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Fifty Famous Curves, Lots of Calculus Questions, And a Few Answers

Summary

Sophisticated calculators have made it easier to carefully sketch more complicated and interesting graphs of equations given in Cartesian form, polar form, or parametrically. These elegant curves, for example, the Bicorn, Catesian Oval, and Freeth's Nephroid, lead to many challenging calculus questions concerning arc length, area, volume, tangent lines, and more. The curves and questions presented are a source for extra, varied AP-type problems and appeal especially to those who learned calculus before graphing calculators.

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1. **Astroid**

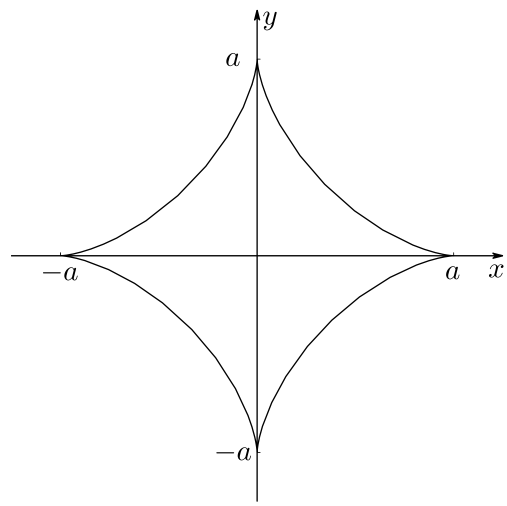
Cartesian Equation:

$x^{2/3} + y^{2/3} = a^{2/3}$

Parametric Equations:

$x(t) = a\cos^3 t$

$y(t) = a\sin^3 t$



Facts:

(a) Also called the tetracuspid because it has four cusps.

(b) Curve can be formed by rolling a circle of radius $a/4$ on the inside of a circle of radius a.

(c) The curve can also be formed as the envelop produced when a line segment is moved with each end on one of a pair of perpendicular axes (glissette).

Calculus Questions:

(a) Find the length of the astroid.

(b) Find the area of the astroid.

(c) Find the equation of the tangent line to the astroid with $t = \theta_0$.

(d) Suppose a tangent line to the astroid intersects the x-axis at X and the y-axis at Y. Find the distance from X to Y.

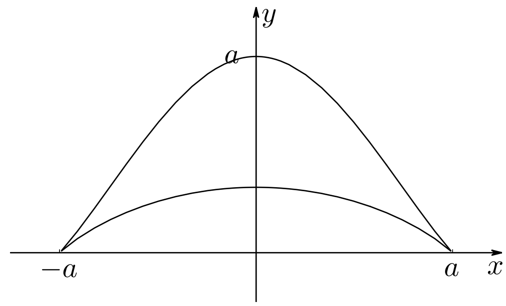
2. **Bicorn**

Cartesian Equation:

$$y^2(a^2 - x^2) = (x^2 + 2ay - a)^2$$

$$y = \frac{2 - 2x^2 - \sqrt{1 - 3x^2 + 3x^4 - x^6}}{3 + x^2}$$

$$y = \frac{2 - 2x^2 + \sqrt{1 - 3x^2 + 3x^4 - x^6}}{3 + x^2}$$

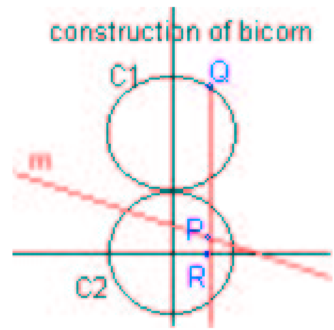


Facts:

(a) The curve has two (bi) horns (corn).

(b) An alternative name is the cocked-hat.

(c) Construction: Consider two tangent circles, C1 and C2, of equal radius. Let Q be a point on C1 and R its projection on the y axis. Let m be the polar inverse (a line) of C1 with respect to Q. The bicorn is formed by the points P which are the intersection of m and QR.



Calculus Questions:

(a) Find the area enclosed by a bicorn.

(b) Find the length of the bicorn curve.

(c) Find the slope of the tangent line at any point along the bicorn.

(d) Are there any points along the bicorn where the slope is ± 1?

(e) Find the area under the bicorn and above the x-axis.

3. **Cardioid**

Cartesian Equation:

$(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2)$

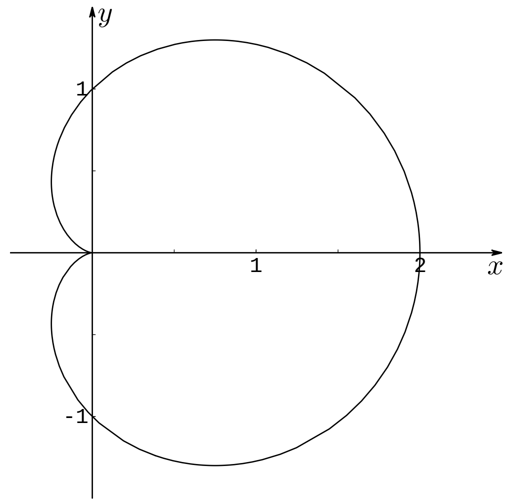
Polar Equation:

$r = 2a(1 + \cos\theta)$

Parametric Equations:

$x(t) = a(2\cos t - \cos(2t))$

$y(t) = a(2\sin t - \sin(2t))$



Facts:

(a) Trace a point on the circle rolling around another circle of equal radius.

(b) There are exactly three parallel tangents at any given point on the cardioid.

(c) The tangents at the ends of any chord through the cusp point are at right angles.

Calculus Questions:

(a) Find the length of any chord through the cusp point.

(b) Find the area enclosed by the cardioid.

(c) Find all the points on the cardioid where the tangent line is vertical, horizontal.

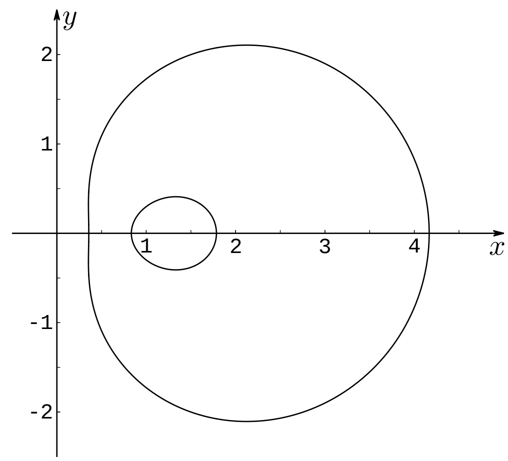
(d) Find the length of the curve.

(e) Find a line $x = k$ that cuts the area in half.

4. **Cartesian Oval**

Cartesian Equation:

$$((1-m^2)(x^2+y^2)+2m^2cx+a^2-m^2c^2)^2$$
$$= 4a^2(x^2+y^2)$$



Facts:

(a) This curve really consists of two ovals. It is the locus of a point P whose distances s and t from two fixed points S and T satisfy $s+mt=a$. When c is the distance between S and T then the curve can be expressed in the form given above.

(b) If $m=\pm 1$, then the curve is a central conic.

(c) If $m=a/c$ then the curve is a limaçon.

Calculus Questions:

(a) Find the area enclosed by the outer oval, inner oval, in between.

(b) Find the equation of the tangent line to an oval at any point.

(c) Find points on the inner and outer oval where the tangent lines have a slope of 1 (the same slope).

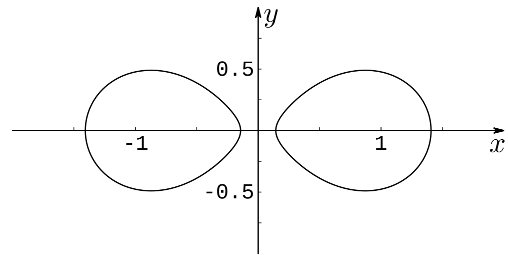
(d) When is the inner oval tangent to the outer oval?

(e) Is it possible for the inner and outer ovals to be tangent at more than one point?

5. **Cassinian Ovals**

Cartesian Equation:

$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) - a^4 + c^4 = 0$



Facts:

(a) The Cassinian ovals are the locus of a point P that moves so that the product of its distances from two fixed points S and T is a constant c^2. The shape of the curve depends on c/a. If $c > a$ then the curve consists of two loops. If $c < a$ then the curve consists of a single loop. If $c = a$ then the curve is a lemniscate.

(b) These curves were first introduced in attempting to describe the movement of the earth relative to the sun. Cassini suggested the Sun traveled around the Earth on one of these ovals, with the Earth at one focus of the oval.

Calculus Questions:

(a) Find the area enclosed by one of the ovals.

(b) Find an expression for the curve in polar form.

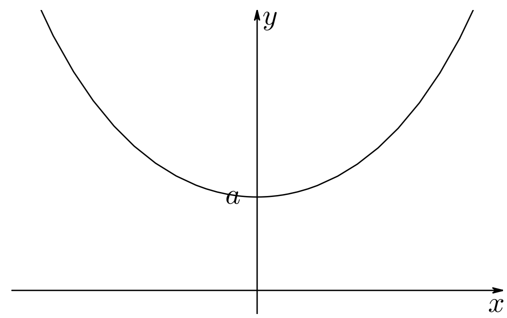
(c) Find the slope of the tangent line to an oval at any point.

6. **Catenary**

Cartesian Equation:

$y = a\cosh(x/a)$

$$y = \frac{e^x + e^{-x}}{2}$$



Facts:

(a) This curve models the force of gravity acting on a perfect, uniform, flexible chain between two supports.

(b) Galileo believed this curve would be a parabola.

(c) The catenary is the locus of the focus of a parabola rolling along a straight line.

Calculus Questions:

(a) Find the length of the curve between two points.

(b) Find the area bounded above by the catenary and below by the x-axis.

(c) Find the equation of the tangent line to the curve at the point $(c, f(c))$.

(d) Let R be the region bounded above by the line $y = 2$ and below by the graph of $y = \cosh x$. Find the volume of the solid obtained by rotating R about the line $y = -1$.

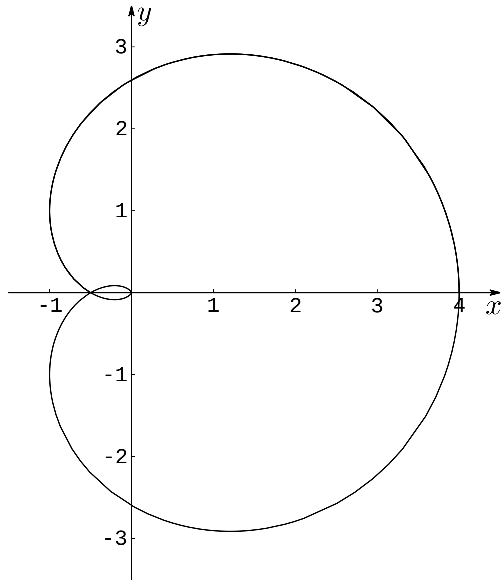
7. **Cayley's Sextic**

Cartesian Equation:

$4(x^2 + y^2 - ax)^3 = 27a^2(x^2 + y^2)^2$

Polar Equation:

$r = 4a\cos^3(\theta/3)$



Facts:

(a) Also called a sinusoidal spiral.

(b) The curve can be formed by a cardioid rolling over another cardioid of the same size.

Calculus Questions:

(a) Find the inner area.

(b) Find the outer area.

(c) Find the length of the curve.

(d) Find the equation of the tangent line to a point on the curve.

(e) Find the points on the curve where the tangent line is horizontal.

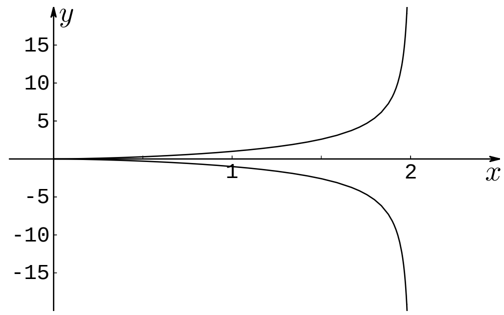
8. **Cissoid of Diocles**

Cartesian Equation:

$$y^2 = \frac{x^3}{2a - x}$$

Polar Equation:

$$r = 2a \tan\theta \sin\theta$$

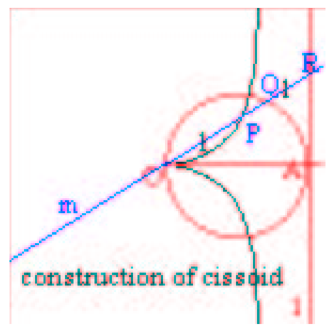


Facts:

(a) From a any point in the plane, there are either one or three tangents to the curve.

(b) Diocles studied the cissoid in an attempt to solve the problem of finding the length of the side of a cube having volume twice that of a given cube.

(c) Construction: Given a circle C with diameter OA and a tangent l through A. Draw lines m through O, cutting the circle C in the point Q and line l at the point R. The cissoid is the set of points P for which $OP = QR$.



(d) If a parabola is rolling over another (equal) parabola, then the path of the vertex is a cissoid.

Calculus Questions:

(a) Find the area bounded by the curve and its asymptote.

(b) Find the equation of the tangent line at any point on the curve.

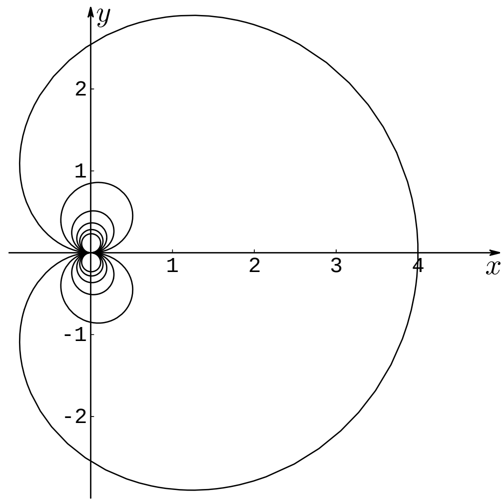
(c) Find the length of the curve from $(0, 0)$ to $(1, 1)$.

(d) Find an equation for a circle tangent to the cissoid and the asymptote.

9. **Cochleoid**

Polar Equation:

$$r = a\,\frac{\sin\theta}{\theta}$$



Calculus Questions:

(a) Find the area of any one loop.

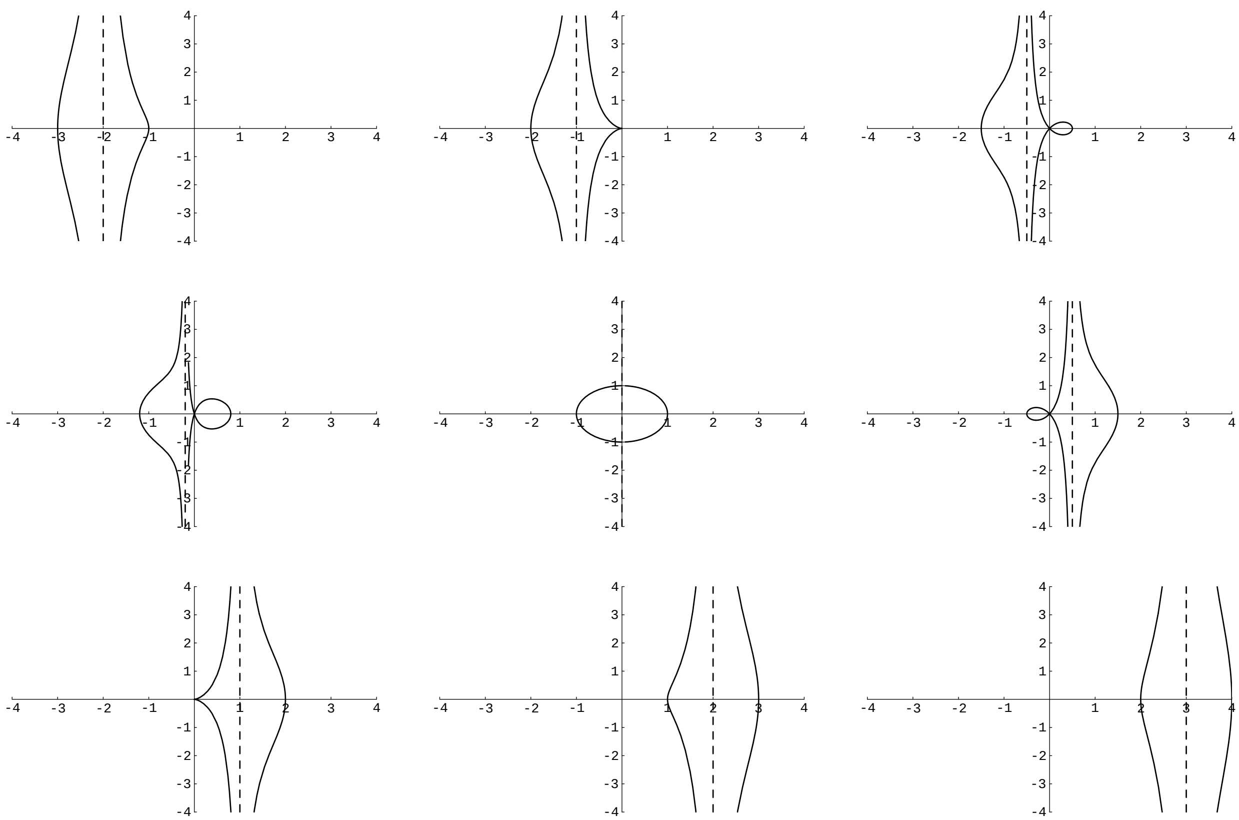
(b) Find the area between two loops.

(c) Find an equation of the tangent line to the curve at any point.

(d) Find the points on the curve where the tangent line is vertical, horizontal.

10. **Conchoid**

Cartesian Equation:	Polar Equation:	Parametric Equations:
$(x-b)^2(x^2+y^2)-a^2x^2=0$	$r = a + b\sec\theta$	$x(t) = a + \cos t$
		$y(t) = a\tan t + \sin t$



Facts:

(a) These curves are also called the *conchoids of Nicomedes*, after the Greek scholar Nicomedes. Called conchoids because the shape of their outer branches resembles a conch shell.

(b) All of these curves (except $a = 0$) have two branches. Both branches approach the vertical asymptote $x = a$ as $x \to a$ from the left or right.

(c) If $a < -1$, both branches are smooth. When a reaches -1, the right branch acquires a sharp point, called a *cusp*.

(d) For $-1 < a < 0$, the cusp turns into a loop, which becomes large as $a \to 0^-$.

(e) If $a = 0$, both branches come together and form a circle.

(f) For $0 < a < 1$, the left branch has a loop, which shrinks to a cusp when $a = 1$. For $a > 1$, the branches become smooth again.

Calculus Questions:

(a) Find the area between either branch and the asymptote.

(b) Find the area of the loop.

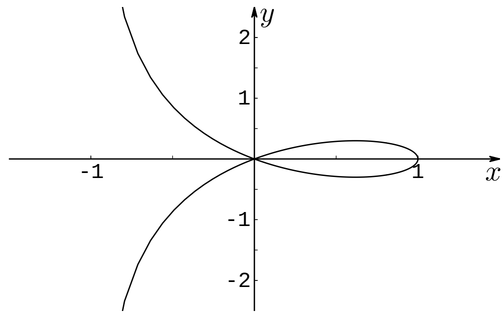
11. **Conchoid of de Sluze**

Cartesian Equation:

$a(x-a)(x^2+y^2) = kx^2$

Polar Equation:

$a(r\cos\theta - a) = k\cos^2\theta$



Calculus Questions:

(a) Find the area of the loop.

(b) Find the area bounded by the curve and the asymptote.

(c) Find the length of the curve around the loop.

(d) Find the equation of a tangent line to the curve at any point.

(e) Let R be the plane region inside the loop. Find the volume of the solid generated when the region R is rotated about the line $x = 2$.

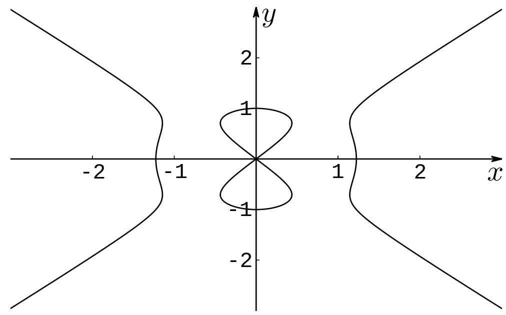
12. **Devil's Curve**

Cartesian Equation:

$y^4 - x^4 + ay^2 + bx^2 = 0$

Polar Equation:

$$r = \sqrt{\frac{25 - 24\tan^2\theta}{1 - \tan^2\theta}}$$



Facts:

(a) Also known as the devil on two sticks.

(b) The constant a is a linear distortion of the curve.

(c) If $b = 25/24$ the curve is called the electric motor curve.

Calculus Questions:

(a) Find the area of one loop.

(b) Find the points on the curve where the tangent line is vertical.

(c) Find the length of the inner curve.

(d) Find the area between the inner and outer curves (and between $y = \pm c$).

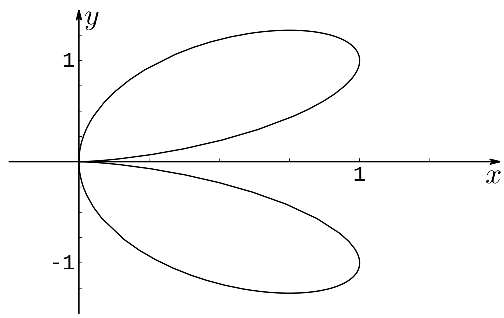
13. **Double Folium**

Cartesian Equation:

$(x^2 + y^2)^2 = 4axy^2$

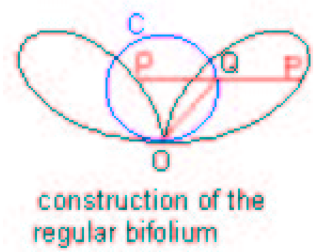
Polar Equation:

$r = 4a\cos\theta\sin^2\theta$



Facts:

(a) Construction: Consider the circle C through the point O. For each point Q on C draw the points P so that $PQ = OQ$. The collection of points P forms the double folium.



Calculus Questions:

(a) Find the area of a loop.

(b) Find the points on the curve where the tangent line is vertical.

(c) Find the area bounded by the loop, a vertical tangent line, and the x-axis.

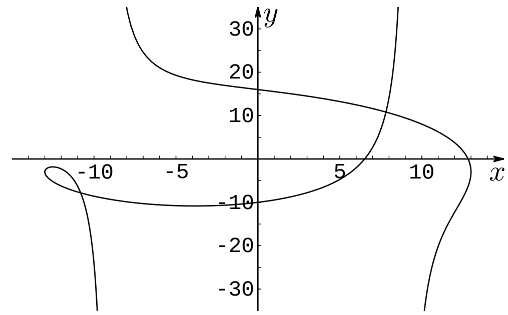
(d) Find the length of a loop.

(e) For $x = c$, $0 < c < 1$, find the point on the x axis where the tangent lines intersect.

14. **Durer's Shell Curves**

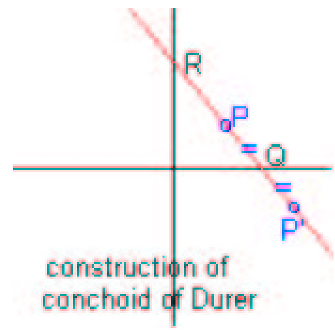
Cartesian Equation:

$(x^2 + xy + ax - b^2)^2 = (b^2 - x^2)(x - y + a)^2$



Facts:

(a) Construction: Consider the two points $Q(q, 0)$ and $R(0, r)$ such that the sum of q and r is constant ($q + r = a$). Draw the points P on a line l through RQ so that PQ is equal to a constant (b). The locus of points P defines the curve.



(b) If $b = 0$, the curve becomes a straight line, $x = 0$.

(c) If $a = 0$, the curve becomes two lines, $x = b/\sqrt{2}$, $x = -b/\sqrt{2}$, together with a circle, $x^2 + y^2 = b^2$.

(d) If $a = b/2$, the curve has a cusp at $(-2a, a)$.

Calculus Questions:

(a) Find the slope of the tangent line to the curve at any point.

(b) Find the points on the curve where the tangent line is horizontal/vertical.

(c) Is there any chance we can find the area of the loop?

(d) How about the length of a piece of the curve?

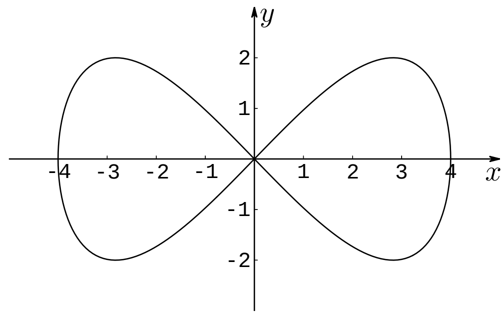
15. **Figure Eight**

Cartesian Equation:

$x^4 = a^2(x^2 - y^2)$

Polar Equation:

$r^2 = a^2 \cos(2\theta) \sec^4 \theta$



Facts:

(a) Also known as the Lemniscate (of Gerono).

Calculus Questions:

(a) Find the area of a loop.

(b) Find the length of the curve.

(c) Find the points on the curve where the tangent line is horizontal/vertical.

(d) Let the region R be the top half of a loop. Find the volume of the solid generated when R is rotated about the x-axis.

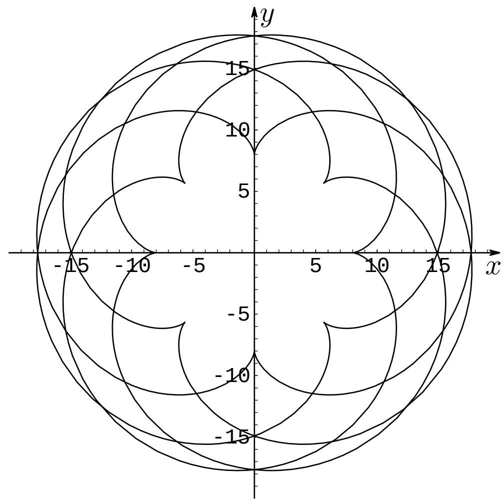
(e) Find the slope of the tangent line to the curve at the point $(0, 0)$.

16. **Epicycloid**

Parametric Equations:

$x(t) = (a+b)\cos t - b\cos((a/b+1)t)$

$y(t) = (a+b)\sin t - b\sin((a/b+1)t)$



Facts:

(a) There are four curves which are closely related: epicycloid, epitrochoid, hypocycloid, and the hypotrochoid. They are all traced by a point P on a circle of radius b which rolls around a fixed circle of radius a.

(b) For an epicycloid, the circle of radius b rolls on the outside of the circle of radius a. The point P is on the edge of the circle of radius b.

(c) If $a = b$: cardioid.

(d) If $a = 2b$, nephroid.

Calculus Questions:

(a) Find the length of the curve if $a = (m-1)b$ where m is an integer (and the area).

(b) Find the slope of the tangent line at any point.

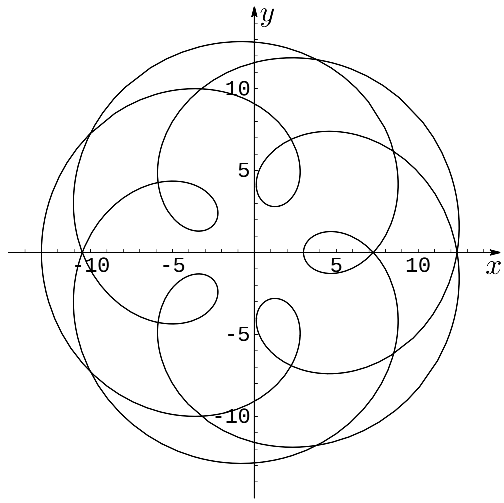
(c) Generate some of the other related curves. Look for some area problems involving loops.

17. **Epitrochoid**

Parametric Equations:

$x(t) = (a + b)\cos t - c\cos((a/b + t)t)$

$y(t) = (a + b)\sin t - c\sin((a/b + 1)t)$



Facts:

(a) The circle of radius b rolls on the outside of the circle of radius a. The point P is at a distance c from the center of the circle of radius b.

Calculus Questions:

(a) Find the area inside a loop.

(b) Find the length of the curve.

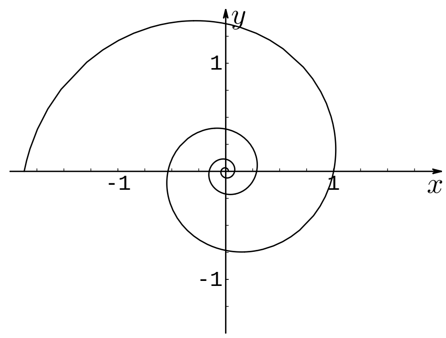
(c) Find the slope of the tangent line to the curve at any point.

(d) Is there any relationship among the points where the tangent line is horizontal?

18. **Equiangular Spiral**

Polar Equation:

$r = ae^{\theta \cot b}$



Facts:

(a) If P is any point on the spiral, then the length of the spiral from P to the origin is finite.

(b) This spiral occurs naturally in many places, for example, sea-shells, where the growth of the organism is proportional to the size of the organism.

(c) The spiral makes a constant angle b with any radius vector. What happens if $b = \pi/2$?

(d) Any radius from the origin intersects the spiral at distances which are in geometric progression.

Calculus Questions:

(a) Suppose a point P is on the curve a distance d from the origin. Find the length of the curve from P to the origin.

(b) Several area problems.

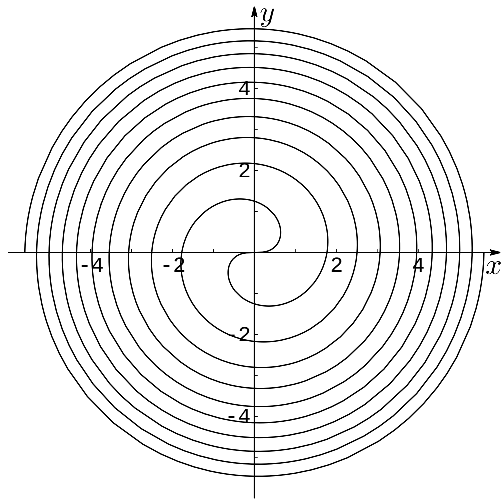
(c) When is the tangent line horizontal?

(d) What's really going on near the origin?

19. **Fermat's Spiral**

Polar Equation:

$r^2 = a^2\theta$



Facts:

(a) For any given positive value of θ there are two corresponding values of r, one is the negative of the other. The resulting spiral is therefore symmetrical about the line $y = -x$.

(b) The people who practice omphaloskepsis adopted this curves as their symbol.

Calculus Questions:

(a) Length of the curve.

(b) Area between two pieces.

(c) Slope of the tangent line to the curve at any point.

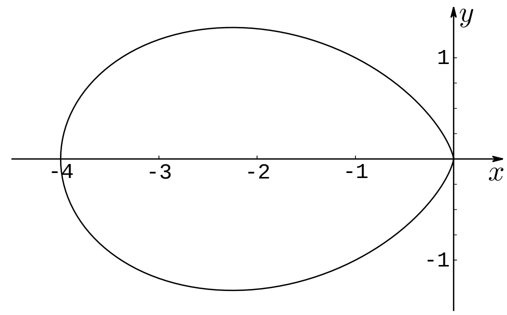
20. **Folium**

Cartesian Equation:

$$(x^2 + y^2)(y^2 + x(x + b)) = 4axy^2$$

Polar Equation:

$$r = -b\cos\theta + 4a\cos\theta\sin^2\theta$$



Facts:

(a) There are three special cases of the folium: the simple folium, the double folium, and the trifolium. They correspond to $b = 4a$, $b = 0$, $b = a$.

Calculus Questions:

(a) Find the area of the loop.

(b) Find the length of the curve.

(c) Find the slope of the tangent line at any point.

(d) Let R be the region enclosed by the folium. Find the volume of the solid obtained by rotating R about the y-axis.

(e) Is there a surface area question in here?

21. **Folium of Descartes**

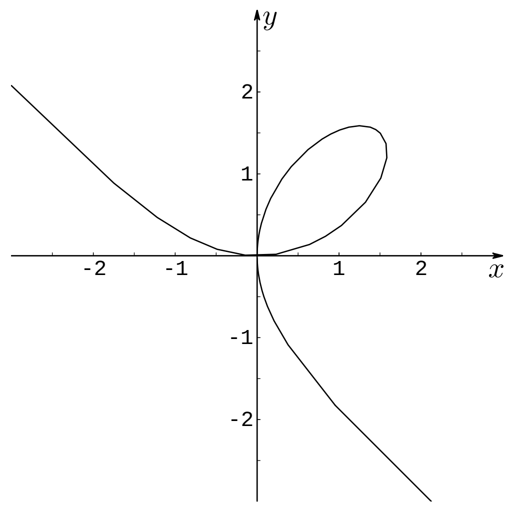
Cartesian Equation:

$x^3 + y^3 = 3axy$

Parametric Equations:

$$x(t) = \frac{3at}{1+t^3}$$

$$y(t) = \frac{3at^2}{1+t^3}$$



Facts:

(a) Descartes believed that the leaf shape was repeated in each quadrant, like the four petals of a flower.

(b) Also known as the noeud de ruban.

Calculus Questions:

(a) Find the equation of the asymptote.

(b) Find an equation of the tangent line to the point when $t = p$.

(c) Try plotting this one on your calculator.

(d) Find the area enclosed by the loop.

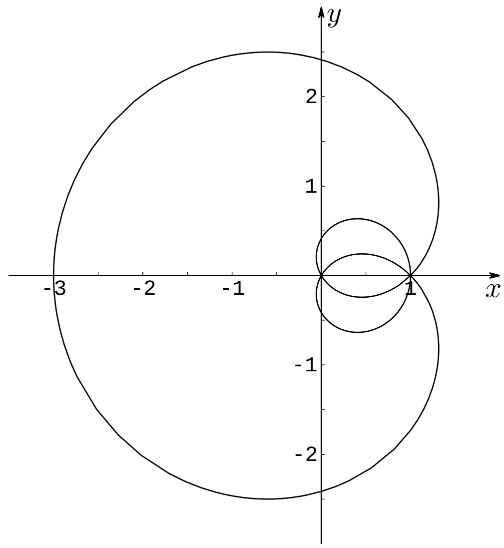
(e) Find the length of the loop.

(f) Revolve the loop about the x-axis. Find the volume of the resulting solid.

22. **Freeth's Nephroid**

Polar Equation:

$r = a(1 + 2\sin(\theta/2))$



Facts:

(a) This curve can be used to construct a regular 7-sided figure. Let P be the node where the curve crosses itself three times. A line l parallel to the y-axis through P cuts the nephroid at Q. The angle POQ is $3\pi/7$.

Calculus Questions:

(a) Lots of interesting area questions.

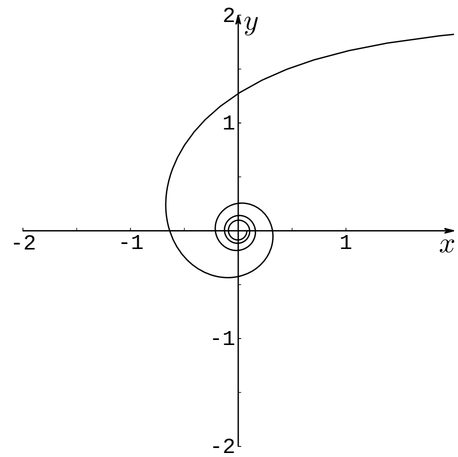
(b) Find the length of the curve.

(c) Where do the tangent lines to the inner loops intersect the y-axis?

23. **Hyperbolic Spiral**

Polar Equation:

$r = a/\theta$



Facts:

(a) The hyperbolic name reflects the relation between the radius and the angle. This is the same relationship between x and y for the hyperbola.

Calculus Questions:

(a) Find the length of the curve between two points.

(b) Find the equation of a tangent line at any point along the curve.

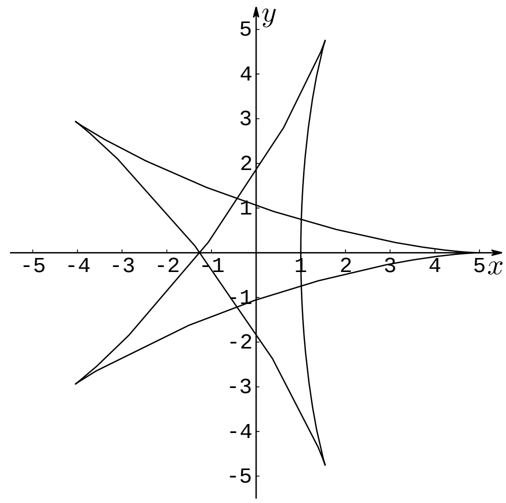
(c) What's going on near the origin?

24. **Hypocycloid**

Parametric Equations:

$x(t) = (a - b)\cos t + b\cos((a/b - 1)t)$

$y(t) = (a - b)\sin t - b\sin((a/b - 1)t)$



Facts:

(a) A circle of radius b rolls on the inside of a circle of radius a. The point P is on the edge of the circle of radius b.

(b) If $a = 3b$, tricuspoid. If $a = 4b$, astroid.

Calculus Questions:

(a) If $a = (n + 1)b$ where n is an integer, find the length of the curve, and find the area.

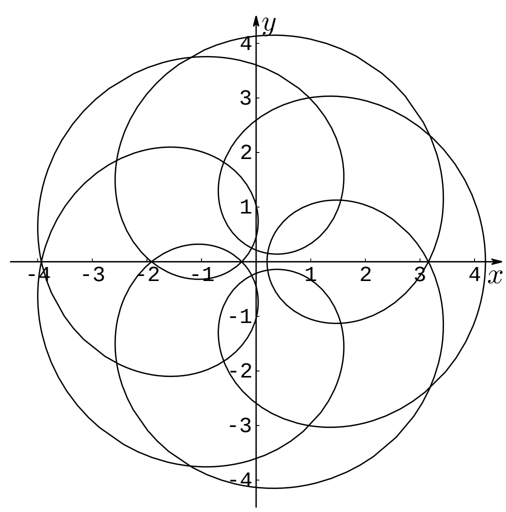
(b) Where does the curve intersect itself?

25. **Hypotrochoid**

Parametric Equations:

$x(t) = (a - b)\cos t + c\cos((a/b - 1)t)$

$y(t) = (a - b)\sin t - c\sin((a/b - 1)t)$



Facts:

(a) The circle of radius b rolls on the inside of the circle of radius a. The point P is at a distance c from the center of the circle of radius b.

26. **Involute of a Circle**

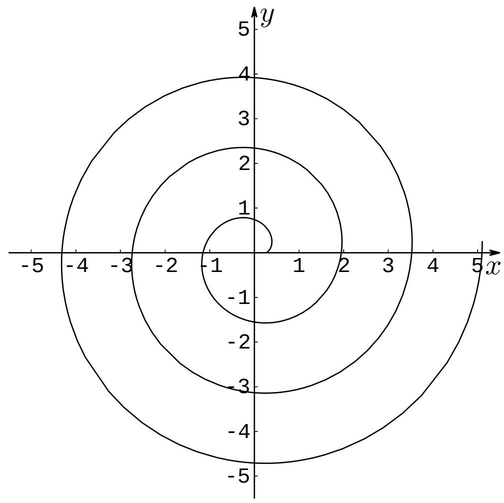
Parametric Equations:

$x(t) = a(\cos t + t\sin t)$

$y(t) = a(\sin t - t\cos t)$

Polar Equation:

$r^2 = \theta^2 + 1$



Facts:

(a) The involute of a circle is the path traced out by a point on a straight line that rolls around a circle.

(b) This curve is used in mechanical engineering for the profile of a gear wheel, in the case of non-parallel axes.

Calculus Questions:

(a) Find the length of the curve between two points.

(b) Find the equation of the tangent line to the curve at any point.

(c) Area questions.

(d) Suppose an object follows this curve with velocity that is linear. What is the rotation speed?

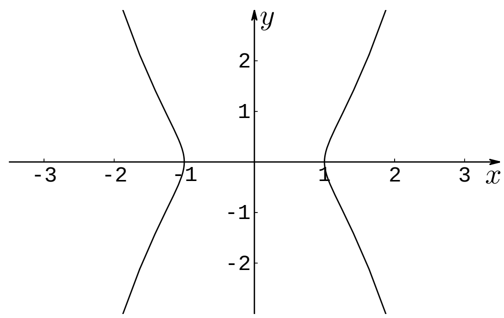
27. **Kampyle of Eudoxus**

Cartesian Equation:

$a^2x^4 = b^4(x^2 + y^2)$

Polar Equation:

$$r = \frac{b^2}{a\cos^2\theta}$$



Facts:

(a) A curve studied by Eudoxus in relation to the classical problem of duplication of the cube.

(b) Eudoxus found formulas for measuring pyramids, cones, and cylinders. His work contains elements of calculus.

Calculus Questions:

(a) Are there asymptotes?

(b) Find the area between the two pieces bounded by $y = \pm c$.

(c) Tangent lines?

(d) Length of the curve between two points?

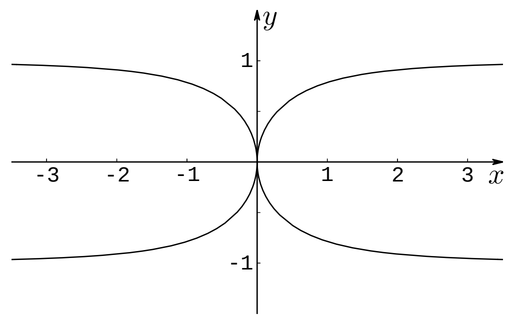
28. **Kappa Curve**

Cartesian Equation:

$y^2(x^2 + y^2) = a^2x^2$

Polar Equation:

$r = a \cot \theta$

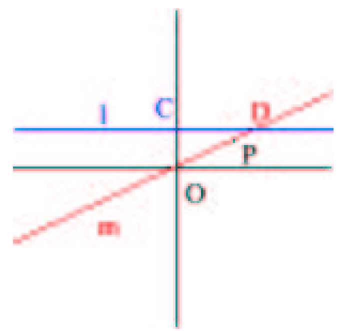


Facts:

(a) Also called Gutschoven's curve

(b) This curve (sort of) resembles the Greek character kappa.

(c) Construction: Let the line l be parallel to the x-axis and intersect the y-axis at the point C. Consider the lines m through O that meets the line l at the point D. The kappa curve is defined as the collection of points P, such that $OP = CD$.



Calculus Questions:

(a) Tangent lines.

(b) Several area questions.

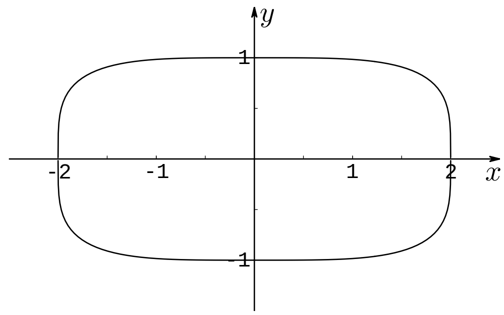
(c) Let R be a region in the top half of the right section. Rotate the region R about the x-axis. Find the volume of the solid of revolution.

(d) Find the area in the triangular region between the curves.

29. **Lamé Curves**

Cartesian Equation:

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$$



Facts:

(a) For even integers n the curve becomes closer to a rectangle as n increases.

(b) For odd integer values of n the curve looks like the even case in the positive quadrant but goes to infinity in both the second and fourth quadrants.

(c) The case $n = 5/2$: The curve has been used for a variety of purposes including bridges and other architectural applications.

Calculus Questions:

(a) Find the area enclosed by the curve.

(b) Find the length of the curve.

(c) Find the slope of the tangent line at any point along the curve.

(d) Let R be the region in the first quadrant. Rotate R about the x-axis and find the volume of the resulting solid.

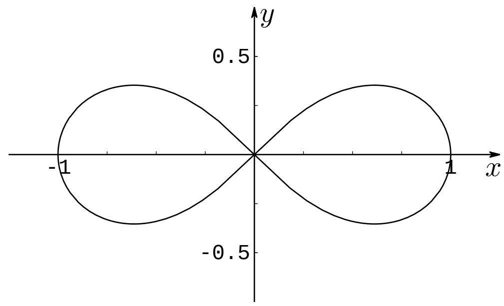
30. **Lemniscate of Bernoulli**

Cartesian Equation:

$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Polar Equation:

$r^2 = a^2 \cos(2\theta)$



Facts:

(a) A special case of a Cassinian Oval.

Calculus Questions:

(a) Find the area of a loop.

(b) Find the length of the curve.

(c) Find the slope of the tangent line to the curve at any point on the curve.

(d) Let R be the top half of a loop. Rotate R about the x-axis and find the volume of the resulting solid.

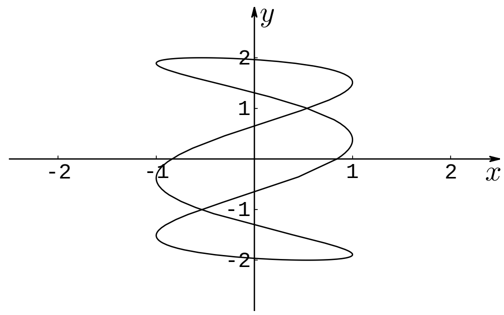
(e) Can we rotate a loop about a line $x = b$ and find the volume of the resulting solid? How about the surface area of the solid?

31. **Lissajous Curves**

Parametric Equations:

$x(t) = a\sin(nt + c)$

$y(t) = b\sin t$



Facts:

(a) These curves have applications in physics, astronomy, and other sciences.

(b) The curves are constructed as a combination of two perpendicular harmonic oscillations. Patterns occur as a result of differences in frequency ratio (n) and phase (c).

(c) $n = 1,\ c = 0:$ straight line $\quad n = 1,\ c = \pi/2:$ ellipse $\quad n = 2,\ c = \pi/2:$ parabola

Calculus Questions:

(a) Are there any generalizations for parameters and symmetry and compartments?

(b) Find the length of this curve.

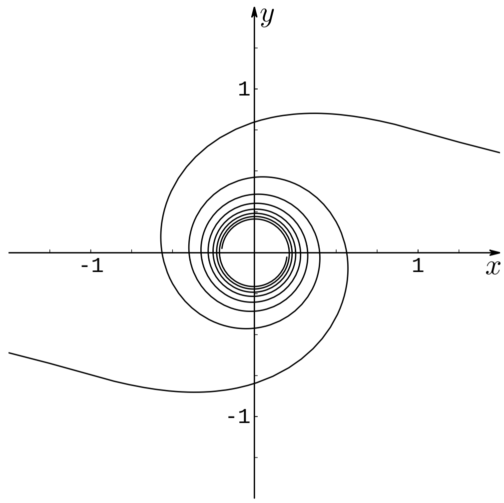
(c) Find the area of a compartment.

(d) Is it possible to find the point where the curve intersects itself?

32. **Lituus**

Polar Equation:

$r^2 = a^2/\theta$



Facts:

(a) The lituus is the locus of a point P moving in such a way that the area of a circular sector remains constant.

(b) Lituus means crook, as in a bishop's crosier.

Calculus Questions:

(a) Find the length of the curve from A to B.

(b) What happens as $\theta \to 0$?

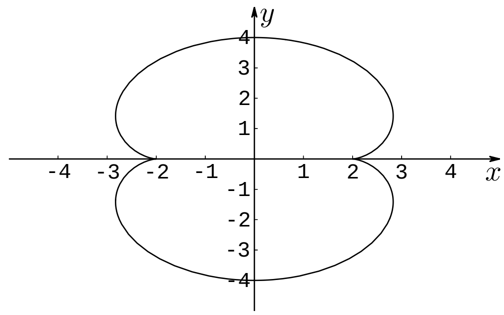
(c) What happens as $\theta \to \infty$?

33. **Nephroid**

Parametric Equations:

$x(t) = a(3\cos t - \cos(3t))$

$y(t) = a(3\sin t - \sin(3t))$



Facts:

(a) The name nephroid means *kidney-shaped*.

(b) Formed by a circle of radius a rolling externally on a fixed circle of radius $2a$.

(c) The curve can be written in Cartesian coordinates as a sixth degree equation.

If $a = 1 : \ (x^2 + y^2 - 4)^3 = 108y^2$

Calculus Questions:

(a) Find the length of the curve.

(b) Find the total area enclosed by the curve.

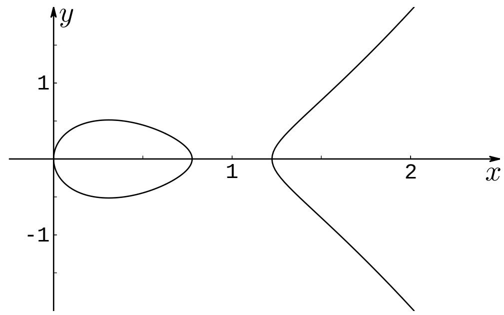
(c) Slope of the tangent line questions?

(d) Solid of revolution questions?

34. **Newton's Diverging Parabolas**

Cartesian Equation:

$ay^2 = x(x^2 - 2bx + c), \ a > 0$



Facts:

(a) Newton gives four classes of cubics. The diverging parabola is the third class. In this case, the "... legs diverge from one another and run out infinitely contrary ways."

(b) These divergent parabolas can be classified by considering the right side of the equation.

All roots real and unequal: Diverging parabola is in the form of a bell with an oval at its vertex (as above).

Two roots are equal: Parabola either Nodated by touching an oval or Punctate by having the oval infinitely small.

Three equal roots: Neilian parabola, or semi-cubical.

One real root: parabola of a bell-like form.

Calculus Questions:

(a) Find the area of the oval.

(b) Find the length of the oval.

(c) Find an expression for the distance from the oval to the parabola.

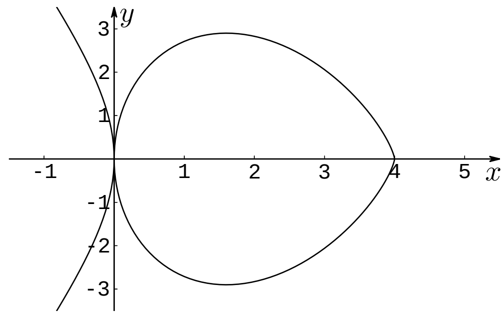
(d) Where does the tangent to the oval intersect the parabola?

(e) Volume questions?

35. **Pearls of Sluze**

Cartesian Equation:

$y^n = k(a-x)^p x^m$



Facts:

(a) A generalization of a teardrop curve (try $n = 2$ and $p = 1$).

(b) Pascal named these curves the Pearls of Sluze.

Calculus Questions:

(a) Find the length of the oval.

(b) Find the area enclosed by the oval.

(c) Find the points where the tangent line is horizontal.

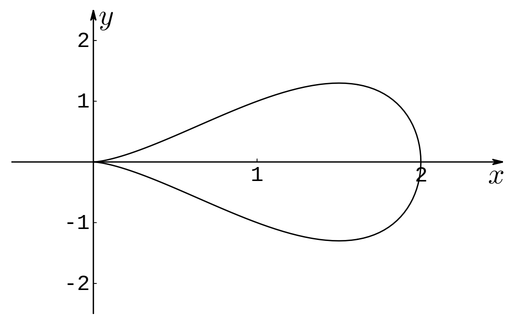
(d) Is there a tangent line at the right edge of the oval?

(e) Volume questions?

36. **Pear-shaped Quartic**

Cartesian Equation:

$b^2y^2 = x^3(a - x)$



Facts:

(a) This curve was studied by G de Longchamps, who had several other curves named after him.

Calculus Questions:

(a) We have to find the area of the pear.

(b) And the length of the curve.

(c) What's going on near the origin?

(d) Where is the tangent line horizontal?

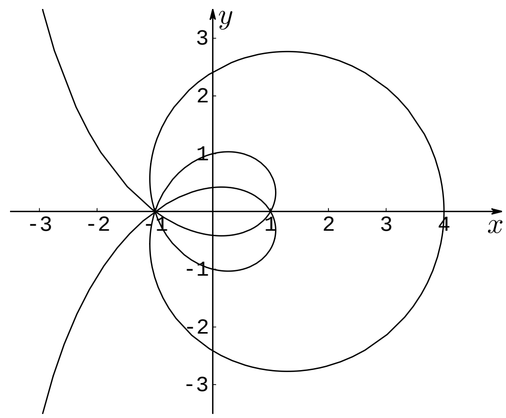
(e) Let R be the top half of the pear. Rotate R about the x-axis and find the volume of the solid of revolution.

37. **Plateau Curves**

Parametric Equations:

$$x(t) = \frac{a\sin((m+n)t)}{\sin((m-n)t)}$$

$$y(t) = \frac{2a\sin(mt)\sin(nt)}{\sin((m-n)t)}$$



Facts:

(a) If $m = 2n$, the Plateau curves become a circle centered at $(1,0)$ with radius 2.

(b) The curve is made up of a number of elliptic forms and asymptotic forms. For other values of m and n, the curve is made up of a number of hyperbolas.

Calculus Questions:

(a) What happens if $m = n$?

(b) There are some interesting curve lengths here.

(c) Where does the curve intersect itself?

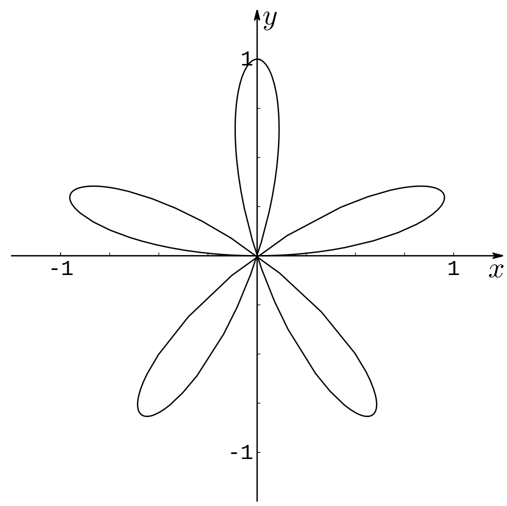
(d) What is (are) the asymptote(s)?

(e) Any chance of finding the area of some of these loops?

38. **Rhodonea Curves**

Polar Equation:

$r = a\sin(k\theta)$



Facts:

(a) If k is an integer, there are k if k is odd, or $2k$ petals if k is even.

(b) If k is irrational, then the number of petals is infinite, the curve does not close.

(c) So, what happens if k is rational?

Calculus Questions:

(a) What happens if $k = 1$?

(b) Find the Cartesian equation.

(c) We really need to find the area of one loop (generalization?).

(d) How about the length of one loop?

(e) Find the slope of the tangent line at any point on the curve, especially at the end of a loop.

(f) Can we determine anything about concavity?

(g) Are there any interesting solids of revolution?

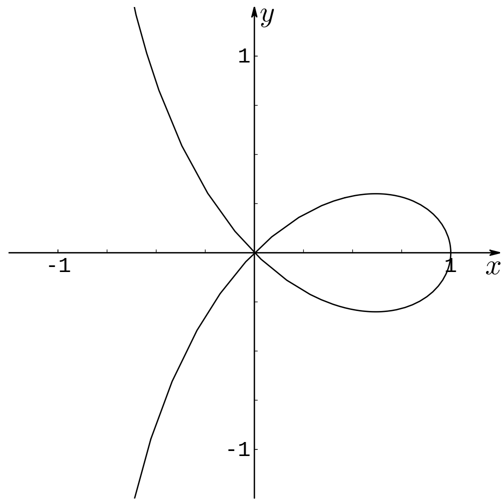
39. **Right Strophoid**

Cartesian Equation:

$$y^2 = \frac{x^2(a-x)}{a+x}$$

Polar Equation:

$$r = \frac{a\cos(2\theta)}{\cos\theta}$$

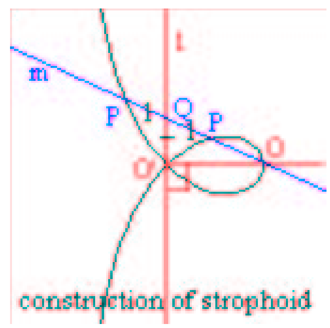


Facts:

(a) Called a right strophoid in order to distinguish from the more general oblique strophoid.

(b) The name means *belt with a twist.*

(c) Construction: Given a line l and a point O (called the pole). Let O'' be the perpendicular projection of O on l. Draw a line m through O and let Q be the intersection of l and m. The strophoid is the collection of points P on m for which $O'Q = PQ$.



Calculus Questions:

(a) Where is the asymptote?

(b) Find the area of the loop.

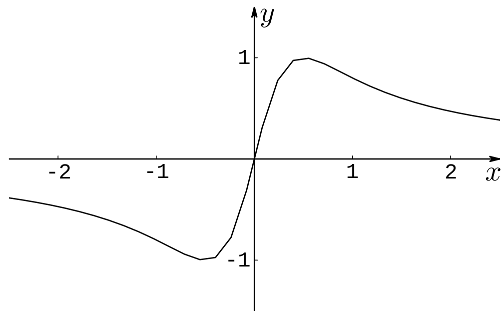
(c) Find the area between the curve and its asymptote.

(d) Where is the tangent line to the curve horizontal?

40. **Serpentine**

Cartesian Equation:

$x^2y + aby - a^2x = 0, \; ab > 0$



Facts:

(a) Newton showed that the curve $f(x, y) = 0$, where $f(x, y)$ is a cubic, can be divided into one of four normal forms. The first of these involves equations of the form

$xy^2 + ey = ax^3 + bx^2 + cx + d.$

This is the hardest case in the classification and the serpentine is one of the subcases of this first normal form.

Calculus Questions:

(a) Where is the tangent line horizontal?

(b) Find a closed form expression for y.

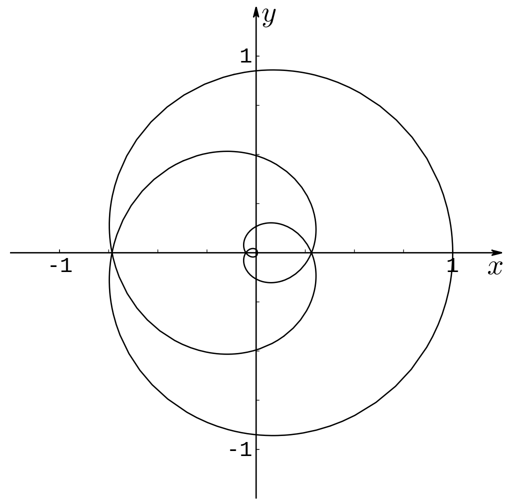
(c) Find the inflection points.

(d) Is the area under the curve in the first quadrant finite?

41. **Sinusoidal Spirals**

Polar Equation:

$r^p = a^p \cos(p\theta)$



Facts:

(a) Any rational number p is acceptable in the formula above.

(b) If $p = -1$, then the graph is a line.
If $p = 1$, then the graph is a circle.
If $p = 1/2$, cardioid. If $p = -1/2$, parabola. If $p = -2$, hyperbola. If $p = 2$, lemniscate.

Calculus Questions:

(a) Find the area of a loop.

(b) Find the length of the curve between two points.

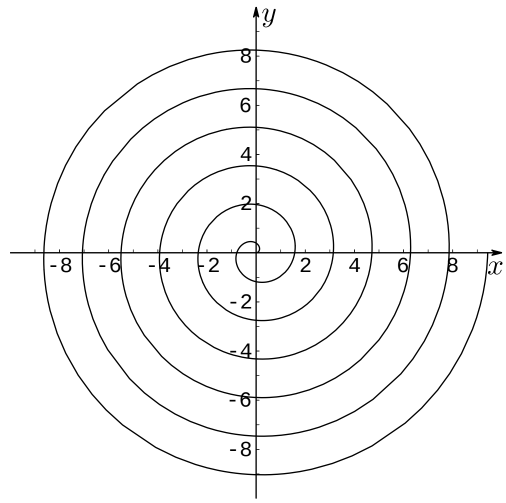
(c) Find the slope of the tangent line at the origin.

(d) Find a value of p so that the graph is symmetric.

42. **Spiral of Archimedes**

Polar Equation:

$r = a\theta$



Facts:

(a) The distance between windings is constant. (Why?)

(b) The curve can be used to convert uniform angular motion into uniform linear motion.

(c) This curve has actually appeared in a certain computer virus.

Calculus Questions:

(a) Find the distance between windings.

(b) Find the length of the curve from any point on the curve to the origin.

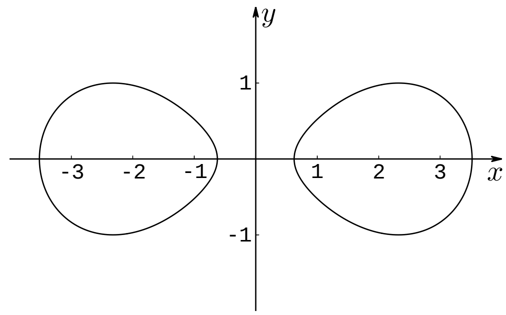
(c) Find the area between windings.

(d) Find the slope of the tangent line to the curve at any point.

43. **Spiric Sections**

Cartesian Equation:

$(r^2 - a^2 + c^2 + x^2 + y^2)^2 = 4r^2(x^2 + c^2)$



Facts:

(a) Curves are obtained by cutting a torus by a plane that is parallel to the line through the center of the hole of the torus.

(b) In the formula above, the torus is formed from a circle of radius a whose center is rotated along a circle of radius r. The value of c gives the distance of the cutting plane from the center of the torus.

(c) If $c = 0$, the curve consists of two circles of radius a whose centers are at $(r, 0)$ and $(-r, 0)$.

Calculus Questions:

(a) What happens if $c = r + a$? How about if $c > r + a$?

(b) Find the area enclosed by a single loop.

(c) Find the length around one loop.

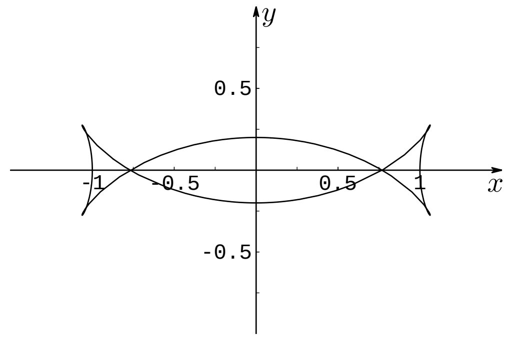
(d) Can we rotate one of these loops around a vertical axis of revolution and find the volume of the resulting solid?

44. **Talbot's Curve**

Parametric Equations:

$$x(t) = \frac{(a^2 + f^2 \sin^2 t)\cos t}{a}$$

$$y(t) = \frac{(a^2 - 2f^2 + f^2 \sin^2 t)\sin t}{b}$$



Facts:

(a) Several different forms. One looks like a football (find the parameters).

(b) In the curve above, there are four cusps and and two nodes.

Calculus Questions:

(a) Find the area of various enclosed pieces.

(b) Find the length of various parts of the curve.

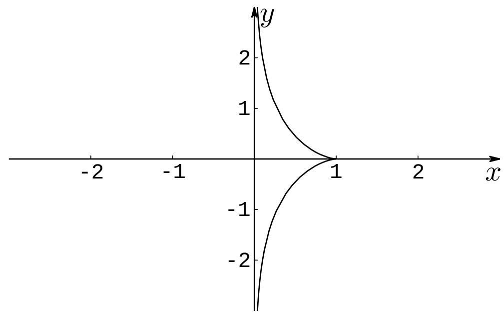
(c) Find the slope of the tangent line at any point on the curve.

45. **Tractrix**

Parametric Equations:

$$x(t) = \frac{1}{\cosh t}$$

$$y(t) = t - \tanh t$$



Facts:

(a) Also called a tractory or equitangential curve.

(b) Leibnitz posed the following problem: What is the path of an object dragged along a horizontal plane by a string of constant length when the end of the string not joined to the object moves along a straight line in the plane?

(c) Hence, the curve is also called the drag curve.

Calculus Questions:

(a) Find the length of the tangent line from the tangent point to the asymptote.

(b) Find the area between the curve and its asymptote.

(c) Find the length of the curve from the x-axis to a point on the curve.

(d) Find the slope of the tangent line at any point along the curve.

46. **Tricuspoid**

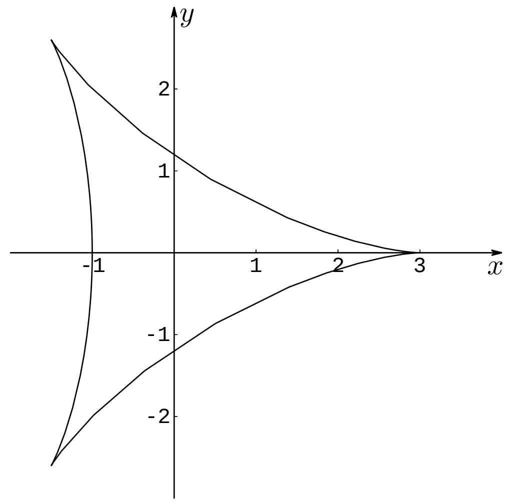
Cartesian Equation:

$(x^2 + y^2 + 12ax + 9a^2)^2 = 4a(2x + 3a)^3$

Parametric Equations:

$x(t) = a(2\cos t + \cos(2t))$

$y(t) = a(2\sin t - \sin(2t))$



Calculus Questions:

(a) Find the length of the curve.

(b) Find the area enclosed by the curve.

(c) Consider the tangent line to the Tricuspoid. Let P and Q be the points where the tangent line intersects the curve again. Find the distance PQ.

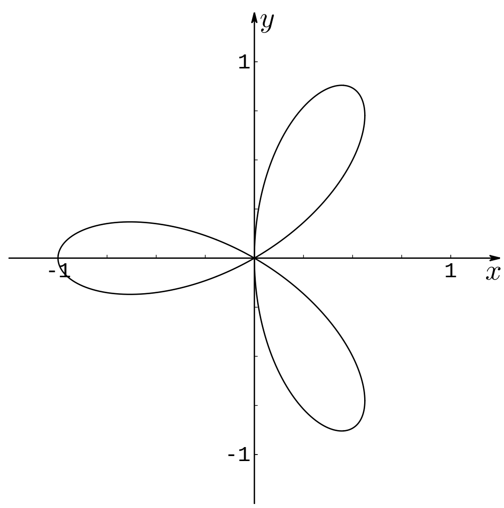
47. **Trifolium**

Cartesian Equation:

$(x^2 + y^2)(y^2 + x(x + a)) = 4axy^2$

Polar Equation:

$r = a\cos\theta(4\sin^2\theta - 1)$



Calculus Questions:

(a) Find the area of one loop.

(b) Find the length of the curve.

(c) Find the slope of the tangent line to any point on the curve.

(d) Find the slope of the tangent line to the curve at the origin.

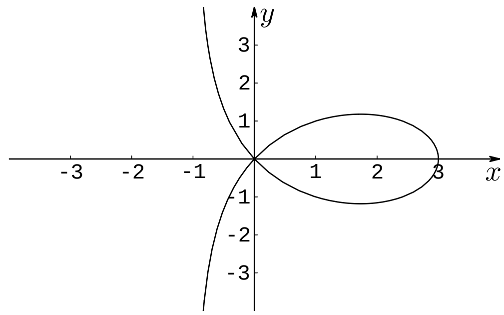
48. **Trisectrix of Maclaurin**

Cartesian Equation:

$y^2(a+x) = x^2(3a-x)$

Polar Equation:

$$r = \frac{2a\sin(3\theta)}{\sin(2\theta)}$$



Facts:

(a) This curve was studied to provide a solution to the problem of trisecting an angle. The curve can be used to trisect angles.

(b) Also called an anallagmatic curve.

Calculus Questions:

(a) Find the area of the loop.

(b) Find the slope of the tangent line(s) to to the curve at the origin.

(c) Find (in general) the point where the curve crosses the x-axis.

(d) Find the length of the curve around the loop.

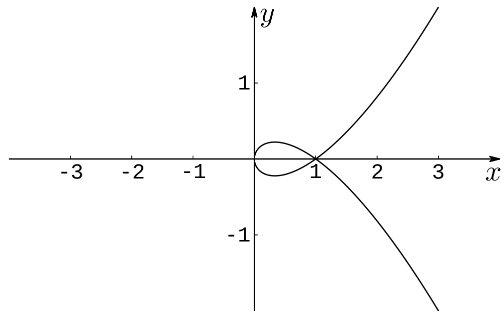
(e) Where is the vertical asymptote?

(f) Find the area between the curve and the vertical asymptote.

49. **Tschirnhaus's Cubic**

Cartesian Equation:

$3ay^2 = x(x-a)^2$



Calculus Questions:

(a) Find the area of the loop.

(b) Find the length of the loop.

(c) Find the slopes of the tangent lines at the point of intersection.

(d) Is there an asymptote? Find the area between the curve and its asymptote.

50. **Witch of Agnesi**

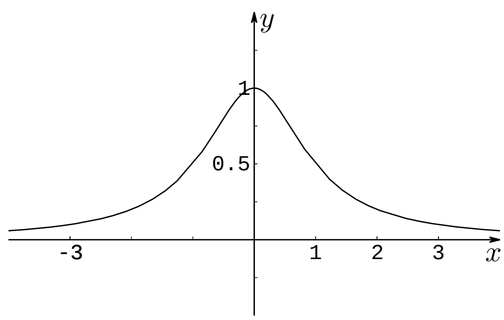
Cartesian Equation:

$y(x^2 + a^2) = a^3$

Parametric Equations:

$x(t) = at$

$y(t) = \dfrac{a}{1+t^2}$



Calculus Questions:

(a) Find the inflection points.

(b) Find an equation of the tangent line to the curve at any point P.

(c) Find the area under the curve.

References:

Famous Curves Index
`http://www-history.mcs.st-andrews.ac.uk/history/Curves/Curves.html`

2d curves
`http://www.2dcurves.com/`

Mathematica notebook: curves.nb