Mechanical Energy and Simple Harmonic Oscillator

8.01

Week 09D1

Summary: Change in Mechanical Energy

Total force: $\vec{\mathbf{F}}^{\text{total}} = \vec{\mathbf{F}}_{\text{c}}^{\text{total}} + \vec{\mathbf{F}}_{\text{nc}}^{\text{total}}$

Total work: $W^{\text{total}} = \int_{\text{initial}}^{\text{final}} \vec{\mathbf{F}}^{\text{total}} \cdot d\vec{\mathbf{r}} = \int_{\text{initial}}^{\text{final}} \left(\vec{\mathbf{F}}_{\text{c}}^{\text{total}} + \vec{\mathbf{F}}_{\text{nc}}^{\text{total}} \right) \cdot d\vec{\mathbf{r}}$

Change in potential energy: $\Delta U^{\text{total}} = -\int_{\text{initial}}^{\text{final}} \vec{\mathbf{F}}_{\text{c}}^{\text{total}} \cdot d\vec{\mathbf{r}}$

Total work done is change in kinetic energy: $W^{\text{total}} = -\Delta U^{\text{total}} + W_{\text{nc}} = \Delta K$

Mechanical Energy Change: $\Delta E^{\text{mechanical}} \equiv \Delta K + \Delta U^{\text{total}}$

Conclusion: $W_{\text{nc}} = \Delta K + \Delta U^{\text{total}}$

Modeling the Motion using Force and Energy Concepts

Force and Newton's Second Law:

- Draw all relevant free body force diagrams
- Identify non-conservative forces.
- Calculate non-conservative work $W_{\text{nc}} = \int_{\text{initial}}^{\text{final}} \vec{\mathbf{F}}_{\text{nc}} \cdot d\vec{\mathbf{r}}.$

Change in Mechanical Energy:

- Choose initial and final states and draw energy diagrams.
- Choose zero point *P* for potential energy for each interaction in which potential energy difference is well-defined.
- Identify initial and final mechanical energy.
- Apply Energy Law. $W_{\text{nc}} = \Delta K + \Delta U^{\text{total}}$

Mechanical Energy Accounting

Initial state:

- Total initial kinetic energy $K_{\text{initial}} = K_{1,\text{initial}} + K_{2,\text{initial}} + \cdots$
- Total initial potential energy $U_{\text{initial}} = U_{1,\text{initial}} + U_{2,\text{initial}} + \cdots$
- Total initial mechanical energy $E_{\text{initial}}^{\text{mechanical}} = K_{\text{initial}} + U_{\text{initial}}$

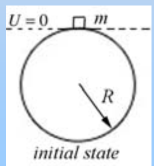
Final state:

- Total final kinetic energy $K_{\text{final}} = K_{1,\text{final}} + K_{2,\text{final}} + \cdots$
- Total final potential energy $U_{\text{final}} = U_{1,\text{final}} + U_{2,\text{final}} + \cdots$
- Total final mechanical energy $E_{\text{final}}^{\text{mechanical}} = K_{\text{final}} + U_{\text{final}}$
- Apply Energy Law: $W_{\text{nc}} = E_{\text{final}}^{\text{mechanical}} - E_{\text{initial}}^{\text{mechanical}}$

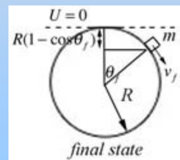
Example: Energy Changes

A small point like object of mass m rests on top of a sphere of radius R. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide. Find an expression for the angle θ with respect to the vertical at which the object just loses contact with the sphere.

Energy Flow diagrams



Initial state



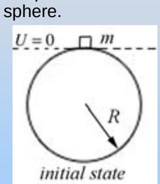
Final State

Example: Energy Changes

A small point like object of mass m rests on top of a sphere of radius R. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide. Find an expression for the angle θ with respect to the vertical at which the object just loses contact with the sphere.

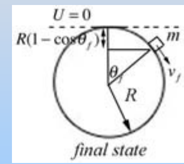
$K_{\text{initial}} \simeq 0$

$U_{\text{initial}} = 0$



$K_{\text{final}} = \frac{1}{2} m v_f^2$

$U_{\text{final}} = -mgR(1-\cos\theta_f)$



$E_{\text{initial}}^{\text{mechanical}} \simeq 0$

$E_{\text{final}}^{\text{mechanical}} = \frac{1}{2} m v_f^2 - mgR(1-\cos\theta_f)$

$$W_{\text{nc}} = 0 = E_{\text{final}}^{\text{mechanical}} - E_{\text{initial}}^{\text{mechanical}} \Rightarrow$$

$$0 = 0 - \left(\frac{1}{2} m v_f^2 - mgR(1-\cos\theta_f)\right) \Rightarrow \quad \frac{1}{2} m v_f^2 = mgR(1-\cos\theta_f)$$

Recall Modeling the Motion: Newton 's Second Law

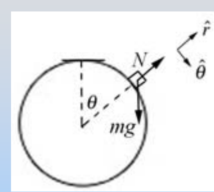
- Define system, choose coordinate system.
- Draw force diagram.
- Newton's Second Law for each direction.
- Example: x-direction $\quad \hat{\mathbf{i}}: \; F_x^{\text{total}} = m \dfrac{d^2 x}{dt^2}.$
- Example: Circular motion $\quad \hat{\mathbf{r}}: \; F_r^{\text{total}} = -m \dfrac{v^2}{R}.$

Example (con't): Free Body Force Diagram

Newton's Second Law

$\hat{\mathbf{r}}: \; N - mg\cos\theta = -m\dfrac{v^2}{R}$

$\hat{\boldsymbol{\theta}}: \; mg\sin\theta = mR\dfrac{d^2\theta}{dt^2}$



Constraint condition:

$N = 0 \text{ at } \theta = \theta_f$

Radial Equation becomes

$mg\cos\theta_f = m\dfrac{v_f^2}{R} \Rightarrow \quad \dfrac{1}{2} m v_f^2 = \dfrac{R}{2} mg\cos\theta_f$

Energy Condition: $\dfrac{1}{2} m v_f^2 = mgR(1-\cos\theta_f)$

Conclusion: $mgR(1-\cos\theta_f) = \dfrac{R}{2} mg\cos\theta_f \Rightarrow \quad \cos\theta_f = \dfrac{2}{3} \Rightarrow \quad \theta_f = \cos^{-1}\left(\dfrac{2}{3}\right)$

Table Problem: Loop-the-Loop

An object of mass *m* is released from rest at a height *h* above the surface of a table. The object slides along the inside of the loop-the-loop track consisting of a ramp and a circular loop of radius *R* shown in the figure. Assume that the track is frictionless. When the object is at the top of the track (point a) it pushes against the track with a force equal to three times it's weight. What height was the object dropped from?

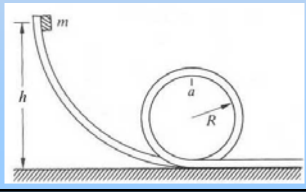
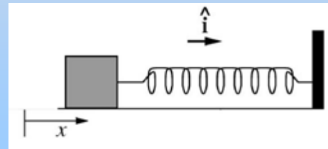


Table Problem: Block-Spring System with Friction

A block of mass m slides along a horizontal surface with speed v_0. At $t = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu_k = bx$ where b is a constant. Find how far the spring has compressed when the block has first come momentarily to rest.



Simple Harmonic Motion

Hooke's Law

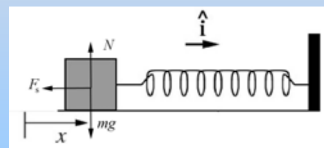
Define system, choose coordinate system.

Draw free-body diagram.

Hooke's Law

$$\vec{\mathbf{F}}_{\text{spring}} = -kx\,\hat{\mathbf{i}}$$

$$-kx = m\frac{d^2x}{dt^2}$$



Concept Question

Which of the following functions $x(t)$ has a second derivative which is proportional to the negative of the function

$$\frac{d^2x}{dt^2} \propto -x?$$

1. $x(t) = \frac{1}{2}at^2$
2. $x(t) = Ae^{t/T}$
3. $x(t) = Ae^{-t/T}$
4. $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$

Concept Question

The first derivative $v_x = dx/dt$ of the sinusoidal function

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

is:

1. $v_x(t) = A\cos\left(\frac{2\pi}{T}t\right)$
2. $v_x(t) = -A\sin\left(\frac{2\pi}{T}t\right)$
3. $v_x(t) = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right)$
4. $v_x(t) = \frac{2\pi}{T}A\cos\left(\frac{2\pi}{T}t\right)$

Simple Harmonic Motion

Equation of Motion: $-kx = m\frac{d^2x}{dt^2}$

Solution: Oscillatory with Period $T = 2\pi\sqrt{\frac{m}{k}}$

Position: $x = A\cos\left(\frac{2\pi}{T}t\right) + B\sin\left(\frac{2\pi}{T}t\right)$

Velocity: $v_x = \frac{dx}{dt} = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right) + \frac{2\pi}{T}B\cos\left(\frac{2\pi}{T}t\right)$

Initial Position at $t = 0$: $x_0 \equiv x(t=0) = A$

Initial Velocity at $t = 0$: $v_{x,0} \equiv v_x(t=0) = \frac{2\pi}{T}B$

General Solution: $x = x_0\cos\left(\frac{2\pi}{T}t\right) + \frac{T}{2\pi}v_{x,0}\sin\left(\frac{2\pi}{T}t\right)$

Period and Angular Frequency

Equation of Motion: $-kx = m\frac{d^2x}{dt^2}$

Solution: Oscillatory with Period T: $x = A\cos\left(\frac{2\pi}{T}t\right) + B\sin\left(\frac{2\pi}{T}t\right)$

x -component of velocity:

$$v_x \equiv \frac{dx}{dt} = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right) + \frac{2\pi}{T}\cos\left(\frac{2\pi}{T}t\right)$$

x -component of acceleration:

$$a_x \equiv \frac{d^2x}{dt^2} = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right) - \left(\frac{2\pi}{T}\right)^2 B\sin\left(\frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{T}\right)^2 x$$

Period:

$$-kx = m\frac{d^2x}{dt^2} = -m\left(\frac{2\pi}{T}\right)^2 x \Rightarrow k = m\left(\frac{2\pi}{T}\right)^2 \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

Angular frequency

$$\omega \equiv \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

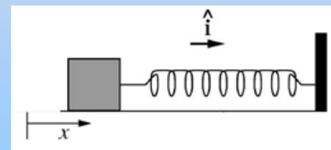
Concept Question: Simple Harmonic Motion

A block of mass m is attached to a spring with spring constant k is free to slide along a horizontal frictionless surface. At $t = 0$ the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest. What is the x -component of the velocity of the block when it first comes back to the equilibrium?

1. $v_x = -x_0 \frac{T}{4}$
2. $v_x = x_0 \frac{T}{4}$
3. $v_x = -\sqrt{\frac{k}{m}} x_0$
4. $v_x = \sqrt{\frac{k}{m}} x_0$

Example: Block-Spring System with No Friction

A block of mass m slides along a frictionless horizontal surface with speed $v_{x,0}$. At $t = 0$ it hits a spring with spring constant k and begins to slow down. How far is the spring compressed when the block has first come momentarily to rest?



Initial and Final Conditions

Initial state: $x_0 = A = 0$ and $v_{x,0} = \frac{2\pi}{T} B$

$$x(t) = \frac{T}{2\pi} v_{x,0} \sin\left(\frac{2\pi}{T} t\right) \qquad v_x(t) = v_{x,0} \cos\left(\frac{2\pi}{T} t\right)$$

First comes to rest when $v_x(t_f) = 0 \Rightarrow \frac{2\pi}{T} t_f = \frac{\pi}{2}$

Since at time $t_f = T/4$

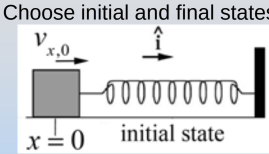
$$\sin\left(\frac{2\pi}{T} t_f\right) = \sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow$$

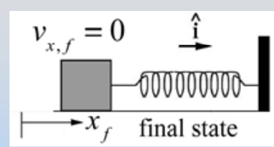
Final position

$$x(t_f) = \frac{T}{2\pi} v_{x,0} = \sqrt{\frac{m}{k}} v_{x,0}$$

Modeling the Motion: Energy

Choose initial and final states:





Change in potential energy: $U(x_f) - U(x_0) = \frac{1}{2} k \left(x_f^2 - x_0^2\right)$

Choose zero point for potential energy: $U(x = 0) = 0$

Potential energy function: $U(x) = \frac{1}{2} k x^2, \; U(x = 0) = 0$

Mechanical energy is constant ($W_{nc} = 0$) $E_{\text{final}}^{\text{mechanical}} = E_{\text{initial}}^{\text{mechanical}}$

Kinetic Energy vs. Potential Energy

State	Kinetic energy	Potential energy	Mechanical energy
Initial $x_0 = 0$ $v_{x,0} > 0$	$K_0 = \frac{1}{2} m v_{x,0}^2$	$U_0 = 0$	$E_0 = \frac{1}{2} m v_{x,0}^2$
Final $x_f > 0$ $v_{x,f} = 0$	$K_f = 0$	$U_f = \frac{1}{2} k x_f^2$	$E_f = \frac{1}{2} k x_f^2$

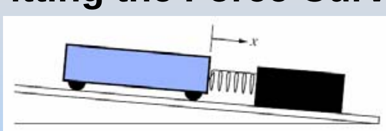
Conservation of Mechanical Energy

$$E_f = E_0 \quad \Rightarrow \quad \frac{1}{2} k x_f^2 = \frac{1}{2} m v_{x,0}^2$$

The amount the spring has compresses when the object first comes to rest is

$$x_f = \sqrt{\frac{m}{k}} v_{x,0}$$

Lecture Demo: Experiment 3 Fitting the Force Curve



Position: $x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$

Velocity $v_x = -\omega_0 C_1 \sin(\omega_0 t) + \omega_0 C_2 \cos(\omega_0 t)$

Angular Frequency: $\omega_0 = \sqrt{k / m_c}$

Initial conditions: $x(t=0) = 0 \Rightarrow C_1 = 0$ $\quad v_x(t=0) = v_1 \Rightarrow C_2 = v_1 / \omega_0$

Lecture Demo: Experiment 3 Fitting the Force Curve

Position: $x = (v_1 / \omega_0) \sin(\omega_0 t)$ Amplitude: $x_{\max} = \omega_0 v_1$

Velocity: $v_x = v_1 \cos(\omega_0 t)$ Acceleration $a_x = -\omega_0 v_1 \sin(\omega_0 t)$

Third Law: $F_{sensor}(t) = -F_{cart}(t) = -m_c a_x = m_c \omega_0 v_1 \sin(\omega_0 t)$

Force on sensor vs. time graph. Best Fit: $F_{sensor}(t) = A_0 \sin(2\pi (t - A_2) / A_1)$

Fit Parameters: $A_2 = \text{offset time}$

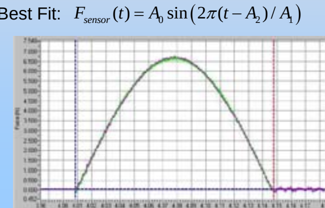
$A_0 = m_c \omega_0 v_1$

$A_1 = 2\pi / \omega_0 = 2\pi \sqrt{m_c / k}$

Spring constant: $k = 4\pi^2 m_c / A_1$

Maximum Displacement:

$x_{\max} = \omega_0 v_1 = A_0 / m_c$



Energy Diagram

Choose zero point for potential energy:

$U(x=0)=0$

Potential energy function:

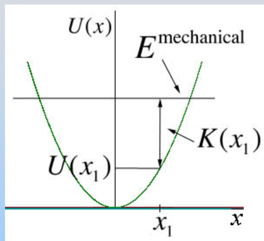
$U(x)=\frac{1}{2}kx^2,\quad U(x=0)=0$

Mechanical energy is represented by a horizontal line since it is a constant

$E^{\text{mechanical}}=K(x)+U(x)=\frac{1}{2}mv_x^2+\frac{1}{2}kx^2$

Kinetic energy is difference between mechanical energy and potential energy (independent of choice of zero point)

$K=E^{\text{mechanical}}-U$



Graph of Potential energy function $U(x)$ vs. x

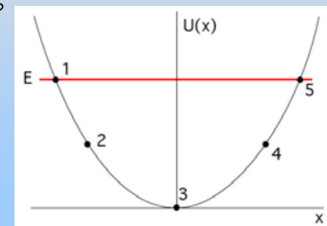
Concept Question: Energy Diagram

The position of a particle is given by

$x(t)=D\cos(\omega t)-D\sin(\omega t),\ D>0$

Where was the particle at $t=0$?

1) 1
2) 2
3) 3
4) 4
5) 5
6) 1 or 5
7) 2 or 4



Concept Question: Energy Diagram 1

A particle with total mechanical energy E has position $x>0$ at $t=0$

1) escapes to infinity in the – x-direction

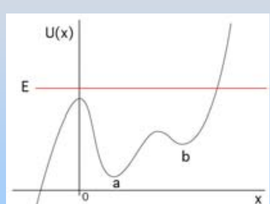
2) approximates simple harmonic motion

3) oscillates around a

4) oscillates around b

5) periodically revisits a and b

6) not enough information



Concept Question: Energy Diagram 2

A particle with total mechanical energy E has position $x>0$ at $t=0$

1) escapes to infinity

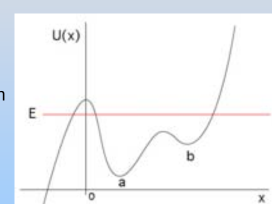
2) approximates simple harmonic motion

3) oscillates around a

4) oscillates around b

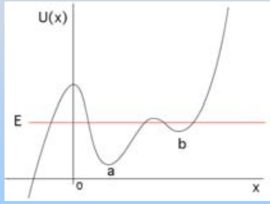
5) periodically revisits a and b

6) not enough information



Concept Question: Energy Diagram 3

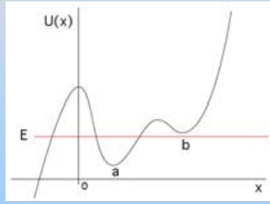
A particle with total mechanical energy E has position $x > 0$ at $t = 0$



1) escapes to infinity
2) approximates simple harmonic motion
3) oscillates around a
4) oscillates around b
5) periodically revisits a and b
6) not enough information

Concept Question: Energy Diagram 4

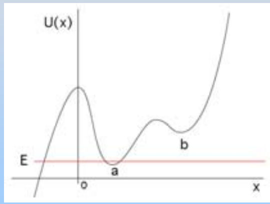
A particle with total mechanical energy E has position $x > 0$ at $t = 0$



1) escapes to infinity
2) approximates simple harmonic motion
3) oscillates around a
4) oscillates around b
5) periodically revisits a and b
6) not enough information

Concept Question: Energy Diagram 5

A particle with total mechanical energy E has position $x > 0$ at $t = 0$



1) escapes to infinity
2) approximates simple harmonic motion
3) oscillates around a
4) oscillates around b
5) periodically revisits a and b
6) not enough information