Lines and Planes in $\mathbb{R}^3$

A **<u>line</u>** in $\mathbb{R}^3$ is determined by a point (a, b, c) on the line and a direction $\vec{v}$ that is parallel[1] to the line. The set of points on this line is given by

$$\{\langle x, y, z \rangle = \langle a, b, c \rangle + t\vec{v}, t \in \mathbb{R}\}$$

This represents that we start at the point (a, b, c) and add all scalar multiples of the vector $\vec{v}$.

The equation

$$\langle x, y, z \rangle = \langle a, b, c \rangle + t\vec{v}$$

is called the **vector equation of the line** (because it consists of vectors).

We can also rewrite this as three separate equation: if $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then (x, y, z) is on the line if

$$
\begin{aligned}
x &= a + tv_1 \\
y &= b + tv_2 \\
z &= c + tv_3
\end{aligned}
$$

are satisfied by the same parameter $t \in \mathbb{R}$. This is called the **parametric equation of the line**. See #1 below.

A **<u>plane</u>** in $\mathbb{R}^3$ is determined by a point (a, b, c) on the plane and two direction vectors $\vec{v}$ and $\vec{u}$ that are parallel to the plane. The fact that we need two vectors parallel to the plane versus one for the line represents that the plane is two dimensional and the line is one dimensional.

The plane is all points (x, y, z) such that the displacement vector from (a, b, c) to (x, y, z) is a sum of scalar multiples of $\vec{v}$ and $\vec{u}$. This set can be written as

$$\{\langle x, y, z \rangle = \langle a, b, c \rangle + t\vec{v} + s\vec{u}, t, s \in \mathbb{R}^3\}$$

This equation

$$\langle x, y, z \rangle = \langle a, b, c \rangle + t\vec{v} + s\vec{u}$$

is sometimes referred to as the parametric equation of the plane. See #3 below.

We discussed briefly that there are many choices for the direction vector(s) that will give the same line or plane: for the line, any scalar multiple of $\vec{v}$ will give the same line, while, for the plane, there are many, many more options. However, it doesn't matter which vectors are chosen (as long as they are parallel to the plane!). Any two vectors will give equations that might look different, but give the same object. See #1 amd #3 below.

There is an important alternate equation for a plane. We know the cross product turns two vectors $\vec{a}$ and $\vec{b}$ into a vector $\vec{a} \times \vec{b}$ that is orthogonal to $\vec{a}$ and $\vec{b}$ and also to any plane parallel to $\vec{a}$ and $\vec{b}$. Alternatively, any vector $\vec{n}$ that is orthogonal to a plane is also orthogonal to any two vectors in the plane.

The outcome of the previous paragraph is this: a plane is (also) determined by a point (a, b, c) on the plane and a vector $\vec{n}$ that is orthogonal to the plane (we use n because *normal* is a synonym for orthogonal). In this formulation, we want all points (x, y, z) such that the vector from (a, b, c) to (x, y, z) is orthogonal to $\vec{n}$. We can write this with the equation

$$\vec{n} \cdot \langle x - a, y - b, z - c \rangle = 0$$

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[1] "Parallel" here and below can be thought of as meaning that if the vector starts at some point on the line/plane/object, then the entire vector lies on the line/plane/object
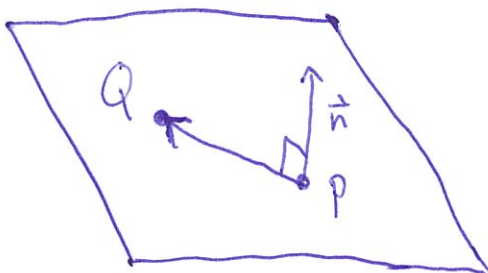
Other ways to write this are ($\vec{n} = \langle n_1, n_2, n_3 \rangle$)

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

$$n_1 x + n_2 y + n_3 z = d$$

for the proper choice of d. An important observation is that the plane is given by a single equation relating x, y, z (called the **implicit equation**), while a line is given by three equations in the parametric equation. See #3 below.

A common exercise is to take some amount of data and find a line or plane that agrees with this data. #1 and #3 are examples of this.


Once we have these objects described, we will want to find the **distance** between them. These are infinite objects, so the distance between them depends on where you look. To specify, whenever we talk about the distance between to objects, we always mean at their closest points.

There is no one formula for this as it depends on the objects involved. It often revolves around finding the length of a projection vector or the area of some parallelogram. There is a list of formulas on Oliver's website at http://sites.fas.harvard.edu/~math21a/supplements/distances/index.html and we will do some more in problem session. However, I encourage you to think about <u>why</u> these formulas are true as opposed to memorizing each one of them. Some examples are #4 and #5.

Examples

1. **Find the equation of the line that contains the points** $(1, 2, 3)$ **and** $(4, 0, 1)$**.**
   We have a point on the line (actually two, but it suffices to pick one). To find the direction of the line, we find the displacement vector from $(1, 2, 3)$ to $(4, 0, 1)$.
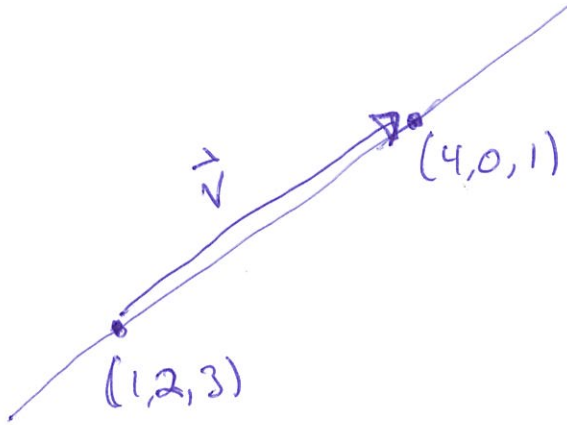
This is $\vec{v} = \langle 4 - 1, 0 - 2, 1 - 3 \rangle = \langle 3, -2, -2 \rangle$.

This gives us the equation

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + t\langle 3, -2, -2 \rangle$$

Follow-up: Convince yourself that

$$
\begin{aligned}
x &= 4 - 3t \\
y &= 2t \\
z &= 1 + 2t
\end{aligned}
$$

describes the same line.

2. **Find the intersection of the lines** $\langle 2, 1, 0 \rangle + t\langle -1, -1, -1 \rangle$ **and** $\langle 3, 0, 5 \rangle + t\langle 2, 0, 6.$
   We convert these two parametric equations *using different parameters*.

   $$x = 2 - t; y = 1 - t; z = -t$$

   and

   $$x = 3 + s; y = 0; z = 5 + 6s$$

   and then set them equal to each other.

   $$
   \begin{aligned}
   2 - t &= 3 + 2s \\
   1 - t &= 0 \\
   -t &= 5 + 6s
   \end{aligned}
   $$

   The second equation gives $t = 1$. Plugging this into the first and second equation both give $s = -1$. Thus, they intersect at the point $(1, 0, -1)$; we can get this point by plugging either parameter into the appropriate equation.

   Follow-up: The set-up gave us three equations in two unknowns. Since we found <u>a</u> solution, we know the lines intersect at a point. If we found <u>no</u> solution, then the lines don't intersect. If we found <u>infinitely many solutions</u>, the lines are the same.

3. **Find the equation of the plane that contains the point** $(1, 3, 0)$ **and the line given by** $x = 3 + 2t$, $y = -4t$, $z = 7 - t$**.**
   Lots of options to start. We know a point on the line is $(1, 3, 0)$. The line has direction $\langle 2, -4, -1 \rangle$, so this lies parallel to the plane. Now we need another direction vector parallel to the plane. Plugging

$t = 0$ into the line equation gives the point $(3, 0, 7)$, which is also on the plane. Then the vector from $(1, 3, 0)$ to $(3, 0, 7)$ will also lie parallel to the plane; this vector is $\langle 2, -3, 7 \rangle$.
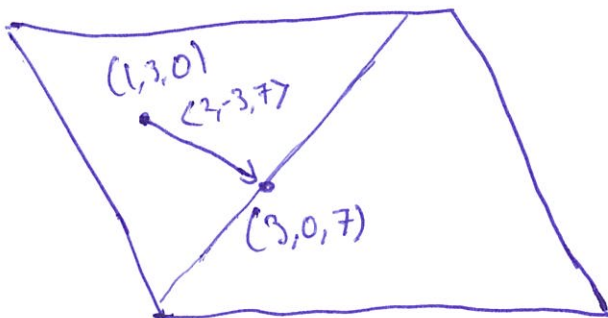
This gives us the following parametric equation of a plane

$$\langle x, y, z \rangle = \langle 1, 3, 0 \rangle + t \langle 2, -4, -1 \rangle + s \langle 2, -3, 7 \rangle$$

To find the implicit formula, we must find a vector orthogonal/normal to the plane. This vector can be computed via the cross product

$$\vec{n} = \langle 2, -4, -1 \rangle \times \langle 2, -3, -7 \rangle = \langle 31, 12, 2 \rangle$$

This gives us with the implicit equation

$$
\begin{aligned}
\langle 31, 12, 2 \rangle \cdot \langle x - 1, y - 3, z \rangle &= 0 \\
31(x - 1) + 12(y - 3) + 2z &= 0 \\
31x + 12y + 2z &= 67
\end{aligned}
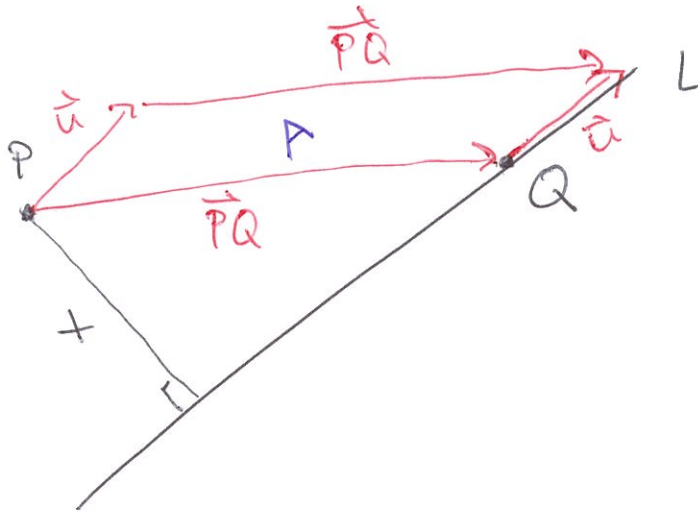$$

Follow-up: Convince yourself that

$$\langle x, y, z \rangle = \langle 3, 0, 7 \rangle + t \langle 4, -7, 6 \rangle + s \langle -2, 3, -7 \rangle$$

and

$$-31(x - 5) + -12(y + 4) + -2(z - 6) = 0$$

give the same plane. This is harder than in #1 because planes give more options. Notice, however, that analyzing the implicit equation makes this clearer.

4. **Find the distance between the point $P = (3, 5, 6)$ and the line given by $\langle 2, 1, 0 \rangle + t \langle 1, 5, 9 \rangle$.**
   We can draw the point and line in a single plane

We want the length x, which happens to be the height of the parallelogram A determined by $\overrightarrow{PQ}$ and $\vec{u}$, the direction vector of the line. This parallelogram has base $|\vec{u}|$, so we have two formulas for its area

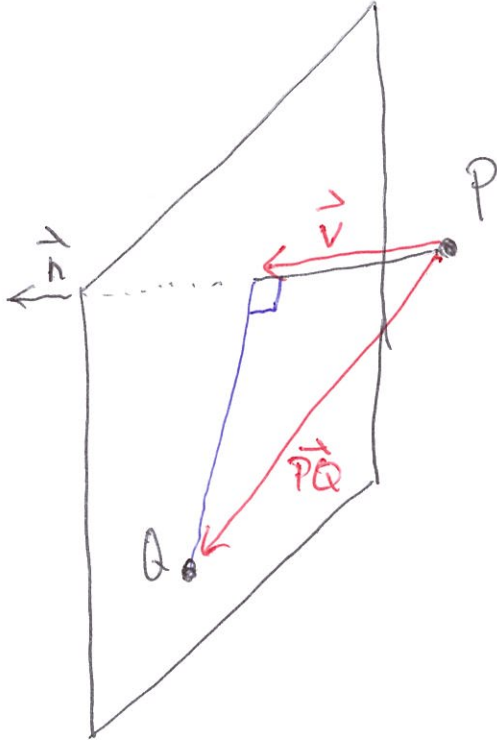$$x|\vec{u}| = \text{area}(A) = |\overrightarrow{PQ} \times \vec{u}|$$

Solving for x gives

$$x = \frac{|\overrightarrow{PQ} \times \vec{u}|}{|\vec{u}|}$$

Taking $Q = (2, 1, 0)$ to be the point on the line, this gives

$$\text{distance} = x = \frac{|\langle 1, 4, 6\rangle \times \langle 1, 5, 9\rangle|}{|\langle 1, 5, 9\rangle} = \frac{|\langle 6, 3, 1\rangle|}{\sqrt{107}} = \frac{\sqrt{46 \cdot 107}}{107} = \frac{\sqrt{4922}}{107}$$

5. **Find the distance between the point $P = (1, 0, -1)$ and the plane $5x + 4y + 3z = 1$.**
   We draw the point and the plane with an additional point Q on the plane.

From the picture, we see that the distance is the length of the vector $\vec{v}$ and that $\vec{v}$ is the projection of $\overrightarrow{PQ}$ onto $\vec{n}$; here, $\vec{n}$ is the normal vector to the plane, which we can tell from the equation is $\langle 5, 4, 3 \rangle$. Thus, the distance is the length of $\text{Pr}_{\vec{n}}(\overrightarrow{PQ})$, which is

$$\text{distance} = |\text{Pr}_{\vec{n}}(\overrightarrow{PQ})| = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

In this case, to find a point on the plane, we can set $x = y = 0$; plugging this into the plane equation gives $z = \frac{1}{3}$, so $(0, 0, \frac{1}{3})$ is on the plane. This means that $\overrightarrow{PQ} = \langle -1, 0, \frac{4}{3} \rangle$. Plugging this in gives

$$\text{distance} = \frac{|\langle -1, 0, \frac{4}{3} \rangle \cdot \langle 5, 4, 3 \rangle|}{|\langle 5, 4, 3 \rangle|} = \frac{\sqrt{2}}{10}$$