5 Integrals to infinity

Philosophy[3]

The only way we can talk about infinity is through limits.

EXAMPLE: we try to make sense of ∞/∞ or $\infty \cdot 0$ but there is no one rule for what this should be. When it comes up as a limit, such as $\lim_{x\to\infty} x^2/e^x$ then at least it is well defined. To evaluate the limit we need to use L'Hôpital's rule or some other means.

EXAMPLE: we try to make sense of $\sum_{n=1}^{\infty} a_n$. There is no already assigned meaning for summing infinitely many things. We **defined** this as a limit, which in each case needs to be evaluated:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \,.$$

It is the same when one tries to integrate over the whole real line. We define this as integrating over a bigger and bigger piece and taking the limit. In fact the definition is even pickier than that. We only let one of the limits of integration go to zero at a time. We define $\int_0^\infty f(x)\,dx$ to be $\lim_{M\to\infty}\int_0^M f(x)\,dx$. In general, for any lower limit b, we can define $\int_b^\infty f(x)\,dx$ to be $\lim_{M\to\infty}\int_b^M f(x)\,dx$. But if we want both limits to be infinite then we define the two parts separately. The value of $\int_{-\infty}^{\infty} f(x)\,dx$ is **defined** to equal

$$\lim_{M\to\infty}\int_b^M f(x)\,dx + \lim_{M\to-\infty}\int_M^b f(x)\,dx \,.$$

If either of these limits does not exist then the whole integral is defined not to exist. At this point you should be bothered by three questions.

1. What is b? Does it matter? How do you pick it?

2. If we get $-\infty+\infty$, shouldn't that possibly be something other than "undefined"?

3. Why do we have to split it up in the first place?

[3] You can skip this section if you care about computing but not about meaning.

The answer to the first question is, pick b to be anything, you'll always get the same answer. That's because if I change b from, say, 3 to 4, then the first of the two integrals loses a piece: $\int_3^4 f(x)\,dx$. But the second integral gains this same piece, so the sum is unchanged. This is true even if one or both pieces is infinite. Adding or subtracting the finite quantity $\int_3^4 f(x)\,dx$ won't change that.

The answer to the second question is that this is really our choice. If we allow infinities to cancel, we have to come up with some very careful rules. If you would like to study this sort of thing, consider being a Math major and taking the Masters level sequence Math 508–509. For the rest of us, we'll avoid it. This will help to avoid the so-called re-arrangement paradoxes, where the same quantity sums to two different values depending on how you sum it.

The last question is also a matter of definition. Consider the sign function

$$f(x) = \text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

On one hand, $\int_{-M}^{M} f(x)\,dx$ is always zero, because the postive and negative parts exactly cancel. On the other hand, $\int_b^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are always undefined. Do we want the answer for the whole integral $\int_{-\infty}^\infty f(x)\,dx$ to be undefined or zero? There is no intrinsically correct choice here but it is a lot safer to have it undefined. If it has a value, one could make a case for values other than zero by centering the integral somewhere else, for example $\int_{7-M}^{7+M} f(x)\,dx$ is always equal to 14.

5.1 Type I Improper integrals and convergence

The central question ofthis section is: how do we tell whether a limit such as $\int_b^\infty f(x)\,dx$ exists, and if so, what the value is?

Case 1: you know how to compute the definite integral

Suppose $\int_b^M f(x)\,dx$ is something for which you know how to compute an explicit formula. The formula will have M in it. You have to evaluate the limit as $M \to \infty$. How do you do that? There is no one way, but that's why we studied limits before. Apply what you know. What about b, do you have to take a limit in b as well? I hope you already knew the answer to that. In this definition, b is any fixed number. You don't take a limit.

Here are some cases you should remember.

Type of integral	Condition for convergence
$\int_b^\infty e^{kx}\,dx$	
$\int_b^\infty x^p\,dx$	
$\int_b^\infty \frac{(\ln x)^q}{x}\,dx$	

You will work out these cases in class: write each as a limit, evaluate the limit, state whether it converges, which will depend on the value of the parameter, k, p or q. Go ahead and pencili them in once you've done this. The second of these especially, is worth remembering because it is not obvious until you do the computation where the break should be between convergence and not.

Case 2: you don't know how to compute the integral

In this case you can't even get to the point of having a difficult limit to evaluate. So probably you can't evaluate the improper integral. But you can and should still try to answer whether the integral has a finite value versus being undefined. This is where the comparison tests come in. You buildup a library of cases where you do know the answer (Case 1) and then for the rest of functions, you try to compare them to functions in your library.

Sometimes a comparison is informative, sometimes it isn't. Suppose that f and g are positive functions and $f(x) \le g(x)$. Consider several pieces of information you might have about these functions.

(a) $\int_b^\infty f(x)\,dx$ converges to a finite value L.

(b) $\int_b^\infty f(x)\,dx$ does not converge.

(c) $\int_b^\infty g(x)\,dx$ converges to a finite value L.

(d) $\int_b^\infty g(x)\,dx$ does not converge.

In which cases can you conclude something about the other function? We are doing this in class. Once you have the answer, either by working it out yourself or from the class discussion, please pencil it in here so you'll have it for later reference. This is essentially the **direct comparison test** at the bottom of page 510 of the textbook.

Even better comparison tests

Here are two key ideas that help your comparison tests work more of the time, based on the fact that the question "convergence or not?" is not sensitive to certain things.

(1) It doesn't matter if $f(x) \leq g(x)$ for every single x as long as the inequality is true from some point onward. For example, if $f(x) \leq g(x)$ once $x \geq 100$, then you can apply the comparison test to compare $\int_b^\infty f(x)\,dx$ to $\int_b^\infty g(x)\,dx$ as long as $b \geq 100$. But even if not, once you compare $\int_{100}^\infty f(x)\,dx$ to $\int_{100}^\infty g(x)\,dx$, then adding the finite quantity $\int_b^{100} f(x)\,dx$ or $\int_b^{100} g(s)\,dx$ will not change whether either of these converges.

(2) Multiplying by a constant does not change whether an integral converges. That's because if $\lim_{M\to\infty} \int_b^M f(x)\,dx$ converges to the finite constant L then $\lim_{M\to\infty} \int_b^M Kf(x)\,dx$ converges to the finite constant KL.

Putting these two ideas together leads to the conclusion that if $f(x) \leq Kg(x)$ from some point onward and $\int_b^\infty g(x)\,dx$ converges, then so does $\int_b^\infty f(x)\,dx$. The theorem we just proved is:

> *If f and g are positive functions on some interval (b, ∞) and if there are some constants M and K such that*
>
> $$f(x) \leq Kg(x) \text{ for all } x \geq K$$
>
> *then convergence of the integral $\int_b^\infty g(x)\,dx$ implies convergence of the integral $\int_b^\infty f(x)\,dx$.*

We used to teach this as the main theorem in this section but students said it was too hard because of the phrase "there exist constants k and M." What are K and M, they would ask, and how do we find them? You can ask about this if you want, but don't worry, it's not required. Instead, just remember, if f is less than any multiple of g from some point on, you can use the comparison test, same as if $f \leq g$.

EXAMPLE: $f(x) = \dfrac{1}{3e^x - 5}$. Actually $f(x) \sim 3e^{-x}$ but all we need to know is that $f(x) \leq 4e^{-x}$ once x is large enough (in this case large enough so that $3e^x \geq 20$). We know that $\int^\infty e^{-x}\,dx$ converges, hence $\int^\infty f(x)\,dx$ converges as well.

5.2 Probability densities

This section is well covered in the book. It is also long: eleven pages. However, it is not overly dense. I will expect you to get most of what you need out of the textbook and just summarize the highlights. I will cover the philosophy in lecture (questions like, "What is probability really?") and stick to the mathematical points here.

A nonnegative continuous function f on a (possibly infinite) interval is a **probability density function** if its integral is 1. If we make a probability model in which some quantity X behaves randomly with this probability density, it means we believe the probability of finding X in any smaller interval $[a, b]$ will equal $\int_a^b f(x)\,dx$. Often the model tells us the form of the function f but not the multiplicative constant. If we know that $f(x)$ should be of the form Cx^{-3} on $[1, \infty)$ then we would need to find the right constant C to make this a probability density, meaning that it makes $\int_1^\infty Cx^{-3}\,dx$ equal to 1.

Several quantities associated with a probability distribution are defined in the book: mean (page 520), variance (page 522), standard deviation (page 522) and median (page 521). Please know these definitions! I will talk in class about their interpretations (that's philosphy again).

There are a zillion different functions commonly used for probability densities. Three of the most common are named n the Chapter: the exponential (page 521), the uniform (page 523) and the normal (page 524). It is good to know how each of these behaves and in what circumstances each would arise as a model in an application.

The exponential distribution

The exponential distribution has a parameter μ which can be any positive real number. Its density is $(1/\mu)e^{-x/\mu}$ on the positive half-line $[0, \infty)$. This is obviously the same as the density Ce^{-Cx} (just take $C = 1/\mu$) but we use the parameter μ rather than C because a quick computation shows that the mean of the distribution is equal to μ: integrate by parts with $u = x$ and $dv = \mu^{-1}e^{-x/\mu}$ to get

$$\int_0^\infty \frac{x}{\mu} e^{-x/\mu}\, dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x/\mu}\, dx = 0 + \left(-\mu e^{-x/\mu}\right)\Big|_0^\infty = \mu\,.$$

Note that when we evaluate these quantities at the endpoints zero and infinity we are really taking a limit for the infinite endpoint.

The exponential distribution has a very important "memoryless" propoerty. If X has an exponential density with any parameter and is interpreted as a waiting time, then once you know it didn't happen by a certain time t, the amount of further time it will take to happen has the same distribution as X had originally. It doesn't get any more or any less likely to happen in the the interval $[t, t+1]$ than it was originally to happen in the interval $[0, 1]$.

The median of the exponential distribution with mean μ is also easy to compute. Solving $\int_0^M \mu^{-1}e^{-x/\mu}\, dx = 1/2$ gives $M = \mu \cdot \ln 2$. When X is a random waiting time, the interpretation is that it is equally likely to occur before $\ln 2$ times its mean as after. So the median is significantly less than the mean.

Any of you who have studied radioactive decay know that each atom acts randomly and independently of the others, decaying at a random time with an exponential distribution. The fraction remaining after time t is the same as the probability that each individual remains undecayed at time t, namely $e^{-t/\mu}$, so another interpretation for the median is the **half-life**: the time at which only half the original substance remains.

The uniform distribution

The uniform distribution on the interval $[a, b]$ is the probability density whose density is a constant on this interval: the constant will be $1/(b - a)$. This is often thought of the least informative distribution if you know that the the quantity must be between the values a and b. The mean and median are both $(a+b)/2$. Example 11 on page 523 of the book discusses why the angle of a spinner should be modeled by a uniform random variable.

The normal distribution

The normal density with mean μ and standard deviation σ is the density

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}\,.$$

The **standard normal** is the one with $\mu = 0$ and $\sigma = 1$. There is a very cool mathematical reason for this formula, which we will not go into. When a random variable is the result of summing a bunch of smaller random variables all acting independently, the result is usually well approximated by a normal. It is possible (though very tricky) to show that the definite integral of this density over the whole real line is in fact 1 (in other words, that we have the right constant to make it a probability density).

Annoyingly, there is no nice antiderivative, so no way in general of computing the probability of finding a normal between specified values a and b. Because the normal is so important in statistical applications, they made up a notation for the antiderivative in the case $\mu = 0, \sigma = 1$:

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx\,.$$

So now you can say that the probability of finding a standard normal between a and b is exactly $\Phi(b) - \Phi(a)$. In the old, pre-computer days, they published tables of values of Φ. It was reasonably efficient to do this because you can get the antiderivative F of any other normal from the one for the standard normal by a linear substition: $F(x) = \Phi((x-\mu)/\sigma)$. Please be sure to read Example 13 on page 525 where this is explained in considerable detail.

5.3 Type II improper integrals

A type II improper integral occurs if we try to integrate $\int_a^b f(x)\,dx$ but somewhere on the interval $[a,b]$ the function f becomes discontiuous. You may not have realized at the time but the definition of the definite integral required f to be continuous on the interval over which you integrate. The most common way that f might fail to be continuous is if it becomes unbounded (e.g., goes to infinity) in which case you can imagine that the Riemann sums defining the integral could be very unstable (for example, there is no upper Riemann sum if the function f does not have a finite maximum).

Here are some examples: 1) integrating $p(x)/q(x)$ on an interval where q has a zero; 2) integrating $\ln(x)$ on an interval containing zero; 3) integrating $\tan x$ on an interval containing $\pi/2$.

Again, the way we handle this is to integrate only over intervals where the function is continuous, then take limits to approach the bad value(s). If the bad value occurs at the endpoint of the interval of integration, it is obvious how to take a limit. Suppose, for example, that f is discontinuous at b. Then define

$$\int_a^b f(x)\,dx := \lim_{c\to b^-} \int_a^c f(x)\,dx\,.$$

Note that this is a one-sided limit. We are not interested in letting c be a little bigger than b, only a little smaller. Similarly, if the discontinuity is at the left endpoint, a, we define

$$\int_a^b f(x)\,dx := \lim_{c\to a^+} \int_c^b f(x)\,dx\,.$$

Notice in both cases I have used the notation ":=" for "is defined as", to emphasize that this is a definition.

If there is a single value c in the interior of the interval, at which f becomes discontinuous, then $\int_a^b f(x)\,dx$ is defined by breaking into two integrals, one from a to c and one from c to b. Each of these has a discontinuity at an endpoint, which we have already discussed how to handle, and we then add the two results. Again, if either one is undefined, then the whole thing is undefined.

$$\int_a^b f(x)\,dx := \lim_{s\to c^-} \int_a^s f(x)\,dx + \lim_{s\to c^+} \int_s^b f(x)\,dx\,.$$

If there is more than one bad point, then we have to break into more than two intervals.

We do the same thing for testing convergence of Type II improper integrals as we did for Type I, namely we find a bunch that we can evaluate exactly and for the rest we compare to one of these. Again, the most useful cases turn out to be powers x^p, with $p = -1$ being the borderline case. One then has to learn the art of comparing to powers. Specifically, if you know p for which $f(x) \sim |x - a|^p$ as $x \to a$, or even that $f(x) \sim C|x - a|^p$ as $x \to a$, then you will be able to determine convergence. We will do some work on this in class but you may also want to check out Examples 4, 5, and 7 on pages 508–511 of the textbook.