-2}.$$

Section 9

Nonhomogeneous Linear Systems

In this final section, we consider a 2×2 linear differential system of the form

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t),$$

where A is a constant matrix with real entries (not all zero) and $\boldsymbol{g}(t)$ is a real vector for each t such that $\boldsymbol{g}(t)$ is not always the zero vector. This type of linear differential system is known as a *nonhomogeneous linear system*.

We will consider two methods for solving nonhomogeneous linear systems, each mimicking the methods employed when solving nonhomogeneous second order differential equations.

Method 1: Integrating Factors

We will consider two different cases, one where the matrix A has two linearly independent eigenvectors and one where A does not have two linearly independent eigenvectors.

Case 1: A has two linearly independent eigenvectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$.

Let $T = \begin{pmatrix} \boldsymbol{v}^{(1)} & \boldsymbol{v}^{(2)} \end{pmatrix}$ be the matrix formed by using the two eigenvectors as columns.

Proposition 9.1. *If r_1 and r_2 are the eigenvalues corresponding to the eigenvectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$, respectively, then*

$$T^{-1}AT = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

(We allow the possibility that $r_1 = r_2$.)

Proof. We know that T is invertible, because the columns are assumed to be linearly independent. Furthermore, since $A\boldsymbol{v}^{(1)} = r_1\boldsymbol{v}^{(1)}$ and $A\boldsymbol{v}^{(2)} = r_2\boldsymbol{v}^{(2)}$, it follows that

$$AT = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} T,$$

and so

$$T^{-1}AT = T^{-1}\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} T = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

□

Define a vector $\boldsymbol{y}$ by setting $\boldsymbol{y} = T^{-1}\boldsymbol{x}$, so that $\boldsymbol{x} = T\boldsymbol{y}$. Then the linear system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$ can be rewritten as

$$T\boldsymbol{y}' = AT\boldsymbol{y} + \boldsymbol{g}(t).$$

If we multiply both sides of this equation by T^{-1} on the left, we have

$$T^{-1}(T\boldsymbol{y}') = T^{-1}(AT)\boldsymbol{y} + T^{-1}\boldsymbol{g}(t),$$

or

$$\boldsymbol{y}' = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \boldsymbol{y} + \underbrace{T^{-1}\boldsymbol{g}(t)}_{\boldsymbol{h}(t)}.$$

For computational purposes, let $\boldsymbol{h}(t) = T^{-1}\boldsymbol{g}(t)$, and let

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}.$$

With these notations, we can write our linear system as follows:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} r_1 y_1 + h_1(t) \\ r_2 y_2 + h_2(t) \end{pmatrix}.$$

This gives rise to two first-order differential equations:

$$\begin{cases} y_1' = r_1 y_1 + h_1(t), \\ y_2' = r_2 y_2 + h_2(t). \end{cases}$$

These two differential equations can be solved using integrating factors. This allows us to solve for $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and then we can determine $\boldsymbol{x}$ using the relationship $\boldsymbol{x} = T\boldsymbol{y}$.

Example 9.2. Solve the nonhomogeneous linear system

$$\boldsymbol{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{pmatrix}, \quad t > 0.$$

Solution. Let A be the matrix that defines this linear system. The eigenvalues for A are $r_1 = 0$ and $r_2 = -5$. We choose corresponding eigenvectors

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

for r_1 and r_2, respectively. Then, the complementary solution (that is, the general solution to the homogeneous system $\boldsymbol{x}' = A\boldsymbol{x}$) is

$$\boldsymbol{x}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}.$$

We let T be the matrix having the eigenvectors as columns. That is, let

$$T = \begin{pmatrix} \boldsymbol{v}^{(1)} & \boldsymbol{v}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Next, let $\boldsymbol{y} = T^{-1}\boldsymbol{x}$ and let

$$\boldsymbol{h}(t) = T^{-1}\boldsymbol{g}(t) = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} + \frac{8}{5} \\ \frac{4}{5} \end{pmatrix}.$$

In order to solve the system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$, we first solve the system $\boldsymbol{y}' = (T^{-1}AT)\boldsymbol{y} + \boldsymbol{h}(t)$. That is, we solve the system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}}_{r_1=0, r_2=-5} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{t} + \frac{8}{5} \\ \frac{4}{5} \end{pmatrix}.$$

This can be written as the following system of equations:

$$\begin{cases} y_1' = \frac{1}{t} + \frac{8}{5}, \\ y_2' = -5y_2 + \frac{4}{5}. \end{cases}$$

The first of these equations can be solved through simple antidifferentiation:

$$y_1 = \ln(t) + \frac{8}{5}t + C_1$$

where C_1 is an arbitrary constant. The second equation can be solved by rewriting the equation as

$$y_2' + 5y_2 = \frac{4}{5}$$

and using an integrating factor of e^{5t}:

$$e^{5t}y_2' + 5e^{5t}y_2 = \frac{4}{5}e^{5t}.$$

This equation is equivalent to

$$\frac{d}{dt}\left[e^{5t}y_2\right] = \frac{4}{5}e^{5t}.$$

We may therefore simply compute an antiderivative to see that

$$e^{5t}y_2 = \frac{4}{25}e^{5t} + C_2,$$

where C_2 is an arbitrary constant. Thus, solving for y_2, we have that

$$y_2 = \frac{4}{25} + C_2e^{-5t}.$$

We need only one solution to the nonhomogeneous equations, and so we may choose $C_1 = 0$ and $C_2 = 0$. Therefore,

$$\boldsymbol{x} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \ln(t) + \frac{8}{5}t \\ \frac{4}{25} \end{pmatrix}.$$

Consequently,

$$\boldsymbol{x} = T\boldsymbol{y} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \ln(t) + \frac{8}{5}t \\ \frac{4}{25} \end{pmatrix} = \begin{pmatrix} \ln(t) + \frac{8}{5}t - \frac{8}{25} \\ 2\ln(t) + \frac{16}{5}t + \frac{4}{25} \end{pmatrix}.$$

This choice of $\boldsymbol{x}$ is a solution to the nonhomogeneous system. We call it a particular solution and label it $\boldsymbol{x}_p$. Observe that

$$\boldsymbol{x}_p = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln(t) + \frac{8}{5}t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{4}{25} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Therefore, the general solution to the nonhomogeneous linear differential system is

$$\boldsymbol{x} = \underbrace{c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}}_{\boldsymbol{x}_c} + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln(t) + \frac{8}{5}t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{4}{25} \begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\boldsymbol{x}_p}.$$

Remark 9.3. In the solution for Example 9.6, we wrote the antiderivative of $\frac{1}{t}$ as $\ln(t)$, without making use of absolute value signs in the argument of the natural logarithm function. In the statement of the problem, it was assumed that $t > 0$, so $\ln(|t|) = \ln(t)$. Note that it was necessary to restrict t to either the interval $(0, \infty)$ or $(-\infty, 0)$, because the components of g would not be continuous on any interval containing $t = 0$ and the solution would not be valid there.

Case 2: A has only one eigenvector $\boldsymbol{v}$.

When we say A has only one eigenvector, we mean that there are not two linearly independent eigenvectors (since if $\boldsymbol{v}$ is an eigenvector, then so too is $k\boldsymbol{v}$ for all nonzero scalars k). This situation can only occur if the matrix A has one real eigenvalue. Let r be this eigenvalue. Let $\boldsymbol{v}$ be an eigenvector for r and let $\boldsymbol{w}$ be a generalized eigenvector for r.

Proposition 9.4. *If r is an eigenvalue with corresponding eigenvector $\boldsymbol{v}$ and generalized eigenvector $\boldsymbol{w}$, then*

$$T^{-1}AT = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix},$$

where $T = \begin{pmatrix} \boldsymbol{v} & \boldsymbol{w} \end{pmatrix}$ is the matrix formed by using the eigenvector and generalized eigenvector as columns.

Example 9.5. Verify Proposition 9.4 for the matrix

$$A = \begin{pmatrix} -3 & -2 \\ 2 & -7 \end{pmatrix}.$$

Solution. First, we find the eigenvalues:

$$\det(A - rI) = \begin{vmatrix} -3-r & -2 \\ 2 & -7-r \end{vmatrix} = r^2 + 10r + 25.$$

This polynomial has one real root, which is $r = -5$, and so the matrix A has $r = -5$ as its only eigenvalue. To find the corresponding eigenvectors, we solve the equation $(A - 5I)\boldsymbol{v} = \mathbf{0}$:

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $v_2 = v_1$, and so

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1.$$

We pick $v_1 = 1$, and so our chosen eigenvector is $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To find a generalized eigenvector, we solve $(A - 5I)\boldsymbol{w} = \boldsymbol{v}$ for $\boldsymbol{w}$:

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It follows that $2w_1 - 2w_2 = 1$, and so $w_1 = \frac{1}{2} + w_2$. Thus,

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + w_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_2.$$

We pick $w_2 = 0$, and so our generalized eigenvector is $\boldsymbol{w} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$.

Now that we have $\boldsymbol{v}$ and $\boldsymbol{w}$, we can construct the matrix T and compute its inverse:

$$T = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}.$$

It remains only to show that $T^{-1}AT$ has the form promised by Proposition 9.4. We compute directly:

$$T^{-1}AT = \underbrace{\begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 2 & -7 \end{pmatrix}}_{\begin{pmatrix} 2 & -7 \\ -10 & 10 \end{pmatrix}} \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix}.$$

This is the desired matrix, and so Proposition 9.4 has been verified in this case.

We wish to solve the nonhomogeneous system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$. The method is similar to that used in Case 1, above. Once again, let $\boldsymbol{y} = T^{-1}\boldsymbol{x}$ and let $\boldsymbol{h}(t) = T^{-1}\boldsymbol{g}(t)$. Then the linear system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$ can be rewritten as

$$\boldsymbol{y}' = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix} \boldsymbol{y} + \boldsymbol{h}(t).$$

If we let

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix},$$

then we can write our linear system as two first-order differential equations:

$$\begin{cases} y_1' = ry_1 + y_2 + h_1(t), \\ y_2' = \qquad ry_2 + h_2(t). \end{cases}$$

We can solve the bottom equation for y_2 and then substitute this solution into the top equation and solve the resulting equation for y_1. Then we compute $\boldsymbol{x}$ using the relationship $\boldsymbol{x} = T\boldsymbol{y}$.

Example 9.6. Find a particular solution for the nonhomogeneous linear system

$$\boldsymbol{x}' = \begin{pmatrix} -3 & -2 \\ 2 & -7 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} t \\ 6 \end{pmatrix}.$$

Solution. Let A be the matrix that defines this linear system. From Example 9.5, we know that $r = -5$ is the only eigenvalue of A and that it corresponds to an eigenvector $\boldsymbol{v}$ and a generalized eigenvector $\boldsymbol{w}$, where

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

Then

$$T = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}.$$

We will solve the linear system

$$\boldsymbol{y}' = \begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix} \boldsymbol{y} + \boldsymbol{h}(t),$$

where $\boldsymbol{y} = T^{-1}\boldsymbol{x}$ and

$$\boldsymbol{h}(t) = T^{-1}\boldsymbol{g}(h) = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} t \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 2t - 12 \end{pmatrix}.$$

We rewrite our linear system as two first-order differential equations:

$$\begin{cases} y_1' = -5y_1 + \ \ y_2 + 6, \\ y_2' = \qquad\quad - 5y_2 + 2t - 12. \end{cases}$$

Both of these first order linear differential equations can be solved using an integrating factor of e^{5t}. First, we solve the bottom equation for y_2 to get

$$y_2 = \frac{1}{5}(2t - 12) - \frac{2}{25} + C_2 e^{-5t}.$$

We choose the constant $C_2 = 0$ and substitute into the top equation to get the differential equation

$$y_1' = -5y_1 + \frac{2}{5}t + \frac{88}{25}.$$

Again we can use an integrating factor of e^{5t} to solve this differential equation:

$$y_1 = \frac{2}{25}t + \frac{86}{125} + C_1 e^{-5t}.$$

We pick $C_1 = 0$ and conclude that

$$\begin{cases} y_1 = \frac{2}{25}t + \frac{86}{125}, \\ y_2 = \frac{2}{5}t - \frac{62}{25}. \end{cases}$$

Therefore,

$$\boldsymbol{x} = T\boldsymbol{y} = \begin{pmatrix} 1 & 1/2 \\ \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{25}t + \frac{86}{125} \\ \\ \frac{2}{5}t - \frac{62}{25} \end{pmatrix} = \begin{pmatrix} \frac{7}{25}t - \frac{69}{125} \\ \\ \frac{2}{25}t + \frac{86}{125} \end{pmatrix},$$

and so

$$\boldsymbol{x} = \frac{1}{25}\begin{pmatrix} 7 \\ 2 \end{pmatrix} t + \frac{1}{125}\begin{pmatrix} -69 \\ 86 \end{pmatrix}.$$

This is a solution to the nonhomogeneous system, and so this can be the particular solution we were asked to find.

Method 2: Undetermined Coefficients

We can use the method of undetermined coefficients provided the entries of $\boldsymbol{g}(t)$ are polynomial, exponential, sinusoidal, or sums or products of these types. The method is the same as the one used to solve linear differential equations, but now the coefficients of terms in the particular solution $\boldsymbol{x}_p$ are undetermined *vectors* instead of scalars.

Exception: If the nonhomogeneous term has the form $\boldsymbol{u}e^{rt}$, where r is an eigenvalue of the matrix A, then the guess should be if the form

$$\boldsymbol{x}_p = \boldsymbol{a}te^{rt} + \boldsymbol{b}e^{rt}.$$

Example 9.7. Solve the nonhomogeneous linear system

$$\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

Solution. As usual, we let A be the matrix defining this linear differential system. The eigenvalues for this matrix are $r_1 = 1$ and $r_2 = -1$ and the corresponding eigenvectors are (respectively)

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Consequently, the complementary solution (which is the general solution to the homogeneous system $\boldsymbol{x}' = A\boldsymbol{x}$) is

$$\boldsymbol{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

We now make a "guess" for our particular solution $\boldsymbol{x}_p$. Because

$$\boldsymbol{g}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t,$$

our first guess might be along the lines of $\boldsymbol{x}_p = \boldsymbol{a}e^t$. However, one of our eigenvalues is $r = 1$, and so this guess is not satisfactory. Instead, we let

$$\boldsymbol{x}_p = \boldsymbol{a}te^t + \boldsymbol{b}e^t,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are undetermined vectors.

We need to determine the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ so that $\boldsymbol{x}_p$ satisfies the relationship $\boldsymbol{x}_p' = A\boldsymbol{x}_p + \boldsymbol{g}(t)$. To that end, we compute $\boldsymbol{x}_p'$ and $A\boldsymbol{x}_p + \boldsymbol{g}(t)$:

$$\boldsymbol{x}_p' = \boldsymbol{a}e^t + \boldsymbol{a}te^t + \boldsymbol{b}e^t,$$

and

$$A\boldsymbol{x}_p + \boldsymbol{g}(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \left(\boldsymbol{a}te^t + \boldsymbol{b}e^t\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

Let

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Then

$$\boldsymbol{x}_p' = \begin{pmatrix} a_1 + a_1 t + b_1 \\ a_2 + a_2 t + b_2 \end{pmatrix} e^t$$

and

$$\begin{aligned} A\boldsymbol{x}_p + \boldsymbol{g}(t) &= \begin{pmatrix} 2a_1 - a_2 \\ 3a_1 - 2a_2 \end{pmatrix} te^t + \begin{pmatrix} 2b_1 - b_2 \\ 3b_1 - 2b_2 \end{pmatrix} e_t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t \\ &= \begin{pmatrix} 2a_1 t - a_2 t + 2b_1 - b2 + 1 \\ 3a_1 t - 2a_2 t + 3b_1 - 2b_2 - 1 \end{pmatrix} e^t. \end{aligned}$$

Since it must be true that $\boldsymbol{x}_p' = A\boldsymbol{x}_p + \boldsymbol{g}(t)$, we have the following system of two equations:

$$\begin{cases} a_1 + a_1 t + b_1 = 2a_1 t - a_2 t + 2b_1 - b2 + 1, \\ a_2 + a_2 t + b_2 = 3a_1 t - 2a_2 t + 3b_1 - 2b_2 - 1. \end{cases}$$

We next rewrite this, grouping together "like" terms:

$$\begin{cases}(-a_1+a_2)t+(a_1-b_1+b_2-1)=0,\\ (-3a_1+3a_2)t+(a_2-3b_1+3b_2+1)=0.\end{cases}$$

Solving this system, we find that

$$a_1=2,\quad a_2=2,\quad b_1=1+x,\quad b_2=x,$$

where x is a free parameter. We may choose x to be any real number, so we pick $x=0$. This results in $b_1=1$ and $b_2=0$, and so

$$\boldsymbol{a}=\begin{pmatrix}2\\2\end{pmatrix}\quad\text{and}\quad\boldsymbol{b}=\begin{pmatrix}1\\0\end{pmatrix}.$$

We conclude that our particular solution is

$$\boldsymbol{x}_p=\begin{pmatrix}2\\2\end{pmatrix}te^t+\begin{pmatrix}1\\0\end{pmatrix}e^t,$$

and consequently,

$$\boldsymbol{x}=\underbrace{c_1\begin{pmatrix}1\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}}_{\boldsymbol{x}_c}+\underbrace{2\begin{pmatrix}1\\1\end{pmatrix}te^t+\begin{pmatrix}1\\0\end{pmatrix}e^t}_{\boldsymbol{x}_p}.$$

Remark 9.8. In Example 9.7, we solved a system of equations to determine the possible entries for our undetermined vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We found that

$$a_1=2,\quad a_2=2,\quad b_1=1+x,\quad b_2=x,$$

where x is a free parameter. We are allowed to choose x to be any real number, and so we chose $x=0$. However, in order to understand better why we are able to chose $x=0$, let us leave x as a free parameter. Then

$$\boldsymbol{a}=\begin{pmatrix}2\\2\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix}=2\boldsymbol{v}^{(1)}$$

and

$$\boldsymbol{b}=\begin{pmatrix}1+x\\x\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}x=\begin{pmatrix}1\\0\end{pmatrix}+x\boldsymbol{v}^{(1)}.$$

Thus, the particular solution has the form

$$x_p=2te^t\boldsymbol{v}^{(1)}+\begin{pmatrix}1\\0\end{pmatrix}e^t+x\boldsymbol{v}^{(1)}e^t.$$

Therefore, the general solution has the form

$$\boldsymbol{x}=\underbrace{c_1\boldsymbol{v}^{(1)}e^t+c_2\boldsymbol{v}^{(2)}e^{-t}}_{\boldsymbol{x}_c}+\underbrace{2te^t\boldsymbol{v}^{(1)}+\begin{pmatrix}1\\0\end{pmatrix}e^t+x\boldsymbol{v}^{(1)}e^t}_{\boldsymbol{x}_p}.$$

Note that $\boldsymbol{x}_c$ and $\boldsymbol{x}_p$ both contain a term of the form

$$(\text{constant})\times\boldsymbol{v}^{(1)}e^t.$$

If we combine these terms, then we have general solution

$$\boldsymbol{x} = \underbrace{(c_1 + x)}_{\widetilde{c_1}} \boldsymbol{v}^{(1)} e^t + c_2 \boldsymbol{v}^{(2)} e^{-t} + 2te^t \boldsymbol{v}^{(1)} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

Since c_1 is an arbitrary constant, so too is $\widetilde{c_1} = c_1 + x$. If we relabel $\widetilde{c_i}$ as c_1, then we have the same general solution as we found in Example 9.7.

The basic idea behind what we see happen here is that the part of the particular solution that looks like $x\boldsymbol{v}^{(1)}e^t$ does not contribute anything new to the general solution to the nonhomogeneous system. In fact, the vector function $x\boldsymbol{v}^{(1)}e^t$ is a solution to the homogeneous system, and so it is already included as part of the complementary solution $\boldsymbol{x}_c$.

If the components of $\boldsymbol{g}(t)$ are sums of functions, then $\boldsymbol{g}(t)$ can be rewritten as the sum of vectors. This allows us to write a guess for $\boldsymbol{x}_p$ that is a sum of guesses for each vector.

Example 9.9. Determine an appropriate "guess" for $\boldsymbol{x}_p$, if

$$\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e^t \\ -e^t + t \end{pmatrix}.$$

Solution. Write

$$\boldsymbol{g}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$$

Then, we choose $\boldsymbol{x}_p$ to be the following

$$\boldsymbol{x}_p = \Big[\boldsymbol{a}te^t + \boldsymbol{b}e^t\Big] + \Big[\boldsymbol{c}t + \boldsymbol{d}\Big],$$

where the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c},$ and $\boldsymbol{d}$ are undetermined.

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