# Independence and Conditional Probability

CS 2800: Discrete Structures, Fall 2014

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# Independence of Events (revisited)

$$\mathrm{P}(A \cap B) \;=\; \mathrm{P}(A)\,\mathrm{P}(B)$$

# Independence of Events (revisited)

$$P(A \cap B) = P(A)\,P(B)$$

Mathematical **definition** of independence

# Independence of Events (revisited)

$$P(A \cap B) = P(A)\,P(B)$$

Mathematical **definition** of independence

A and B are independent if and only if this relation holds

# WTF?

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Why does this even make sense?

# Independence of Events (revisited)

$$P(A \cap B) = P(A)P(B)$$

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if and only if

$$P(B) = \frac{P(A \cap B)}{P(A)}$$

(assuming $P(A) \neq 0$)

# Independence of Events (revisited)

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if and only if

conditional probability

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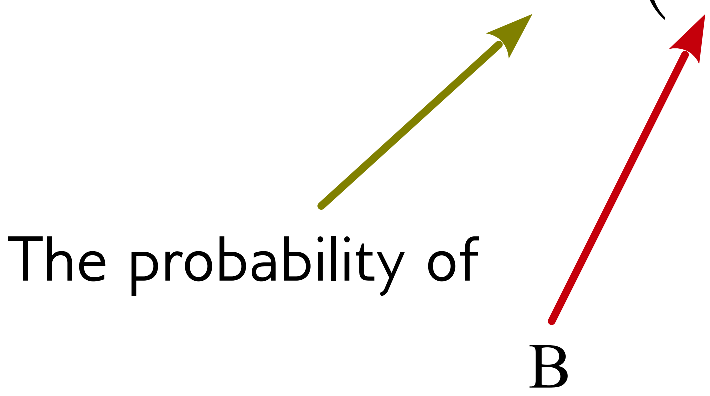

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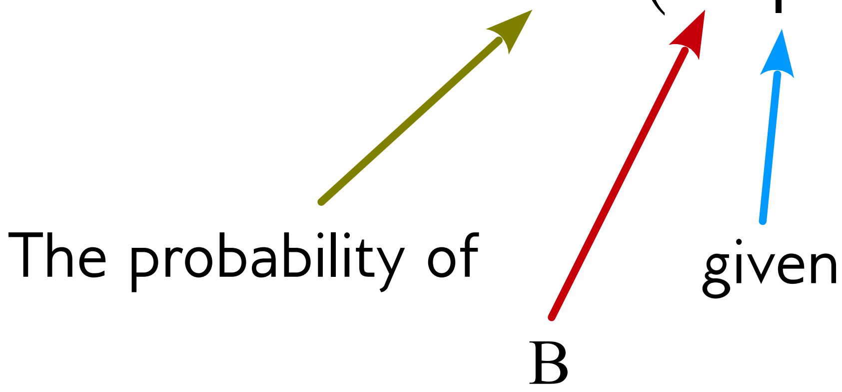

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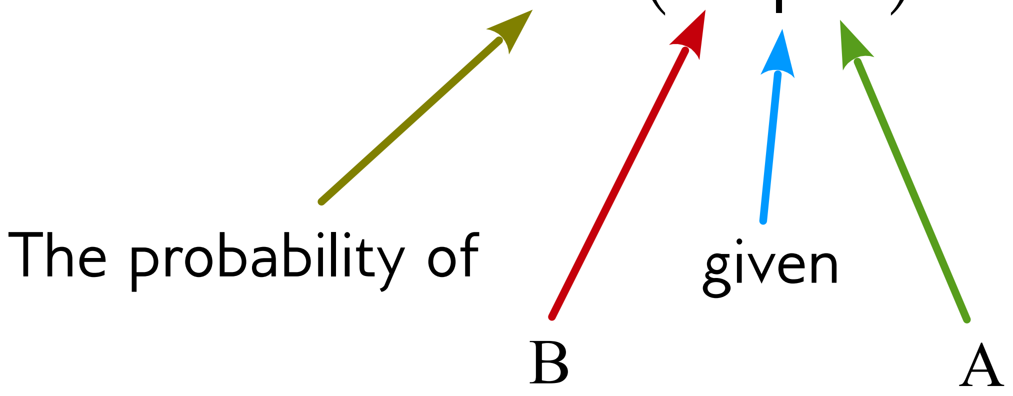

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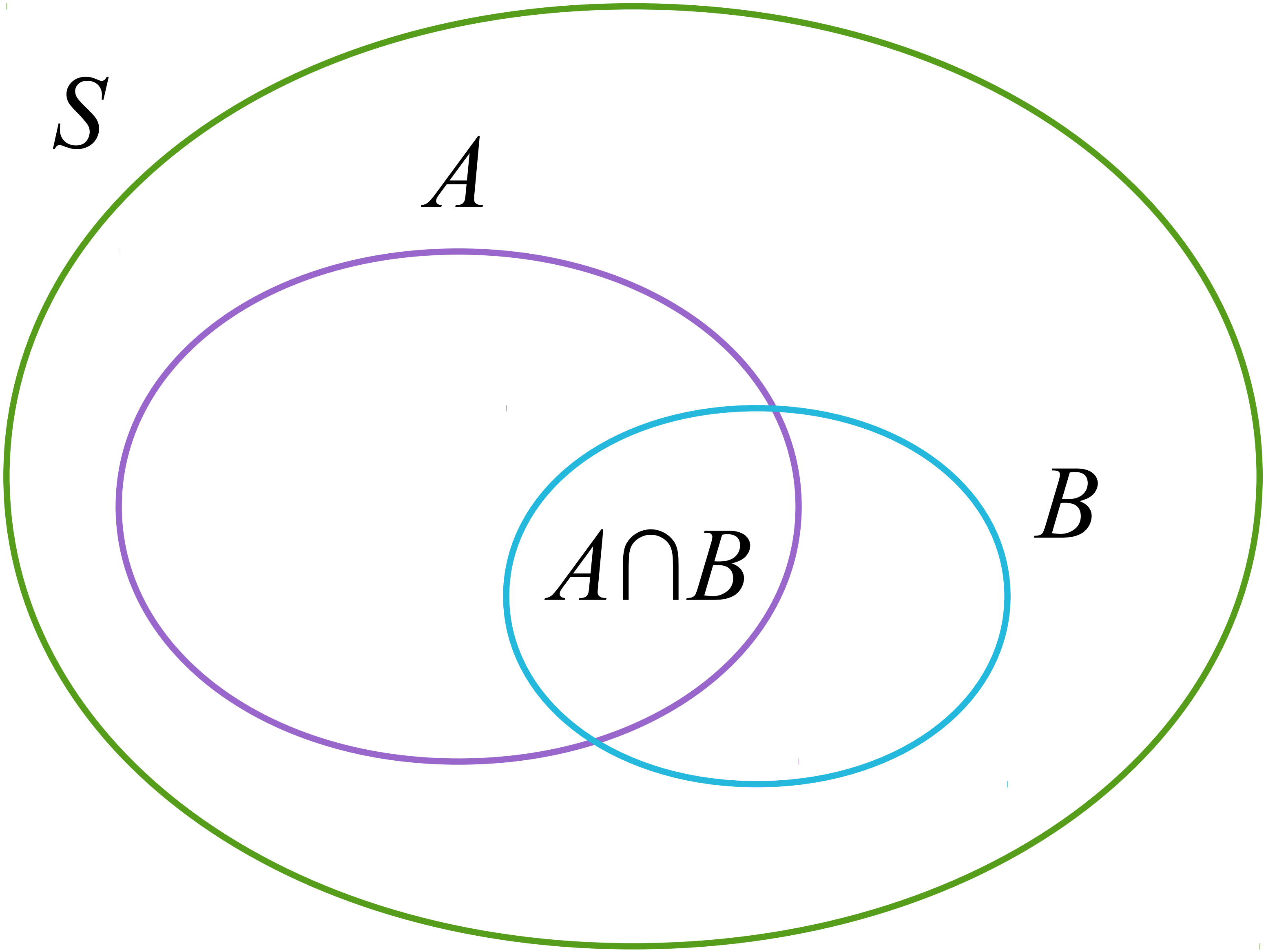
# WTF #2?

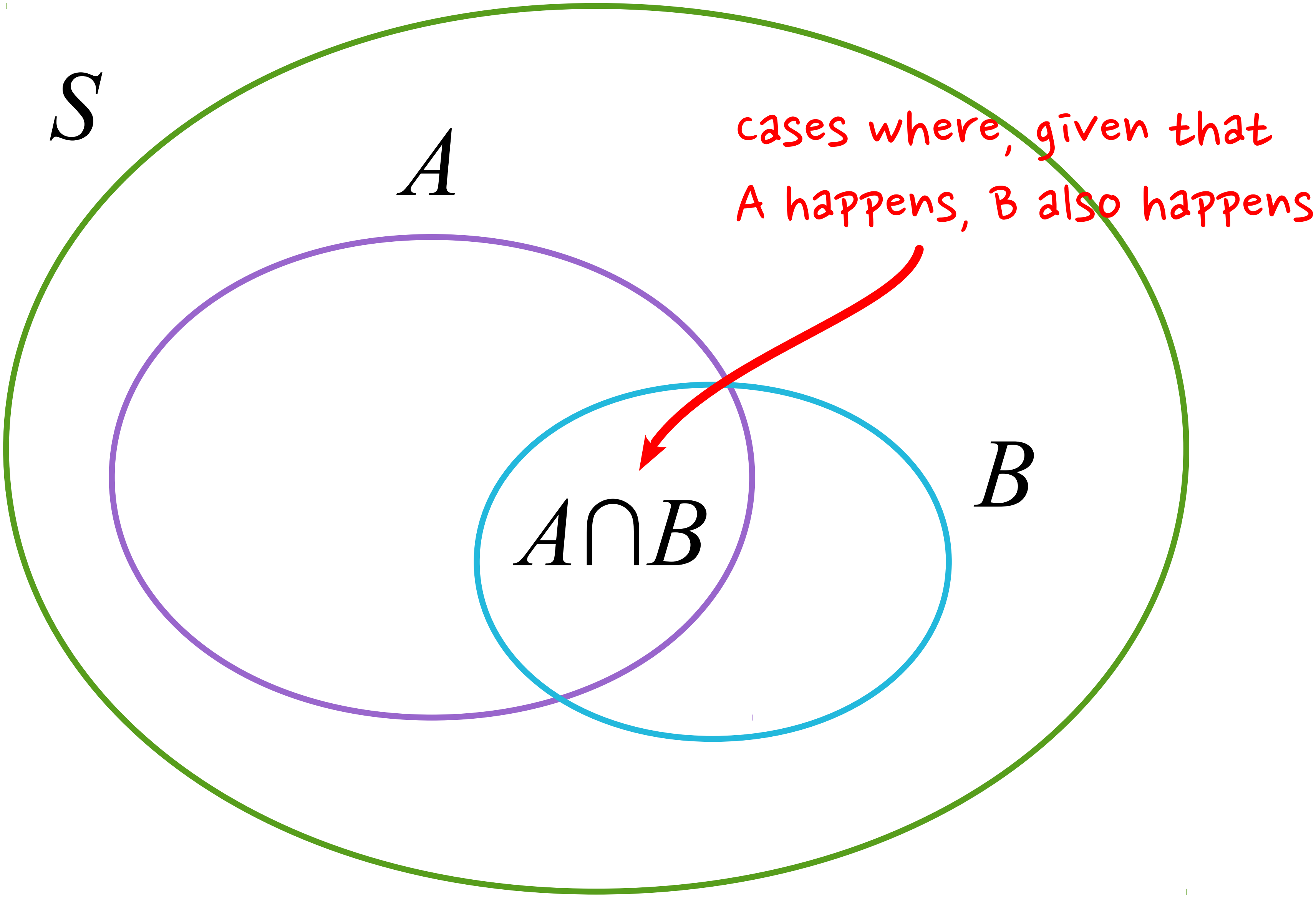
Why does this make sense?

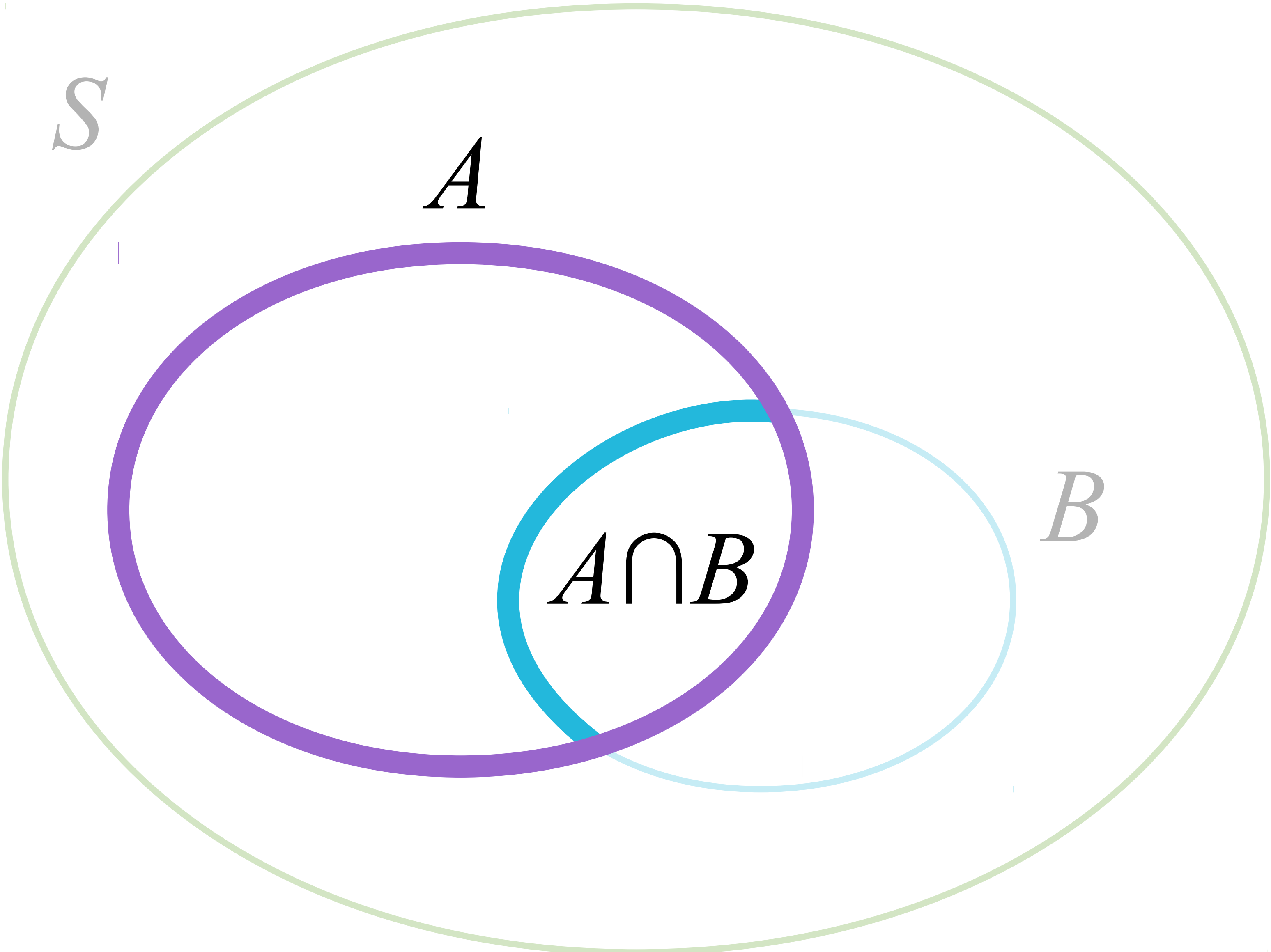
Intuitively, $\mathrm{P}(B \mid A)$ is the probability that event $B$ occurs, given that event $A$ has already occurred

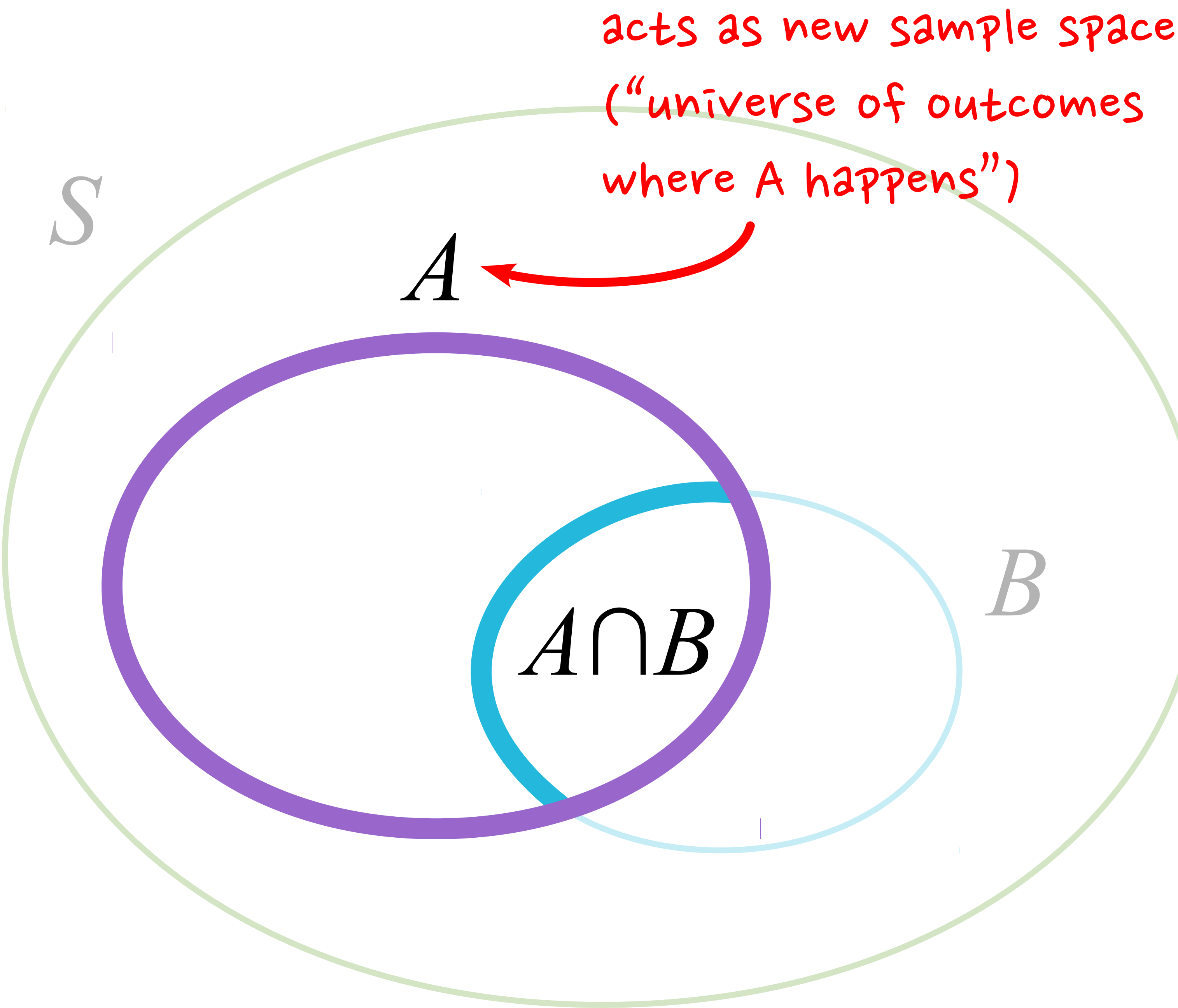
(this is NOT the formal math definition)

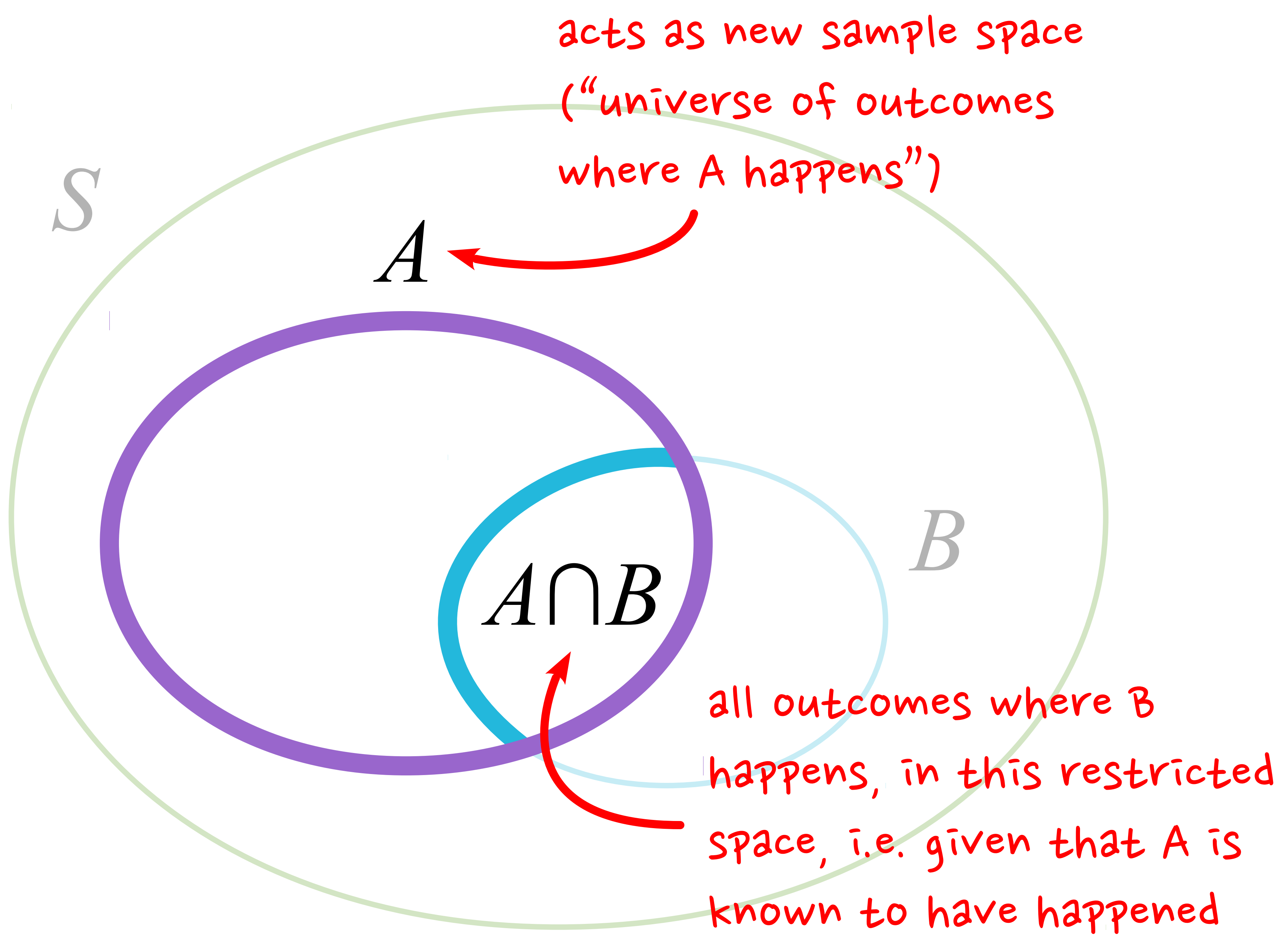
($A$ and $B$ need not actually occur in temporal order)


S
A
B
A∩B


S
A
cases where, given that
A happens, B also happens
B
A∩B


S
A
$A \cap B$
B


acts as new sample space
("universe of outcomes
where A happens")
S
A
A∩B
B


acts as new sample space
("universe of outcomes
where A happens")
S
A
$A \cap B$
B
all outcomes where B
happens, in this restricted
space, i.e. given that A is
known to have happened

# Thought for the Day #1

If the conditional probability $\mathrm{P}(B \mid A)$ is defined as $\mathrm{P}(A \cap B) / \mathrm{P}(A)$, and $\mathrm{P}(A) \neq 0$, then show that $(A, \mathrm{Q})$, where $\mathrm{Q}(B) = \mathrm{P}(B \mid A)$, is a valid probability space satisfying Kolmogorov's axioms.

# Independence of Events (revisited)

$$P(A \cap B) = P(B \mid A)\ P(A)$$

(by definition)

$$P(A \cap B) = P(B)\ P(A)$$

(if independent)

# Independence of Events (revisited)

In other words, assuming $\mathrm{P}(A) \neq 0$, $A$ and $B$ are independent if and only if

$$\mathrm{P}(B \mid A) \ = \ \mathrm{P}(B)$$

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(Intuitively: the probability of $B$ happening is unaffected by whether $A$ is known to have happened)

(Note: $A$ and $B$ can be swapped, if $\mathrm{P}(B) \neq 0$)

# Bayes' Theorem

Assuming $\mathrm{P}(A), \mathrm{P}(B) \neq 0$,

$$P\left(A \middle| B\right) = \frac{P\left(B \middle| A\right) P\left(A\right)}{P\left(B\right)}$$

# Bayes' Theorem

Assuming $\mathrm{P}(A), \mathrm{P}(B) \neq 0,$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

since $\mathrm{P}(A \mid B)\,\mathrm{P}(B) = \mathrm{P}(A \cap B) = \mathrm{P}(B \mid A)\,\mathrm{P}(A)$
(by definition of conditional probability)

# Bayes' Theorem

Prior probability of A

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# Bayes' Theorem

Assuming $\mathrm{P}(A), \mathrm{P}(B) \neq 0$,

Prior probability of A

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Posterior probability of A, given evidence B

since $\mathrm{P}(A \mid B)\,\mathrm{P}(B) = \mathrm{P}(A \cap B) = \mathrm{P}(B \mid A)\,\mathrm{P}(A)$
(by definition of conditional probability)

# How do we estimate $\mathrm{P}(B)$?

- Theorem of Total Probability (special case):

If $\mathrm{P}(A) \neq 0$ or $1$,

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}((B \cap A) \cup (B \cap A')) \\ &= \mathrm{P}(B \cap A) + \mathrm{P}(B \cap A') && \text{(Axiom 3)} \\ &= \mathrm{P}(B \mid A)\,\mathrm{P}(A) + \mathrm{P}(B \mid A')\,\mathrm{P}(A') && \text{(Definition of conditional probability)} \end{aligned}$$

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- Suppose:
  - **1 in 1000** people carry a disease, for which there is a pretty reliable test

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  - Probability of a false positive (non-carrier tests positive) is **5%**

# Example: Medical Diagnosis

- Suppose:
  - **1 in 1000** people carry a disease, for which there is a pretty reliable test
  - Probability of a false negative (carrier tests negative) is **1%** (so probability of carrier testing positive is **99%**)
  - Probability of a false positive (non-carrier tests positive) is **5%**
- A person just tested positive. What are the chances (s)he is a carrier of the disease?

# Example: Medical Diagnosis

- Priors:
  - P(*Carrier*) = 0.001
  - P(*NotCarrier*) = 1 - 0.001 = 0.999

# Example: Medical Diagnosis

- Priors:
  - $P(Carrier) = 0.001$
  - $P(NotCarrier) = 1 - 0.001 = 0.999$

- Conditional probabilities:
  - $P(Positive \mid Carrier) = 0.99$
  - $P(Positive \mid NotCarrier) = 0.05$

# Example: Medical Diagnosis

$$\mathrm{P}(Carrier \mid Positive)$$

$$= \frac{\mathrm{P}(Positive \mid Carrier)\ \mathrm{P}(Carrier)}{\mathrm{P}(Positive)}$$

(by Bayes' Theorem)

# Example: Medical Diagnosis

$$P(\mathit{Carrier} \mid \mathit{Positive})$$

$$= \frac{P(\mathit{Positive} \mid \mathit{Carrier})\ P(\mathit{Carrier})}{P(\mathit{Positive})}$$

(by Bayes' Theorem)

$$= \frac{P(\mathit{Positive} \mid \mathit{Carrier})\ P(\mathit{Carrier})}{\left( \begin{array}{c} P(\mathit{Positive} \mid \mathit{Carrier})\ P(\mathit{Carrier}) \\ + P(\mathit{Positive} \mid \mathit{NotCarrier})\ P(\mathit{NotCarrier}) \end{array} \right)}$$

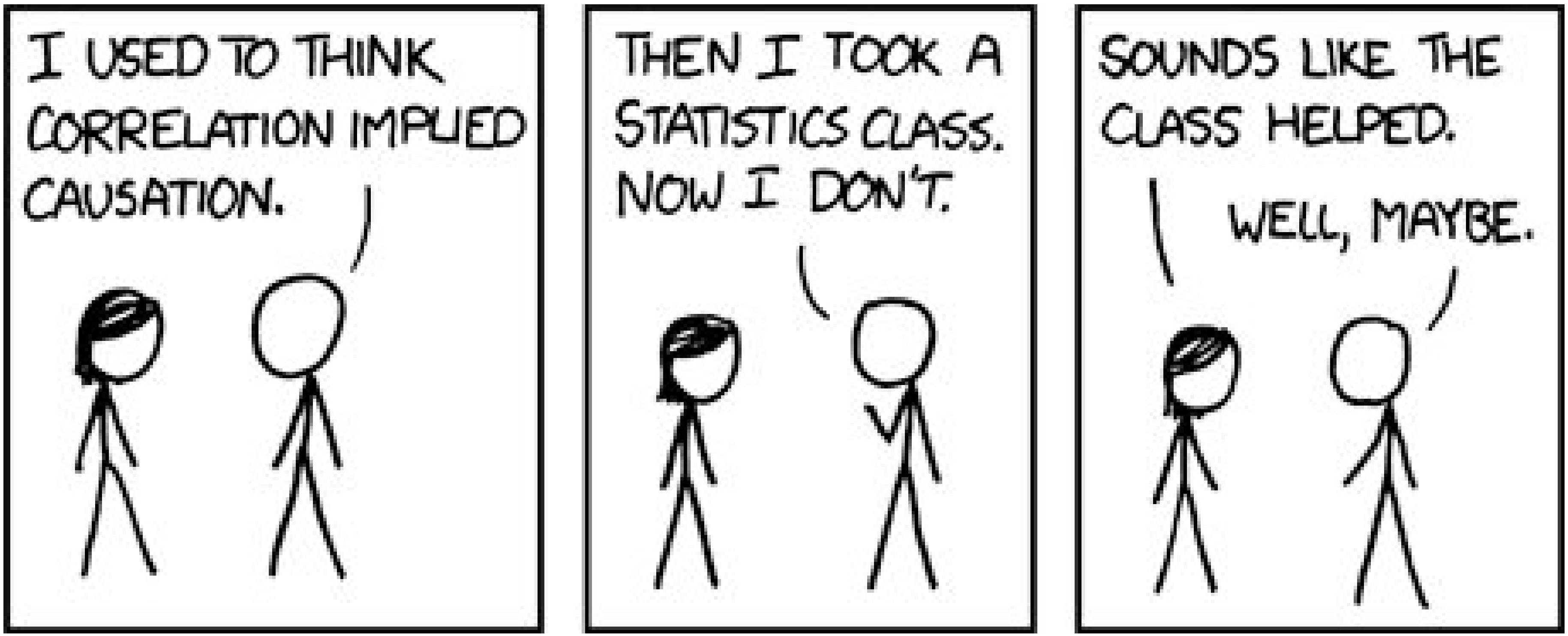
(by Theorem of Total Probability)

# Example: Medical Diagnosis

$$\frac{P(\textit{Positive} \mid \textit{Carrier})\ P(\textit{Carrier})}{\left(\begin{array}{c} P(\textit{Positive} \mid \textit{Carrier})\ P(\textit{Carrier}) \\ +\ P(\textit{Positive} \mid \textit{NotCarrier})\ P(\textit{NotCarrier}) \end{array}\right)}$$

$$= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.05 \times 0.999}$$

$$= 0.0194$$


I USED TO THINK CORRELATION IMPLIED CAUSATION.
THEN I TOOK A STATISTICS CLASS. NOW I DON'T.
SOUNDS LIKE THE CLASS HELPED.
WELL, MAYBE.

XKCD