As stated in the previous lesson, when changing from a function to its inverse the inputs and outputs of the original function are switched. This is because when we find an inverse function, we are taking some original function and solving for its input x; so what used to be the input becomes the output, and what used to be the output becomes the input.

$$f(x) = \frac{-11}{1+3x}$$

$$\boldsymbol{f = \frac{-11}{1+3x}}$$

$$\boldsymbol{(1+3x)\cdot f = -11}$$

$$\boldsymbol{f + 3xf = -11}$$

$$\boldsymbol{3xf = -11 - f}$$

$$\boldsymbol{x = \frac{-11-f}{3f}}$$

$$f^{-1}(x) = \frac{-11-x}{3x}$$

Given to the left are the steps to find the inverse of the original function f. These steps illustrates the changing of the inputs and the outputs when going from a function to its inverse. We start out with f (the output) isolated and x (the input) as part of the expression, and we end up with x isolated and f as the input of the expression. Keep in mind that once x is isolated, we have basically found the inverse function.

Since the inputs and outputs of a function are switched when going from the original function to its inverse, this means that the domain of the original function f is the range of its inverse function f^{-1}. This also means that the range of the original function f is the domain of its inverse function f^{-1}.

In this lesson we will review how to find an inverse function (as shown above), and we will also review how to find the domain of a function (which we covered in Lesson 18). In addition to those two topics which we've already covered in previous lessons, we'll also show how to find the range of a function algebraically, either by finding the inverse of the function first and then using its domain, or by making an input/output table.

Remember from Lesson 18 there are two ways the domain of a function can be restricted. One way is to have a function that is defined by a fraction, and the other is to have a function that is defined by a square root.

When a function is defined by a fraction, the denominator of that fraction cannot be equal to zero

- if $f(x) = \frac{1}{x+2}$, then the denominator $x + 2 \neq 0$, so $x \neq -2$; therefore the domain of f is $(-\infty, -2) \cup (-2, \infty)$
- this means the range of f^{-1} will also be $(-\infty, -2) \cup (-2, \infty)$, because the domain of an original function f is the range of its inverse function f^{-1}

Since the domain of a function is the range of its inverse, and the range of a function is the domain of its inverse, one way to find the range of an original function is to find its inverse function, and the find the domain of its inverse.

Example 1: List the domain and range of the following function. Then find the inverse function and list its domain and range.

$$f(x) = \frac{1}{x+2}$$

As stated above, the denominator of fraction can never equal zero, so in this case $x + 2 \neq 0$. That means $x \neq -2$, so the domain is all real numbers except -2.

Domain of f: $\boldsymbol{(-\infty, -2) \cup (-2, \infty)}$

Also as stated above, the domain of a function and the range of its inverse are always the same, because when we go from function to its inverse we switch the inputs and outputs. So, if the domain of f is all real numbers except -2, the range of f^{-1} is the same.

Range of f^{-1}: $\boldsymbol{(-\infty, -2) \cup (-2, \infty)}$

To find the range of the original function $f(x) = \frac{1}{x+2}$, I will find its inverse function first. That is because the range of f will be the same as the domain of f^{-1}, just like the domain of f was the same as the range of f^{-1}. To find the inverse function, I will follow the same steps I used in Lesson 27 (change to an equation, solve for x, express as an inverse):

$$f(x) = \frac{1}{x+2}$$

$$f = \frac{1}{x+2}$$

$$f(x+2) = 1$$

$$xf + 2f = 1$$

$$xf = 1 - 2f$$

$$x = \frac{1-2f}{f}$$

$$\boldsymbol{f^{-1}(x) = \frac{1-2x}{x}}$$

Now that I have the inverse function, and I can see that the inverse function is rational just like the original function f, I can find its domain by simply stating that the denominator cannot equal zero. In this case $x \neq 0$, which means the domain of f^{-1} is all real numbers except 0.

Domain of $\boldsymbol{f^{-1}}$: $\boldsymbol{(-\infty, 0) \cup (0, \infty)}$

And once again, if this is the domain of f^{-1}, this is also the range of f.

Range of $\boldsymbol{f}$: $\boldsymbol{(-\infty, 0) \cup (0, \infty)}$

LON-CAPA Problem:

List the domain and range of the function $f(x) = \frac{10-x}{x+3}$, then find the inverse function $f^{-1}(x)$ and list its domain and range. List the domain and range in interval notation.

Domain of $\boldsymbol{f}$:

Range of $\boldsymbol{f^{-1}}$:

$\boldsymbol{f^{-1}(x) =}$

Domain of $\boldsymbol{f^{-1}}$:

Range of $\boldsymbol{f}$:

Example 2: List the domain and range of each of the following functions. Then find the inverse function and list its domain and range.

a. $f(x) = -\frac{2}{1-x}$

$1 - x \neq 0$

$1 \neq x$

Domain of f:

$(-\infty, 1) \cup (1, \infty)$

Range of f:

$(-\infty, 0) \cup (0, \infty)$

Keep in mind that after finding the domain of f, I would find the inverse function next, then I'd find the domain of f^{-1} in order to get the range of f.

$f = -\frac{2}{1-x}$

$f = \frac{-2}{1-x}$

$(1-x)f = -2$

$f - fx = -2$

$f + 2 = fx$

$\frac{f+2}{f} = x$

$f^{-1}(x) = \frac{x+2}{x}$

$x \neq 0$

Domain of f^{-1}:

$(-\infty, 0) \cup (0, \infty)$

Range of f^{-1}:

$(-\infty, 1) \cup (1, \infty)$

b. $f(x) = \frac{5}{2x} + 7$

Domain of f:

Range of f:

$f = \frac{5}{2x} + 7$

$2xf = 5 + 14x$

$2xf - 14x = 5$

$x(2f - 14) = 5$

$x = \frac{5}{2f - 14}$

$f^{-1}(x) = \frac{5}{2x-14}$

Domain of f^{-1}:

Range of f^{-1}:

c. $f(x) = \frac{2-x}{x+1}$

$x + 1 \neq 0$

$x \neq -1$

Domain of f:

$(-\infty, -1) \cup (-1, \infty)$

Range of f:

$(-\infty, -1) \cup (-1, \infty)$

$f = \frac{2-x}{x+1}$

$f(x+1) = 2 - x$

$xf + f = 2 - x$

$xf + x = 2 - f$

$x(f+1) = 2 - f$

$x = \frac{2-f}{f+1}$

$f^{-1}(x) = \frac{2-x}{x+1}$

$x + 1 \neq 0$

$x \neq -1$

Domain of f^{-1}:

$(-\infty, -1) \cup (-1, \infty)$

Range of f^{-1}:

$(-\infty, -1) \cup (-1, \infty)$

The function $f(x) = \frac{2-x}{x+1}$ is a special case where the function and its inverse are identical.

d. $f(x) = \frac{x+5}{7-2x}$

Domain of f:

Range of f:

$f = \frac{x+5}{7-2x}$

$f(7-2x) = x + 5$

$7f - 2xf = x + 5$

$7f - 5 = x + 2xf$

$7f - 5 = x(1 + 2f)$

$\frac{7f-5}{1+2f} = x$

$f^{-1}(x) = \frac{7f-5}{1+2f}$

Domain of f^{-1}:

Range of f^{-1}:

Another way that the domain of a function could be restricted is by having a function that is defined by a radical with an even root, such as a square root.

When a function is defined by a square root, the radicand of that square root cannot be negative

- if $f(x) = \sqrt{x-2}$, then the radicand $x - 2 \geq 0$, so $x \geq 2$; therefore the domain of f is $[2, \infty)$
- this means the range of f^{-1} will also be $[2, \infty)$

As we'll see on the next example, sometimes finding the inverse function does nothing to help us determine its domain, or the range of the original function.

Example 3: List the domain and range of the following function. Then find the inverse function and list its domain and range.

$f(x) = \sqrt{x-2}$; this function is defined by a square root, so the radicand must be non-negative

$$x - 2 \geq 0$$

$$x \geq 2$$

Domain of f: $[2, \infty)$

Range of f:

$$f = \sqrt{x-2}$$

$$f^2 = x - 2$$

$$f^2 + 2 = x$$

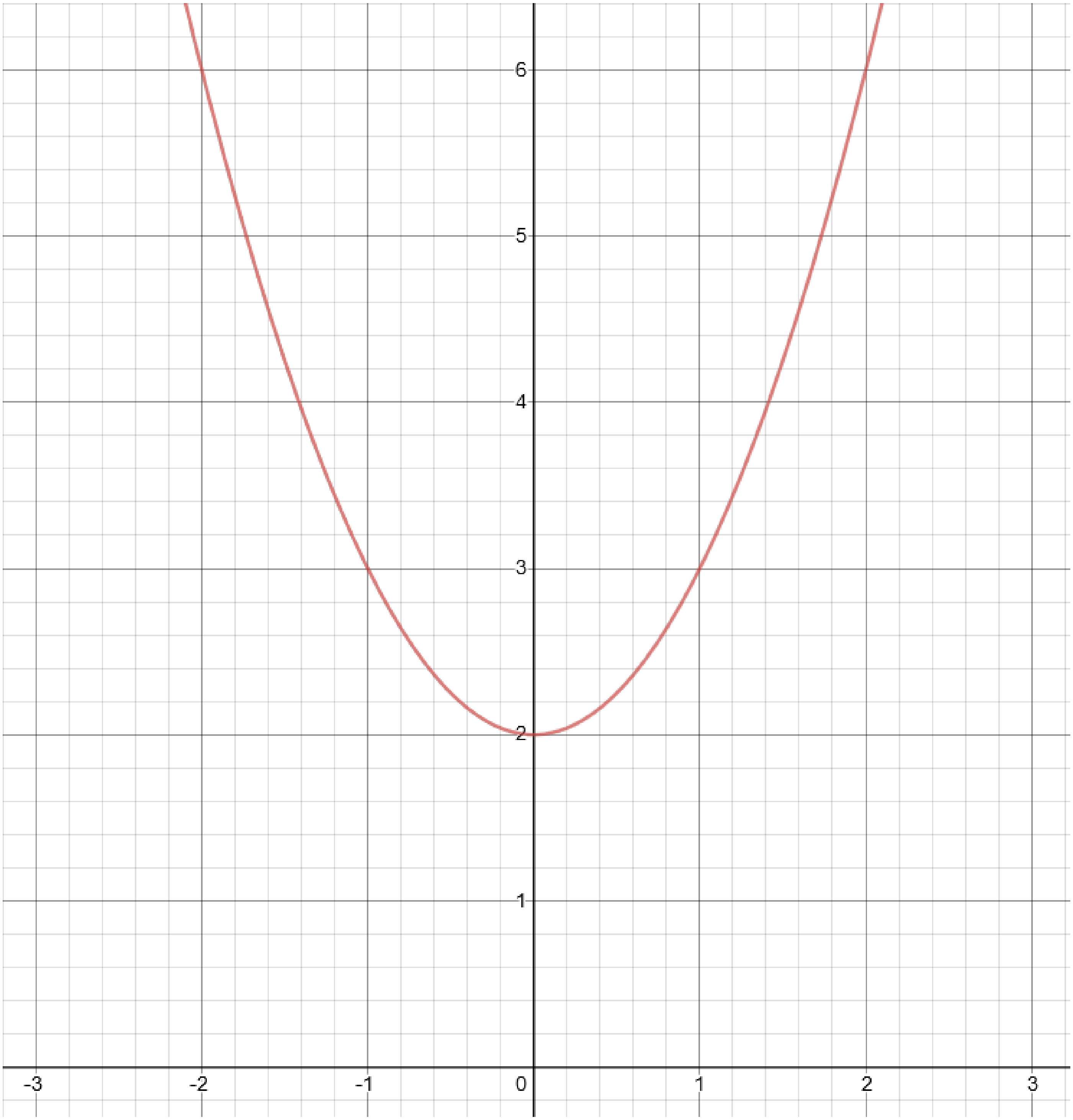
$\boldsymbol{f^{-1}(x)} =$ $\boldsymbol{x^2 + 2}$; in this case the inverse function is quadratic, so it doesn't help us in determining the domain of f^{-1} or the range of f. Therefore we'll need to find those using some other method.

Domain of f^{-1}:

Range of f^{-1}: $[2, \infty)$

(remember that the domain of f is the range of f^{-1})

Below is the graph of $x^2 + 2$:



As we saw in Lesson 27, while this is the graph of a function (a quadratic function), this is not the graph of a one-to-one function because it does not pass the horizontal line test. In order to make this graph one-to-one, we need to restrict its domain. However it's not obvious how that should be done, and that is the issue we run into with this quadratic function, as well as other quadratic functions.

Since finding the inverse function didn't help us in determining the range of the function $f(x) = \sqrt{x-2}$, I will try using an input/output table to determine the range of f and the domain of its inverse..

The domain of the function f is $[2, \infty)$, so that is why I started at $x = 2$ and proceeded to plug in larger and larger x-values from there. Since f is defined by a square root, some inputs don't produce nice outputs.

Inputs	**Outputs**
x	$f(x) = \sqrt{x-2}$
2	$f(2) = \sqrt{2-2} = \sqrt{0} = 0$
3	$f(3) = \sqrt{3-2} = \sqrt{1} = 1$
4	$f(4) = \sqrt{4-2} = \sqrt{2}$
5	$f(5) = \sqrt{5-2} = \sqrt{3}$
6	$f(6) = \sqrt{6-2} = \sqrt{4} = 2$
11	$f(11) = \sqrt{11-2} = \sqrt{9} = 3$
18	$f(18) = \sqrt{18-2} = \sqrt{16} = 4$
27	$f(27) = \sqrt{27-2} = \sqrt{25} = 5$

The outputs for the function f start at 0 ($f(2) = 0$) and proceeded to get larger and larger from there. So that means the range of f is $[0, \infty)$. This makes sense because a square root should produce only non-negative outputs.

This input/output table shows that as I plug in x-values (inputs) from the domain, such as $2, 3, 4, 5, 6, \ldots$, I get function values (outputs) that start at 0 and get larger and larger $(1, 2, 3, 4, 5, \ldots)$. So range of $f(x) = \sqrt{x-2}$ is $[0, \infty)$. That means the domain of f^{-1} is also $[0, \infty)$.

$\boldsymbol{f(x) = \sqrt{x-2}}$ **Domain of $\boldsymbol{f}$:** $[2, \infty)$ **Range of $\boldsymbol{f}$:** $\boldsymbol{[0, \infty)}$

$\boldsymbol{f^{-1}(x) =}$ $\boldsymbol{x^2 + 2}$ **Domain of $\boldsymbol{f^{-1}}$:** $\boldsymbol{[0, \infty)}$ **Range of $\boldsymbol{f^{-1}}$:** $\boldsymbol{[2, \infty)}$

As shown in this example, the inverse of a square root function is a quadratic function. And since the domain of a quadratic function is usually unrestricted, we had to use another method to find its domain and the range of the original function.

Example 4: List the domain and range of each of the following functions. Then find the inverse function and list its domain and range.

a. $f(x) = -\sqrt{3-x}$

Domain of $\boldsymbol{f}$:

Range of $\boldsymbol{f}$:

$\boldsymbol{f^{-1}(x) =}$

Domain of $\boldsymbol{f^{-1}}$:

Range of $\boldsymbol{f^{-1}}$:

Inputs	**Outputs**
x	$f(x) = -\sqrt{3-x}$

b. $f(x) = 1 - \sqrt{1+x}$

Domain of f:

Range of f:

$f = 1 - \sqrt{1+x}$

$f - 1 = -\sqrt{1+x}$

$(f-1)^2 = \left(-\sqrt{1+x}\right)^2$

$(f-1)^2 = 1 + x$

$(f-1)^2 - 1 = x$

$$f^{-1}(x) = (x-1)^2 - 1$$

Domain of f^{-1}:

Range of f^{-1}:

c. $f(x) = 3 - \sqrt{\frac{x+2}{5}}$

Domain of f:

Range of f:

$f = 3 - \sqrt{\frac{x+2}{5}}$

$f - 3 = -\sqrt{\frac{x+2}{5}}$

$(f-3)^2 = \left(-\sqrt{\frac{x+2}{5}}\right)^2$

$(f-3)^2 = \frac{x+2}{5}$

$5(f-3)^2 = x + 2$

$5(f-3)^2 - 2 = x$

$$f^{-1}(x) = 5(x-3)^2 - 2$$

Domain of f^{-1}:

Range of f^{-1}:

Ways to find the range of a function (or the domain of its inverse):

1. find the inverse of the function, and then find the domain of the inverse (this is what I did on Examples 1 & 2)
   a. I will use this method anytime the original function is rational (defined by a fraction) or quadratic (as we'll see on Example 5)
2. use an input/output table (this is what I did in Examples 3 & 4)
   a. I will **ONLY** use this method when the original function is defined by a square root

There are other options as well such as graphing which you're welcome to use.

Remember that one-to-one functions and their inverses never change direction, they are either **ALWAYS** increasing or **ALWAYS** decreasing. This is why using an input/output table is a good option, because once you determine whether the outputs are increasing or decreasing, they will **ALWAYS** continue to move in that direction.

Inputs	**Outputs**
x	$f(x) = 3 - \sqrt{\frac{x+2}{5}}$
-2	$f(-2) = 3 - \sqrt{\frac{-2+2}{5}} = 3$
-1	$f(-1) = 3 - \sqrt{\frac{-1+2}{5}} \approx 2.55$
0	$f(0) = 3 - \sqrt{\frac{0+2}{5}} \approx 2.37$
3	$f(3) = 3 - \sqrt{\frac{3+2}{5}} = 2$
18	$f(18) = 3 - \sqrt{\frac{18+2}{5}} = 1$
43	$f(43) = 3 - \sqrt{\frac{43+2}{5}} = 0$

The input/output table on the left goes along with Example 4 part c. It shows some inputs and outputs for the function

$f(x) = 3 - \sqrt{\frac{x+2}{5}}$. The domain of that function is $[-2, \infty)$, so -2 is the smallest input of the function. $f(-2)$ produces an output of 3. From there the inputs get larger $(-1, 0, \dots)$ and the outputs get smaller $(2.55, 2.37, 2, 1, 0, \dots)$, so the range is $(-\infty, 3]$. This demonstrates that one-to-one functions, such as

$f(x) = 3 - \sqrt{\frac{x+2}{5}}$, are either always decreasing (like this function) or always increasing, but they never change direction.

Remember that when changing from a function to its inverse, the inputs and outputs of the original function are switched. This means the following are true:

- **the domain of a function is the range of its inverse**
- **the range of a function is the domain of its inverse**

Also keep in mind that **ONLY** one-to-one functions have an inverse. So a function such as $f(x) = \pm\sqrt{x+2}$ would not have an inverse (and could not be an inverse) because it is not one-to-one ($f(7) = 3$ and -3). So to make it a one-to-one function, we need to restrict it to either $f(x) = \sqrt{x+2}$ or $f(x) = -\sqrt{x+2}$. We will see how to determine which restriction to go with in the next example.

Example 5: List the domain and range of each of the following functions. Then find the inverse function and list its domain and range.

a. $f(x) = 5 - x^2; x \geq 0$

Domain of *f*:

Range of *f*:

$f = 5 - x^2$

$x^2 = 5 - f$

$x = \pm\sqrt{5 - f}$

b. $f(x) = x^2 + 3; x \leq 0$

Domain of *f*:

Range of *f*:

$f = x^2 + 3$

$f - 3 = x^2$

$\pm\sqrt{f - 3} = x$

At this point we need to determine whether to keep the + sign or the – sign (keeping both means this would not be a one-to-one function). Use the inequalities provided with original function to determine which sign to keep.

a. $\boldsymbol{f^{-1}(x) = \sqrt{5 - x}}$

Domain of f^{-1}:

Range of f^{-1}:

b. $\boldsymbol{f^{-1}(x) =}$

Domain of f^{-1}:

Range of f^{-1}:

c. $f(x) = 3(x-2)^2 + 4; x \leq 2$

Since $x \leq 2$, the domain of f will be $(-\infty, 2]$.

Domain of f:

$(\infty, \mathbf{2}]$

Range of f:

$[\mathbf{4}, \infty)$

Keep in mind that once again, after finding the domain of f, I would find the inverse function next, then I'd find the domain of f^{-1} in order to get the range of f.

$f = 3(x-2)^2 + 4$

$f - 4 = 3(x-2)^2$

$\frac{f-4}{3} = (x-2)^2$

$\pm\sqrt{\frac{f-4}{3}} = x - 2$

$2 \pm \sqrt{\frac{f-4}{3}} = x$

Since $x \leq 2$, we keep the minus sign, not the plus sign.

$$\boldsymbol{f^{-1}(x) = 2 - \sqrt{\frac{x-4}{3}}}$$

$\frac{x-4}{3} \geq 0$

$x - 4 \geq 0$

$x \geq 4$

Domain of f^{-1}:

$[\mathbf{4}, \infty)$

Range of f^{-1}:

$(\infty, \mathbf{2}]$

d. $f(x) = -(x+5)^2 - 2; x \geq -5$

Domain of f:

Range of f:

$$\boldsymbol{f^{-1}(x) =}$$

Domain of f^{-1}:

Range of f^{-1}:

When the original function you're given is quadratic, like of the functions from Example 5, the inequalities that are given with the function tell you two things:

1. The inequality tells you what the domain of the function will be. The domain of the function $f(x) = 5 - x^2$ from Example 5 part a. is $[0, \infty)$ because we were told that $x \geq 0$. On Example 5 part c., we were given the function $f(x) = 3(x-2)^2 + 4$, along with the inequality $x \leq 2$. The domain of that function is $(-\infty, 2]$, because the inequality tells us that x must be less than or equal to 2.

2. The inequality also tells you which sign (+ or –) to keep in front of the square root in the inverse function. On Example 5 part a., the inverse of $f(x) = 5 - x^2$ is $f^{-1}(x) = \sqrt{5-x}$ because we were told that $x \geq 0$; the inverse of $f(x) = 3(x-2)^2 + 4$ on Example 5 part c. is $f^{-1}(x) = 2 - \sqrt{\frac{x-4}{3}}$ because we were told that $x \leq 2$.

Answers to Exercises:

1. $D: (-\infty, -2) \cup (-2, \infty), R: (-\infty, 0) \cup (0, \infty);\ f^{-1}(x) = \frac{1-2x}{x}, D: (-\infty, 0) \cup (0, \infty), R: (-\infty, -2) \cup (-2, \infty)$

2a. $D: (-\infty, 1) \cup (1, \infty), R: (-\infty, 0) \cup (0, \infty);\ f^{-1}(x) = \frac{x+2}{x}, D: (-\infty, 0) \cup (0, \infty), R: (-\infty, 1) \cup (1, \infty)$

2b. $D: (-\infty, 0) \cup (0, \infty), R: (-\infty, 7) \cup (7, \infty);\ f^{-1}(x) = \frac{5}{2x-14}, D: (-\infty, 7) \cup (7, \infty), R: (-\infty, 0) \cup (0, \infty)$

2c. $D: (-\infty, -1) \cup (-1, \infty), R: (-\infty, -1) \cup (-1, \infty);\ f^{-1}(x) = \frac{2-x}{x+1}, D: (-\infty, -1) \cup (-1, \infty), R: (-\infty, -1) \cup (-1, \infty)$

2d. $D: \left(-\infty, \frac{7}{2}\right) \cup \left(\frac{7}{2}, \infty\right), R: \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right);\ f^{-1}(x) = \frac{7x-5}{1+2x}, D: \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right), R: \left(-\infty, \frac{7}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$

3. $D: [2, \infty), R: [0, \infty);\ f^{-1}(x) = x^2 + 2, D: [0, \infty), R: [2, \infty)$

4a. $D: (-\infty, 3], R: (-\infty, 0];\ f^{-1}(x) = 3 - x^2, D: (-\infty, 0], R: (-\infty, 3]$

4b. $D: [-1, \infty), R: (-\infty, 1];\ f^{-1}(x) = (x-1)^2 - 1, D: (-\infty, 1], R: [-1, \infty)$

4c. $D: [2, \infty), R: (-\infty, 3];\ f^{-1}(x) = 5(x-3)^2 + 2, D: (-\infty, 3], R: [2, \infty)$

5a. $D: [0, \infty), R: (-\infty, 5];\ f^{-1}(x) = \sqrt{5-x}, D: (-\infty, 5], R: [0, \infty)$

5b. $D: (-\infty, 0], R: [3, \infty);\ f^{-1}(x) = \sqrt{x-3}, D: [3, \infty), R: (-\infty, 0]$

5c. $D: (-\infty, 2], R: [4, \infty);\ f^{-1}(x) = 2 - \sqrt{\frac{x-4}{3}}, D: [4, \infty), R: (-\infty, 2]$

5d. $D: [-5, \infty), R: (-\infty, -2];\ f^{-1}(x) = -5 + \sqrt{-x-2}, D: (-\infty, -2], R: [-5, \infty)$