Math 3C Homework 3 Solutions

Ilhwan Jo and Akemi Kashiwada

ilhwanjo@math.ucla.edu, akashiwada@ucla.edu

Assignment: Section 12.3 Problems 2, 7, 8, 9, 11, 13, 15, 18, 21, 22, 29, 31, 32

2. You draw three cards from a standard deck of 52 cards. Find the probability that the third card is a club given that the first two cards were clubs.

Solution

Let A be the event that the third card is a club and let B the event that the first two cards were clubs. Then the probability that all three cards are clubs is

$$P(A \cap B) = \frac{\binom{13}{3}}{\binom{52}{3}},$$

and the probability of the first two cards being clubs is

$$P(B) = \frac{\binom{13}{2}}{\binom{52}{2}}.$$

So the conditional probability that the third card is a club given that the first two were clubs is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{\binom{13}{3}}{\binom{52}{3}}}{\frac{\binom{13}{2}}{\binom{52}{2}}} = \frac{11}{50}.$$

Another way of doing this problem is noticing that the probability of the first card being a club is $\frac{13}{52}$ and the probability of the second card being a club is $\frac{12}{51}$. As 2 clubs have already been removed from the deck, there are only 11 clubs left out of the 50 remaining cards. So the probability that the third card is a club given that the two we already removed are clubs is $\frac{11}{50}$.

7. You roll two fair dice. Find the probability that the first die is a 4 given that the sum is 7.

Solution

Let A denote the event that the first die shows 4 and B denote the event that the sum of the dice is 7. Notice that for any number the first die shows, there is only one number the second die can show to make the sum 7 (e.g. if the first die shows 5 then the second die must show 2 to make the sum 7). So there are a total of 6 ways to make the sum of the two dice equal 7 so

$$P(B) = \frac{6}{36}.$$

Now, if the first die shows 4 there is only one way to make the sum of both dice equal 7 which means

$$P(A \cap B) = \frac{1}{36}.$$

Therefore, the probability that the first die shows 4 given that the sum is 7 is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}.$$

8. You roll two fair dice. Find the probability that the first die is a 5 given that the minimum of the two numbers is 3.

Solution

Let A denote the event that the first die shows 5 and B denote the event that the minimum of both dice is 3. If the minimum of both dice is 3, then at least one of the two dice must be 3 and both must be greater than or equal to 3. So

$$P(B) = \frac{7}{36}.$$

The probability that the first die shows 5 and the second shows 3 is

$$P(A \cap B) = \frac{1}{36}.$$

So the probability that the first die is 5 given that the minimum of both dice is 3 is

$$P(A|B) = \frac{1/36}{7/36} = \frac{1}{7}.$$

9. You toss a fair coin three times. Find the probability that the first coin is heads given that at least one head occurred.

Solution

Let A be the event that the first coin is heads and B be the event that at least one head occurred. This means B^c is the event that no heads occur which happens with probability $\frac{1}{8}$. So we get

$$P(B) = 1 - P(B^c) = 1 - \frac{1}{8} = \frac{7}{8}.$$

If the first coin is a head, then the other two flips can be either heads or tails to satisfy the requirement that at least one head occurred. Given that we know that

$$P(A \cap B) = \frac{4}{8}.$$

So the probability that the first coin is a heads given that at least one head occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{4/8}{7/8} = \frac{4}{7}.$$

11. A screening test for a disease shows a positive test result in 90% of all cases when the disease is actually present and in 15% of all cases when it is not. Assume that the prevalence of the disease is 1 in 100. If the test is administered to a randomly chosen individual, what is the probability that the result is negative?

Solution

One way of solving this problem is to find the probability that the result is positive then subtracting that from 1. Let A be the event that the result is negative so A^c is the probability that the result is positive. Also let B_1 be the event that the person is infected and B_2 be the event that the person is not infected. Using the law of total probability we know that the probability that the result is positive is

$$P(A^c) = P(A^c|B_1)P(B_1) + P(A^c|B_2)P(B_2).$$

¿From the given information, we know that $P(B_1) = 1/100$, $P(B_2) = 99/100$, $P(A^c|B_1) = 0.90$ and $P(A^c|B_2) = 0.15$. So

$$P(A^c) = 0.90(1/100) + 0.15(99/100) = 0.1575$$

Since A is the event that the test result is negative and we can calculate that by

$$P(A) = 1 - P(A^c) = 1 - 0.1575 = 0.8425.$$

Another other way of solving this problem is to directly calculate the probability the result is negative. Using the same definitions of A, B_1, B_2 as above, we can use the law of total probability to get

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2).$$

Then $P(B_1) = 1/100$ and $P(B_2) = 99/100$. Now the probability that the test is negative given that the person is infected is $P(A|B_1) = 1 - 0.90 = 0.10$. Similarly the probability that the result is negative if the person is not infected is $P(A|B_2) = 1 - 0.15 = 0.85$. So

$$P(A) = 0.10(1/100) + 0.85(99/100) = 0.8425.$$

13. A drawer contains three bags numbered 1-3. Bag 1 contains two blue balls, bag 2 contains 2 green balls, and bag 3 contains one blue and one green ball. You choose one bag at random and take out one ball. Find the probability that the ball is blue.

Solution

Let A be the event that we choose a blue ball and let B_i be the event that bag i is chosen (that is, B_1 is the event that we choose bag 1, B_2 is the event that we choose bag 2 and B_3 is the event that we choose bag 3). Using the law of total probability we see that

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3).$$

Since each bag is equally likely of being chosen, $P(B_i) = \frac{1}{3}$ for $i = 1, 2, 3$. If we choose bag 1, then the probability of getting a blue ball is 1 and if we choose bag 2 the probability of getting a blue ball is 0. But if we choose bag 3, there is one of each color ball so the probability of getting the blue ball is $\frac{1}{2}$. Therefore, we get

$$P(A) = (1)\frac{1}{3} + (0)\frac{1}{3} + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

15. You pick two cards from a standard deck of 52 cards. Find the probability that the second card is an ace. Compare this to the probability that the first card is an ace.

Solution

Let A be the event that the second card is an ace, B_1 be the event that the first card is an ace and B_2 be the event that the first card is not an ace. Since the B_i partition the sample space, we can use the law of total probability to get

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2).$$

The probability of the first card being an ace is $\frac{4}{52} = \frac{1}{13}$ so the probability of the first card not being an ace is $1 - \frac{1}{13} = \frac{12}{13}$. If the first card was an ace, then there are only 3 aces left out of the 51 cards so $P(A|B_1) = \frac{3}{51} = \frac{1}{17}$. Otherwise, if the first card was not an ace, there are still 4 aces left so $P(A|B_2) = \frac{4}{51}$. So we have

$$P(A) = \left(\frac{1}{17}\right)\left(\frac{1}{13}\right) + \left(\frac{4}{51}\right)\left(\frac{12}{13}\right) = \frac{1}{13}.$$

Notice that we said the probability of the first card being an ace was $P(B_1) = \frac{4}{51} = \frac{1}{13}$ so the probability of the second card being an ace is the same as the probability of the first card being an ace.

18. Suppose that you have a batch of red- and white-flowering pea plants where all three genotypes CC, Cc, and cc are equally represented. You pick one plant at random and cross it with a white-flowering pea plant. What is the probability that the offspring will have red flowers?

Solution

Let A be the event that the offspring has red flowers, B_1 the event that you pick a plant of genotype CC, B_2 the event that you pick a plant of genotype Cc, and B_3 the event that you pick a plant of genotype cc. Since they are equally represented, $P(B_i) = \frac{1}{3}$. Using the law of total probability,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3).$$

Since $P(A|B_1) = 1, P(A|B_2) = \frac{1}{2}, P(A|B_3) = 0$,

$$P(A) = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{2}.$$

21. You are dealt on card from a standard deck of 52 cards. If A denotes the event that the card is a spade and if B denotes the event that the card is an ace, determine whether A and B are independent.

Solution

$P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{13}$. The probability $P(A \cap B)$ that the card is the spade ace is $\frac{1}{52}$. Hence

$$P(A \cap B) = \frac{1}{52} = P(A)P(B)$$

and A and B are independent.

22. You are dealt two cards from a standard deck of 52 cards. If A denotes the event that the first card is an ace and B denotes the event that the second card is an ace, determine whether A and B are independent.

Solution

From problem **15**, we know that $P(A) = P(B) = \frac{1}{13}$. The probability $P(A \cap B)$ that the two cards are aces is $\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{13 \cdot 17}$. Hence

$$P(A \cap B) \neq P(A)P(B)$$

and A and B are not independent.

29. Assume that the probability that an insect species lives more than five days is 0.1. Find the probability that in a sample of size 10 of the this species at least one insect will still be alive after 5 days.

Solution

The probability that an insect lives more than five days is independent of the other insects. So the probability that all of the ten insects will die within 5 days is $(0.9)^{10}$. The probability that at least one insect will be alive after 5 days is therefore $1 - (0.9)^{10}$.

31. A screening test for a disease shows a positive result in 95% of all cases when the disease is actually present and in 10% of all cases when it is not. If the prevalence of the disease is 1 in 50, and an individual tests positive, what is the probability that the individual actually has the disease?

Solution

Let A denote the probability that the individual has the disease and B denote the probability that the test shows a positive result. Then we need to find $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Since the prevalence is $\frac{1}{50}$,

$$P(A \cap B) = 0.95 \cdot \frac{1}{50} = 0.019,$$

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^C)P(A^C) \\ &= 0.95 \cdot \frac{1}{50} + 0.1 \cdot \left(1 - \frac{1}{50}\right) = 0.117. \end{aligned}$$

Therefore, $P(A|B) = \frac{0.019}{0.117} \approx 0.1624$.

32. A screening test for a disease shows a positive result in 95% of all cases when the disease is actually present and in 10% of all cases when it is not. If a result is positive, the test is repeated. Assume that the second test is independent of the first test. If the prevalence of the disease is 1 in 50, and an individual tests positive twice, what is the probability that the individual actually has the disease?

Solution

Let A denote the probability that the individual has the disease, B_1 the event that the first result is positive, and B_2 the event that the second result is positive. We need to find $P(A|B_1 \cap B_2) = \frac{P(A \cap B_1 \cap B_2)}{P(B_1 \cap B_2)}$. We assume that the two tests are independent. What this means is the following *conditional independency*:

$$P(B_1 \cap B_2|A) = P(B_1|A)P(B_2|A) = (0.95)^2$$
$$P(B_1 \cap B_2|A^C) = P(B_1|A^C)P(B_2|A^C) = (0.1)^2.$$

So, we have

$$\begin{aligned}
P(B_1 \cap B_2 \cap A) &= P(B_1 \cap B_2|A)P(A) = P(B_1|A)P(B_2|A)P(A) \\
&= (0.95)^2 \cdot \frac{1}{50} = 0.01805 \\
P(B_1 \cap B_2) &= P(B_1 \cap B_2|A)P(A) + P(B_1 \cap B_2|A^C)P(A^C) \\
&= (0.95)^2 \cdot \frac{1}{50} + (0.1)^2 \cdot \left(1 - \frac{1}{50}\right) = 0.02785.
\end{aligned}$$

Therefore, $P(A|B_1 \cap B_2) = \frac{0.01805}{0.02785} \approx 0.6481$.