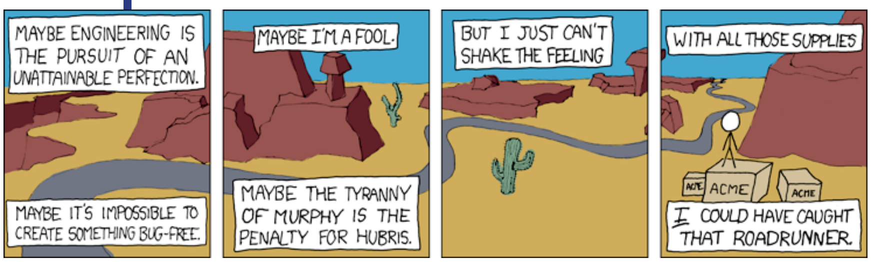
Centroids



The golfer guessed that his ball landed 20 feet off the fairway.
Of course, that was just a rough estimate.

THE UNIVERSITY OF MEMPHIS
Department of Civil Engineering

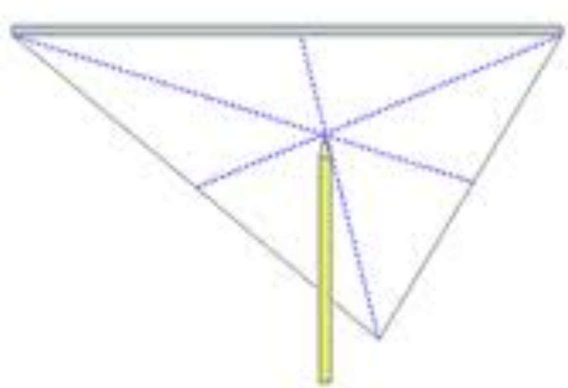
Centroids

- When we dealt with distributed loads, we found the magnitude of the force generated by the loading as the area under the loading curve.
- I gave you the location of the line of action of the force for both a rectangular shape and a right-triangular shape.

Centroids

- In this meeting, we are going to find out just why that line of action was located where it was.
- The line of action was located through the centroidial axis of the loading diagram.
- If we took a centroidial axis in every direction, their intersection point would be known as the centroid

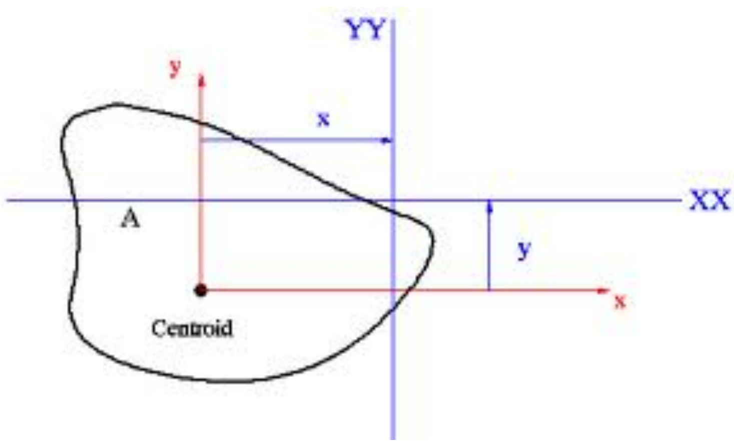
Centroid as Balance Point



Centroids

- By common practice, we refer to the centroidal axis as the centroid but to keep the confusion down we will often speak of a x-centroid or a y-centroid referring to the coordinate along that axis where the centroidal axis intersects the coordinate axis.

Centroidal Axis

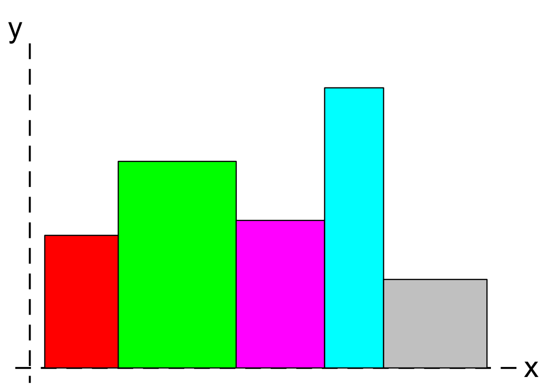


Centroids

- We are going to look at two mathematical techniques for locating this centroidal axis or centroid.
- First we will look at what a centroid physically represents

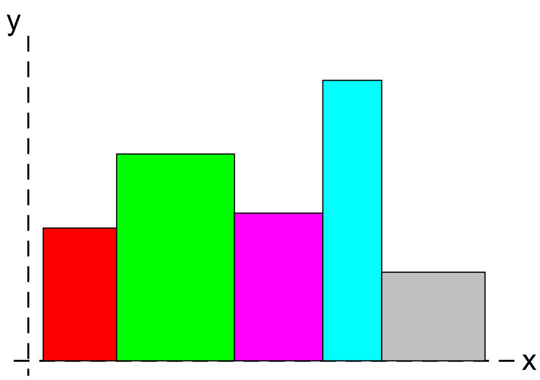
Centroids

- Consider that we have a series of rectangular loads along an axis



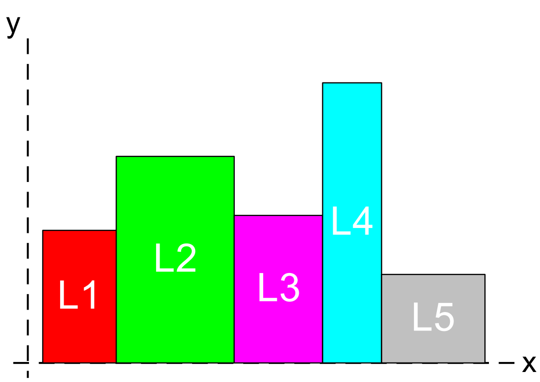
Centroids

- We would like to replace this loading with a single point force for analysis purposes



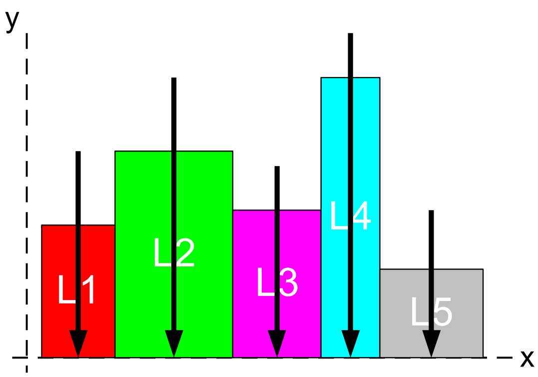
Centroids

- We would label each load



Centroids

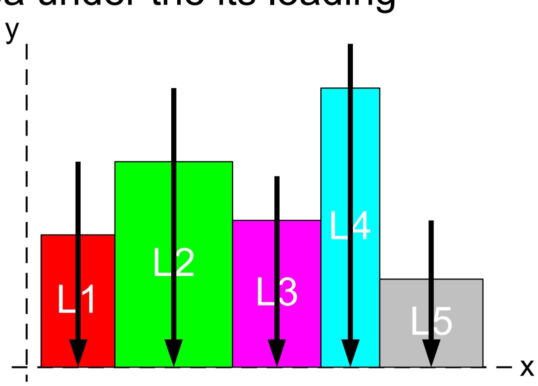
- Then find the area of each loading, giving us the force which is located at the center of each area



Centroids

- The force generated by each loading is equal to the area under the its loading diagram so

$$F_n = A_{L_n}$$



Centroids

- The force generated by each loading is equal to the area under the loading diagram so

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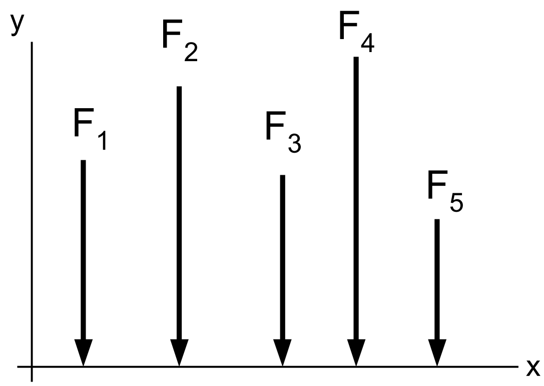
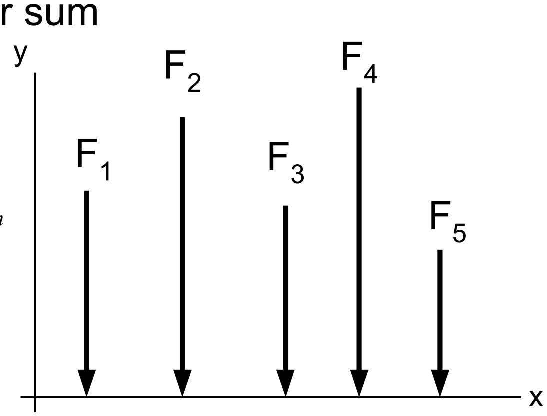


Centroids

- The total force generated by all these forces is just their sum

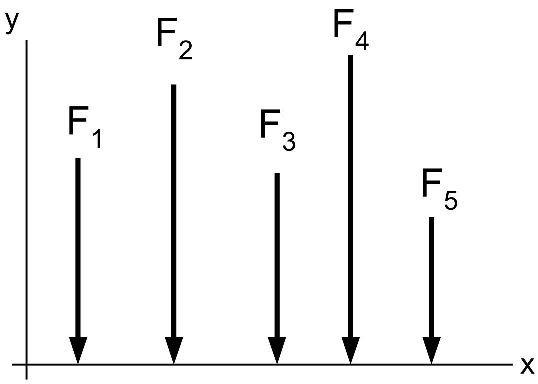
$$F = \sum F_n = \sum A_{L_n}$$



Centroids

- But we want to replace these forces with a single force with the same net effect on the system

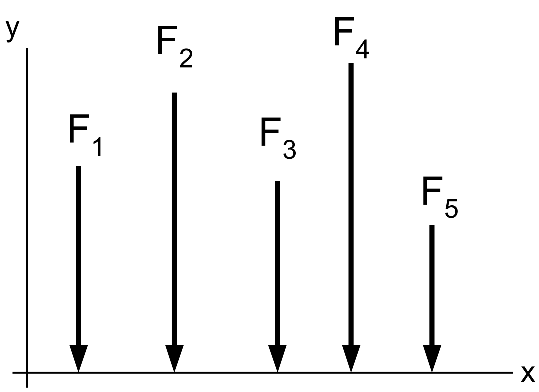
$$F = \sum F_n = \sum A_{L_n}$$



Centroids

- That would mean that it would have to produce the same moment about any point on the system

$$F = \sum F_n = \sum A_{L_n}$$



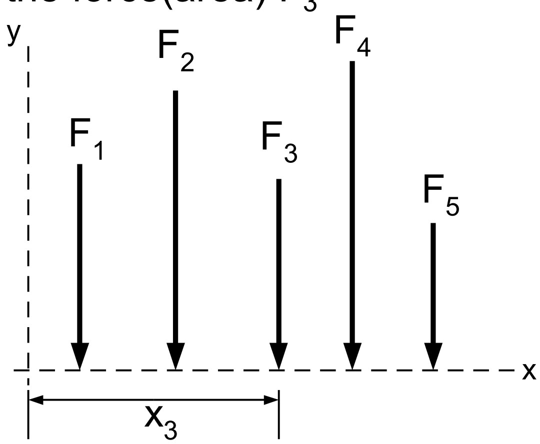
Centroids

- If we choose the origin for the moment center, the moment of each of these forces about the origin is equal to the x-distance to the line of action of the force times the force

Centroids

- For example, for the force(area) F_3

$$M_3 = x_3 F_3$$



Centroids

- The moment for any force is

$$M_n = x_n F_n$$

Centroids

- Which can be replaced by

$$M_n = x_n A_{L_n}$$

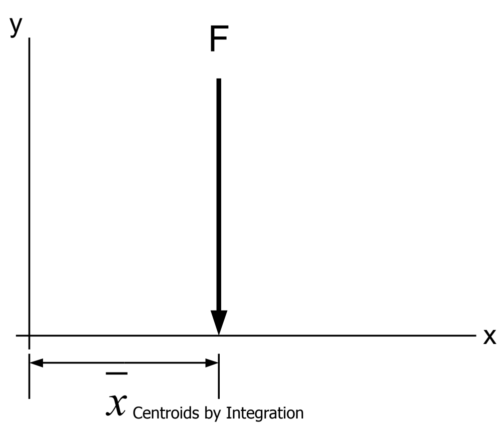
Centroids

- And the total moment of all the forces(areas) about the origin is

$$M_{total} = \sum_{i=1}^{n} x_i A_{L_i}$$

Centroids

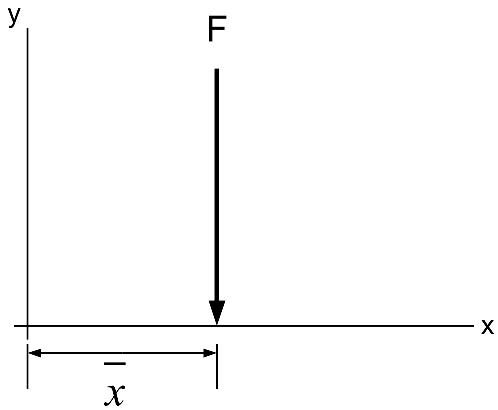
- If we now look at the effect of the total force



Centroids

- The value $\bar{x}$ is known as the x-centroid of the loading

$$M = \bar{x}F$$

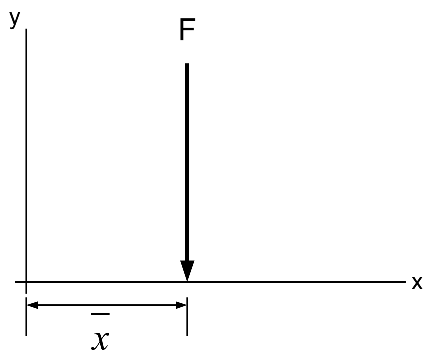


Centroids

- If we substitute the sums developed earlier for the total force and the total moment

$$M = \bar{x}F$$

$$\sum x_i F_i = \bar{x} \sum F_i$$



Centroids

- And isolating the centroid we have

$$\bar{x} = \frac{\sum x_i F_i}{\sum F_i} \text{ or } \frac{\sum x_i A_i}{\sum A_i}$$

Centroids

- A general definition for the x-centroid of a series of n areas would be

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

Centroids

- If the areas represent forces, the centroid represents a center of force

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i F_i}{\sum_{i=1}^{n} F_i}$$

Centroids

- If we are considering weight, and x is the axis parallel to the ground, the xbar would be the center of gravity.

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i F_i}{\sum_{i=1}^{n} F_i}$$

Centroids

- If the areas represent masses, the centroid represents the center of mass

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}$$

Centroids

- If the areas are just areas, the centroid represents the center of area or centroid

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

Centroids

- This is the general formulation for finding the x-centroid of *n* areas

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

31 Centroids by Integration Wednesday, November 7, 2012

Centroids

- The same type of formula could be found for finding the y centroid

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i}$$

32 Centroids by Integration Wednesday, November 7, 2012

Centroids

- Remember that the x_i is the x-distance to the centroid of the i^{th} area

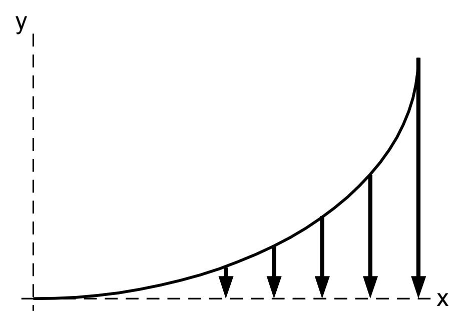
$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

Centroids from Functions

- So far, we have been able to describe the forces (areas) using rectangles and triangles.
- Now we have to extend that to loadings and areas that are described by mathematical functions.

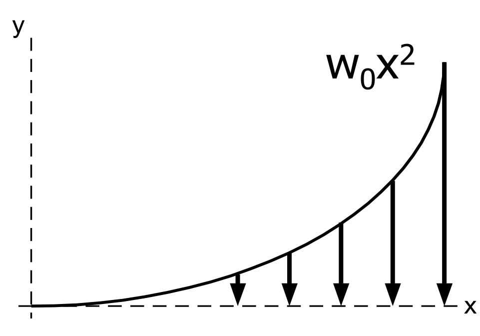
Centroids from Functions

- For example



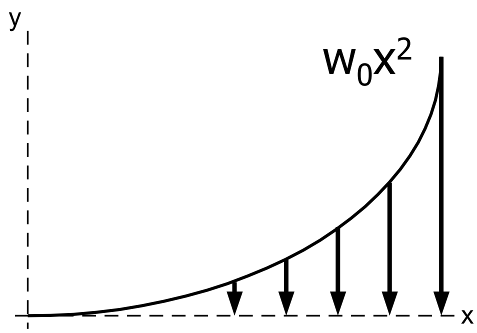
Centroids from Functions

- This is a distributed load that at any x has a load intensity of w_0x^2



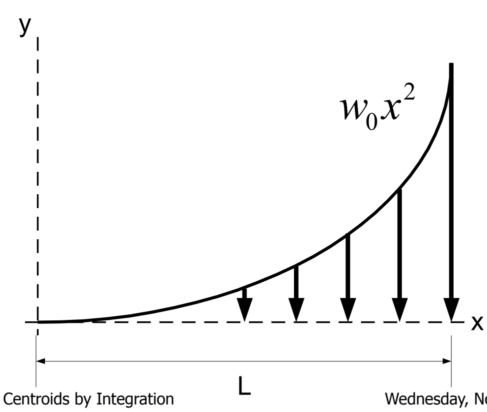
Centroids from Functions

- w_0 is a proportionality constant that will have units to make sure that the units of the product w_0x^2 will be in force per length units



Centroids from Functions

- If we had this type of loading over a distance L, how would we find the equivalent point force and its location?



Riemann Sums

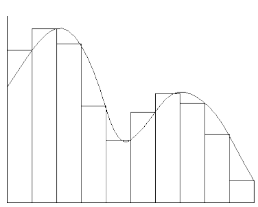
Let y = f(x) be any function the integral will approximately equal

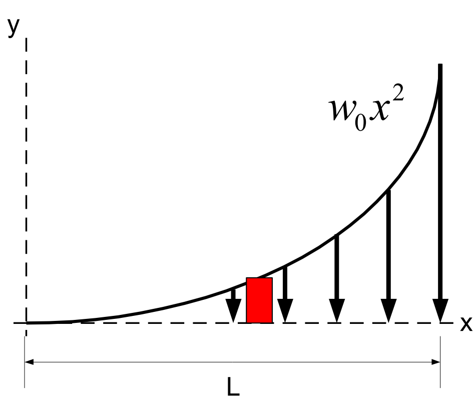
$$\int_a^b f(x)dx = \frac{b-a}{n}\left[f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\right]$$

where x_1, x_2, x_3, ... , x_n are the midpoints of the n rectangles.

Centroids from Functions

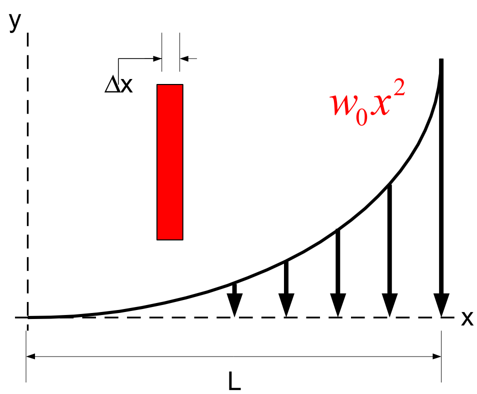
- We could generate a series of rectangles to lay over the curve





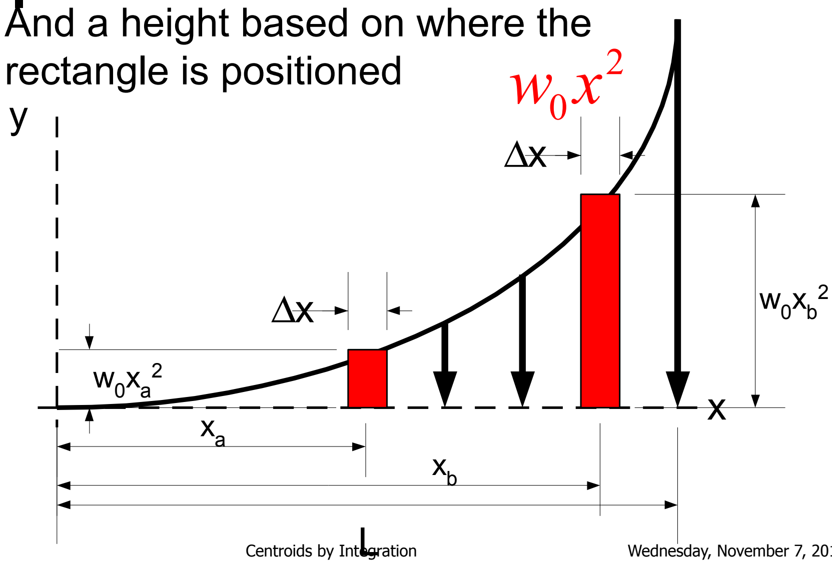
Centroids from Functions

- Each rectangle will have some width Δx



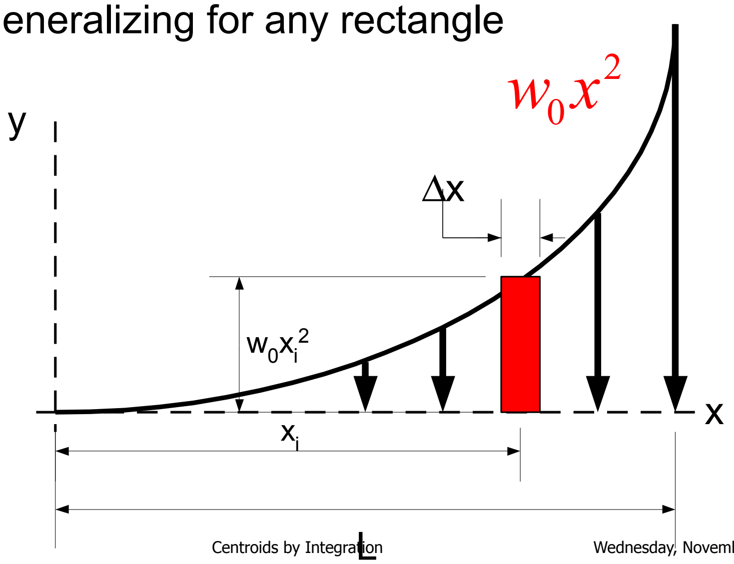
Centroids from Functions

- And a height based on where the rectangle is positioned



Centroids from Functions

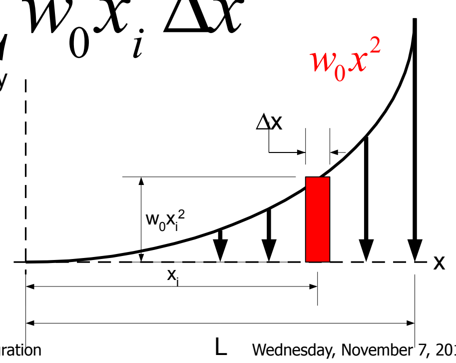
- Generalizing for any rectangle



Centroids from Functions

- So for n-rectangles the approximate area under the curve would be

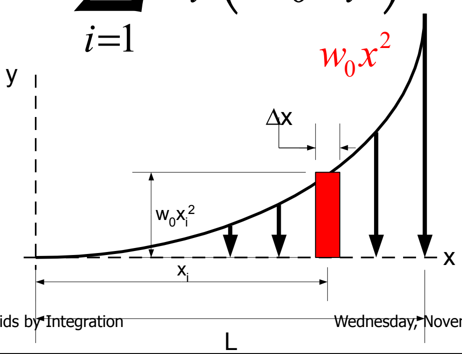
$$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} w_0 x_i^2 \Delta x$$



Centroids from Functions

- The total moment generated by these areas would be

$$M=\sum_{i=1}^{n} x_i A_i=\sum_{i=1}^{n} x_i\left(w_0 x_i^2\right)\Delta x$$



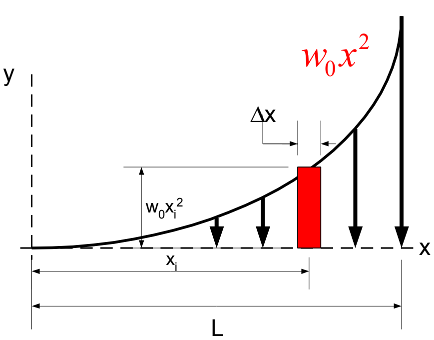
Centroids from Functions

- And the location of the centroid would be

$$\sum_{i=1}^{n} x_i A_i=\bar{x}\sum_{i=1}^{n} A_i$$

$$\sum_{i=1}^{n} x_i\left(w_0 x_i^2\right)\Delta x=\bar{x}\sum_{i=1}^{n}\left(w_0 x_i^2\right)\Delta x$$

$$\bar{x}=\frac{\sum_{i=1}^{n} x_i\left(w_0 x_i^2\right)\Delta x}{\sum_{i=1}^{n}\left(w_0 x_i^2\right)\Delta x}$$

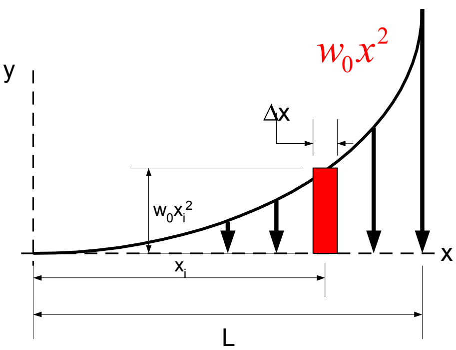


Centroids from Functions

- The general form would be

$$\sum_{i=1}^{n} x_i A_i = \bar{x} \sum_{i=1}^{n} A_i$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$



Centroids from Functions

- So for any loading that we can break up into n individual loadings with known centroids, the centroid of the composite would be equal to

$$\sum_{i=1}^{n} x_i A_i = \bar{x} \sum_{i=1}^{n} A_i$$

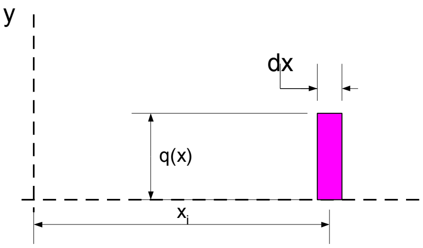
$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

Centroids from Functions

- If we reduce the width of the rectangles to a differential size, the summation become an integral

$$\bar{x} = \frac{\int_0^L x_i q(x_i)\,dx}{\int_0^L q(x_i)\,dx}$$

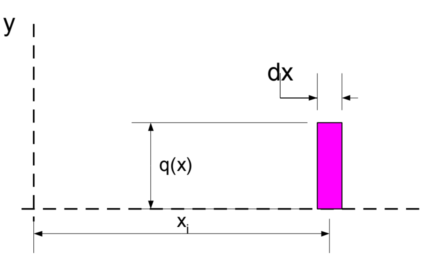
$q(x_i)$ represents a general loading function



Centroids from Functions

- If we can define the height of the loading diagram at any point x by the function q(x), then we can generalize out summations of areas by the quotient of the integrals

$$\bar{x} = \frac{\int_0^L x_i q(x_i)\,dx}{\int_0^L q(x_i)\,dx}$$



Centroids from Functions

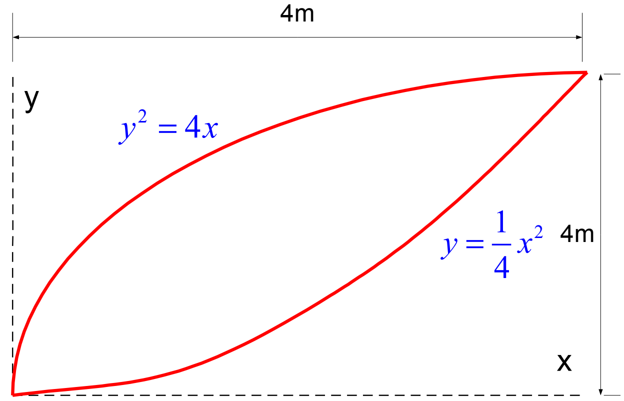
- This is the general form for the integral to locate the centroid

$$\bar{x} = \frac{\int_A xq(x)\,dx}{\int_A q(x)\,dx}$$

Centroids from Functions

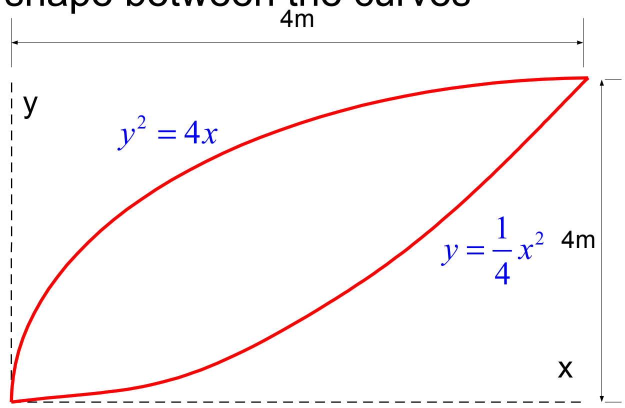
- It isn't always quite that simple
- You have to be careful in
  - *Knowing the height of your rectangular section*
  - *Knowing the limits of integration*
  - *Making the correct integration*

An Example



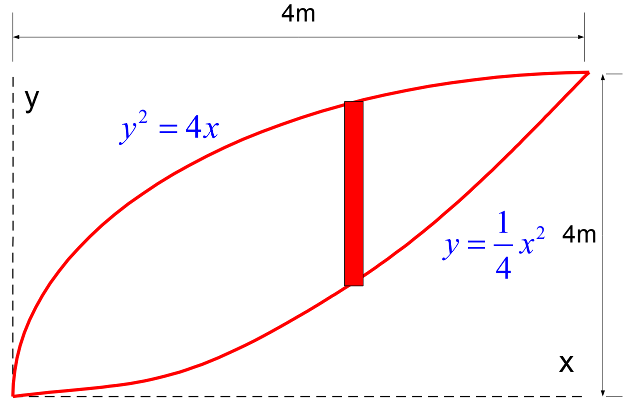
An Example

- We need to locate the x and y centroids of the shape between the curves



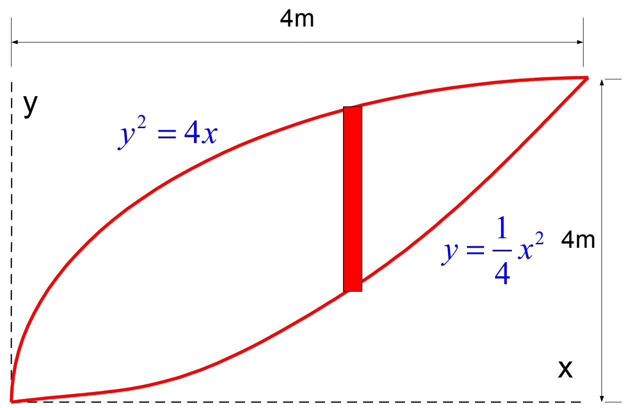
An Example

- First we will sketch a representative rectangle



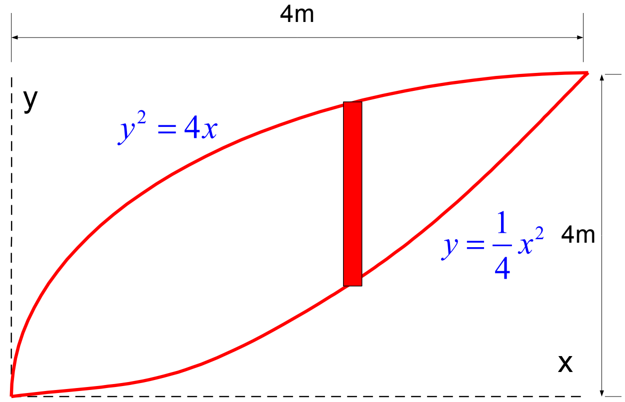
An Example

- Determine the height of the rectangle



An Example

- Determine the width of the rectangle



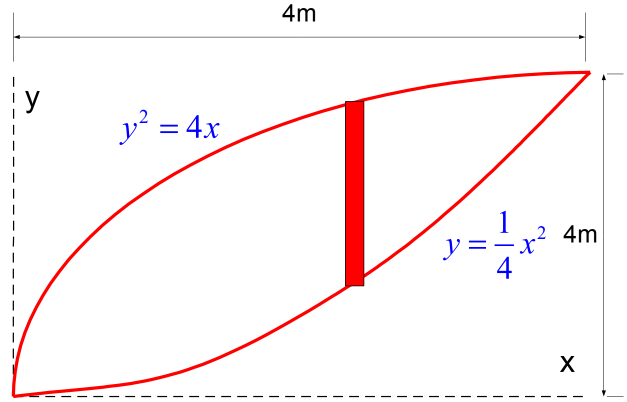
An Example

- So the area of the rectangle becomes

$$dA = \left(\sqrt{4x} - \frac{x^2}{4} \right) dx$$

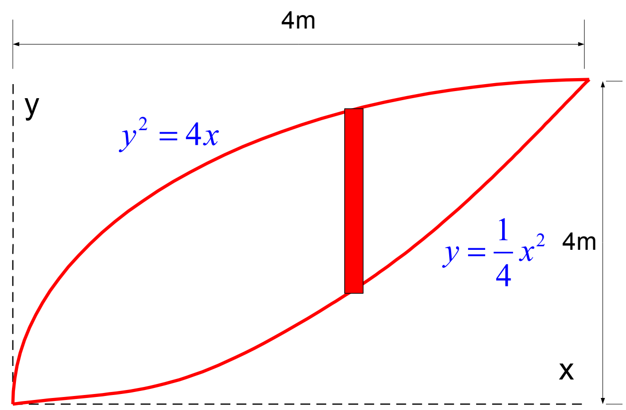
An Example

- The moment arm is the distance from the the moment center (in this case the origin)



An Example

- The limits of integration will be the beginning and ending points of x



An Example

- So when we set up the integral form for the centroid we have

$$\bar{x} = \frac{\int_{0m}^{4m} x\left(\sqrt{4x} - \frac{x^2}{4}\right)dx}{\int_{0m}^{4m}\left(\sqrt{4x} - \frac{x^2}{4}\right)dx}$$

An Example

- Integrating

$$\bar{x} = \frac{\int_{0m}^{4m}\left(2x^{\frac{3}{2}} - \frac{x^3}{4}\right)dx}{\int_{0m}^{4m}\left(2x^{\frac{1}{2}} - \frac{x^2}{4}\right)dx} = \frac{\left.\frac{4}{5}x^{\frac{5}{2}} - \frac{x^4}{16}\right|_{0m}^{4m}}{\left.\frac{4}{3}x^{\frac{3}{2}} - \frac{x^3}{12}\right|_{0m}^{4m}}$$

An Example

- Substituting at the upper and lower limits

$$\bar{x} = \frac{\left(\frac{4}{5}(4m) - \frac{(4m)^{\frac{5}{2}}}{16}\right) - \left(\frac{4}{5}(0m) - \frac{(0m)^{\frac{5}{2}}}{16}\right)}{\left(\frac{4}{3}(4m) - \frac{(4m)^{\frac{3}{2}}}{12}\right) - \left(\frac{4}{3}(0m) - \frac{(0m)^{\frac{3}{2}}}{12}\right)}$$

An Example

- Evaluating

$$\bar{x} = 1.8m$$

Points to Remember

- Draw the rectangle you are going to use
- Be careful that you take the correct distance from the correct axis
- You may want to always use x or y as the variable of integration; be very careful here, it is only to the center of the differential side that you can assume the moment arm goes to

Homework

- Problem 9-7
- Problem 9-12
- Problem 9-15