

# **ME 130 Applied Engineering Analysis**

## **Chapter 3**

# **Application of First Order Differential Equations in Mechanical Engineering Analysis**

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# Chapter Outlines

- Review solution method of first order ordinary differential equations
- Applications in fluid dynamics
  - Design of containers and funnels
- Applications in heat conduction analysis
  - Design of heat spreaders in microelectronics
- Applications in combined heat conduction and convection
  - Design of heating and cooling chambers
- Applications in rigid-body dynamic analysis

# Part 1

## Review of Solution Methods for First Order Differential Equations

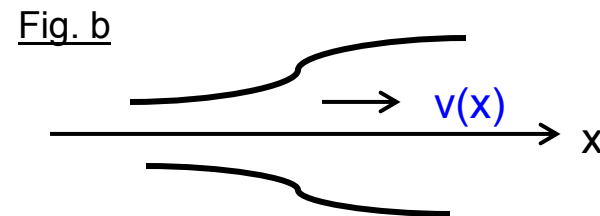
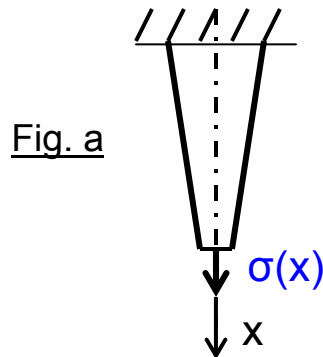
In “real-world,” there are many physical quantities that can be represented by functions involving **only one of the four variables** e.g., (x, y, z, t)

Equations involving highest order derivatives of order one = **1<sup>st</sup> order differential equations**

### Examples:

Function  $\sigma(x)$  = the stress in a uni-axial stretched tapered metal rod (Fig. a), or

Function  $v(x)$  = the velocity of fluid flowing a straight channel with varying cross-section (Fig. b):



Mathematical modeling using differential equations involving these functions are classified as **First Order Differential Equations**

## Solution Methods for First Order ODEs

### A. Solution of linear, **homogeneous** equations (p.48):

Typical form of the equation:

$$\frac{du(x)}{dx} + p(x)u(x) = 0 \quad (3.3)$$

The **solution**  $u(x)$  in Equation (3.3) is:

$$u(x) = \frac{K}{F(x)} \quad (3.4)$$

where  $K$  = constant to be determined by given condition, and the function  $F(x)$  has the form:

$$F(x) = e^{\int p(x)dx} \quad (3.5)$$

in which the function  $p(x)$  is given in the differential equation in Equation (3.3)

**B. Solution of linear, Non-homogeneous equations (P. 50):**

Typical differential equation:

$$\frac{du(x)}{dx} + p(x)u(x) = g(x) \quad (3.6)$$

The appearance of function  $g(x)$  in Equation (3.6) makes the DE **non-homogeneous**

The solution of ODE in Equation (3.6) is similar by a little more complex than that for the homogeneous equation in (3.3):

$$u(x) = \frac{1}{F(x)} \int F(x) g(x) dx + \frac{K}{F(x)} \quad (3.7)$$

Where function  $F(x)$  can be obtained from Equation (3.5) as:

$$F(x) = e^{\int p(x) dx}$$

**Example** Solve the following differential equation (p. 49):

$$\frac{du(x)}{dx} - (\sin x)u(x) = 0 \quad (a)$$

with condition  $u(0) = 2$

**Solution:**

By comparing terms in Equation (a) and (3.6), we have:  $p(x) = -\sin x$  and  $g(x) = 0$ .

Thus by using Equation (3.7), we have the solution:

$$u(x) = \frac{K}{F(x)}$$

in which the function  $F(x)$  is:  $F(x) = e^{\int p(x)dx} = e^{\cos x}$ , leading to the solution:

$$u(x) = Ke^{-\cos x}$$

Since the given condition is  $u(0) = 2$ , we have:  $2 = Ke^{-\cos(0)} = K(e^{-1}) = \frac{K}{e} = \frac{K}{2.7183}$

, or  $K = 5.4366$ . Hence the solution of Equation (a) is:

$$u(x) = 5.4366 e^{-\cos x}$$

**Example:** solve the following differential equation (p. 51):

$$\frac{du(x)}{dx} + 2u(x) = 2 \quad (a)$$

with the condition:  $u(0) = 2$  (b)

**Solution:**

By comparing the terms in Equation (a) and those in Equation (3.6), we will have:  $p(x) = 2$  and  $g(x) = 2$ , which leads to:

$$F(x) = e^{\int p(x)dx} = e^{\int 2dx} = e^{2x}$$

By using the solution given in Equation (3.7), we have:

$$u(x) = \frac{1}{F(x)} \int F(x) g(x) dx + \frac{K}{F(x)} = \frac{1}{e^{2x}} \int (e^{2x})(2) dx + \frac{K}{e^{2x}} = 1 + \frac{K}{e^{2x}}$$

We will use the condition given in (b) to determine the constant K:

$$u(x) \Big|_{x=0} = 2 = 1 + \frac{K}{e^{2x}} \Big|_{x=0} = 1 + K \rightarrow K = 1$$

Hence, the solution of Equation (a) with the condition in (b) is:

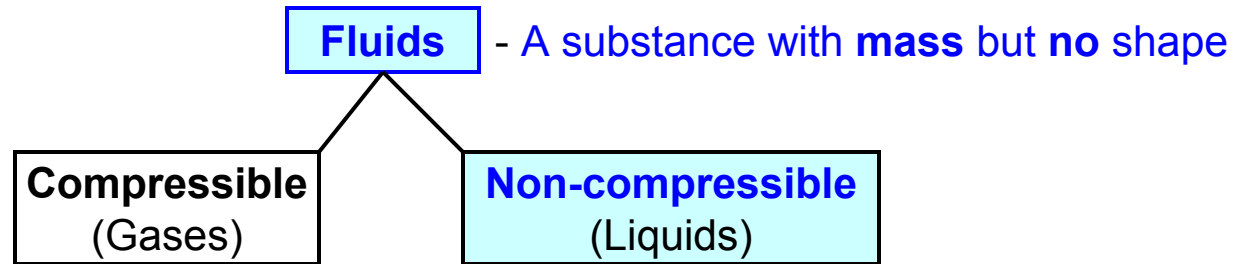
$$u(x) = 1 + \frac{1}{e^{2x}} = 1 + e^{-2x}$$

## **Part 2**

# **Application of First Order Differential Equation to Fluid Mechanics Analysis**



# Fundamental Principles of Fluid Mechanics Analysis

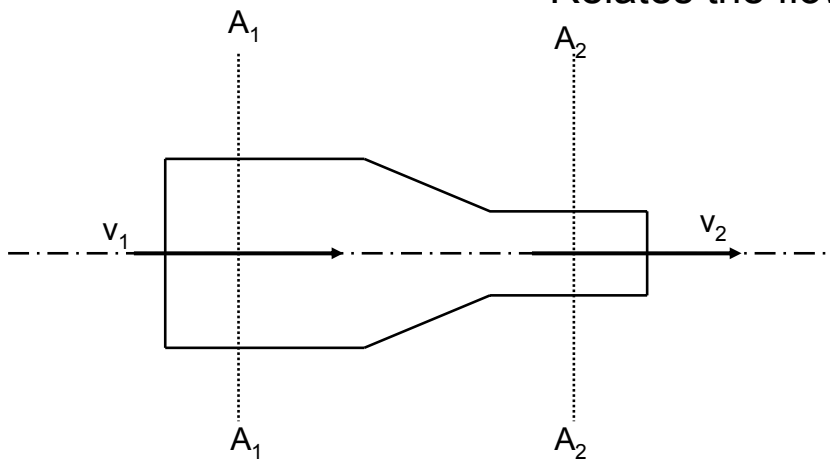


## Moving of a fluid requires:

- A **conduit**, e.g., tubes, pipes, channels
- Driving **pressure**, or by **gravitation**, i.e., difference in “**head**”
- Fluid flows with a **velocity v** from higher pressure (or elevation) to lower pressure (or elevation)

## The law of continuity

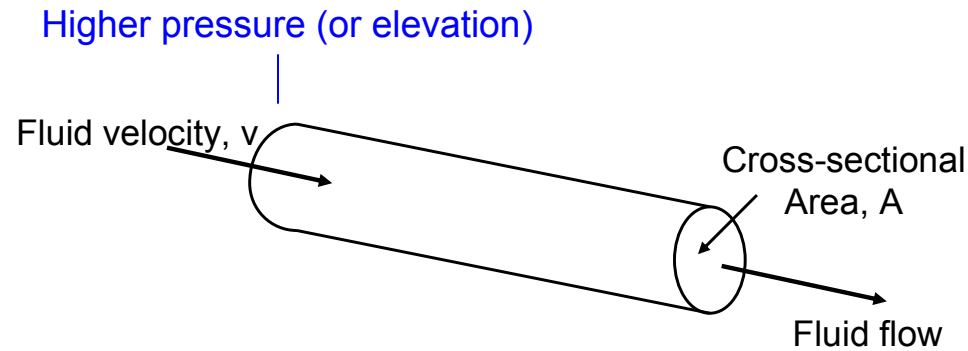
- Derived from Law of conservation of mass
- Relates the flow velocity ( $v$ ) and the cross-sectional area ( $A$ )



The **rate of volumetric flow** follows the rule:

$$q = A_1 v_1 = A_2 v_2 \quad \text{m}^3/\text{s}$$

## Terminologies in Fluid Mechanics Analysis



**The total mass flow,  $Q = \rho A v \Delta t$  (g) (3.8a)**

**Total mass flow rate,  $\dot{Q} = \frac{Q}{\Delta t} = \rho A v$  (g/sec) (3.8b)**

**Total volumetric flow rate,  $\dot{V} = \frac{\dot{Q}}{\rho} = A v$  ( $m^3$ /sec) (3.8c)**

**Total volumetric flow,  $V = \dot{V} \Delta t = A v \Delta t$  ( $m^3$ ) (3.8d)**

in which  $\rho$  = mass density of fluid ( $g/m^3$ ),

$A$  = Cross-sectional area ( $m^2$ ),

$v$  = Velocity ( $m/s$ ), and  $\Delta t$  = duration of flow (s).

The units associated with the above quantities are (g) for grams, (m) for meters and (sec) for seconds.

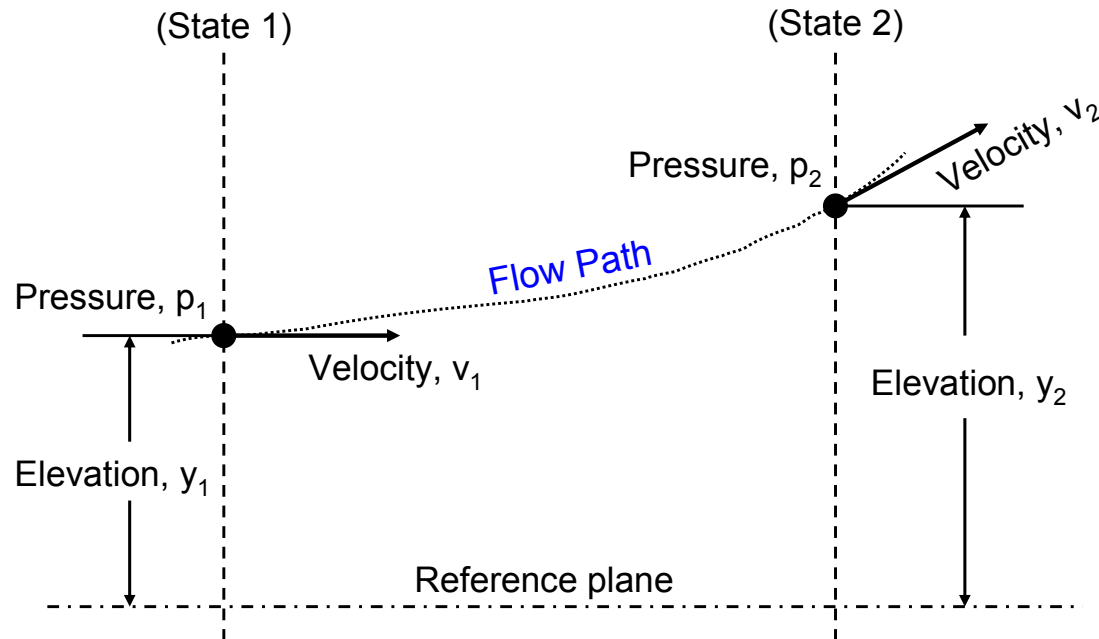
- In case the velocity varies with time, i.e.,  $v = v(t)$ :

Then the change of **volumetric flow** becomes:  $\Delta V = A v(t) \Delta t$ , (3.8e)

with  $\Delta t$  = time duration for the fluid flow

## The Bernoullis Equation

(The mathematical expression of the law of physics relating the **driving pressure** and **velocity** in a moving non-compressible fluid)

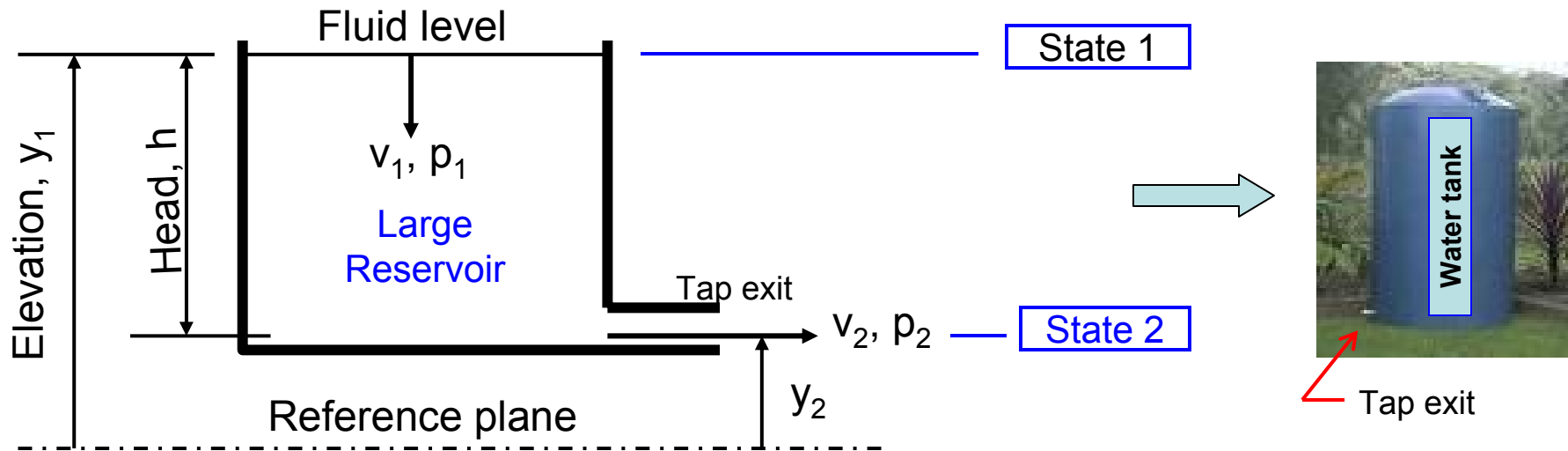


Using the Law of conservation of energy, or the First Law of Thermodynamics, for the energies of the fluid at State 1 and State 2, we can derive the following expression relating **driving pressure (p)** and the resultant **velocity of the flow (v)**:

The **Bernoullis Equation**:

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + y_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + y_2 \quad (3.10)$$

Application of **Bernoulli's equation** in liquid (water) flow in a **LARGE** reservoir:



From the **Bernoulli's** equation, we have:

$$\frac{v_1^2 - v_2^2}{2g} + \frac{p_1 - p_2}{\rho g} + (y_1 - y_2) = 0 \quad (3.10)$$

If the difference of elevations between State 1 and 2 is not too large, we can have:  $p_1 \approx p_2$

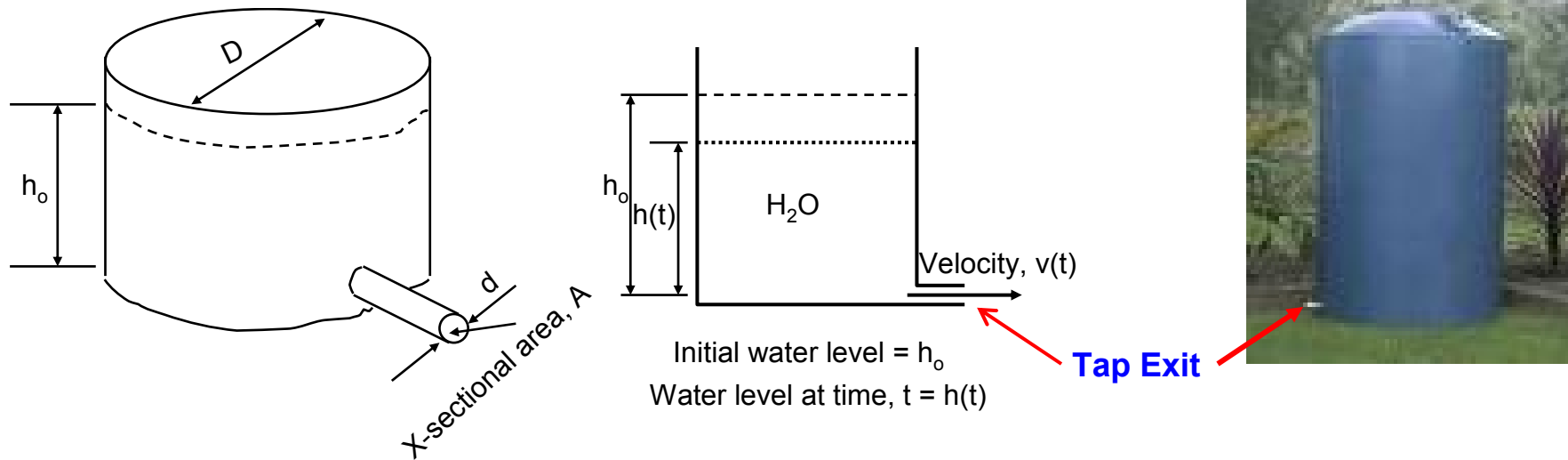
Also, because it is a LARGE reservoir (or tank), we realize that  $v_1 \ll v_2$ , or  $v_1 \approx 0$

Equation (3.10) can be reduced to the form:  $-\frac{v_2^2}{2g} + 0 + h = 0$  with  $h = y_1 - y_2$

from which, we may express the **exit velocity** of the liquid at the tap to be:

$$v_2 = \sqrt{2gh} \quad (3.11)$$

## Application of 1<sup>st</sup> Order DE in Drainage of a Water Tank



Use the **law of conservation of mass**:

The total volume of water leaving the tank during  $\Delta t$  ( $\Delta V_{\text{exit}}$ ) =  
The total volume of water supplied by the tank during  $\Delta t$  ( $\Delta V_{\text{tank}}$ )

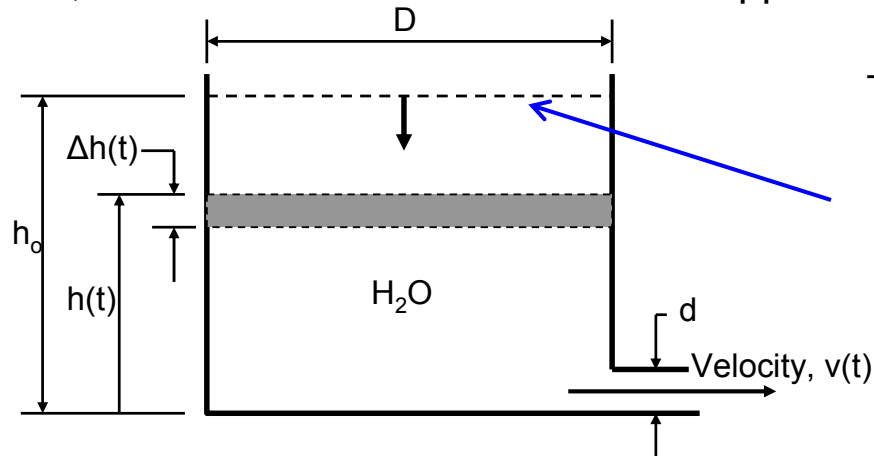
We have from Equation (3.8e):  $\Delta V = A v(t) \Delta t$ , in which  $v(t)$  is the velocity of moving fluid

Thus, the volume of water leaving the tap exit is:

$$\Delta V_{\text{exit}} = A v(t) \Delta t = \left( \frac{\pi d^2}{4} \right) \sqrt{2g h(t)} \Delta t \quad (\text{a})$$

with  $v(t) = \sqrt{2g h(t)}$  Given in Equation (3.11)

Next, we need to formulate the water supplied by the tank,  $\Delta V_{\text{tank}}$ :



The initial water level in the tank is  $h_0$   
 The water level keeps dropping after the tap exit is opened, and the reduction of Water level is CONTINUOUS

Let the water level at time  $t$  be  $h(t)$

We let  $\Delta h(t)$  = amount of drop of water level during time increment  $\Delta t$

Then, the volume of water LOSS in the tank is: 
$$\Delta V_{\text{tank}} = -\frac{\pi D^2}{4} \Delta h(t) \quad (b)$$

(**Caution:** a “-” sign is given to  $\Delta V_{\text{tank}}$  b/c of the LOSS of water volume during  $\Delta t$ )

The total volume of water leaving the tank during  $\Delta t$  ( $\Delta V_{\text{exit}}$ ) in Equation (a) =  
 The total volume of water supplied by the tank during  $\Delta t$  ( $\Delta V_{\text{tank}}$ ) in Equation (b):

Equation (a):      Equation (b):

$$\frac{\pi d^2}{4} \sqrt{2g h(t)} \Delta t = -\frac{\pi D^2}{4} \Delta h(t) \quad (c)$$

By re-arranging the above: 
$$\frac{\Delta h(t)}{\Delta t} = -[h(t)]^{1/2} \left( \frac{d^2}{D^2} \right) \sqrt{2g} \quad (d)$$

If the process of draining is indeed CONTINUOUS, i.e.,  $\Delta t \rightarrow 0$ , we will have Equation (d) expressed in the “differential” rather than “difference” form as follows:

$$\frac{dh(t)}{dt} = -\sqrt{2g} \left( \frac{d^2}{D^2} \right) \sqrt{h(t)} \quad (3.13)$$

with an initial condition of  $h(0) = h_o$  (f)

Equation (3.13) is the 1<sup>st</sup> order differential equation for the draining of a water tank.

The solution of Equation (3.13) can be done by separating the function  $h(t)$  and the variable  $t$  by re-arranging the terms in the following way:

$$\frac{dh(t)}{\sqrt{h(t)}} = -\sqrt{2g} \left( \frac{d^2}{D^2} \right) dt$$

Upon integrating on both sides:  $\int h^{-1/2} dh = -\sqrt{2g} \left( \frac{d^2}{D^2} \right) \int dt + c$  where  $c$  = integration constant

from which, we obtain the solution of Equation (3.13) to be:  $2h^{1/2} = -\sqrt{2g} \left( \frac{d^2}{D^2} \right) t + c$

The constant  $c = 2\sqrt{h_o}$  is determined from the initial condition in Equation (f).

The complete solution of Equation (3.13) with the initial condition in Equation (f) is thus:

$$h(t) = \left[ -\sqrt{\frac{g}{2}} \left( \frac{d^2}{D^2} \right) t + \sqrt{h_o} \right]^2 \quad (g)$$

The solution in Equation (g) will allow us to determine the water level in the tank at any given instant, t.

The time required to drain the tank is the time  $t_e$ .

Mathematically, it is expressed as  $h(t_e) = 0$ :  $0 = \left[ -\sqrt{\frac{g}{2}} \left( \frac{d^2}{D^2} \right) t_e + \sqrt{h_o} \right]^2$

We may solve for  $t_e$  from the above expression to be:

$$t_e = \frac{D^2}{d^2} \sqrt{\frac{2h_o}{g}} \quad s$$

Numerical example:

Tank diameter,  $D = 12'' = 1 \text{ ft.}$

Drain pipe diameter,  $d = 1'' = 1/12 \text{ ft.}$

Initial water level in the tank,  $h_o = 12'' = 1 \text{ ft.}$

Gravitational acceleration,  $g = 32.2 \text{ ft/sec.}$

The time required to empty the tank is:

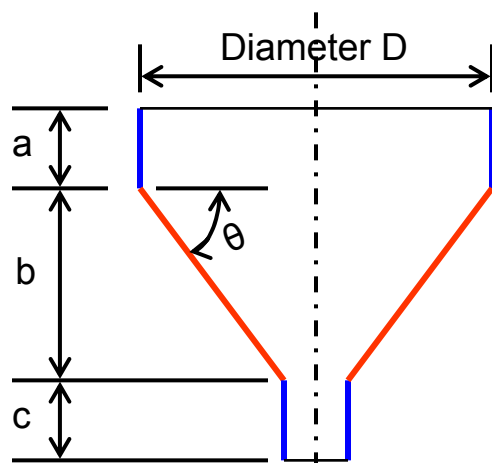
$$t_e = \left( \frac{1}{\frac{1}{12}} \right)^2 \sqrt{\frac{2 \times 1}{32.2}} = 35.89 \quad \text{seconds}$$

So, now you know how to determine the time required to drain a "fish tank" a "process tank" or a "swimming pool," Or do you?



## Application of 1<sup>st</sup> Order DE in Drainage of Tapered Funnels

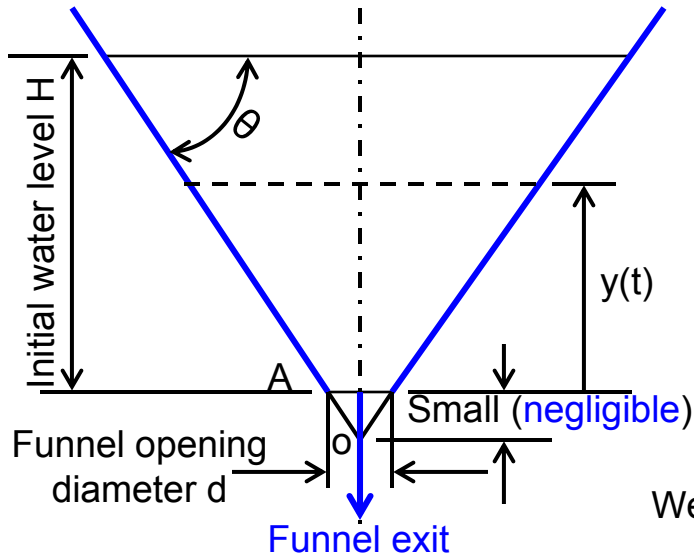
Tapered funnels are common piece of equipment used in many process plants, e.g., wine bottling



Design of tapered funnels involves the determination of configurations, i.e. the tapered angle, and the diameters and lengths of sections of the funnel for the intended liquid content.

It is also required the determination on the time required to empty the contained liquid.

## Formulations on Water Level in Tapered Funnels



The **physical solution** we are seeking is the water level at given time  $t$ ,  $y(t)$  after the water is let to flow from the exit of the funnel.

The “real” funnel has an outline of **frustum cone** with smaller circular end at “A” allowing water flow.

In the subsequent analysis, we assume the funnel has an **outline shape of “right cone”** with its tip at “O.”

We assume the initial water level in the funnel =  $H$

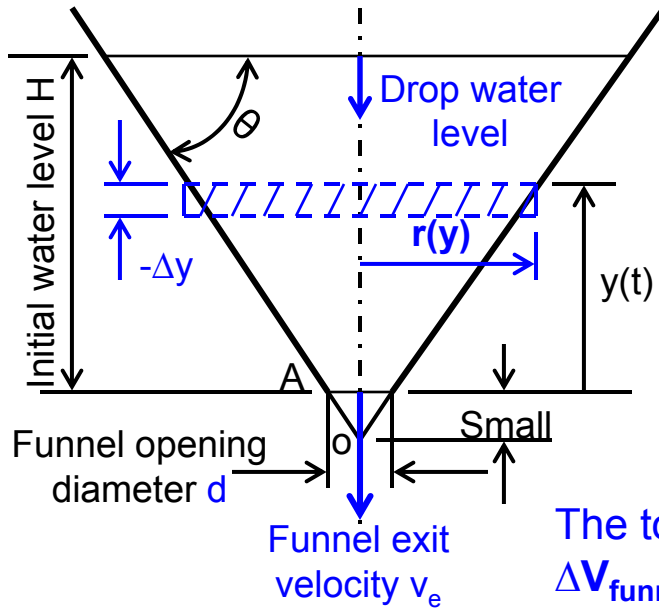
Once the funnel exit is open, and water begin to flow, the water level in the funnel at time  $t$  is represented by the function  $y(t)$ .

We will use the same principle to formulate the expression for  $y(t)$  as in the straight tank:

The total volume of water leaving the tank during  $\Delta t$  ( $\Delta V_{\text{exit}}$ ) =

The total volume of water supplied by the tank during  $\Delta t$  ( $\Delta V_{\text{funnel}}$ )

## Determine the Instantaneous Water Level in a Tapered Funnel:



The total volume of water leaving the tank during  $\Delta t$ :

$$(\Delta V_{\text{exit}}) = (A_{\text{exit}}) (v_e) (\Delta t)$$

But from Equation (3.11), we have the exit velocity  $v_e$  to be:

$$v_e = \sqrt{2 g y(t)}$$

which leads to:

$$\Delta V_{\text{exit}} = \frac{\pi d^2}{4} \sqrt{2 g y(t)} \Delta t \quad (\text{a})$$

The total volume of water supplied by the tank during  $\Delta t$ :

$\Delta V_{\text{funnel}} =$  volume of the cross-hatched in the diagram

$$\Delta V_{\text{funnel}} = - [\pi (r(y)^2)] (\Delta y) \quad (\text{b})$$

A “-ve” sign to indicate decreasing  $\Delta V_{\text{funnel}}$  with increasing  $y$

From the diagram in the left, we have:

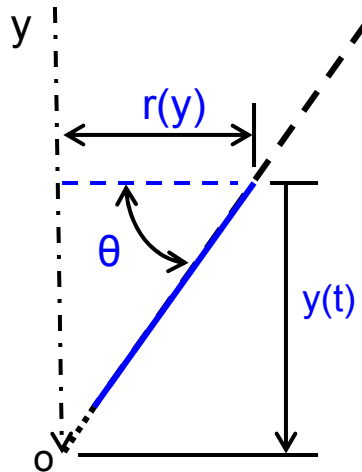
$$r(y) = \frac{y(t)}{\tan \theta} \quad (\text{c})$$

Hence by equating (a) and (b) with  $r(y)$  given in (c), we have:

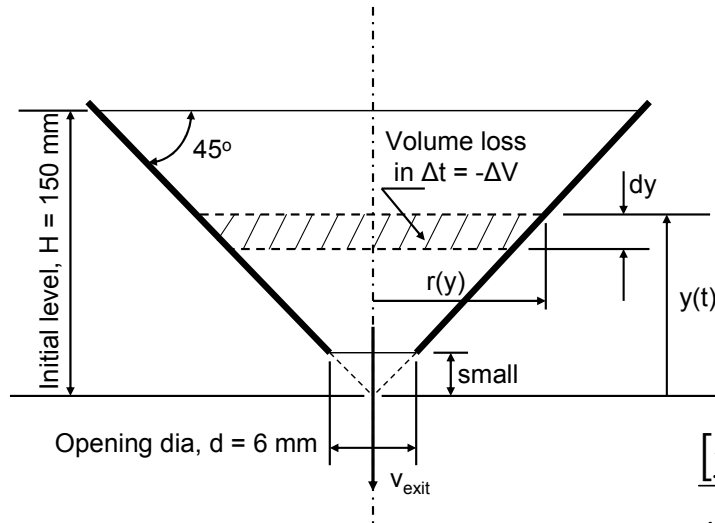
$$\frac{\pi d^2}{4} \sqrt{2 g y(t)} \Delta t = -\pi \frac{[y(t)]^2}{\tan^2 \theta} \Delta y \quad (\text{d})$$

For a CONTINUOUS variation of  $y(t)$ , we have  $\Delta t \rightarrow 0$ , we will have the differential equation for  $y(t)$  as:

$$\frac{[y(t)]^2}{\tan^2 \theta} \frac{dy(t)}{dt} = -\frac{d^2}{4} \sqrt{2g} \quad (\text{e})$$



## Drainage of a Tapered Funnel (Section 3.4.3, p. 58):



To determine the time required to empty the funnel with initial water level of 150 mm and with the dimensions shown in the figure.

- This is a special case of the derivation of a general tapered funnel with  $\theta = 45^\circ$ .
- We thus have the differential equation similar to Equation (e) with  $\tan\theta = \tan 45^\circ = 1$ :

$$\frac{[y(t)]^3}{\sqrt{y(t)}} \frac{dy(t)}{dt} + \frac{d^2}{4} \sqrt{2g} = 0 \quad (\text{a})$$

(b)

with condition:  $y(t)|_{t=0} = H = 150 \text{ mm}$

Equation (a) is a 1<sup>st</sup> order DE, and its solution is obtained by integrating both sides w.r.t variable  $t$ :

$$\frac{2}{5} y^{5/2} = -\frac{d^2}{4} \sqrt{2g} t + c \quad (\text{c})$$

The integration constant  $c$  is determined by using Equation (b)  $\rightarrow c = 2H^{5/2}/5$ , which leads to the complete solution of :

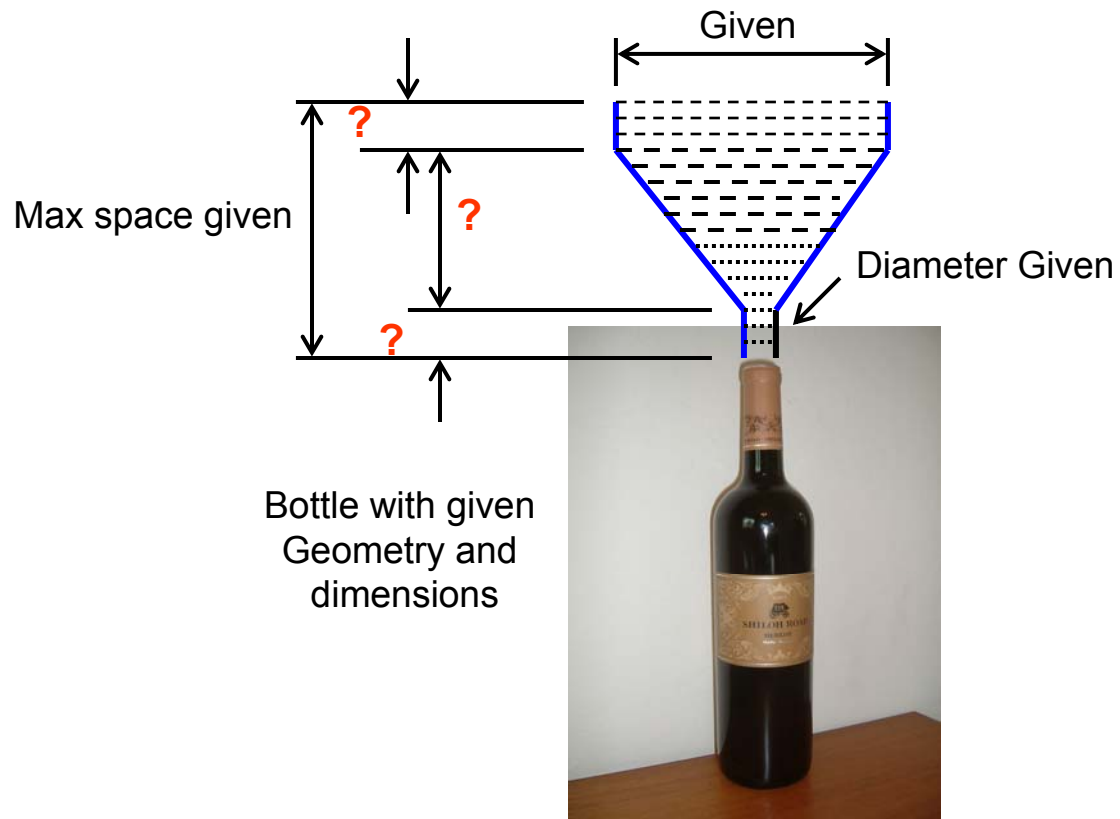
$$[y(t)]^{5/2} = H^{5/2} - \frac{5d^2}{8} \sqrt{2g} t \quad (\text{d})$$

If we let time required to empty (drain) the funnel to be  $t_e$  with  $y(t_e) = 0$ , we will solve Equation (e) With these conditions to be:

$$t_e = \frac{8H^{5/2}}{5d^2 \sqrt{2g}} \quad (\text{e})$$

**Example** on the drainage of a funnel in a winery: To design a funnel that will fill a wine bottle

**Design objective:** To provide **SHORTEST time** in draining the funnel for fastest bottling process



Refer to Problem 3.15 on p. 77

## Part 3

# Applications of First Order Differential Equations to Heat Transfer Analysis

### The Three Modes of Heat Transmission:

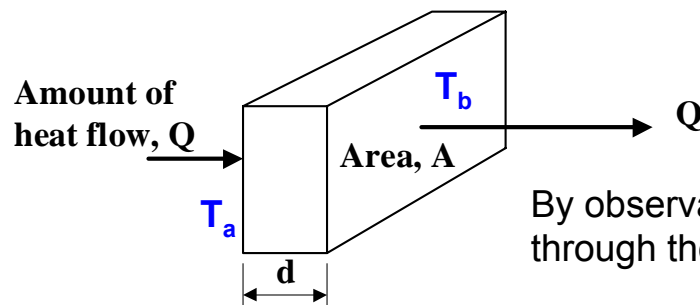
- Heat conduction in solids
- Heat convection in fluids
- Radiation of heat in space

## Review of Fourier Law for Heat Conduction in Solids

- Heat flows in SOLIDS by conduction
- Heat flows from the part of solid at higher temperature to the part with low temperature - a situation similar to water flow from higher elevation to low elevation
- Thus, there is definite relationship between **heat flow (Q)** and the **temperature difference ( $\Delta T$ )** in the solid
- Relating the Q and  $\Delta T$  is what the Fourier law of heat conduction is all about

### Derivation of Fourier Law of Heat Conduction:

**A solid slab:** With the left surface maintained at temperature  $T_a$  and the right surface at  $T_b$



Heat will flow from the left to the right surface if  $T_a > T_b$

By observations, we can formulate the total amount of heat flow (**Q**) through the thickness of the slab as:

$$Q \propto \frac{A(T_a - T_b)t}{d}$$

where  $A$  = the area to which heat flows;  $t$  = time allowing heat flow; and  $d$  = the distance of heat flow

Replacing the  $\propto$  sign in the above expression by an = sign and a constant  $k$ , leads to:

$$Q = k \frac{A(T_a - T_b)t}{d} \quad (3.14)$$

The **constant  $k$**  in Equation (3.14) is “**thermal conductivity**” – treated as a property of the solid material with a unit: Btu/in-s- $^{\circ}$ F of W/m- $^{\circ}$ C

The amount of total heat flow in a solid as expressed in Equation (3.14) is useful, but make less engineering sense without specifying the area  $A$  and time  $t$  in the heat transfer process.

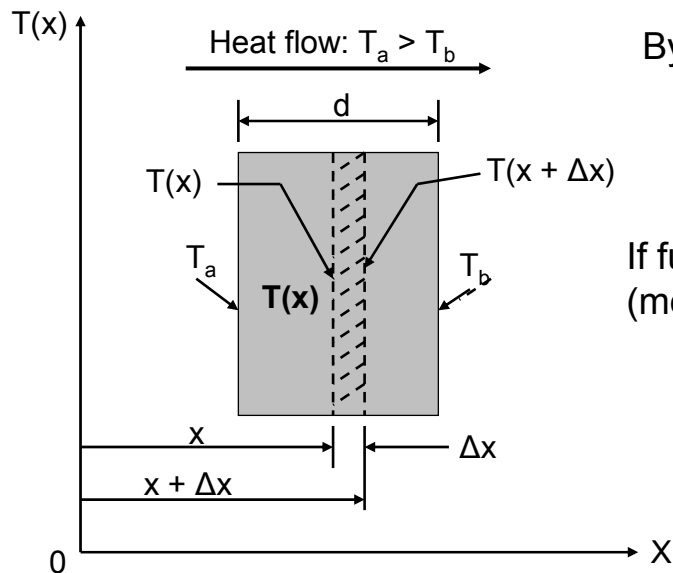
Consequently, the “**Heat flux**” ( $q$ ) – a sense of the intensity of heat conduction is used more frequently in engineering analyses. From Equation (3.14), we may define the heat flux as:

$$q = \frac{Q}{At} = k \frac{(T_a - T_b)}{d} \quad (3.15)$$

with a unit of: Btu/in<sup>2</sup>-s, or **W/cm<sup>2</sup>**

We realize Equation (3.15) is derived from a situation of heat flow through a thickness of a slab with distinct temperatures at both surfaces.

In a situation the **temperature variation in the solid is CONTINUOUS**, by function **T(x)**, as illustrated below:



By following the expression in Equation (3.15), we will have:

$$q = k \frac{T(x) - T(x + \Delta x)}{\Delta x} = -k \frac{T(x + \Delta x) - T(x)}{\Delta x} \quad (3.16)$$

If function **T(x)** is a **CONTINUOUS** varying function w.r.t variable  $x$ , (meaning  $\Delta x \rightarrow 0$ ), We will have the following from Equation (3.16):

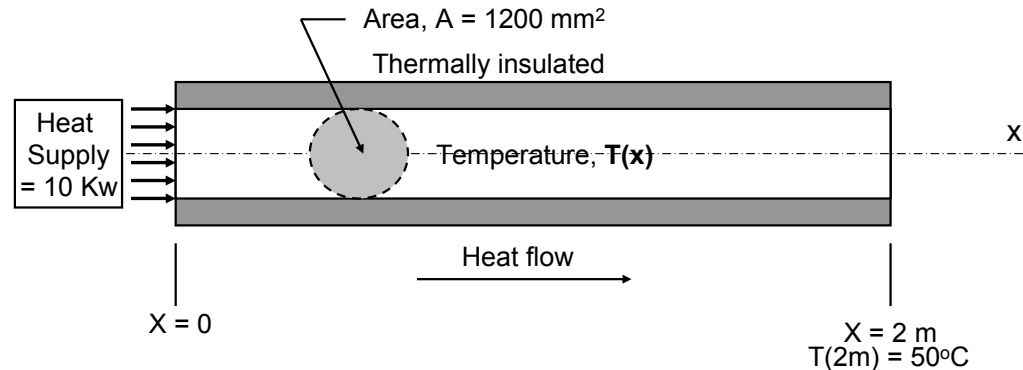
$$q(x) = \lim_{\Delta x \rightarrow 0} \left[ -k \frac{T(x + \Delta x) - T(x)}{\Delta x} \right] = -k \frac{dT(x)}{dx} \quad (3.17)$$

**Equation (3.17) is the mathematical expression of Fourier Law of Heat Conduction in the x-direction**



**Example 3.7** (p. 63):

A metal rod has a **cross-sectional area 1200 mm<sup>2</sup>** and **2m in length**. It is thermally insulated in its circumference, with one end being in contact with a **heat source supplying heat at 10 kW**, and the other end maintained at 50°C. Determine the temperature distribution in the rod, if the thermal conductivity of the rod material is **k = 100 kW/m-°C**.



Solution:

The **total heat flow Q per unit time t (Q/t)** in the rod is given by the heat source to the left end, i.e. **10 kW**.

Because heat flux is  $q = Q/(At)$  as shown in Equation (3.15), we have  **$(Q/t) = qA = 10\text{ kW}$**

But the Fourier Law of heat conduction requires  $q(x) = -k \frac{dT(x)}{dx}$  as in Equation (3.17), we thus have:

$$Q = qA = -kA \frac{dT(x)}{dx} \quad \longrightarrow \quad \frac{dT(x)}{dx} = -\frac{Q}{kA} = -\frac{10}{100(1200 \times 10^{-6})} = -83.33 \text{ } ^\circ\text{C}/\text{m} \quad (\text{a})$$

Expression in (a) is a 1<sup>st</sup> order differential equation, and its solution is:  $T(x) = -83.33x + c$  (b)

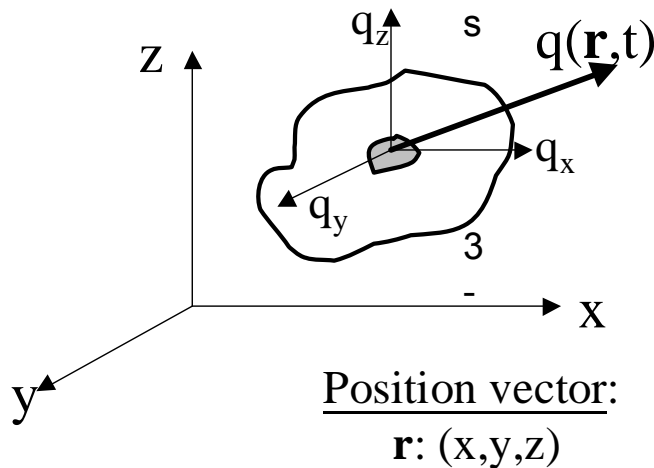
If we use the condition:  $T(2) = 50^\circ\text{C}$ , we will find  $c = 216.67$ , which leads to the complete solution:

$$T(x) = 216.67 - 83.33x$$

## Heat Flux in Space

### Expressions in 3-dimensional form

- Heat flows in the **direction of decreasing temperature** in a solid
- In solids with temperature variations in all direction, heat will flow in ALL directions
- So, in general, there can be 3-dimensional heat flow in solids
- This leads to 3-dimensional formulation of heat flux
- Heat flux  $\mathbf{q}(\mathbf{r},t)$  is a vectorial quantity, with  $\mathbf{r}$  = position vector, representing (x, y, z)



The magnitude of vector  $\mathbf{q}(\mathbf{r},t)$  is:

$$q(x, y, z, t) = \sqrt{q_x^2 + q_y^2 + q_z^2} \quad (3.21)$$

with the components along respective x-, y- and z-coordinates:

$$q_x = -k_x \frac{\partial T(x, y, z, t)}{\partial x}$$

$$q_y = -k_y \frac{\partial T(x, y, z, t)}{\partial y}$$

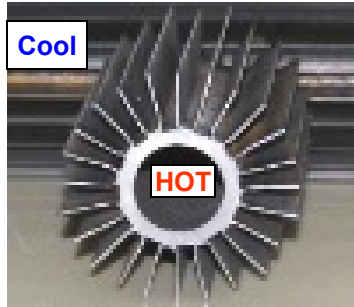
$$q_z = -k_z \frac{\partial T(x, y, z, t)}{\partial z}$$

In general, the heat flux vector in the Fourier Law of heat conduction can be expressed as:

$$\mathbf{q}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t) \quad (3.20)$$

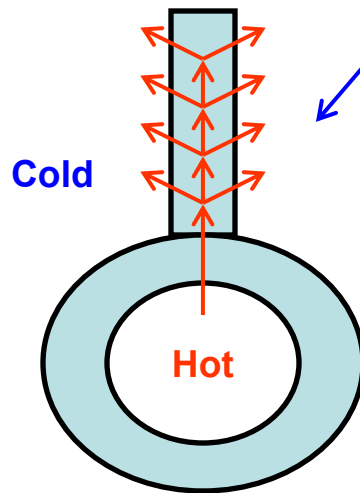
## Heat Flux in a 2-D Plane

Tubes with **longitudinal fins** are common in many heat exchangers and boilers for effective heat exchange between the hot fluids inside the tube to cooler fluids outside:



The heat inside the tube flows along the plate-fins to the cool contacting fluid outside.

It is desirable to analyze how effective heat can flow in the cross-section of the fin.

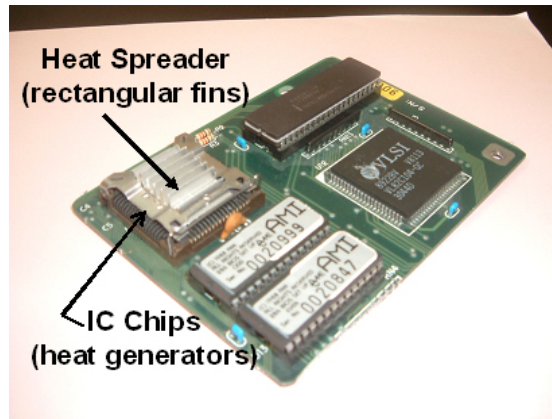


Cross-section of a tube with longitudinal fins with only one fin shown

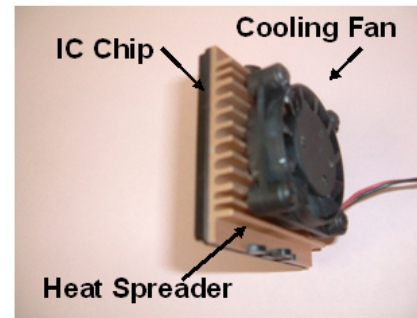
Fins are also used to conduct heat from the hot inside of an **Internal combustion engine** to the outside cool air in a motor cycle:



# Heat Spreaders in Microelectronics Cooling

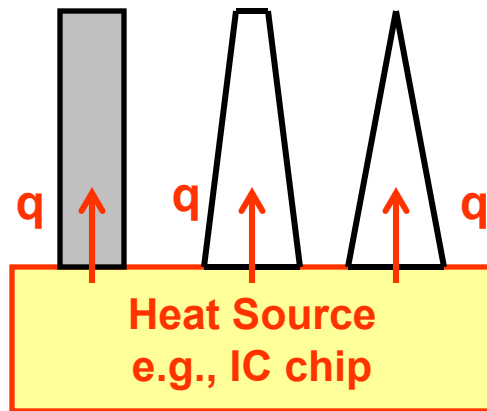


(a) A typical printed circuit board



(b) Dissipation of heat in an IC chip

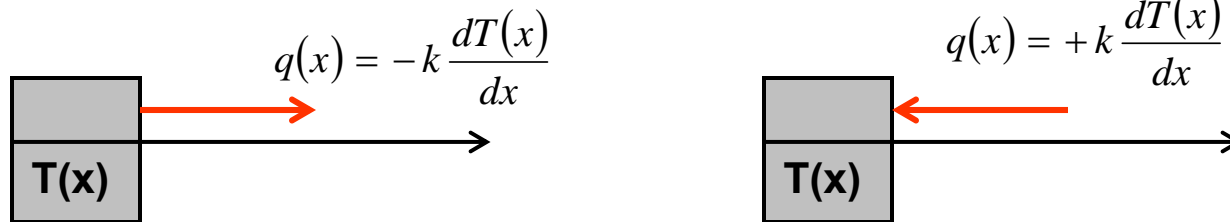
## Heat spreader of common cross-sections



Heat flow in a 2-dimensional plane, by Fourier Law in x-y plane

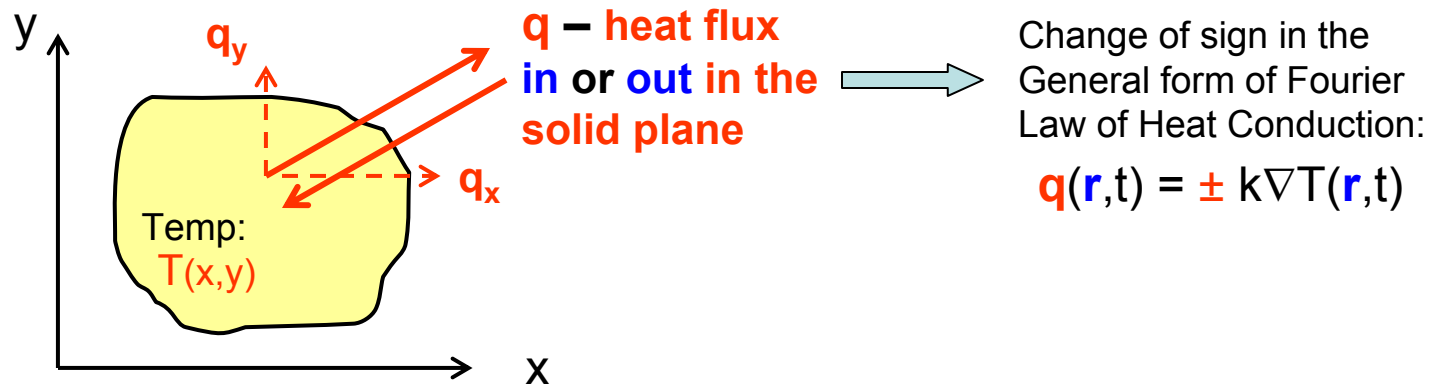
## Fourier Law of Heat Conduction in 2-Dimensions

For **one-dimensional** heat flow:



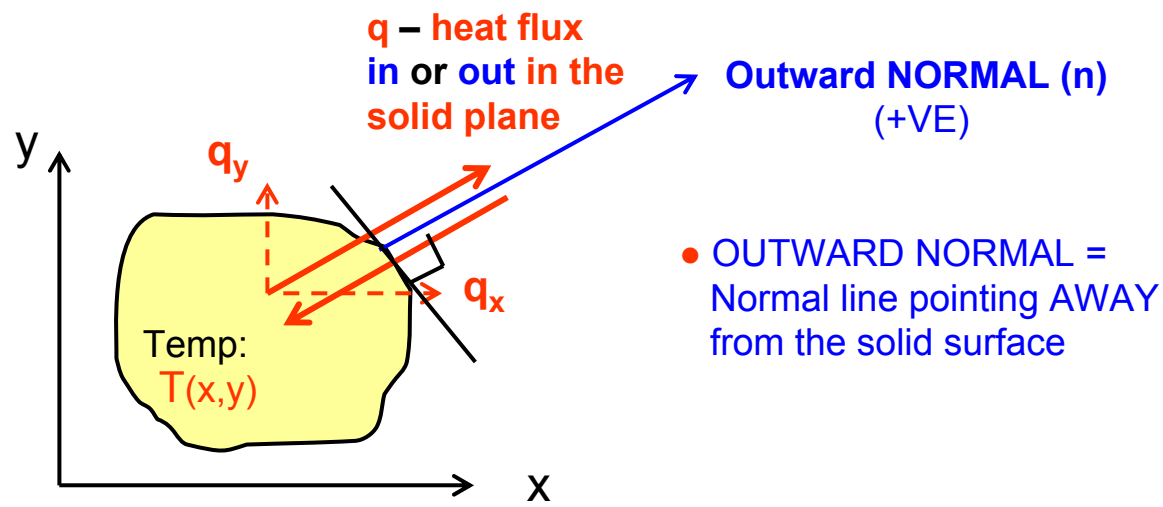
**NOTE:** The **sign** attached to  $q(x)$  changes with **change of direction** of heat flow!!

For **two-dimensional** heat flow:



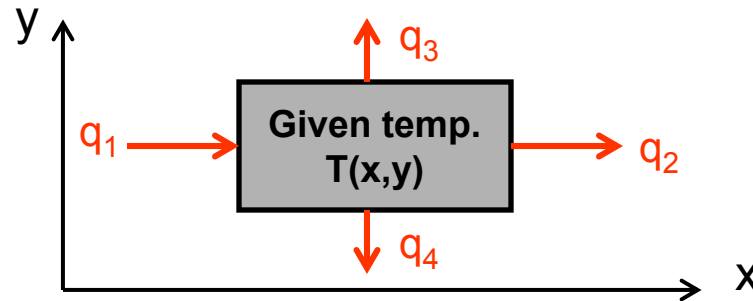
**Question:** How to assign the **CORRECT** sign in heat flux??

# Sign of Heat Flux



Sign of Outward Normal ( $n$ )	$q$ along $n$ ?	Sign of $q$ in Fourier Law
+	Yes	-
+	No	+
-	Yes	+
-	No	-

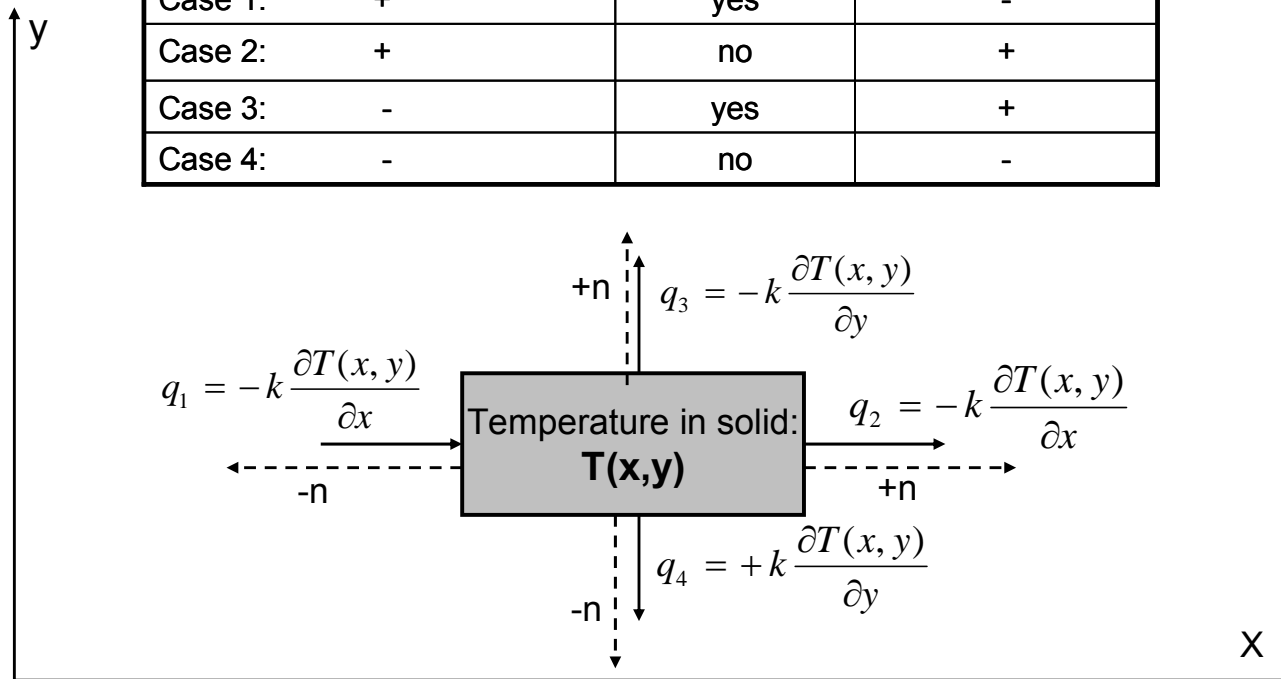
**Example:** Express the heat flux across the four edges of a rectangular block with correct +ve or -ve sign (p.66)



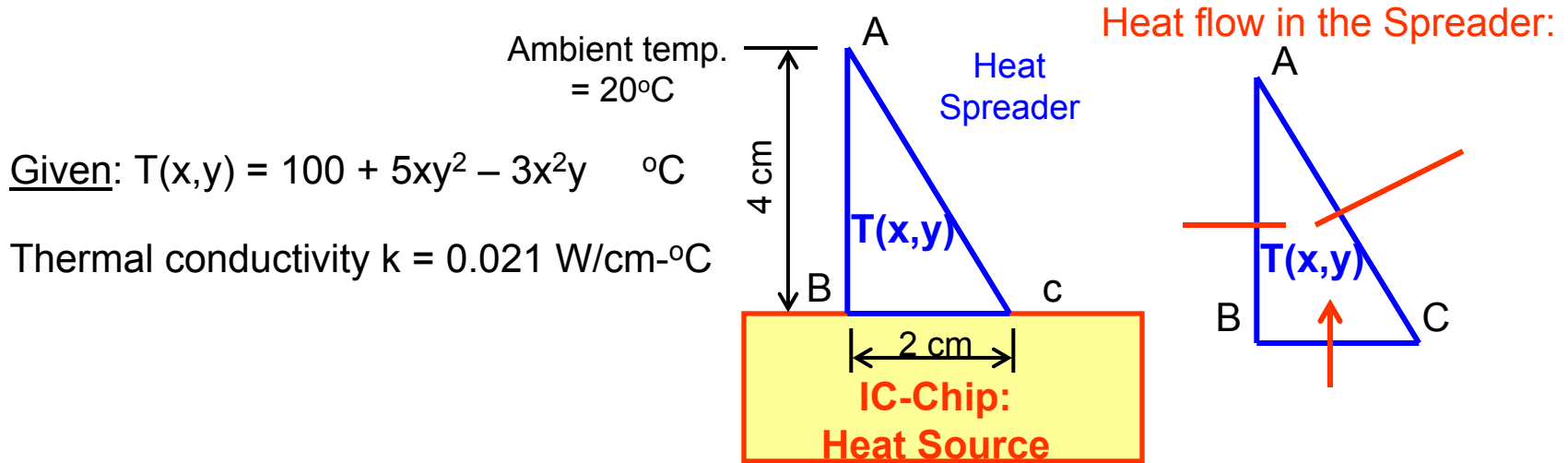
The direction of heat fluxes is prescribed. Thermal conductivity of the material  $k$  is given.

**Solution:**

Sign of outward normal, $n$	$q$ along $n$ ?	Sign of $q$ in Fourier law
Case 1: +	yes	-
Case 2: +	no	+
Case 3: -	yes	+
Case 4: -	no	-

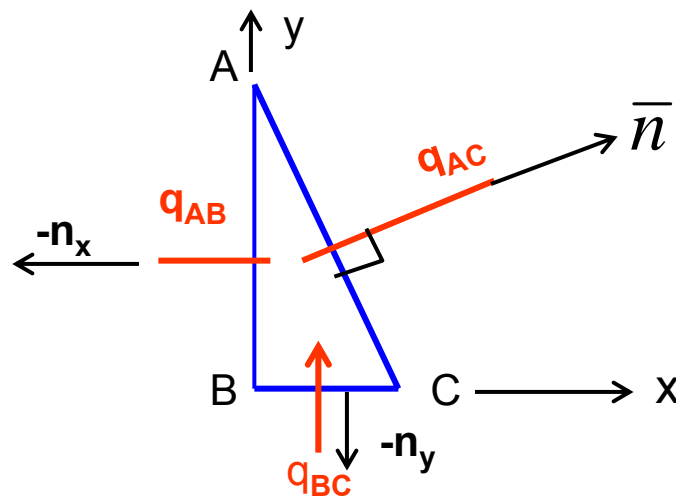


**Example 3.8** Heat fluxes leaving a heat spreader of half-triangular cross-section. (p.66)



**Solution:**

Set coordinate system and Identify outward normals:





### A. Heat flux across surface BC:

The direction of heat flow is known (from heat source to the spreader)

-ve n  $\rightarrow$  Case 3 or 4;  $q_{BC}$  is not along n  $\rightarrow$  Case 4 with -ve sign

$$q_{bc} = -k \frac{\partial T(x, y)}{\partial y} \Big|_{y=0} = -0.021 \frac{\partial(100 + 5xy^2 - 3x^2y)}{\partial y} \Big|_{y=0} = -0.021(10xy - 3x^2) \Big|_{y=0} = 0.063x^2 \text{ w/cm}^2$$

### B. Heat flux across surface AB:

We need to verify the direction of heat flow across this surface first:

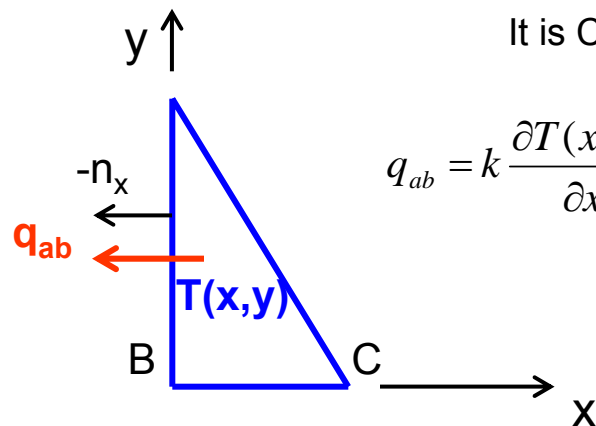
Checking the temperature at the two terminal points of the Edge AB:

At Point B ( $x = 0$  and  $y = 0$ ) the corresponding temperature is:

$$(100 + 5xy^2 - 3x^2y) \Big|_{\substack{x=0 \\ y=0}} = 100^\circ \text{C} > 20^\circ \text{C}, \text{ the ambient temperature}$$

The same temperature at terminal A

So, heat flows from the spreader to the surrounding.



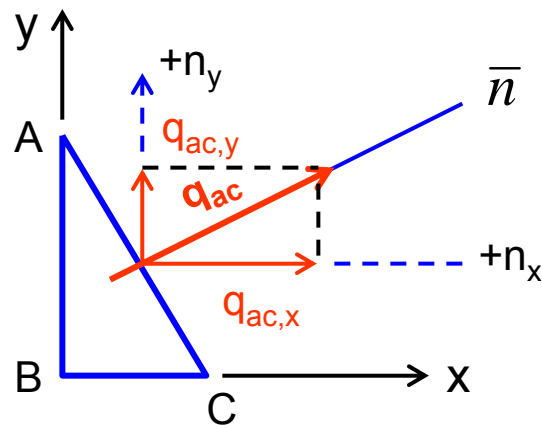
It is Case 3 in Fourier law, we thus have:

$$q_{ab} = k \frac{\partial T(x, y)}{\partial x} \Big|_{x=0} = 0.021 \frac{\partial(100 + 5xy^2 - 3x^2y)}{\partial x} \Big|_{x=0} = 0.105y^2 \text{ w/cm}^2$$

### C. Heat flux across surface AC:

Surface AC is an **inclined surface**, so we have a situation as illustrated below.

Use the same technique as in Case B, we may find the temperature at both terminal A and C to be  $100^\circ\text{C} > 20^\circ\text{C}$  in ambient. So heat leaves the surface AC to the ambient.



Based on the direction of the components of the heat flow and outward normal, we recognize **Case 1** for both  $q_{ac,x}$  and  $q_{ac,y}$ . Thus we have:

$$q_{ac,x} = -k \frac{\partial T(x, y)}{\partial x} \Big|_{\substack{x=1 \\ y=2}} = -0.021(5y^2 - 6xy) \Big|_{\substack{x=1 \\ y=2}} = -0.021(20 - 12) = -0.168 \text{ w/cm}^2$$

and

$$q_{ac,y} = -k \frac{\partial T(x, y)}{\partial y} \Big|_{\substack{x=1 \\ y=2}} = -0.021(10xy - 3x^2) \Big|_{\substack{x=1 \\ y=2}} = -0.021(20 - 3) = -0.357 \text{ w/cm}^2$$

The heat flux across the mid-point of surface AC at  $x = 1 \text{ cm}$  and  $y = 2 \text{ cm}$  is:

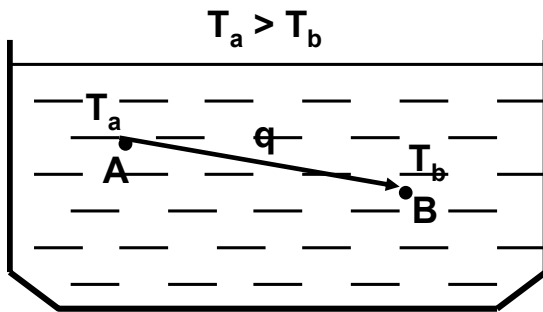
$$\vec{q}_{ac} = \vec{q}_{ac,x} + \vec{q}_{ac,y} = \sqrt{(-0.168)^2 + (-0.357)^2} = 0.3945 \text{ w/cm}^2$$

## Review of Newton's Cooling Law for Heat Convection in Fluids

- Heat flow (transmission) in fluid by **CONVECTION**
- Heat flow from higher temperature end to low temperature end
- **Motion** of fluids causes heat convection
- As a rule-of-thumb, the amount of heat transmission by convection is proportional to the **velocity** of the moving fluid

### Mathematical expression of heat convection – The Newton's Cooling Law

A fluid of non-uniform temperature in a container:



Heat flows from  $T_a$  to  $T_b$  with  $T_a > T_b$ . The heat flux between A and B can be expressed by:

$$q \propto (T_a - T_b) = h(T_a - T_b) \quad (3.22)$$

where **h = heat transfer coefficient** (W/m<sup>2</sup>-°C)

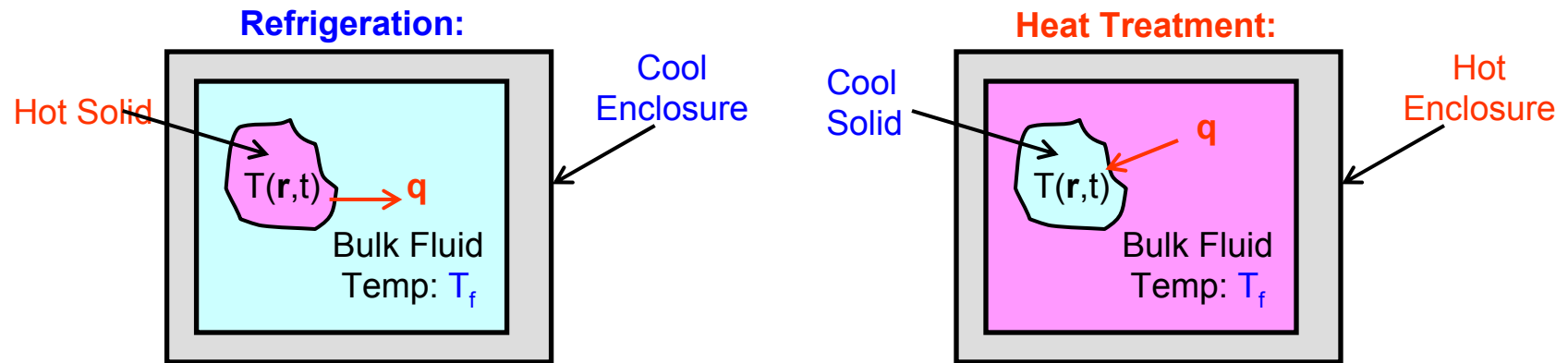
The heat transfer coefficient  $h$  in Equation (3.22) is normally determined by empirical expression, with its values relating to the **Reynolds number (Re)** of the moving fluid. The Reynolds number is expressed as:

$$\text{Re} = \frac{\rho L v}{\mu}$$

with  $\rho$  = mass density of the fluid;  $L$  = characteristic length of the fluid flow, e.g., the diameter of a circular pipe, or the length of a flat plate;  $v$  = velocity of the moving fluid;  $\mu$  = dynamic viscosity of the fluid

## Heat Transfer in Solids Submerged in Fluids

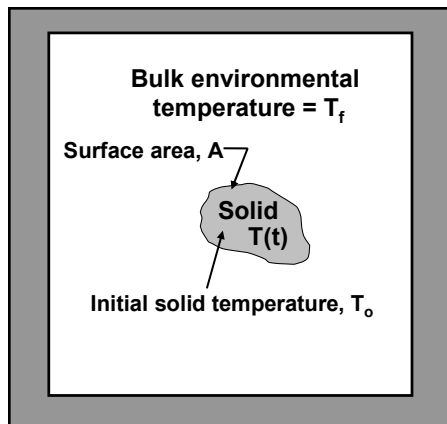
- There are numerous examples of which solids are in contact with fluids at different temperatures.



- In such cases, there is heat flow between the contacting solid and fluid.
- But the physical laws governing heat flow in solids is the Fourier Law and that in fluids by the Newton's Cooling Law

So, mathematical modeling for the contacting surface in this situation requires the use of **both** Fourier Law and Newton's Cooling Law:

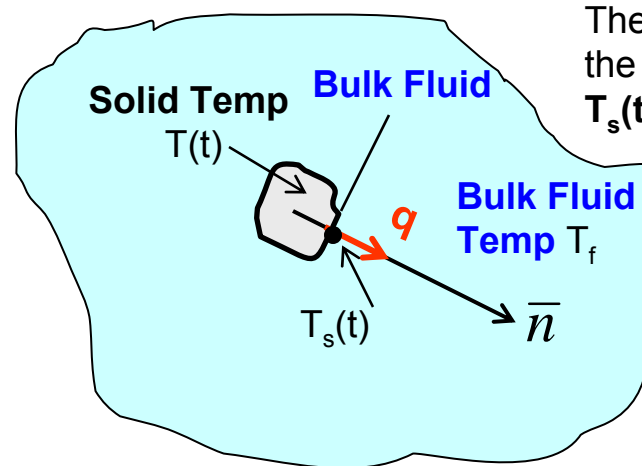
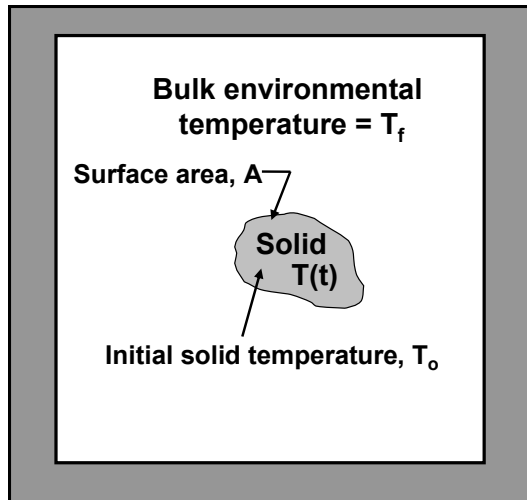
### Mathematical Modeling of Small Solids in Refrigeration and Heating



We will formulate a **simplified case** with **assumptions** on:

- the solid is initially at temperature  $T_0$
- the solid is so **small** that it has **uniform temperature**, but its temperature varies with time  $t$ , i.e.,  **$T = T(t)$**
- the time  $t$  begins at the instant that the solid is submerged in the fluid at a different temperature  $T_f$
- **variation of temp. in the solid is attributed by the heat supplied or removed by the fluid**

## Derivation of Math Model for Heat Transfer in Solids Submerged in Fluids



The solid is small, so the surface temperature  $T_s(t) = T(t)$ , the solid temperature

Heat flows in the fluid, when  $T(t) \neq T_f$ , the bulk fluid temp.

Heat flows in the fluid follows the **Newton Cooling Law** expressed in Equation (3.22), i.e.:

$$q = h [T_s(t) - T_f] = h [T(t) - T_f] \quad (a)$$

where  $h$  = heat transfer coefficient between the solid and the bulk fluid

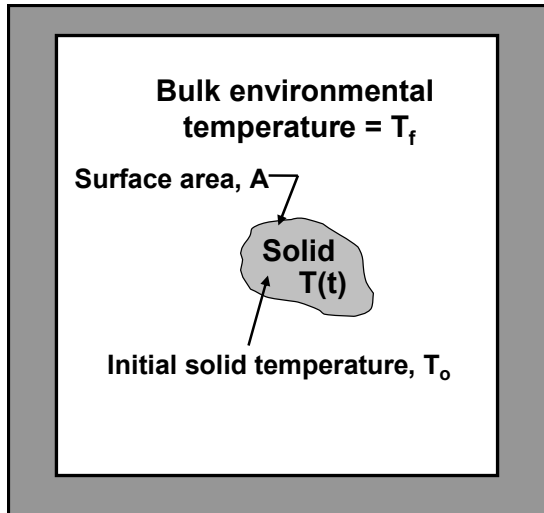
From the First Law of Thermodynamics, the heat required to produce temperature change in a solid  $\Delta T(t)$  during time period  $\Delta t$  can be obtained by the principle:

Change in internal energy of the small solid during $\Delta t$	=	Net heat flow from the small solid to the surrounding fluid during $\Delta t$
$- \rho c V \Delta T(t)$	=	$Q = q A_s \Delta t = h A_s [T(t) - T_f] \Delta t$ (b)

where  $\rho$  = mass density of the solid;  
 $V$  = volume of the solid;

$c$  = specific heat of solid  
 $A_s$  = contacting surface between solid and bulk fluid

From Equation (b), we express the rate of temperature change in the solid to be:



$$\frac{\Delta T(t)}{\Delta t} = -\frac{h}{\rho c V} A_s [T(t) - T_f] \quad (c)$$

Since  $h$ ,  $\rho$ ,  $c$  and  $V$  on the right-hand-side of Equation (c) are constant, we may lump these three constant to let:

$$\alpha = \frac{h}{\rho c V} \quad (d)$$

with a unit ( $/m^2 \cdot s$ )

Equation (c) is thus expressed as:

$$\frac{\Delta T(t)}{\Delta t} = -\alpha A_s [T(t) - T_f] \quad (e)$$

Since the change of the temperature of the submerged solid  $T(t)$  is CONTINUOUS with respect to time  $t$ , i.e.,  $\Delta t \rightarrow 0$ , and if we replace the contact surface area  $A_s$  to a generic symbol  $A$ , we can express

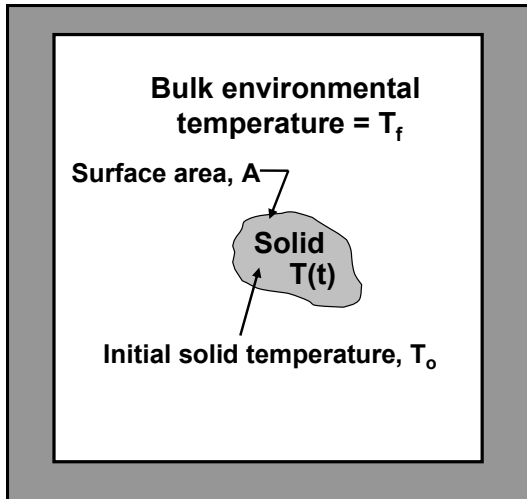
Equation (e) in the form of a 1<sup>st</sup> order differential equation:

$$\frac{dT(t)}{dt} = -\alpha A [T(t) - T_f] \quad (3.23)$$

with an initial condition:

$$T(t)|_{t=0} = T(0) = T_0 \quad (3.23a)$$

**Example 3.9:** Determine the time required to cool down a solid object initially at 80°C to 8°C. It is placed in a refrigerator with its interior air maintained at 5°C. If the coefficient  $\alpha = 0.002/\text{m}^2\text{-s}$  and the contact area between the solid and the cool air in the refrigerator is  $A = 0.2 \text{ m}^2$ .



**Solution:**

We have  $T_o = 80^\circ\text{C}$ ,  $T_f = 5^\circ\text{C}$ ,  $\alpha = 0.002/\text{m}^2\text{-s}$  and  $A = 0.2 \text{ m}^2$

Substituting the above into Equation (3.23) will lead to the following 1<sup>st</sup> order differential equation:

$$\frac{dT(t)}{dt} = -(0.002)(0.2)[T(t) - 5] = -0.0004[T(t) - 5] \quad (\text{a})$$

with the condition:  $T(0) = 80^\circ\text{C} \quad (\text{b})$

Equation (a) can be re-written as:  $\frac{dT(t)}{T(t) - 5} = -0.0004 dt \quad (\text{c})$

Integrating both sides of Equation (c):  $\int \frac{dT(t)}{T(t) - 5} = -0.0004 \int dt + c_1 \quad (\text{d})$

Leads to the solution:  $T(t) - 5 = e^{-0.0004t + c_1} = ce^{-0.0004t} \quad (\text{e})$

The integration constant  $c$  in Equation (e) can be obtained by the condition  $T(0) = 80^\circ\text{C}$  in Equation (b) with  $c = 75$ . consequently, the solution  $T(t)$  is:

$$T(t) = 5 + 75e^{-0.0004t} \quad (\text{f})$$

If  $t_e =$  required time for the solid to drop its temperature from 80°C to 8°C, we should have:

$$T(t_e) = 8 = 5 + 75e^{-0.0004t_e} \quad (\text{g})$$

Solve Equation (g) for  $t_e = 8047 \text{ s}$  or  $2.24 \text{ h}$

**What would you do if the required time to cool down the solid is too long?**

## Part 4

# Applications of First Order Differential Equations to Kinematic Analysis of Rigid Body Dynamics

We will demonstrate the application of 1<sup>st</sup> order differential equation in rigid body dynamics using Newton's Second Law

$$\sum \mathbf{F} = m\mathbf{a}$$



## Rigid Body Motion Under Strong Influence of Gravitation:

There are many engineering systems that involve dynamic behavior under strong influence of gravitation. Examples such as::



Rocket launch



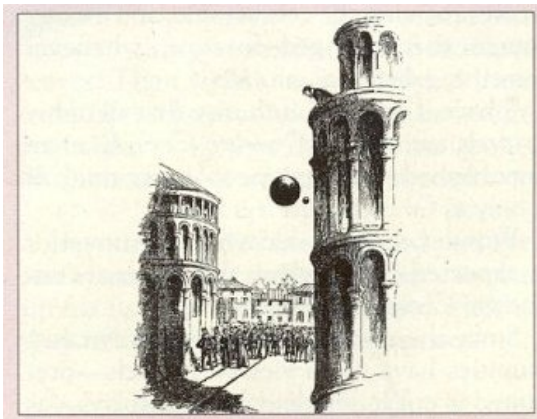
The helicopter



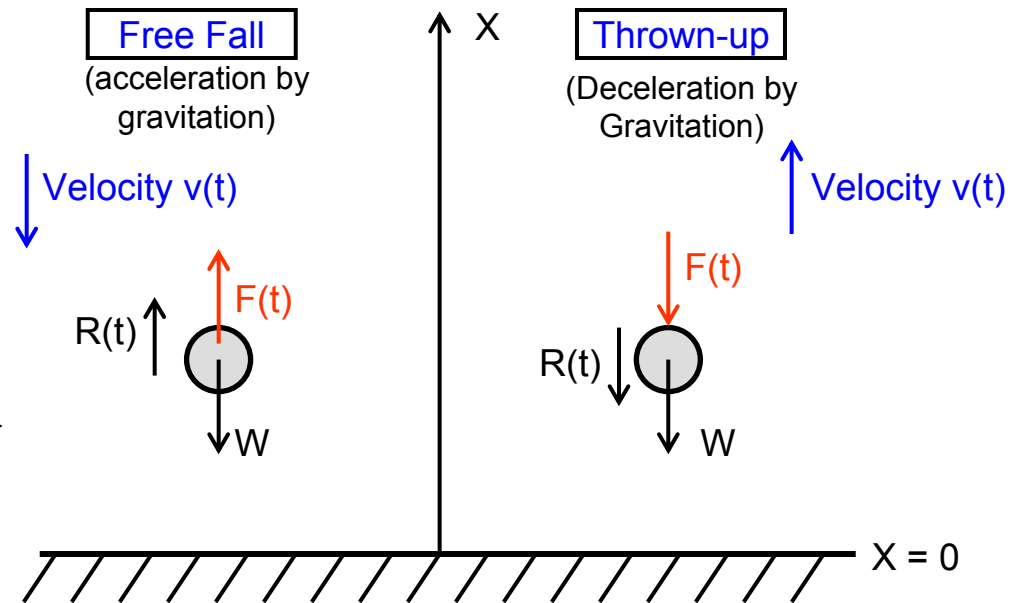
The paratroopers

## A Rigid Body in Vertical Motion

Galileo Galilei



Math  
Modeling



### Solution sought:

- The instantaneous position  $x(t)$
- The instantaneous velocity  $v(t)$
- The maximum height the body can reach, and the required time with initial velocity  $v_0$  in the “thrown-up” situation

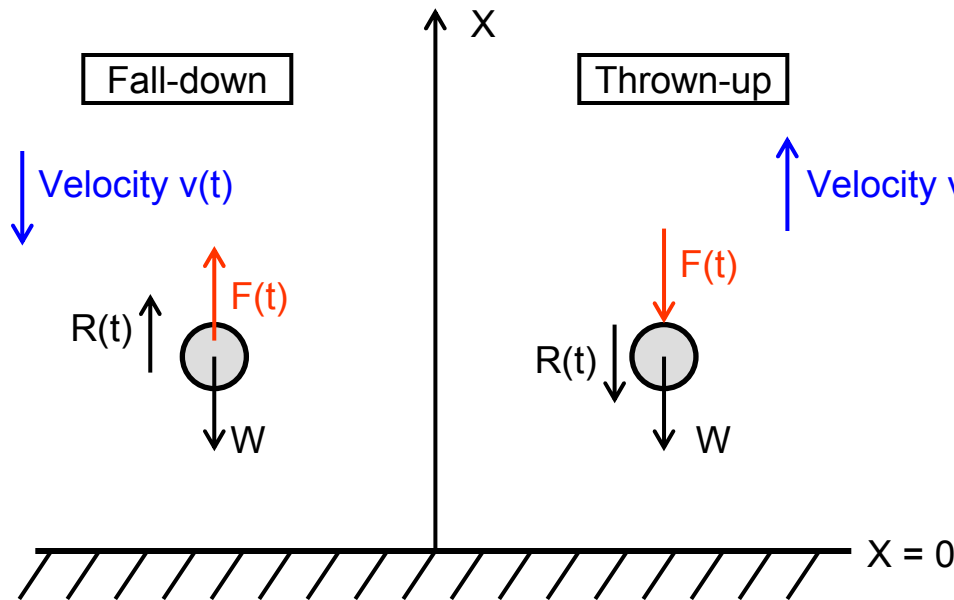
Galileo’s free-fall experiment from the leaning tower in Pisa, Italy, December 1612

These solutions can be obtained by first deriving the mathematical expression (a differential equation in this case), and solve for the solutions

By kinematics of a moving solid:

If the **instantaneous position** of the solid is expressed as  $x(t)$ , we will have:  $v(t) = \frac{dx(t)}{dt}$  to be the **instantaneous velocity**, and  $a(t) = \frac{dv(t)}{dt}$  to be the **instantaneous acceleration** (or deceleration)

## Derivation of Math Expression for Free-Fall of a Solid:



Referring to the [Left](#)-half of the diagram:

Forces acting on the falling solid at time  $t$  include:

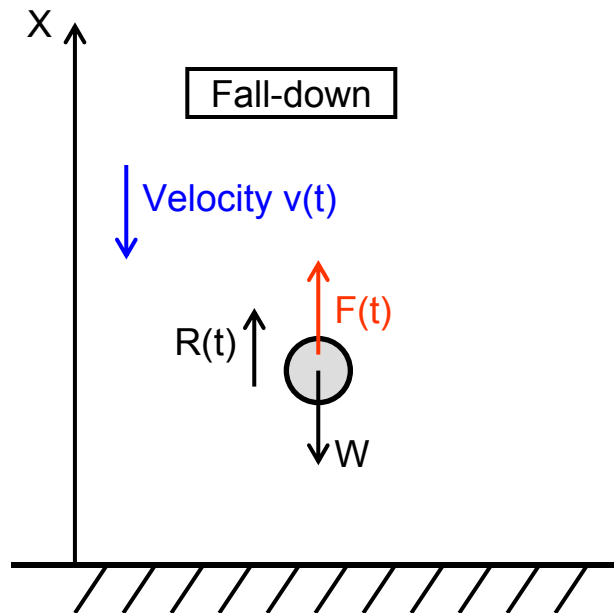
$W$  = the weight

$R(t)$  = the resistance of the air on the falling solid

$F(t)$  = the dynamic (inertia) force due to changing velocity of the fall under gravitational acceleration ( $g$ )

- (1) The weight of the body,  $\mathbf{w} = m\mathbf{g}$ , in which  $m$  = mass of the body, and  $g$  = gravitational acceleration ( $g = 9.81 \text{ m/s}^2$ ). This force always points towards the Earth.
- (2) The resistance encountered by the moving body in the medium such as air,  $\mathbf{R}(t) = c \mathbf{v}(t)$ , in which  $c$  is the proportional constant determined by experiments and  $v(t)$  is the instantaneous velocity of the moving body.  $R(t)$  act opposite to the direction of motion.
- (3) The dynamic (or inertia) force,  $\mathbf{F}(t) = m\mathbf{a}(t)$ , in which  $\mathbf{a}(t)$  is the acceleration (or deceleration with a negative sign) of the solid at time  $t$  – the [Newton's Second Law](#)  
[One should notice that  \$F\(t\)\$  carries a sign that is opposite to the acceleration](#)  
 (Tell me your personal experience??)

## Case A: Free-Fall of a solid:



The forces acting on the falling solid should be in **equilibrium at time t**:

By using a sign convention of forces along +ve x-axis being +ve, we have:

$$\sum F_x = 0$$

Leads to: 
$$\sum F_x = -W + R(t) + F(t) = 0$$

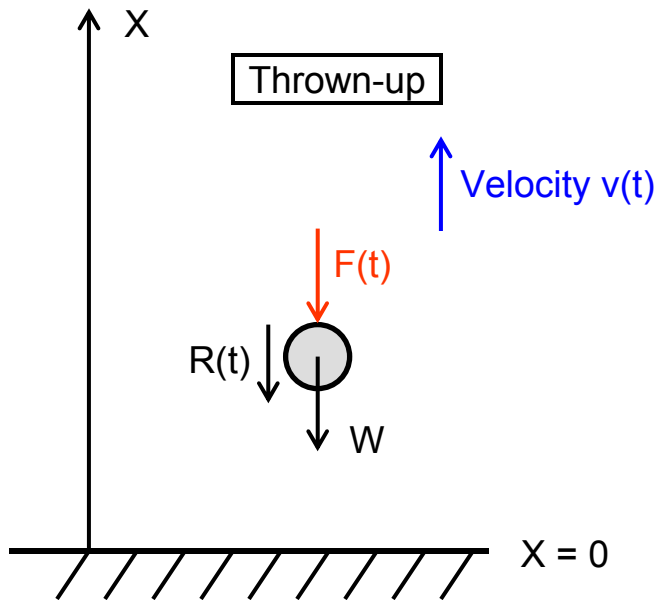
which yields to the following **1st order differential equation**:

$$\frac{dv(t)}{dt} + \frac{c}{m}v(t) = g \quad (3.29)$$

with an initial condition:  $v(t)|_{t=0} = v(0) = 0 \quad (a)$



## Case B: Throw-up of a solid with initial velocity $v_0$ :



As in the case of “Free-fall,” the forces acting on the Up-moving solid should be in **equilibrium at time  $t$** :

By using a sign convention of forces along +ve x-axis Being +ve for the forces, we have:

$$\sum F_x = 0$$

Leads to: 
$$\sum F_x = -W - R(t) - F(t) = 0$$

with  $W = mg$ ,  $R(t) = cv(t)$ , and  $F(t) = ma(t) = m \frac{dv(t)}{dt}$

A 1<sup>st</sup> order differential equation is obtained:

$$\frac{dv(t)}{dt} + \frac{c}{m}v(t) = -g \quad (3.25)$$

with an initial condition:

$$v(t)|_{t=0} = v(0) = v_0 \quad (a)$$

The solution of Equation (3.25) is obtained by comparing it with the [typical 1st order differential equation](#) in Equation (3.6) with solution in Equation (3.7):

$$\frac{dv(t)}{dt} + p(t)u(t) = g(t) \quad (3.6)$$

with a solution:

$$v(t) = \frac{1}{F(t)} \int F(t) g(t) dt + \frac{K}{F(t)} \quad (3.7)$$

The present case has:

$$F(t) = e^{\int p(t) dt}, \quad p(t) = \frac{c}{m} \quad \text{and} \quad g(t) = -g$$

Consequently, the solution of Equation (3.25) has the form:

$$v(t) = \frac{1}{e^{\frac{ct}{m}}} \int e^{\frac{ct}{m}} (-g) dt + \frac{K}{e^{\frac{ct}{m}}} = -\frac{mg}{c} + Ke^{-\frac{c}{m}t} \quad (3.26)$$

In which the constant K is determined by the given initial condition in Equation (a), with:

$$K = v_o + \frac{mg}{c} \quad (b)$$

The complete solution of Equation (3.25) with the substitution of K in Equation (b) into Equation (3.26):

$$v(t) = -\frac{mg}{c} + \left( v_o + \frac{mg}{c} \right) e^{-\frac{c}{m}t} \quad (3.27)$$

The instantaneous position of the rigid body at time  $t$  can be obtained by:

$$x(t) = \int_0^t v(t) dt \quad (3.27a)$$

where the velocity function  $v(t)$  is given in Equation (3.27)

The time required for the rigid body to reach the maximum height  $t_m$  is the time at which the upward velocity of the body reduced to zero. Mathematically, it is expressed as:

$v(t_m) = 0$  in Equation (3.27):

$$v(t_m) = 0 = -\frac{mg}{c} + \left( v_o + \frac{mg}{c} \right) e^{-\frac{c}{m}t_m}$$

Solve  $t_m$  from the above equation, resulting in:

$$t_m = \frac{m}{c} \ln \left( 1 + \frac{v_o c}{mg} \right) \quad (3.28)$$

### Example 3.10



An armed paratrooper with ammunitions weighing 322 lbs jumped with initial velocity from an airplane at an attitude of 10,000 feet with negligible side wind.

Assume the air resistance  $R(t)$  the paratrooper encountered with is:  $R(t) = c[V(t)]^2$  in which the coefficient  $c = 15$ . Determine:

- The appropriate equation for the instantaneous descending velocity of the paratrooper, and
- The function of the descending velocity  $v(t)$
- The time required to land
- The impact velocity upon landing

### Solution:

(a) Differential equation for the velocity  $v(t)$ :

The total mass of the falling body,  $m = 322/32.2 = 10$  slug, and the air resistance,  $R(t) = cv(t) = 15[v(t)]^2$   
The instantaneous descending velocity,  $v(t)$  can be obtained by using Equation (3.29) as:

$$\frac{dv(t)}{dt} + \frac{15[v(t)]^2}{10} = 32.2 \quad (a)$$

or in another form:  $10 \frac{dv(t)}{dt} = 322 - 15[v(t)]^2 \quad (b)$

with the condition:  $v(0) = 0 \quad (c)$

(b) The solution of Equation (b) with the condition in Equation (c) is:

$$v(t) = \frac{4.634(e^{13.9t} - 1)}{e^{13.9t} + 1} \quad (d)$$

(Refer to P. 75 of the printed notes for procedure to the above solution)



(c) The descending distance of the paratrooper can be obtained by Equation (3.27a):

$$x(t) = \int_0^t v(t) dt = \int_0^t \frac{4.634(e^{13.9t} - 1)}{e^{13.9t} + 1} dt$$

The above integral is not available in math handbook, and a numerical solution by computer is required.

Once the expression of  $x(t)$  is obtained, we may solve for the time required for the paratrooper to reach the ground from a height of 10,000 feet by letting:

$$x(t_g) = 10000$$

in which  $t_g$  is the time required to reach the ground.

Another critical solution required in this situation is the velocity of the rigid body upon landing (i.e. the **impact velocity** of the paratrooper). It can be obtained by evaluating the velocity in Equation (d) at time  $t_m$ :

$$v(t_g) = \frac{4.634(e^{13.9t_g} - 1)}{e^{13.9t_g} + 1}$$