

# Chapter 01 - Measurement of Interest

## Section 1.1 - Introduction

**Definition:** **Interest** is the compensation that a borrower of capital pays to a lender of capital for its use.

**Definition:** The **principal** is the amount of money initially borrowed. This money accumulates over time. The difference between the initial amount and the amount returned at the end of the period is called **interest**.

## Section 1.2 - Basics

**Definition:** The **accumulation function**,  $a(t)$ , describes the accumulated value at time  $t$  of initial investment of 1.

**Properties** of the accumulation function:

(1)  $a(0) = 1$

(2)  $a(t)$  is generally increasing function of time.

(3) If interest accrues continuously then  $a(t)$  will be a continuous function.

**Definition:** The **amount function**,  $A(t)$ , gives the accumulated value of an initial investment of  $k$  at time  $t$ , i.e.

$$A(t) = ka(t).$$

It follows that:

$$A(0) = k,$$

and the interest earned during the  $n$ th period from the date of investment is:

$$I_n = A(n) - A(n - 1) \quad \text{for } n = 1, 2, \dots .$$

## Section 1.3 - Rates of Interest

**Definition:** The **effective rate of interest**,  $i$ , is the amount that 1 invested at the beginning of the period will earn during the period when the interest is paid at the end of the period.

That is,

$$\begin{aligned}i &= a(1) - a(0) \quad \text{or} \\(1 + i) &= a(1).\end{aligned}$$

The quantity  $i$  is **always a decimal value** even though it is often expressed as a percent, i.e.

$$6\% \text{ interest} \qquad i = .06$$

Note that

$$i = a(1) - a(0) = \frac{A(1) - A(0)}{A(0)} = \frac{I_1}{A(0)}$$

The **effective rate of interest** is the interest earned in the period divided by the principal at the beginning of the period.

Thus by extension,

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}$$

is the **effective rate of interest during the  $n$ th period** from the date of investment. Also,

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} \quad \text{for } n = 1, 2, \dots$$

**Example:** Consider the accumulation function

$$a(t) = (.05)t^2 + 1.$$

Here  $a(0) = 1$  and

$$i_1 = a(1) - a(0) = .05 + 1 - 1 = .05.$$

is the effective interest rate for period one.

For the  $n$ th period, the effective interest rate is:

$$\begin{aligned}i_n &= \frac{a(n) - a(n-1)}{a(n-1)} \\&= \frac{(.05)(n)^2 + 1 - [(.05)(n-1)^2 + 1]}{(.05)(n-1)^2 + 1} \\&= \frac{(.10)n - (.05)}{(.05)(n-1)^2 + 1}\end{aligned}$$

$n$	$i_n$
1	.05
2	.143
3	.208

If \$100 is invested, how much interest will be earned over three periods?

$$\text{Ans} = 100[a(3) - a(0)] = 100[(.05)(3)^2 + 1 - 1] = \$45.$$

## Section 1.4 - Simple Interest

Under **simple interest**,

$$a(t) = 1 + (n - 1)i \quad \text{for } n - 1 \leq t < n \quad \text{and } n = 1, 2, 3, \dots$$

The effective rate of interest for the  $n$ th period is

$$\begin{aligned} i_n &= \frac{a(n) - a(n-1)}{a(n-1)} \\ &= \frac{(1 + in) - (1 + i(n-1))}{1 + i(n-1)} \\ &= \frac{i}{1 + i(n-1)}. \end{aligned}$$

This is a decreasing function of  $n$ .

Why is it decreasing?

**Answer:** The same amount of interest is credited to the account for each period. Since the amount in the account is increasing, the effective rate is

$$\frac{i}{a(n-1)} = \frac{\text{constant}}{\text{increasing function}}$$

which will be a decreasing function. Simple interest can be extended to **partial periods** by letting

$$a(t) = 1 + it \quad \text{for all } t \geq 0.$$

This extension is justified when interest will be paid for partial periods and accumulation increments have an additive nature.

Additive accumulation increments means:

$$[a(t + s) - a(0)] = [a(t) - a(0)] + [a(s) - a(0)]$$

or

$$a(t + s) = a(t) + a(s) - 1 \quad \text{since } a(0) = 1$$

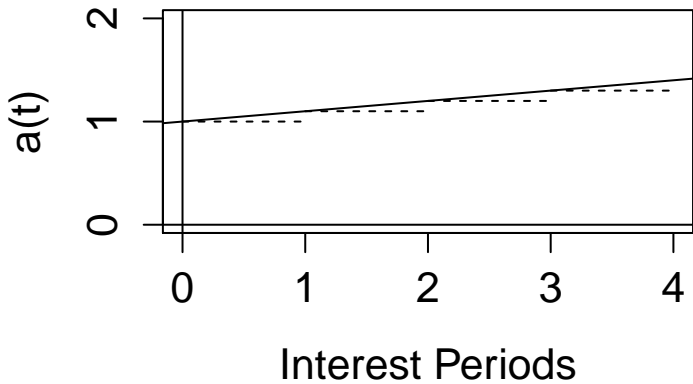
In this case,

$$\begin{aligned} a'(t) &= \lim_{s \rightarrow 0} \left[ \frac{a(t + s) - a(t)}{s} \right] \\ &= \lim_{s \rightarrow 0} \left[ \frac{a(t) + a(s) - 1 - a(t)}{s} \right] \\ &= \lim_{s \rightarrow 0} \left[ \frac{a(s) - a(0)}{s} \right] \\ &= a'(0) \end{aligned}$$

Thus the derivative is the same for all  $t$ , i.e.  $a(t)$  is a straight line and

$$a(t) = 1 + a'(0)t = 1 + it \quad \text{for all } t \geq 0.$$





**Example:** An account is receiving 6% simple interest. If \$100 is invested, how long must it remain invested until the account contains \$200?

$$200 = 100(1 + (.06)t)$$

$$2 = 1 + (.06)t$$

$$.06t = 1$$

$$t = \frac{1}{.06} \text{ or } 16\frac{2}{3} \text{ periods.}$$

In ten periods this account contains

$$A(10) = 100(1 + (.06)(10)) = 160 \text{ dollars}$$

## Section 1.5 - Compound Interest

With compound interest, all interest earned in previous periods is reinvested to earn interest in subsequent periods.

$$\begin{aligned}a(1) &= (1 + i)a(0) = (1 + i) \\a(2) &= (1 + i)a(1) = (1 + i)^2 \\a(3) &= (1 + i)a(2) = (1 + i)^3 \quad \text{etc.}\end{aligned}$$

Extending this to include partial periods produces

$$a(t) = (1 + i)^t \quad \text{for all } t \geq 0.$$

We will see shortly that this accumulation function is also justified in terms of a **multiplicative property of the accumulation function**,

$$a(t + s) = a(t)a(s) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

The effective interest rate for the  $n$ th period is

$$\begin{aligned}i_n &= \frac{a(n) - a(n-1)}{a(n-1)} \\&= \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} \\&= 1 + i - 1 \\&= i.\end{aligned}$$

Thus the effective interest rate is the same in every period.

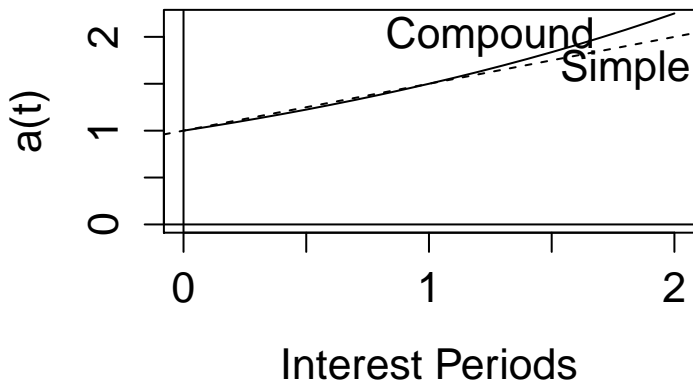
For Simple Interest, the accumulation increment

$$a(t+s) - a(t) \quad \text{does not depend on } t$$

For Compound Interest, the relative growth

$$\frac{a(t+s) - a(t)}{a(t)} \quad \text{does not depend on } t$$

# Accumulation Functions



Note that  $a(0) = 1$  and  $a(1) = 1 + i$ , for both simple and compound interest accumulation functions.

When  $0 < t < 1$ ,  $a_{si}(t) > a_{ci}(t)$ , and when  $t > 1$ ,  $a_{si}(t) < a_{ci}(t)$ .

## Aside:

This accumulation function is also justified in terms of a **multiplicative property of the accumulation function**,

$$a(t + s) = a(t)a(s) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

This property says that the accumulation at the end of  $t + s$  periods is the same as if the accumulation over  $t$  periods was reinvested for  $s$  periods under the same original plan.

$$\begin{aligned} a'(t) &= \lim_{s \rightarrow 0} \left[ \frac{a(t + s) - a(t)}{s} \right] \\ &= a(t) \lim_{s \rightarrow 0} \left[ \frac{a(s) - a(0)}{s} \right] \\ &= a(t) a'(0) \end{aligned}$$

In other words, the **relative growth is always the same**, or

$$\frac{d}{dt} \ln(a(t)) = \frac{a'(t)}{a(t)} = a'(0) \quad \text{does not depend on } t.$$

It follows that

$$\int_0^t \frac{d}{dr} \ln(a(r)) dr = \int_0^t a'(r) dr$$

$$\ln(a(r)) \Big|_0^t = a'(r) t$$

$$\ln(a(t)) - \ln(1) = a'(0)t.$$

Note also that

$$\ln(a(1)) = \ln(1 + i) = a'(0)1$$

and thus

$$\ln(a(t)) = (\ln(1 + i))t = \ln((1 + i)^t)$$

or

$$a(t) = (1 + i)^t.$$

**End Aside**

**Example:** An account is receiving 6% compound interest. How long must \$100 be invested for the account to contain \$200?

$$200 = 100(1 + .06)^t$$

$$2 = (1 + .06)^t$$

$$\ln(2) = t \ln(1.06)$$

$$t = \frac{\ln(2)}{\ln(1.06)} = 11.896 \text{ periods.}$$

In 10 periods, the account value is:

$$A(10) = 100(1.06)^{10} = 179.08 \text{ dollars.}$$



### Example:

A 35-year-old investor deposits \$25,000 in an account earning 5% compound interest until retirement at age 65. What is the value of the account at retirement?

$$A(30) = 25000(1.05)^{30} = 108,048.55 \text{ dollars.}$$

Note also that between ages 35 and 45, the account grows by:

$$A(10) - A(0) = 25000[(1.05)^{10} - 1] = 15,722.36 \text{ dollars.}$$

Between ages 45 and 55, it grows by:

$$A(20) - A(10) = 25000[(1.05)^{20} - (1.05)^{10}] = 25,610.08 \text{ dollars.}$$

Between ages 55 and 65, it grows by

$$A(30) - A(20) = 25000[(1.05)^{30} - (1.05)^{20}] = 41,716.11 \text{ dollars.}$$

Because interest from earlier years also earns interest in later years, the fund grows most rapidly toward the end.

## Example:

A depositor puts \$10,000 in a bank account. It earns an effective annual interest of  $i$  during the first year and an effective annual of  $(i - .05)$ , the second year. After two years the account has a balance of \$12,093.75. What would the account contain after 3 years if the annual effective rate is  $(i + .09)$  for each of the three years?

## Section 1.6 - Present Values

An asset of one grows to

$$a(1) = (1 + i)1$$

at the end of one period. So  $(1 + i)$  is called the **accumulation factor**. What would you need to invest today to achieve a value of one after one interest period?

$$a(1) = 1 = (1 + i)(\text{what value?})$$

Solving this equation for the unknown value yields

$$v = \frac{1}{(1 + i)}.$$

The value  $v$  is called the **discount factor**. If you are going to receive an asset of  $k$  at the end of one period, its **present value** is

$$kv = \frac{k}{(1 + i)}.$$

When set to be in correspondence with one another, accumulation and discounting are **reciprocal processes**, i.e. the discount function and the accumulation function satisfy

$$a(t) \cdot d(t) = 1,$$

that is,

$$d(t) = \frac{1}{a(t)} = [a(t)]^{-1}.$$

So under **simple interest**

$$a(t) = (1 + it)$$

$$d(t) = \frac{1}{(1 + it)}.$$

Under **compound interest**

$$a(t) = (1 + i)^t$$

$$d(t) = (1 + i)^{-t} = \nu^t$$

The **properties** of a discount function are:

- ▶  $d(0) = \frac{1}{a(0)} = 1$
- ▶  $d(1) = \frac{1}{a(1)} = \frac{1}{(1+i)} = \nu$
- ▶  $d(t)$  is typically a decreasing function of  $t$ .

## Exercise 1-17:

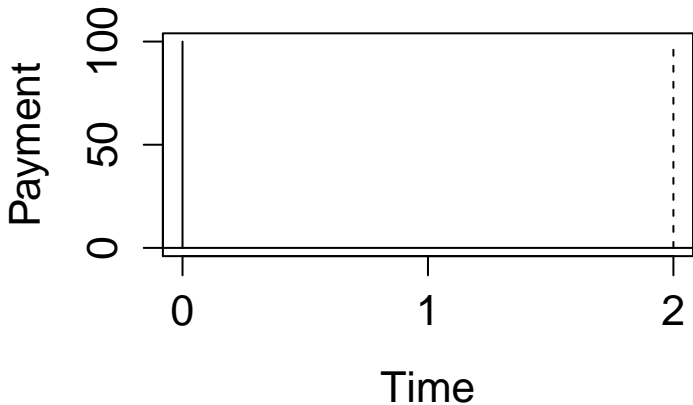
Two sets of grandparents of a newborn agreed to invest immediately to fund \$20,000 per year for four years of college at age 18 for the child. Grandparents A agreed to cover the first two years of college with grandparents B covering the last two years. The account has an effective rate of 6% per annum. What should each of the grandparents contribute to the fund today to meet this need?

Grandparents A:

$$20000v^{18} + 20000v^{19} = 7,006.88 + 6,610.26 = 13,617.14 \text{ dollars}$$

Grandparents B:

$$20000v^{20} + 20000v^{21} = 6,236.09 + 5,883.11 = 12,119.20 \text{ dollars}$$



Which would you rather have:

**Case A:** \$100 Today? Or

**Case B:** Receive \$100 two years from today?

The answer is clear - you would prefer to have the money NOW!  
Money in your possession today is more valuable to you (today) than the same amount of money received in the future.

So what value should you place on this \$100 you will receive in two years? You value it as though it is the amount that you would need to invest today to have \$100 in two years. Therefore its value is

$$\$100v^2 = \$100 \left( \frac{1}{1+i} \right)^2$$

So this future payment is viewed relative to some interest rate  $i$ .

### Time Value of Money:

The **present value** of a future payment is discounted according to some interest rate  $i$ .

In order to be compared, **all payments must be viewed from the same time point and therefore must be accumulated or discounted to that time point via an interest rate  $i$ .**



## Section 1.7 - Effective Rate of Discount

The effective rate of interest,  $i$ , satisfies:

$$A(0) + iA(0) = A(1) \quad \text{or}$$

$$i = \frac{A(1) - A(0)}{A(0)} = \frac{a(1) - a(0)}{a(0)}.$$

Likewise, the effective rate of discount,  $d$ , satisfies:

$$A(1) - dA(1) = A(0) \quad \text{or}$$

$$d = \frac{A(1) - A(0)}{A(1)} = \frac{a(1) - a(0)}{a(1)}.$$

By extension, during the  $n^{\text{th}}$  period, the effective rate of discount is:

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{a(n) - a(n-1)}{a(n)}$$

When do the two rates  $i$  and  $d$  correspond to the same movement of assets in opposite directions (equivalent in this sense)?

$$(1 + i)A(0) = A(1) \quad \text{and}$$

$$(1 - d)A(1) = A(0) \quad \text{or}$$

$$(1 + i)(1 - d) = 1.$$

Solving this equation for  $d$  yields:

$$d = \frac{i}{(1 + i)} = i\nu$$

and solving it for  $i$  yields:

$$i = \frac{d}{(1-d)}.$$

This shows that

$$1 - d = \frac{1}{(1+i)} = \nu,$$

$$d = 1 - \nu \quad \text{or}$$

$$\nu + d = 1.$$

In addition,

$$d = i\nu = i(1-d) \quad \text{or}$$

$$i - d = id > 0.$$

For **compound discount**,

$$d(t) = v^t = (1 - d)^t.$$

Clearly

$$d(t)a(t) = v^t(1 + i)^t = 1 \quad \text{or}$$

$$d(t) = \frac{1}{a(t)} = [a(t)]^{-1}.$$

For **simple discount**:

$$d(t) = (1 - dt) \quad \text{for } 0 < t < \frac{1}{d}.$$

**Exercise 01-25** If  $i$  and  $d$  are equivalent rates of simple interest and simple discount over  $t$  periods, show that  $i - d = idt$ .

**Exercise:** Give a faculty member a  $100i\%$  raise and require each to pay  $100i\%$  into a retirement account. What is the faculty member's new salary? What should the raise be to make it even out?

## Section 1.8 - Nominal Rates

Interest is quoted in terms of an annual rate, but frequently is compounded over shorter intervals. For example, an 8% interest rate when compounded quarterly means 2% percent interest is added to the principal at the end of each quarter thus the effective annual rate  $i$  is determined by:

$$(1 + i) = \left(1 + \frac{.08}{4}\right)^4 \quad \text{or} \quad i = .0824.$$

which is larger than 8%.

In this example, 8% is the **nominal annual rate (APR)** and 8.24% is the **effective annual rate (APY)**.

In general, suppose a nominal annual rate of  $i^{(m)}$  is compounded over  $m$  equal segments of the year. Then the effective annual interest rate  $i$  is determined by

$$(1 + i) = \left(1 + \frac{i^{(m)}}{m}\right)^m \quad \text{or} \quad i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1.$$

Moreover, we can find the nominal annual rate which achieves a fixed effective annual rate via

$$i^{(m)} = m \left[ (1 + i)^{\frac{1}{m}} - 1 \right].$$

The accumulation function in this setting is

$$a(t) = \left( 1 + \frac{i^{(m)}}{m} \right)^{mt}.$$

Suppose  $i^{(m)}$  does not depend on  $m$ , that is the nominal level stays the same, no matter how many times the account is compounded. Call this value  $i^{(1)}$ . In calculus we learned that

$$\lim_{m \rightarrow \infty} \left( 1 + \frac{i^{(1)}}{m} \right)^m = e^{i^{(1)}}.$$



As  $m \rightarrow \infty$ , the account is said to be **compounded continuously**.  
 When  $m \rightarrow \infty$  and  $i^{(m)} = i^{(1)}$  for all  $m$  (i.e. the nominal rate does not change as  $m$  changes), the effective annual rate is

$$i = e^{i^{(1)}} - 1 \quad \text{and} \quad i^{(1)} = \ln(1 + i)$$

where  $i^{(1)}$  is the nominal annual interest rate.

**Example:** Compound a fixed 5% nominal rate ( $i^{(1)} = .05$  for all  $m$ ).

Period	$m$	$i$ (effective rate)
Annually	1	.05
Semi-annually	2	.050625
Quarterly	4	.050945
Monthly	12	.051162
Daily	365	.051267
Continuously	$\rightarrow \infty$	.051271

In a similar fashion, discounting can also be applied  $m$  times each period. If  $d^{(m)}$  denotes the **nominal discount rate** in a complete period, the **effective discount rate** for a complete period when convertible  $m$  times per period is:

$$d = 1 - \left(1 - \frac{d^{(m)}}{m}\right)^m$$

or we can solve the relationship for

$$d^{(m)} = m \left[1 - (1 - d)^{\frac{1}{m}}\right] = m \left[1 - v^{\frac{1}{m}}\right].$$

The discount function is

$$d(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{mt} \quad \text{for all } t > 0.$$

When the nominal discount rate is the same for all  $m$ , i.e.  $d^{(m)} = d^{(1)}$ , and  $m \rightarrow \infty$  (the discounts are convertible continuously), the effective rate of discount per period is

$$d = 1 - e^{-d^{(1)}}.$$

Also note that for the effective rates of interest and discount to match, that is

$$a(t)d(t) = 1$$

then

$$\left(1 + \frac{i^{(m)}}{m}\right)^{mt} = \left(1 - \frac{d^{(p)}}{p}\right)^{-pt}.$$

even when interest is compounded *m*thly per period and discount is convertible *p*thly per period.

**Example:** Find the accumulated value of \$500 at the end of three years

(a) if the nominal rate of interest is 5% convertible semiannually.

$$500\left(1 + \frac{.05}{2}\right)^6 = 579.85$$

(b) if the nominal rate of discount is 7% convertible every two years.

$$\frac{500}{(1 - .14)^{1.5}} = 626.93$$

## Exercise 1-23:

Find the present value of \$5000 to be paid at the end of 25 months at a discount of 8% convertible quarterly.

(a) Assume compound discount throughout:

$8\frac{1}{3}$  periods with 2% discount per period

$$\begin{aligned} 5000v^{\frac{25}{3}} &= 5000(1 - .02)^{\frac{25}{3}} \\ &= 4225.27 \end{aligned}$$

(b) Assume compound discount for whole periods and simple discount for fractional periods.

$$5000(1 - .02)^8 \left(1 - \frac{.02}{3}\right) = 4225.46$$

**Exercise 1-29:** Given  $i^{(m)} = .1844144$  and  $d^{(m)} = .1802608$  are nominal annual rates producing effective annual rates, find  $m$ .

## Section 1.9 - Force of Interest (Discount)

Earlier we described the effective rate of interest for the  $n^{\text{th}}$  period to be

$$\frac{A(n) - A(n - 1)}{A(n - 1)} = \frac{\text{Change in Amount during the period}}{\text{Amount entering the period}}.$$

representing the relative growth in the amount during that period. As we ignore the discrete period boundaries and focus on the amount function  $A(t)$  at all  $t > 0$ , it is natural to describe the relative growth in this amount function at time  $t$  with

$$\frac{A'(t)}{A(t)},$$

the instantaneous rate of change in the amount function relative to its current size.

**Definition:** The force of interest  $\delta_t$  at time  $t > 0$  is

$$\delta_t = \lim_{h \rightarrow 0} \frac{A(t+h) - A(t)}{hA(t)} = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}.$$

## Examples

Simple Interest:

$$a(t) = 1 + it \quad \delta_t = \frac{i}{1 + it} \quad (\text{decreasing in } t)$$

Compound Interest:

$$a(t) = (1 + i)^t \quad \delta_t = \ln(1 + i) \quad (\text{constant in } t)$$

Earlier Example:

$$a(t) = (.05)t^2 + 1 \quad \delta_t = \frac{(.1)t}{(.05)t^2 + 1}$$

Note that

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt} \ln(A(t)) = \frac{d}{dt} \ln(a(t)).$$

Replacing  $t$  with  $r$  and integrating yields

$$\begin{aligned} \int_0^t \delta_r dr &= \int_0^t \frac{d}{dr} \ln(A(r)) dr = \ln(A(r)) \Big|_0^t \\ &= \ln(A(t)) - \ln(A(0)) = \ln\left(\frac{A(t)}{A(0)}\right). \end{aligned}$$

Thus

$$e^{\int_0^t \delta_r dr} = \frac{A(t)}{A(0)} = \frac{a(t)}{a(0)} = a(t).$$

Also

$$\begin{aligned} \int_0^t A(r) \delta_r dr &= \int_0^t A'(r) dr = A(r) \Big|_0^t \\ &= A(t) - A(0). \end{aligned}$$



A **Constant Force of Interest** means

$$\delta_t = \delta \quad \text{for all } t > 0$$

In this case

$$\int_0^t \delta_r dr = \delta r \Big|_0^t = \delta t$$

and

$$a(t) = e^{\delta t}.$$

Since

$$a(1) = 1 + i = e^{\delta},$$

$$\delta = \ln(1 + i),$$

where  $i$  is the **effective rate of interest** and  $\delta$  is the **nominal rate of interest** when interest is compounded continuously.

So when a problem is described as having a constant force of interest,

$$a(t) = e^{\delta t} = (1 + i)^t \quad \text{and} \quad \delta = \ln(1 + i)$$

where  $i$  is the effective interest rate,  $\delta$  is the nominal interest rate and there is continuous compounding of interest.

The force of discount is defined as

$$\delta'_t = -\frac{d'(t)}{d(t)},$$

where the minus sign is included because  $d'(t)$  is typically negative.

Because  $d(t) = [a(t)]^{-1}$ ,

$$\begin{aligned}\delta'_t &= \frac{-\frac{d}{dt}[a(t)]^{-1}}{[a(t)]^{-1}} = \frac{[a(t)]^{-2}a'(t)}{[a(t)]^{-1}}, \\ &= \frac{a'(t)}{a(t)} = \delta_t.\end{aligned}$$

so we dropped the prime, because these two are the same quantity.  
Also note that

$$d(t) = [a(t)]^{-1} = e^{-\int_0^t \delta_r dr}.$$

A **constant force of discount** means the same thing as a constant force of interest, i.e.

$$d(t) = [a(t)]^{-1} = e^{-\delta t}$$

where  $\delta = \ln(1 + i) = -\ln(1 - d)$ , for  $i$  the effective rate of interest and  $d$  the effective rate of discount.

**Example:** What is the account balance after two years when a \$500 deposit is made in an account with  $\delta_t = (.02) + (.03)t^2$ ?

$$\int_0^2 [.02 + .03t^2] dt = [(.02)t + (.01)t^3] \Big|_0^2 = .04 + .08 = .12$$

$$A(2) = 500e^{.12} = 563.75$$

**Exercise 1-46:** You are given  $\delta_t = 2/(t - 1)$  for  $2 \leq t \leq 10$ . For any one-year interval between  $n$  and  $n + 1$ , with  $n = 2, \dots, 9$ , calculate the equivalent  $d^{(2)}$ .

$$\int_n^{n+1} \frac{2}{(t-1)} dt = 2 \ln(t-1) \Big|_n^{n+1} = 2 \ln \left( \frac{n}{n-1} \right)$$

$$e^{-2 \ln \left( \frac{n}{n-1} \right)} = \left( 1 - \frac{d^{(2)}}{2} \right)^2$$

$$\left( 1 - \frac{d^{(2)}}{2} \right) = \frac{n-1}{n} \quad \text{or} \quad d^{(2)} = \frac{2}{n}$$

**Exercise:** A fund earns interest at a force of interest of  $\delta_t = kt$ . A deposit of 100 at time 0 will grow to 250 at the end of five years. Find  $k$ .

## Section 1.10 - Varying Interest (Discount)

For many investments the interest rates vary from one period to the next, where the periods are of fixed and equal length. The compound accumulation function over  $n$  periods is then

$$a(n) = (1 + i_1)(1 + i_2) \cdots (1 + i_n) = \prod_{j=1}^n (1 + i_j),$$

where  $i_j$  is the effective interest rate for the  $j^{\text{th}}$  interest period. In similar fashion, if the discount rates vary from period to period, the discount function over  $n$  periods is

$$d(n) = (1 - d_1)(1 - d_2) \cdots (1 - d_n) = \prod_{j=1}^n (1 - d_j)$$

where  $d_j$  is the effective rate of discount during the  $j^{\text{th}}$  period.

**Example:** A bank recently advertised a variable interest rate CD that earns 1.6% APR for the first six months, 1.8% for the second six months, 2.0% for the third and 2.2% for the fourth. What is the effective annual interest rate for the CD?

$$\begin{aligned} a(2) &= \left(1 + \frac{.016}{2}\right)\left(1 + \frac{.018}{2}\right)\left(1 + \frac{.020}{2}\right)\left(1 + \frac{.022}{2}\right) \\ &= 1.03854 \end{aligned}$$

Thus we set

$$1.03854 = (1 + i)^2 \quad \text{and get} \quad i = .019089.$$



**Exercise:** At a certain interest rate the present values of the following are equal:

(a) 200 at the end of 5 years and 500 at the end of 10 years

(b) 400.94 at the end of 5 years.

At the same interest rate, 100 invested now and 120 invested at the end of 5 years accumulates to  $P$  at the end of 10 years. Find  $P$ .

**Exercise:** An investor puts 100 into fund X and 100 into fund Y. Fund Y earns compound interest at the annual rate of  $j > 0$  and fund X earns simple interest at the annual rate of  $(1.05)j$ . At the end of 2 years, the amount in fund Y is equal to the amount in fund X. Calculate the amount in fund Y at the end of 5 years.

## Appendix to Chapter 01 - Time Value of Money Principles

- (1) Establish a vision point (some time  $t = t_0$ ) at which values of payments are to be evaluated.
  - (a) accumulate all earlier payments to the vision point
  - (b) discount all future payments to the vision point
- (2) Only combine payments after they are all projected to the same (a common) vision point
- (3) The value of a payment stream at the time  $t = t_0$  is the **sum** of all the (projected to  $t_0$ ) payment values
- (4) Two or more investments (payment streams) are only comparable when they are all projected to one common vision point.

The two most common vision points are:

**PV = Present Value** = (vision point is today, i.e.  $t = 0$ )

**FV = Future Value** = (vision point is the end of the last interest period, i.e.  $t = n$ )