Chapter 1

Review of fractions

Vocabulary

- Whole numbers
- Integers
- Fraction
- Numerator
- Denominator
- Rational numbers
- Fractions in simplest form
- Equivalent fractions
- Reciprocal

1.1 Introduction to fractions

Growing up, the first numbers we encounter are *whole numbers*: $0, 1, 2, \ldots$. These numbers (with the possible exception of zero) are as concrete as a number can be. While I may not be able to picture a "three," I have a clear idea of what three books mean, three dollars are, three fingers, etc. The whole numbers, together with their opposites ("negative whole numbers") are called *integers*.

As soon as we start dividing whole numbers, though, we encounter the problem that the quotient of two whole numbers may not be a whole number.

A *fraction* is a symbolic way of writing a quotient, which is the result of dividing two numbers. In this way, the operation of division is "built into" the notion of a fraction.

For example, $\frac{1}{2}$ is a symbol representing the number that results by performing the operation $1 \div 2$.

Some things to notice right away: A fraction is *one symbol* consisting of two numbers separated by a bar (the bar representing the operation of division). One number is written above the bar—it is called the *numerator*. The other number, written below the bar, is called the *denominator*. The two numbers play different roles. After all, division is not commutative: $1 \div 2$ does not give the same result as $2 \div 1$. So $\frac{1}{2}$ is not the same as $\frac{2}{1}$.

For the first few chapters of this book, most of the numbers we will encounter will be *rational numbers*. A rational number is the result of dividing two integers. Said differently, a rational number is a number which can be written as a fraction whose numerator and denominator are both integers. Remember that since division by zero causes very fundamental problems, the denominator of a fraction representing a rational number must not be zero.

1.2 Decimal representation

Fractions are not the only way to represent the result of a division. Using long division, the quotient of two numbers can be expressed in decimal notation. Here is a simple example:

$$\begin{array}{r} 0.5 \\ 2\overline{\smash{)}1.0} \end{array}$$

Changing from fraction to decimal notation

To change a fraction to a number in decimal form, perform long division of the numerator by the denominator.

Be sure to practice this, as there are several different ways of performing long division depending on in which country you went to school!

One of the unfortunate features of decimal representations of numbers is that they may not terminate nicely, as the previous simple example did. For example, $1 \div 3 = \frac{1}{3} = 0.\overline{3} = 0.33333\ldots$.

A basic fact of rational numbers, however, is that their decimal representation must either terminate or repeat.

1.2.1 Exercises

Convert the following fractions to decimals.

1. $\frac{3}{4}$

2. $\frac{5}{11}$

3. $\frac{2}{7}$

Throughout the text, starred exercises are those that might be slightly more challenging, or explore a topic in greater depth.

4. (*) Write the number $0.\overline{123}$ in fraction notation.

   (Hint: Represent the number $0.\overline{123}$ by the letter N. Since N has three digits repeating, multiply N by 1000. What is $1000N$? Using these values, subtract $1000N - N = 999N$. Then divide by 999 to "solve for N.")

5. (*) Use the hint in the previous exercise to write the following repeating decimals using fraction notation.
   (a) $0.\overline{123456}$
   (b) $3.\overline{14}$

1.3 Other conventions: Mixed numbers

You may recall that fractions whose numerator is smaller (in magnitude) than the denominator is called a *proper fraction.* For this reason, a fraction whose numerator is greater than or equal to the denominator is sometimes called an *improper fraction.* However, there is nothing improper about improper fractions at all—we will work with them routinely. In fact, in most circumstances, it is better to work with improper factions than their alternative.

However, there is another common way of expressing improper fractions. These are what are called *mixed numbers.* A mixed number has an integer part and a proper fraction part.

To convert from an improper fraction to a mixed number, perform the indicated division. The quotient will be the integer part of the mixed number. The remainder will be the numerator of the proper fraction part; the denominator is the same as the denominator of the original improper fraction.

Example 1.3.1. *Convert* $\frac{22}{7}$ *to a mixed number.*

Answer. $22 \div 7 = 3\ R\ 1$. *So*

$$\frac{22}{7} = 3\frac{1}{7}.$$

The answer is $3\frac{1}{7}$.

Notice that $3\frac{1}{7}$ actually means $3 + \frac{1}{7}$. This is an unfortunate notation, since normally the absence of a symbol for an operation between two numbers implies multiplication. But at this point, the notation is a historical fact of life.

To convert from a (positive) mixed number to an improper fraction, multiply the integer part by the denominator of the fraction part and add the numerator of the fraction part; the result will be the numerator of the improper fraction. The denominator of the improper fraction is the same as the denominator of the fraction part of the mixed number.

Example 1.3.2. *Convert* $5\frac{2}{3}$ *to an improper fraction.*

Answer. *To obtain the new numerator, first multiply the denominator by the integer part:* $3 \times 5 = 15$*. Then add the numerator of the fraction part:* $15 + 2 = 17$*.*

The denominator is the same as the denominator of the fraction part, in this case 3*.*

The answer is $\frac{17}{3}$*.*

We will not insist that improper fractions be converted to mixed numbers! In most cases, we will not work with mixed numbers at all.

1.3.1 Exercises

Change the following improper fractions to mixed numbers.

1. $\frac{19}{5}$

2. $\frac{100}{3}$

Change the following mixed numbers to improper fractions.

3. $4\frac{1}{8}$

4. $2\frac{3}{10}$

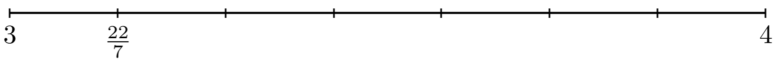
1.4 Graphical representation of fractions

You may remember from your youth seeing your math teacher drawing pictures of pizzas on the board to illustrate a way to represent fractions. Throughout this text, the most convenient way to represent numbers , including fractions, will be using a *number line.* In this representation, every number will correspond to a geometrical point.

The key features of a number line are: (1) it extends infinitely in both directions; (2) one direction is designated the positive direction (to the right) while the other is the negative direction (to the left); (3) there is one distinguished point on the line, representing the number 0; and (4) there is a “unit distance” with length one, which allows us to mark off all the points representing integers.

Proper fractions (at least the positive ones) are then represented by points lying between those labeled 0 and 1. The denominator tells us how many units to subdivide the segment between the points labeled 0 and 1; the numerator tells us how many of these sub-units to count from 0.

Improper fractions are handled similarly. For these fractions, it is more convenient to represent the number as a mixed number. Instead of dividing the segment between the points representing 0 and 1, we divide the segment starting at the point representing the integer part of the number and the next point representing an integer away from the point representing 0. For example, thinking of the improper fraction $\frac{22}{7}$ as the mixed number $3\frac{1}{7}$, we can represent this number with a point $\frac{1}{7}$ of a unit between 3 and 4:



1.5 Equivalent fractions and fractions in simplest form

One of the most important features of fractions is that two different-looking fractions might represent the same number. When you think about it, this shouldn't be a big surprise. After all, there are many different division problems that give the same result. For example, $9 \div 3$ is the same as $6 \div 2$—both are 3. That's one way to see that the fractions $\dfrac{9}{3}$ and $\dfrac{6}{2}$ are different ways of symbolizing the same number.

Two fractions are called *equivalent* if they represent the same number.

The basic principle we will need to keep in mind is the following: **Multiplying or dividing both the numerator and the denominator of a fraction by the same non-zero number gives an equivalent fraction**. In fact, this procedure amounts to multiplying a number by 1, which of course does not change the number.

There are two main reasons that we will be interested in equivalent fractions: writing fractions with common denominators, and writing fractions in simplest form.

1.5.1 Writing fractions with common denominators

As we will see below, there are many situations when we would like to write two fractions in an equivalent way so that they have the same denominator.

Example 1.5.1. *Write the fractions* $\dfrac{3}{10}$ *and* $\dfrac{7}{15}$ *with a common denominator.*

Answer. *There are two major steps in writing two fractions using a common denominator.*

Step 1. *Find a common denominator.*

We will find the least common multiple of the two denominators 10 *and* 15*. That is, we will find the smallest whole number which is multiple of both 10 and 15.*

Multiples of 10: $10, 20, 30, 40, \ldots$

Multiples of 15: $15, 30, 45, 60, \ldots$

$LCM(10, 15) = 30.$

Step 2. *For each of the fractions, decide what number is needed to multiply the original denominator in order to obtain the new, common denominator. Then multiply both the numerator and the denominator of the fractions by this number to obtain the equivalent fraction with the common denominator.*

For the fraction $\dfrac{3}{10}$*, what number do we need to multiply the original denominator* 10 *by to obtain the new common denominator* 30*?* $30 \div 10 = 3$*. So:*

$$\frac{3}{10} = \frac{3 \times 3}{10 \times 3} = \frac{9}{30}.$$

Likewise, for the fraction $\dfrac{7}{15}$*, what number do we need to multiply* 15 *by to obtain* 30*?* $30 \div 15 = 2$*. So:*

$$\frac{7}{15} = \frac{7 \times 2}{15 \times 2} = \frac{14}{30}.$$

The answer is $\dfrac{9}{30}$ *and* $\dfrac{14}{30}$.

1.5.2 Writing fractions in simplest form

A fraction is said to be in *simplest form* when the numerator and denominator have no factors in common (except 1).

For example, the fraction $\frac{8}{52}$ is not in simplest form, since 4 is a factor of 8 (since $8 = 4 \times 2$) and 4 is also a factor of 52 (since $52 = 4 \times 13$). When dealing with large numbers as numerators and denominators, it is sometimes helpful to see their *prime factorizations.* We will not emphasis that here; most of the time we will be able to see common factors without much trouble.

Example 1.5.2. *Write the fraction* $\frac{8}{52}$ *in simplest form.*

Answer. *We saw above that the numerator and the denominator have a common factor of* 4*. To write the fraction in simplest form, we will apply the opposite procedure we used above in writing fractions with a common denominator: we will* ***divide both the numerator and denominator by the common factor****.*

$$\frac{8}{52} = \frac{8 \div 4}{52 \div 4} = \frac{2}{13}.$$

Notice that 2 *and* 13 *have no factors in common, so* $\frac{2}{13}$ *is in simplest formand equivalent to the original fraction* $\frac{8}{52}$*.*

The answer is $\frac{2}{13}$*.*

To repeat, multiplying or dividing both the numerator and denominator by the same nonzero number results in an equivalent fraction.

1.5.3 Exercises

Write the following fractions in simplest form.

1. $\frac{6}{8}$

2. $\frac{20}{25}$

3. $\frac{18}{6}$

4. $\frac{118}{177}$

5. $\frac{14}{10}$

1.6 Operations with fractions: Multiplying and dividing

Now that we have reviewed the basic properties of fractions, we will review the rules for performing arithmetic operations using this symbolism. We will begin with multiplying and dividing. That might seem strange if you think that adding and subtracting are "easier" operations to work with, but since fraction notation is based on the operation of division, it should not be too hard to believe that multiplying and diving are much more suited to the notation than adding and subtracting.

1.6.1 Multiplying fractions

Multiplying whole numbers has a clear relationship to the operation of addition. For example, 2×3 means the same as "three added together two times," or $3+3$. However, if we want to extend the operation of multiplication to fractions, negative numbers, and other more exotic numbers, we have to make sure that certain basic properties are preserved, like the *commutative* and *associative* properties (which, in the case of whole number multiplication, are just easy consequences of the corresponding properties for addition). In addition, multiplication and addition must be related by the *distributive* property.

We will not review these properties here. We only mention them to indicate that the rules for multiplying (and dividing) fractions are not arbitrary, but are carefully constructed so that all our basic operations interact in the same way we expect them to do based on our experience with whole numbers.

Multiplying fractions

The product of two fractions is a new fraction whose numerator is the product of the two numerators and whose denominator is the product of the two denominators.

Example 1.6.1. *Multiply:* $\frac{2}{3} \times \frac{4}{5}$.

Answer.

$$\begin{gathered} \frac{2}{3} \times \frac{4}{5} \\ \frac{2 \times 4}{3 \times 5} \\ \frac{8}{15}. \end{gathered}$$

The answer is $\dfrac{8}{15}$.

Note that the answer is in simplest form. That doesn't always happen, as the next example shows.

Example 1.6.2. *Multiply:* $\dfrac{1}{4}\times\dfrac{2}{3}$.

Answer. *The two fractions we begin with are in simplest form. Multiplying,*

$$\begin{array}{rl}
\dfrac{1}{4}\times\dfrac{2}{3} & \\
\dfrac{1\times 2}{4\times 3} & \\
\dfrac{2}{12} & \textit{(Note that the numerator and denominator have a common factor of } 2\textit{)} \\
\dfrac{2^{\div 2}}{12^{\div 2}} & \\
\dfrac{1}{6}. &
\end{array}$$

The answer, in simplest form, is $\dfrac{1}{6}$.

We use one more example as a reminder that whole numbers can be thought of as fractions, most easily as "itself divided by 1".

Example 1.6.3. *Multiply:* $15\times\dfrac{4}{7}$.

Answer.

$$\begin{array}{r}
15\times\dfrac{4}{7} \\
\dfrac{15}{1}\times\dfrac{4}{7} \\
\dfrac{15\times 4}{1\times 7} \\
\dfrac{60}{7}.
\end{array}$$

The final answer, $\frac{60}{7}$*, is in simplest form. According to our convention, we will not bother to change it to a mixed number. (If you wanted to, it would be* $8\ \frac{4}{7}$*.)*

1.6.2 Dividing fractions

To divide fractions, we will take advantage of the fact that division is the "opposite of," or more precisely the *inverse operation* of multiplication. In order to take advantage of this fact, we recall the idea of the *reciprocal* of a number.

Two numbers are *reciprocals* if their product is 1.

In practice, to find the reciprocal of a number written as a fraction, we interchange the numerator and the denominator.

Example 1.6.4. *The reciprocal of* $\frac{2}{3}$ *is* $\frac{3}{2}$.

After all,

$$\frac{2}{3}\cdot\frac{3}{2}=\frac{6}{6}=1.$$

Example 1.6.5. *The reciprocal of* 4 *(*$=\frac{4}{1}$*) is* $\frac{1}{4}$.

After all,

$$\frac{4}{1}\cdot\frac{1}{4}=\frac{4}{4}=1.$$

Dividing fractions

The quotient of two fractions is the same as the product of the first fraction by the reciprocal of the second fraction.

So, for example, we will rewrite $\frac{3}{4}\div\frac{2}{3}$ as $\frac{3}{4}\cdot\frac{3}{2}$.

Example 1.6.6. *Divide:* $\frac{7}{8}\div\frac{3}{4}$.

Answer.

$$\begin{array}{rl}
\frac{7}{8}\div\frac{3}{4} & \\
\frac{7}{8}\cdot\frac{4}{3} & \textit{Rewriting as a product} \\
\frac{7\times 4}{8\times 3} & \\
\frac{28}{24} & \\
\frac{28^{\div 4}}{24^{\div 4}} & \\
\frac{7}{6}. &
\end{array}$$

The answer is $\frac{7}{6}$.

Example 1.6.7. *Divide:* $15\div\frac{2}{3}$.

Answer.

$$\begin{aligned}
&15 \div \frac{2}{3} && \\
&\frac{15}{1} \div \frac{2}{3} && \textit{Rewriting the whole number as a fraction} \\
&\frac{15}{1} \cdot \frac{3}{2} && \textit{Rewriting as a product} \\
&\frac{15 \times 3}{1 \times 2} && \\
&\frac{45}{2}. &&
\end{aligned}$$

The answer is $\frac{45}{2}$.

Example 1.6.8. *Divide:* $\frac{9}{2} \div 15$.

Answer.

$$\begin{aligned}
&\frac{9}{2} \div \frac{15}{1} && \\
&\frac{9}{2} \cdot \frac{1}{15} && \textit{Rewriting as a product} \\
&\frac{9 \times 1}{2 \times 15} && \\
&\frac{9}{30} && \\
&\frac{9^{\div 3}}{30^{\div 3}} && \textit{Reducing} \\
&\frac{3}{10}. &&
\end{aligned}$$

The answer is $\frac{3}{10}$.

Keep in find that division is also be indicated by the fraction bar, as the following example illustrates.

Example 1.6.9. *Divide:* $\frac{\frac{5}{4}}{12}$.

Answer.

$$\begin{aligned}
&\frac{5}{4} \div \frac{12}{1} && \\
&\frac{5}{4} \cdot \frac{1}{12} && \textit{Rewriting as a product} \\
&\frac{5 \times 1}{4 \times 12} && \\
&\frac{5}{48}. &&
\end{aligned}$$

The answer is $\frac{5}{48}$.

1.6.3 Exercises

Perform the indicated operation.

1. $\frac{3}{4} \cdot \frac{2}{5}$

2. $\frac{2}{3} \div \frac{4}{5}$

3. $\frac{17}{2} \cdot \frac{4}{3}$

4. $15 \cdot \frac{1}{5}$

5. $\frac{3}{8} \div 18$

6. $\frac{\frac{7}{12}}{\frac{14}{3}}$

1.7 Operations with fractions: Adding and subtracting

Unlike multiplication and division, addition and subtraction does not follow a rule that sounds like "do the operation to the top and bottom separately." Instead, they follow a rule of a different pattern, one which we will see many times ahead. Adding and subtracting fractions requires that the fractions be "of the same kind." For example, we will have a way to understand adding sixths to sixths, but adding sixths to fifths will require that we express them as fractions "of the same kind.".

Adding and subtracting fractions

To add or subtract two fractions:

1. Write the two fractions, using equivalent fractions if necessary, with a common denominator.
2. Add (or subtract) the numerators, while keeping the same (common) denominator.

Example 1.7.1. *Add:* $\frac{3}{4} + \frac{3}{8}$.

Answer. *The two fractions are not written with a common denominator. The least common denominator is* 8*.*

$$\begin{aligned} &\frac{3}{4} + \frac{3}{8} \\ &\frac{3^{\times 2}}{4^{\times 2}} + \frac{3^{\times 1}}{8^{\times 1}} && \textit{Rewrite using common denominator} \\ &\frac{6}{8} + \frac{3}{8} \\ &\frac{6+3}{8} \\ &\frac{9}{8}. \end{aligned}$$

The answer is $\frac{9}{8}$*.*

Example 1.7.2. *Subtract:* $2 - \frac{4}{7}$.

Answer. *We will rewrite the whole number* 2 *as* $\frac{2}{1}$*. The least common denominator for the two fractions is* 7*.*

$$\begin{aligned}
&2-\frac{4}{7}\\
&\frac{2}{1}-\frac{4}{7} && \textit{Rewrite the whole number as a fraction}\\
&\frac{2^{\times 7}}{1^{\times 7}}-\frac{4^{\times 1}}{7^{\times 1}} && \textit{Rewrite using common denominator}\\
&\frac{14}{7}-\frac{4}{7}\\
&\frac{14-4}{7}\\
&\frac{10}{7}.
\end{aligned}$$

The answer is $\frac{10}{7}$.

Example 1.7.3. *Subtract:* $\frac{5}{6}-\frac{1}{8}$.

Answer. *The least common denominator of the two fractions is* 24.

$$\begin{aligned}
&\frac{5}{6}-\frac{1}{8}\\
&\frac{5^{\times 4}}{6^{\times 4}}-\frac{1^{\times 3}}{8^{\times 3}} && \textit{Rewrite using common denominator}\\
&\frac{20}{24}-\frac{3}{24}\\
&\frac{20-3}{24}\\
&\frac{17}{24}.
\end{aligned}$$

The answer is $\frac{17}{24}$.

1.7.1 Exercises

Perform the indicated operation.

1. $\frac{8}{7}+\frac{4}{5}$

2. $\frac{5}{6}-\frac{2}{3}$

3. $\frac{4}{5}+\frac{3}{10}$

4. $\frac{5}{12} - \frac{1}{3}$

5. $5 - \frac{2}{3}$

6. $1\frac{2}{3} - \frac{3}{4}$.

1.8 Chapter summary

- A fraction is a way of representing the result of dividing two numbers.
- A fraction can be changed to the corresponding decimal representation by performing long division.
- The only time we will use mixed numbers will be when we want to represent a fraction graphically.
- Multiplying or dividing BOTH the numerator AND the denominator of a fraction by the same nonzero number will result in an equivalent fraction.
- Dividing two fractions is performed as multiplication of the first fraction by the reciprocal of the second fraction.
- Adding and subtracting fractions (unlike multiplying and dividing) require that both fractions have the same (common) denominator.
- Final answers involving fractions should always be expressed in simplest form, but may be improper fractions.