Chapter 13. Vector-Valued Functions and Motion in Space

13.2. Integrals of Vector Functions; Projectile Motion

Definition. A differentiable vector function $\mathbf{R}(t)$ is an *antiderivative* of a vector function $\mathbf{r}(t)$ on in interval I if $d\mathbf{R}/dt = \mathbf{r}$ at each point of I. The *indefinite integral* of $\mathbf{r}$ with respect to t is the **set** of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \{\mathbf{R} \mid \mathbf{R}'(t) = \mathbf{r}(t)\} = \mathbf{R}(t) + \mathbf{C}.$$

Note. Whereas **antiderivatives are functions**, **indefinite integrals are sets**—indefinite integrals are sets of antiderivatives. We will use set notation sometimes, but often will abbreviate the set notation with the "$+\mathbf{C}$" which is similar to how indefinite integrals were dealt with in Calculus 1. Also similar to Calculus 1, we see in the following definition that **definite integrals are numbers**.

Definition. If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the *definite integral* of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Examples. Page 738, number 4; and page 739, number 15.

Note. Suppose an object (a "projectile") is given an initial velocity $\mathbf{v}_0$ and is then only acted on by the force of gravity (so we ignore frictional drag, for example). We assume that $\mathbf{v}_0$ makes an angle α with the horizontal.

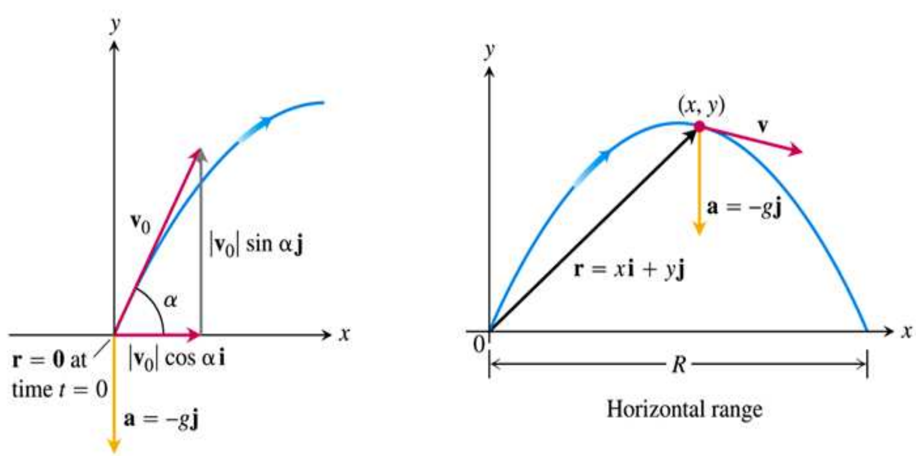

Figure 13.10, page 735

Then

$$\mathbf{v}_0 = (|\mathbf{v}_0| \cos\alpha)\mathbf{i} + (|\mathbf{v}_0| \sin\alpha)\mathbf{j} = (v_0 \cos\alpha)\mathbf{i} + (v_0 \sin\alpha)\mathbf{j}.$$

Suppose the initial position is $\mathbf{r}_0 = \mathbf{0}$. Newton's Second Law of Motion says that the force acting on the projectile is equal to the projectile's mass m times its acceleration ("$F = ma$"), or $m(d^2\mathbf{r}/dt^2)$ where $\mathbf{r}$ is the projectile's position vector and t is time. With this gravitational force as the only force, $-mg\mathbf{j}$, then

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j} \text{ and } \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}$$

where g is the acceleration due to gravity. We find $\mathbf{r}$ as a function of t by solving the initial value problem:

Differential Equation: $\dfrac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}$

Initial Conditions: $\mathbf{r} = \mathbf{r_0}$ and $\dfrac{d\mathbf{r}}{dt} = \mathbf{v}_0$ when $t = 0$.

We get by integration and use of initial conditions first that $\dfrac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0$ and then that $\mathbf{r} = -\dfrac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0$. Expanding $\mathbf{v}_0$ and $\mathbf{r}_0$ gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + (v_0\cos\alpha)t\mathbf{i} + (v_0\sin\alpha)t\mathbf{j} + \mathbf{0}$$

or

$$\mathbf{r} = (v_0\cos\alpha)t\mathbf{i} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

The angle α is the projectile's *launch angle* and v_0 is the projectile's initial speed. As parametric equations, we have

$$x = (v_0\cos\alpha)t \text{ and } y = (v_0\sin\alpha)t - \frac{1}{2}gt^2.$$

Example. Page 739, number 22.

Note. We can easily find the the maximum height, range, and flight time of a projectile. We get:

Maximum Height: $y_{\text{max}} = \dfrac{(v_0 \sin\alpha)^2}{2g}$

Flight Time: $t = \dfrac{2v_0 \sin\alpha}{g}$

Range: $R = \dfrac{v_0^2}{g} \sin 2\alpha.$

Example. Page 740, number 32.