BINOMIAL OPTION PRICING AND BLACK-SCHOLES

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1. Introduction

This paper aims to investigate the assumptions under which the binomial option pricing model converges to the Black-Scholes formula. The results are not original; the paper mostly follows the outline of Cox, Ross, and Rubenstein[1]. However, the convergence is treated in greater detail than I have found elsewhere in the literature. This exercise clarifies the assumptions behind the binomial model and subsequent convergence results.

2. The Binomial Model

We begin by defining the binomial option pricing model. Suppose we have an option on an underlying with a current price S. Denote the option's strike by K, its expiry by T, and let r be one plus the continuously compounded risk-free rate.

We model the option's price using a branching binomial tree over n discrete time periods. Let u represent one plus a positive return on the underlying's value over a single period and similarly let d represent a negative return. Denote the single-period interest rate by r_n and let π be the risk-neutral probability; i.e. $r_n = \pi u + (1-\pi)d$[1]. Then the binomial model for the price C of the option is given by

$$C = \frac{1}{r_n^n}\sum_{k=1}^{n}\binom{n}{k}\pi^k(1-\pi)^{n-k}\max(0, u^k d^{n-k}S - K). \tag{1}$$

This model can be interpreted as follows. At each discrete time step, the underlying may increase or decrease in value, by u or d respectively, as controlled by independent Bern(p) random variables. Therefore, after n time steps, the underlying will have made k up moves where $k \sim \mathcal{B}(n,p)$. The value of the option at expiry is then $P = \max(0, u^k d^{n-k}S - K)$. This is the payoff of the option. The expected value of the option at expiry using this model follows by the law of the unconscious statistician:

$$\mathbb{E}[P] = \sum_{k=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}P(k).$$

[1]We implicitly assume $d \le r_n \le u$ making $0 \le \pi \le 1$. This assumption is reasonable; it amounts to a statement that there exists a future state of the world in which holding the underlying asset yields a profit.

We might naively think that we could price an option by discounting this expected value. This is not the case! No-arbitrage constraints[2] instead force us to substitute the risk-neutral probability π for the true probability p. Accordingly, we may view the binomial model as the discounted expected payoff of the option in a risk-neutral world:

$$C = \frac{1}{r_n^n}\mathbb{E}_\pi\left[\max(0, u^k d^{n-k}S - K)\right] = \frac{1}{r_n^n}\mathbb{E}_\pi[P].$$

Note that the binomial model is contingent upon model parameters u, d, and n. Clearly modeling n discrete time steps is imprecise; in real-world trading, underlying price moves are effectively a continuous process. The purpose of the paper is essentially to investigate the limiting behavior of this model as $n \to \infty$. Choice of u and d is more open. After some algebraic preliminaries in section 3, we will consider these parameters in section 4. Then, in section 5, we will see how binomial pricing converges in the limit to the Black-Scholes formula.

3. Algebraic Considerations

The object of this section is merely to algebraically re-formulate the model we have introduced. Knowledgeable readers will see the pattern of Black-Scholes begin to emerge.

Proposition 3.1. *Let $a = \min_k P(k) > 0$. Then for some $\zeta \in [0, 1)$ we have*

$$a = \frac{\log(K/S) - n\log d}{\log(u/d)} + \zeta.$$

Proof. By definition of a, $Su^a d^{n-a} > K$. Solving for a we have

$$a\log u + (n-a)\log d + \log S > \log K$$

$$a(\log u - \log d) > \log K - \log S - n\log d$$

$$a > \frac{\log(K/S) - n\log d}{\log(u/d)}.$$

Introducing an error term, it follows that for some $\zeta \geq 0$,

$$a = \frac{\log(K/S) - n\log d}{\log(u/d)} + \zeta.$$

Furthermore (because a was defined to be minimal) we see that $\zeta < 1$. $\square$

Proposition 3.2. *Let $a = \min_k P(k) > 0$ and define $\pi^* = (u/r_n)\pi$ then*

$$C = S\mathcal{B}(-a; n, \pi^*) - Kr_n^{-n}\mathcal{B}(-a; n, \pi).$$

[2]The no-arbitrage constraints of the binomial model are beyond the scope of this paper. See [1] for details. The idea is to construct a portfolio of the underlying and riskless asset that replicates the returns of the option for a single period. Then, extrapolate to the n-period case by induction.

Proof. Let $a = \min_k P(k) > 0$. Note $P(k)$ is monotone increasing, so we may re-write the binomial model as

$$C = \frac{1}{r_n^n} \sum_{k=a}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} (u^k d^{n-k} S - K)$$

$$= S r_n^{-n} \sum_{k=a}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} u^k d^{n-k} - K r_n^{-n} \sum_{k=a}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k}.$$

Recall that π is risk-neutral. Substituing $\pi^* = (u/r_n)\pi$ we see that $1 - \pi^* = r_n^{-1}(1-\pi)d$ and therefore

$$r_n^{-n} \sum_{k=a}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} u^k d^{n-k} = \sum_{k=a}^{n} \binom{n}{k} \left[\frac{\pi u}{r_n}\right]^k \left[\frac{(1-\pi)d}{r_n}\right]^{n-k} = \sum_{k=a}^{n} \binom{n}{k} \pi^{*k} (1-\pi^*)^{n-k}.$$

We therefore see that

$$C = S \sum_{k=a}^{n} \binom{n}{k} \pi^{*k} (1-\pi^*)^{n-k} - K r_n^{-n} \sum_{k=a}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k}.$$

$$= S(1 - \mathcal{B}(a; n, \pi^*)) - K r_n^{-n} (1 - \mathcal{B}(a; n, \pi))$$

$$= S\mathcal{B}(-a; n, \pi^*) - K r_n^{-n} \mathcal{B}(-a; n, \pi).$$

The last equality follows from the symmetry of the binomial distribution. □

4. Statistical Considerations

We now turn our attention to the model parameters u and d. There are many plausible choices available to us, each of which leads to slightly different binomial models. See Chance[5] for a discussion and comparison of many proposals. There does not appear to be a final word yet in the literature on the selection of binomial model parameters. Chance gives us the modest observation that "binomial option pricing is a remarkably flexible procedure." The reader may also be interested in Liesen and Reimer[6], who suggest that parameter choice has implications for the model's rate of convergence

For our purposes, we will adopt the parameter choices of Cox, Ross, and Rubenstein[1]. In particular, we introduce a new paramter σ and take $u = e^{\sigma\sqrt{T/n}}$ and $d = 1/u$. This may appear unhelpful; we have merely renamed our u and d parameters in terms of σ. But we will see at the end of this section that the parameter σ has a meaningful interpretation.

Proposition 4.1. *Let $u = e^{\sigma\sqrt{T/n}}$ and let q satisfy $q \log u + (1-q) \log d < \infty$. If $k \sim \mathcal{B}(n, q)$ then as $n \to \infty$,*

$$\mathbb{E}\left[\log\left(\frac{S_n^*}{S}\right)\right] = \nu T, \quad \operatorname{Var}\left[\log\left(\frac{S_n^*}{S}\right)\right] \to \sigma^2 T.$$

Proof. By our hypothesized condition on q, there must be some $\nu \in \mathbb{R}$ such that

$$\nu = q \log u + (1-q) \log d.$$

In other words, for some ν we have

$$q = \frac{\nu + \log(u)}{2\log(u)} = \frac{\nu + \sigma\sqrt{T/n}}{2\sigma\sqrt{T/n}} = \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\frac{T}{n}}.$$

First we will examine the mean of log returns. Recall that $k \sim \mathcal{B}(n,q)$, so by proposition $C.1$, $\mathbb{E}[k] = nq$. Note (by definition) that $S_n^*/S = u^k d^{n-k}$ and by linearity,

$$\mathbb{E}\left[\log\left(S_n^*/S\right)\right] = \mathbb{E}[k\log(u/d) + n\log d] = n(q\log(u/d) + \log(d)).$$
$$= n(2q-1)\log(u) = n\left((\nu/\sigma)\sqrt{T/n}\right)\left(\sigma\sqrt{T/n}\right) = \nu T.$$

Now we will examine the variance. Since $k \sim \mathcal{B}(n,q)$ by proposition $C.2$ we have $\mathrm{Var}[k] = nq(1-q)$. Then by basic algebra,

$$\mathrm{Var}\left[\log S_n^*/S\right] = \mathrm{Var}\left[k\log(u/d) + n\log d\right]$$
$$= nq(1-q)\log(u/d)^2 = 4q(1-q)\log(u)^2 n$$
$$= 2q(2-2q)\sigma^2(T/n) = \left(1 + \frac{\nu}{\sigma}\sqrt{T/n}\right)\left(1 - \frac{\nu}{\sigma}\sqrt{T/n}\right)\sigma^2 T.$$
$$= \left(1 - \frac{\nu^2}{\sigma^2}(T/n)\right)\sigma^2 T = \sigma^2 T - \frac{\nu^2 T^2}{\sigma^2 n}.$$

And so we may conclude that

$$\lim_{n\to\infty} \mathrm{Var}\left[\log\left(\frac{S_n^*}{S}\right)\right] = \sigma^2 T.$$

□

From this we see that under the mild condition $q\log u + (1-q)\log d < \infty$, the log return of the underlying has variance $\sigma^2 T$. In particular both the risk neutral measure π and π^* (as well as the physical measure p) satisfy this condition.

Proposition 4.2. *Let $u = e^{\sigma\sqrt{T/n}}$ and let $\nu = q\log(u) + (1-q)\log(d) < \infty$. If $k \sim \mathcal{B}(n,q)$ then as $n \to \infty$,*

$$\log(S_n^*/S) \xrightarrow{d} \mathcal{N}(\nu T, \sigma^2 T).$$

Proof. By definition, we have

$$\log\left(\frac{S_n^*}{S}\right) = \log\left(u^k d^{n-k}\right) = (2k-n)\log(u) = (2k-n)\sigma\sqrt{T/n}.$$

By basic algebra we see that

$$\mathbb{P}\left(\log(S_n^*/S) \le x\right) = \mathbb{P}\left((2k-n)\sigma\sqrt{T/n} \le x\right) = \mathbb{P}\left(k \le \frac{x\sqrt{n}}{2\sqrt{\sigma^2 T}} + \frac{n}{2}\right).$$

Let $z = \frac{x\sqrt{n}}{2\sqrt{\sigma^2 T}} + \frac{n}{2}$. Because $k \sim \mathcal{B}(n, q)$ we have

$$\mathbb{P}\left(\log(S_n^*/S) \leq x\right) = \mathcal{B}(z, n, q).$$

Let y be such that $z = y\sqrt{np(1-q)} + nq$. Some algebra shows us that

$$q = \frac{\nu + \log(u)}{2\log(u)} = \frac{\nu + \sigma\sqrt{T/n}}{2\sigma\sqrt{T/n}} = \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\frac{T}{n}}.$$

And substituting this value of q, a bit more algebra shows us that as $n \to \infty$,

$$y = \frac{z - nq}{\sqrt{nq(1-q)}} = \frac{\frac{x\sqrt{n}}{2\sqrt{\sigma^2 T}} - \frac{\nu\sqrt{Tn}}{2\sigma}}{\sqrt{\frac{n}{4} - \frac{\nu^2 T}{4\sigma^2}}} = \frac{\frac{x\sqrt{n}}{\sqrt{\sigma^2 T}} - \frac{\nu\sqrt{Tn}}{\sigma}}{\sqrt{n - \frac{\nu^2 T}{\sigma^2}}}$$

$$= \frac{\frac{x\sqrt{n}}{\sqrt{T}} - \nu\sqrt{Tn}}{\sqrt{n\sigma^2 - \nu^2 T}} = \frac{x\sqrt{n} - \nu T\sqrt{n}}{\sqrt{nT\sigma^2 - \nu^2 T^2}} = \frac{x - \nu T}{\sqrt{\sigma^2 T - \nu^2 T^2/n}} \to \frac{x - \nu T}{\sigma\sqrt{T}}.$$

It follows by the central limit theorem that

$$\lim_{n\to\infty} \mathbb{P}\left(\log(S_n^*/S) \leq x\right) = \lim_{n\to\infty} \mathcal{B}(z, n, q) = \mathcal{N}(y; 0, 1) = \mathcal{N}(x; \nu T, \sigma^2 T).$$

$\square$

Regarding notation: we have seen that $\log(S_n^*/S)$ converges to a normal under many probability measures q. Moreover, these normals share a common variance $\sigma^2 T$; they are therefore fully characterized by the parameter ν. We have consistenly adopted the notation $\mathbb{E}_q$ to denote expectation of a discrete process under the Bernoulli measure with parameter q. We will use $\mathbb{E}_\nu$ to denote expectation of a continuous process under the normal measure with paramter ν derived by the limiting behavior of a discrete process under q.

Before moving on, we digress to make an historical observation. Cox, Ross, and Rubenstein initially chose their u and d parameters with the intent of fitting their model to the empirical process of underlying asset returns. This is unnecessary; any choices of u, d and risk neutral measure satisfying no-arbitrage constraints are admissable. But their choice gives us an interpretation of σ. If we assume that asset returns are distributed like $\log\mathcal{N}(\mu, \sigma^2)$, then by the preceeding proposition, σ is the volatility of the underlying asset. Although evidence shows that asset returns are not log-normal, we may interpret σ as a crude estimate of volatility.

5. The Black-Scholes Formula

Proposition 5.1. *The discount factor r_n^{-n} is constant in n; in particular $r_n^{-n} = e^{-T\log r}$.*

Proof. Let n_y denote the number of periods in a year. Then by definition we have $r_n = r^{1/n_y}$ where r is the n-period return. Recalling that T is the time (in years) to expiry of the option, $T = n/n_y$,

$$\frac{1}{r_n^n} = r^{-n/n_y} = r^{-T} = e^{-T\log r}.$$

This is clearly constant as n varies. □

Proposition 5.2. *Under the risk-neutral measure π,*

$$\mathbb{E}_\nu\left[\log\left(\frac{S^*}{S}\right)\right] = \left(\log r - \frac{\sigma^2}{2}\right)T.$$

Proof. By definition of the binomial model,

$$\mathbb{E}_\pi\left[\frac{S_k}{S_{k-1}}\right] = \pi u + (1-\pi)d.$$

And because S_n is independent of S_{n-1},

$$\mathbb{E}_\pi(S_n^*/S) = \mathbb{E}_\pi \prod_{k=1}^n S_k/S_{k-1} = \prod_{k=1}^n \mathbb{E}_\pi[S_k/S_{k-1}] = (\pi u + (1-\pi)d)^n.$$

Therefore by definition of π and proposition 5.1,

$$\mathbb{E}_\pi[S_n^*/S] = r_n^n = e^{T\log r}.$$

And so we have

$$\log\left(\mathbb{E}_\pi[S_n^*/S]\right) = T\log r.$$

By continuity, and proposition 4.1 respectively,

$$T\log r = \lim_{n\to\infty}\log\left(\mathbb{E}_\pi[S_n^*/S]\right) = \log\left(\lim_{n\to\infty}\mathbb{E}_\pi[S_n^*/S]\right) = \log\left(\mathbb{E}_\nu[S^*/S]\right).$$

And because S^*/S is lognormally distributed (proposition 4.2) by proposition $B.1$

$$\log\left(\mathbb{E}_\nu(S^*/S)\right) = \mathbb{E}_\nu[\log(S^*/S)] + \frac{1}{2}\mathrm{Var}_\nu[\log(S^*/S)].$$

It follows that

$$\mathbb{E}_\nu[\log(S^*/S)] = T\log r - \frac{\sigma^2 T}{2}.$$

□

Proposition 5.3. *Under the measure $\pi^* = (u/r_n)\pi$,*

$$\mathbb{E}_{\nu^*}\left[\log\left(\frac{S^*}{S}\right)\right] = \left(\log r + \frac{\sigma^2}{2}\right)T.$$

Proof. By definition of π^*,

$$\pi^* = \frac{u}{r_n}\left(\frac{r_n - d}{u-d}\right).$$

Rearranging this equation shows us that

$$r_n = \left[(1/u)\pi^* + (1/d)(1-\pi^*)\right]^{-1}.$$

And therefore we have

$$r^T = \left[(1/u)\pi^* + (1/d)(1-\pi^*)\right]^{-n}.$$

Now running our model in reverse, note that

$$\mathbb{E}_{\pi^*}\left[S_{k-1}/S_k\right] = (1/u)\pi^* + (1/d)(1-\pi^*).$$

We sketch the remaining details of the proof, which is quite similar to the proof for π given above:

$$\log\left(\mathbb{E}_{\nu^*}\left[S/S^*\right]\right) = \lim_{n\to\infty} \log\left(\mathbb{E}_{\pi^*}\left[S/S_n\right]\right) = \log(r^{-T}) = -T\log r.$$

The inverse of a lognormal distribution is lognormal, so we have

$$-T\log r = \mathbb{E}_{\nu^*}\left[\log\left(S/S^*\right)\right] + \frac{1}{2}\mathrm{Var}_{\nu^*}[\log(S/S^*)]$$

And we see that

$$\mathbb{E}_{\nu^*}\left[\log\left(S^*/S\right)\right] = T\log r + \frac{\sigma^2 T}{2}.$$

$\square$

Proposition 5.4. *Let* $a = \min_k P(k) > 0$ *and* $-a = d_p\sqrt{np(1-p)} - np$. *Then*

$$\lim_{n\to\infty} d_p = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S}{K}\right) + \mathbb{E}_p\left[\log\left(\frac{S^*}{S}\right)\right]\right).$$

Proof. By proposition 3.1,

$$a = \frac{\log(K/S) - n\log d}{\log(u/d)} + \zeta.$$

Therefore by our definition of d_p, we have

$$d_p\sqrt{np(1-p)} - np = \frac{\log(S/K) + n\log d}{\log(u/d)} - \zeta.$$

And with some algebra, we see that

$$d_p = \frac{\log(S/K) + (\log d + p\log(u/d))n}{\log(u/d)\sqrt{np(1-p)}} - \frac{\zeta}{\sqrt{np(1-p)}}.$$

Substituting the values of the mean and variance of the binomial process we have

$$d_p = \frac{\log(\frac{S}{K}) + \mathbb{E}_p\left[\log\left(\frac{S_n^*}{S}\right)\right]}{\mathrm{Std}_p\left[\log\left(\frac{S_n^*}{S}\right)\right]} - \frac{\zeta}{\sqrt{np(1-p)}}.$$

By proposition 3.1, $0 < \zeta < 1$. Therefore the second term above vanishes as $n \to \infty$. By proposition 4.1 we may substitute the variance $\sigma^2 T$ leaving us with

$$\lim_{n\to\infty} d_p = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S}{K}\right) + \lim_{n\to\infty}\mathbb{E}_p\left[\log\left(\frac{S_n^*}{S}\right)\right]\right)$$

$\square$

Theorem 5.5 (Black-Scholes)**.** *Let* $d_{1,2} = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S}{K}\right) + \left(\log r \pm \frac{\sigma^2}{2}\right)T\right]$. *Then*

$$C = S\mathcal{N}(d_1) - Ke^{-\log(r)T}\mathcal{N}(d_2).$$

Proof. By proposition 3.2, we have

$$C = S\mathcal{B}(-a; n, \pi^*) - Kr_n^{-n}\mathcal{B}(-a; n, \pi).$$

Taking the limit as $n \to \infty$,

$$\begin{aligned} C &= S\left(\lim_{n\to\infty} \mathcal{B}(-a; n, \pi^*)\right) - K\left(\lim_{n\to\infty} r_n^{-n}\right)\left(\lim_{n\to\infty} \mathcal{B}(-a; n, \pi)\right). \\ &= S\mathcal{N}(d_1) - Ke^{-\log(r)T}\mathcal{N}(d_2). \end{aligned}$$

The limiting expressions are replaced via propositions $A.2$ and 5.1. The values of d_1 and d_2 are obtained via proposition 5.5. □

Appendix A. Log-normal Statistics

Proposition A.1. *Let* $X \sim \log\mathcal{N}(\mu, \sigma^2)$. *Then*

$$\mathbb{E}[X] = e^{\mu+\sigma/2}.$$

Proof. By definition, $X = e^Y$ where $Y \sim \mathcal{N}(\mu, \sigma^2)$. Then by the expectation rule and subsequent algebra,

$$\begin{aligned} \mathbb{E}[X] = \mathbb{E}[e^Y] &= \int_{-\infty}^{\infty} e^y \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} dy \\ &= \int_{-\infty}^{\infty} e^{u+\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} du \\ = e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{2\sigma^2 u - u^2}{2\sigma^2}} du &= e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(u-\sigma^2)^2+\sigma^4}{2\sigma^2}} du \\ &= e^{\mu+\sigma^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(u-\sigma^2)^2}{2\sigma^2}} du = e^{\mu+\sigma^2/2}. \end{aligned}$$

The last equality follows because the integral is taken over a normal density and therefore must integrate to 1; this can be proved directly by the polar coordinates method of Gauss. □

Appendix B. Binomial Statistics

Proposition B.1. *Let* k *be binomially distributed with parameters* n *and* p. *Then*

$$\mathbb{E}[k] = np.$$

Proof. Let $q = 1 - p$. Then

$$\begin{aligned} \mathbb{E}[k] = \sum_{k=1}^{n} kPr(k) &= \sum_{k=1}^{n} k\binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k q^{n-k} \\ = np\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} &= np\sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k} \\ &= np(p+q)^{n-1} = np. \end{aligned}$$

The last reduction is attained by applying the binomial theorem (noting that $p+q=1$). Alternatively, we may interpret the last sum as summing over the probabilities of a binomial distribution with parameters $n-1$ and p, which must of course sum to 1. $\square$

Proposition B.2. *Let k be binomially distributed with parameters n and p. Then*

$$\mathrm{Var}[k] = np(1-p).$$

Proof. Let $q = 1-p$. Then

$$\mathrm{Var}[k] = \mathbb{E}[k^2] - \mathbb{E}[k]^2 = \sum_{k=1}^{n} k^2 Pr(k) - (np)^2 = \sum_{k=1}^{n} k^2 \binom{n}{k} p^k q^{n-k} - (np)^2$$

$$= \sum_{k=1}^{n} kn \binom{n-1}{k-1} p^k q^{n-k} - (np)^2 = np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} - (np)^2$$

$$= np \sum_{k=0}^{n-1} (k+1) \binom{n-1}{k} p^k q^{(n-1)-k)} - (np)^2$$

$$= np \left(\sum_{k=0}^{n-1} k \binom{n-1}{k} p^k q^{(n-1)-k)} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k)} \right) - (np)^2$$

$$= np \left(\sum_{k=1}^{n-1} (n-1) \binom{n-2}{k-1} p^k q^{(n-1)-k)} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k} \right) - (np)^2$$

$$= np \left((n-1)p \sum_{k=1}^{n-1} \binom{n-2}{k-1} p^{k-1} q^{(n-2)-(k-1))} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k} \right) - (np)^2$$

$$= np \left((n-1)p \sum_{k=1}^{n-1} \binom{n-2}{k-1} p^{k-1} q^{(n-2)-(k-1))} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k} \right) - (np)^2.$$

$$= np \left((n-1)p(p+q)^{n-2} + (p+q)^{n-1} \right) - (np)^2$$

$$= np(np+1-p) - (np)^2 = np(1-p).$$

$\square$

Appendix C. Properties of the CRR parameters

Lemma C.1. *If p is the physical measure, π is the risk-neutral measure, and $\pi^* = (u/r_n)\pi$,*

$$\lim_{n\to\infty} p = \lim_{n\to\infty} \pi = \lim_{n\to\infty} \pi^* = \frac{1}{2}$$

Proof. Recall from proposition 4.1 that

$$p = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\frac{T}{n}}.$$

From this is clearly follows that $p \to 1/2$ as $n \to \infty$. By definitions and proposition 5.1,

$$\pi = \frac{r_n - d}{u - d} = \frac{e^{-(T/n)\log r} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}.$$

Clearly both the numerator and denominator vanish as $n \to \infty$. Therefore by l'Hôpital,

$$\lim_{n\to\infty} \pi = \lim_{n\to\infty} \frac{d}{dn}\frac{e^{-(T/n)\log r} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}.$$

Taking derivatives with respect to (continuous) n gives us

$$\frac{d}{dn}e^{\sigma\sqrt{T/n}} = -\frac{\sigma\sqrt{T}e^{\sigma\sqrt{T/n}}}{2n^{3/2}}.$$

And with a bit of algebra we have

$$\pi = \frac{-2T\log(r)e^{-(T/n)\log r}/\sqrt{n} + \sigma\sqrt{T}e^{-\sigma\sqrt{T/n}}}{\sigma\sqrt{T}e^{\sigma\sqrt{T/n}} + \sigma\sqrt{T}e^{-\sigma\sqrt{T/n}}}$$

The first term in the numerator goes to zero and therefore we have

$$\lim_{n\to\infty} \pi = \lim_{n\to\infty} \frac{\sigma\sqrt{T}e^{-\sigma\sqrt{T/n}}}{\sigma\sqrt{T}e^{\sigma\sqrt{T/n}} + \sigma\sqrt{T}e^{-\sigma\sqrt{T/n}}} = \frac{1}{2}.$$

Finally, by continuity we observe that

$$\lim_{n\to\infty} \frac{u}{r_n} = \lim_{n\to\infty} e^{\sigma\sqrt{T/n}}e^{(T/n)\log r} = \lim_{n\to\infty} e^{\sigma\sqrt{T/n}+(T/n)\log r} = 1.$$

It follows that $\pi^* \to 1/2$ as $n \to \infty$. $\square$

References

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