Zero-Coupon Bonds (Pure Discount Bonds)

- By Eq. (1) on p. 23, the price of a zero-coupon bond that pays F dollars in n periods is

$$F/(1+r)^n, \tag{9}$$

where r is the interest rate per period.

- Can be used to meet future obligations as there is no reinvestment risk.

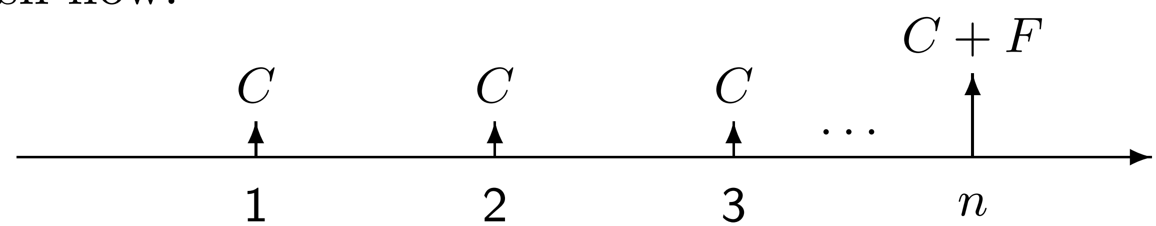
Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value.
- It will be quoted as 20.83.[a]
- If the bond matures in 10 years instead of 20, its price would be 45.64.

[a] Only one fifth of the par value!

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value, and C denotes the coupon.
- Cash flow:



- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.[a]

[a] "You see, Daddy didn't bake the cake, and Daddy isn't the one who gets to eat it. But he gets to slice the cake and hand it out. And when he does, little golden crumbs fall off the cake. And Daddy gets to eat those," wrote Tom Wolfe (1931–) in *Bonfire of the Vanities* (1987).

Pricing Formula

$$
\begin{aligned}
P &= \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} \\
&= C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}. \qquad (10)
\end{aligned}
$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- Note $C = Fc/m$ when c is the annual coupon rate.
- Price P can be computed in $O(1)$ time.

Yields to Maturity

- It is the r that satisfies Eq. (10) on p. 61 with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$\begin{aligned} & 5 \times \frac{1 - [\,1 + (0.15/2)\,]^{-2 \times 10}}{0.15/2} + \frac{100}{[\,1 + (0.15/2)\,]^{2 \times 10}} \\ = \; & 74.5138 \end{aligned}$$

  percent of par.
- So 15% is the yield to maturity if the bond sells for 74.5138.[a]

[a]Note that the coupon rate 10% is less than the yield 15%.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]

[a] CNN, December 21, 2001.

Price Behavior (2)[a]

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate c is above the market interest rate r;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

[a]Consult the text for proofs.

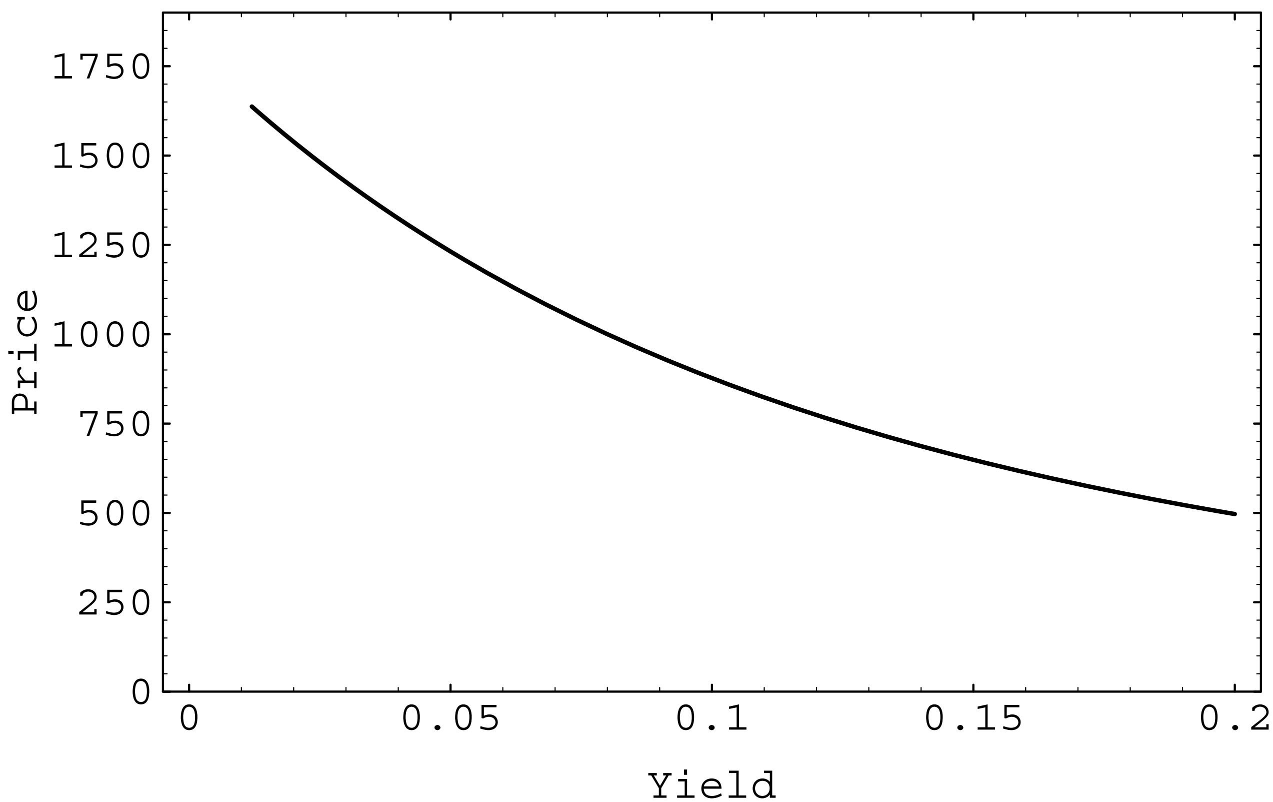
9% Coupon Bond

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

Price Behavior (3): Convexity



Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)$$

- But if d_1 or d_2 is 31, we need to change it to 30 before applying the above formula.

Day Count Conventions: 30/360 (concluded)

- An equivalent formula without any adjustment is (check it)

$$
\begin{aligned}
&360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\
&+ \max(30 - d_1, 0) + \min(d_2, 30).
\end{aligned}
$$

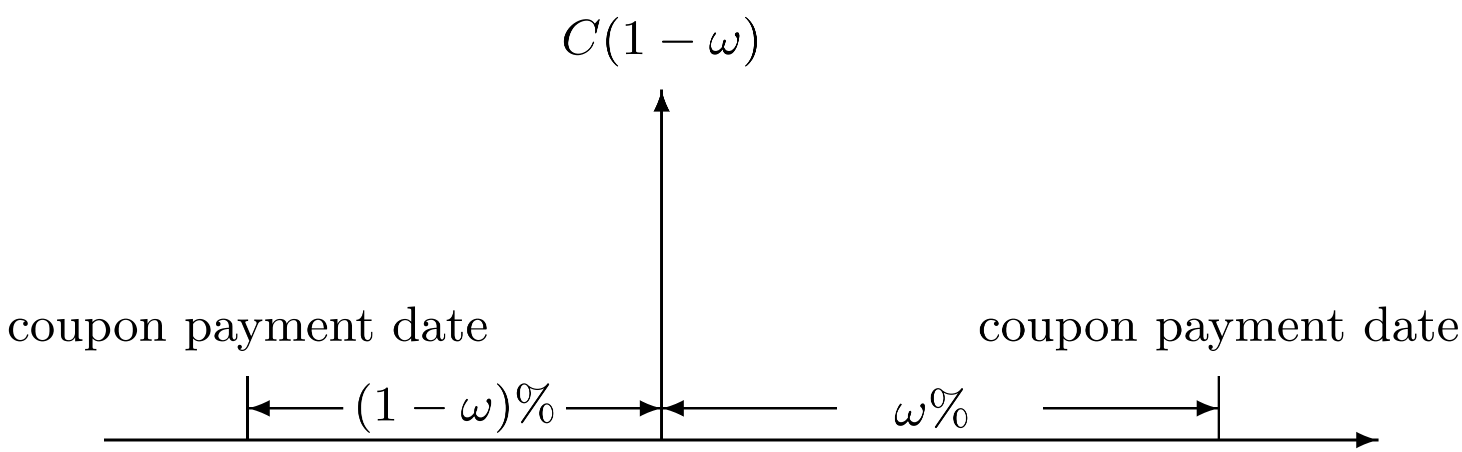
- Many variations regarding 31, Feb 28, and Feb 29.

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \tag{11}$$

Full Price (continued)



Full Price (concluded)

- The price is now calculated by

$$\begin{aligned} \text{PV} &= \frac{C}{\left(1+\frac{r}{m}\right)^{\omega}} + \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+1}} \cdots \\ &= \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \qquad (12) \end{aligned}$$

Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price *plus* the accrued interest (AI).
- The accrued interest equals

$$C \times \frac{\begin{array}{c}\text{number of days from the last}\\ \text{coupon payment to the settlement date}\end{array}}{\text{number of days in the coupon period}} = C \times (1 - \omega).$$

Accrued Interest (concluded)

- The yield to maturity is the r satisfying Eq. (12) on p. 73 when PV is the invoice price:

$$\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}.$$

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Example ("30/360") (concluded)

- The accrued interest is $(10/2) \times (1 - \frac{60}{180}) = 3.3333$ per \$100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (12) on p. 73 with
  - $\omega = 60/180$,
  - $m = 2$,
  - $F = 100$,
  - $C = 5$,
  - PV$= 111.2891 + 3.3333$,
  - $r = 0.03$.

Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Now suppose the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates.
- Then its yield to maturity is less than the coupon rate.
  - The short reason: Exponential growth to C is replaced by linear growth, hence "overpaying."

Bond Price Volatility

“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy

Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.

Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-\frac{\frac{\partial P}{\partial y}}{P}.$$

Price Volatility of Bonds

- The price volatility of a level-coupon bond is

$$-\frac{(C/y)\, n - \left(C/y^2\right)\left((1+y)^{n+1} - (1+y)\right) - nF}{(C/y)\left((1+y)^{n+1} - (1+y)\right) + F(1+y)}.$$

  - F is the par value.
  - C is the coupon payment per period.
  - Formula can be simplified a bit with $C = Fc/m$.

- For bonds without embedded options,

$$-\frac{\frac{\partial P}{\partial y}}{P} > 0.$$

- What is the volatility of the bond in Eq. (12) on p. 73?

Macaulay Duration[a]

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,

$$\mathrm{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} i \, \frac{C_i}{(1+y)^i}.$$

- The Macaulay duration, in periods, is equal to

$$\mathrm{MD} = -(1+y) \, \frac{\partial P}{\partial y} \frac{1}{P}. \tag{13}$$

[a]Macaulay (1938).

MD of Bonds

- The MD of a level-coupon bond is

$$\text{MD} = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \qquad (14)$$

- It can be simplified to

$$\text{MD} = \frac{c(1+y)\left[\,(1+y)^n - 1\,\right] + ny(y-c)}{cy\left[\,(1+y)^n - 1\,\right] + y^2},$$

where c is the period coupon rate.

- The MD of a zero-coupon bond equals n, its term to maturity.
- The MD of a level-coupon bond is less than n.

Remarks

- Equations (13) on p. 84 and (14) on p. 85 hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y.
  - That is, if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- But for securities whose maturity actually decreases as a result, the price volatility (*as originally defined*) may decrease.

How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.

Conversion

- For the MD to be year-based, modify Eq. (14) on p. 85 to
$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\,\frac{C}{\left(1+\frac{y}{k}\right)^{i}}+\frac{n}{k}\,\frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$
where y is the *annual* yield and k is the compounding frequency per annum.

- Equation (13) on p. 84 also becomes
$$\text{MD}=-\left(1+\frac{y}{k}\right)\frac{\partial P}{\partial y}\,\frac{1}{P}.$$

- By definition, MD (in years) $=\dfrac{\text{MD (in periods)}}{k}$.

Modified Duration

- Modified duration is defined as

$$\text{modified duration} \equiv -\frac{\partial P}{\partial y}\frac{1}{P} = \frac{\text{MD}}{(1+y)}. \tag{15}$$

- By the Taylor expansion,

$$\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.$$

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Modified Duration of a Portfolio

- The modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

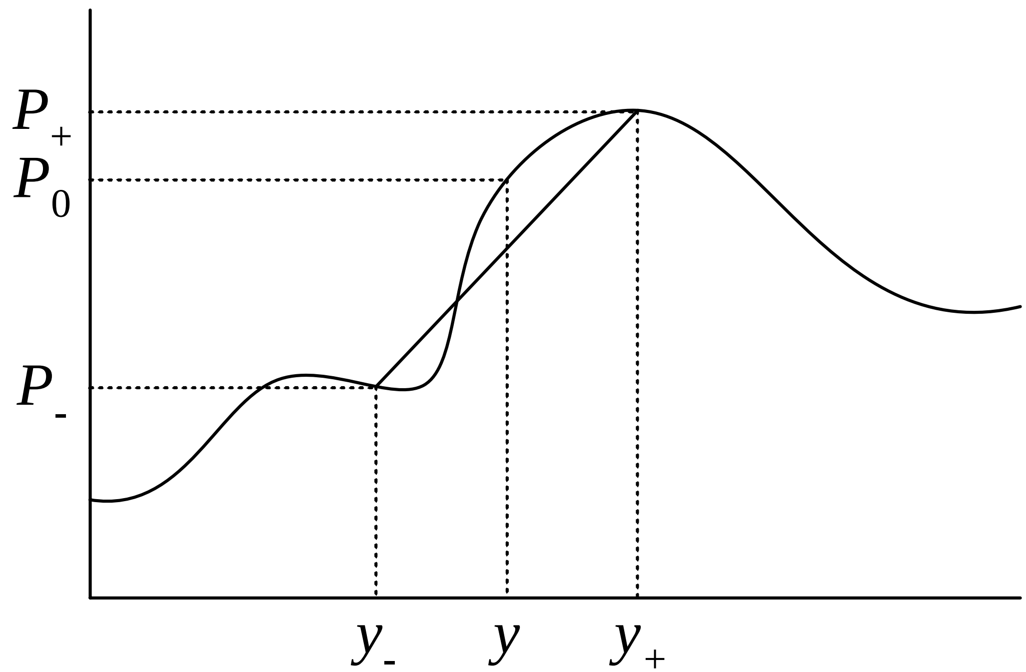
  - D_i is the modified duration of the ith asset.
  - ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}.$$

  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.



P_+
P_0
P_-
y_-
y
y_+

Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use
$$\frac{P_0 - P_+}{P_0 \, \Delta y}.$$
  - More economical but theoretically less accurate.

The Practices

- Duration is usually expressed in percentage terms — call it $D_{\%}$ — for quick mental calculation.[a]
- The percentage price change expressed in percentage terms is then approximated by

$$-D_{\%} \times \Delta r$$

  when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 20% if $D_{\%} = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.
- $D_{\%}$ in fact equals modified duration (prove it!).

[a]Neftci (2008), "Market professionals do not like to use decimal points."

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}.$$

- The approximate *dollar* price change is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

- One can hedge a bond with a dollar duration D by bonds with a dollar duration $-D$.

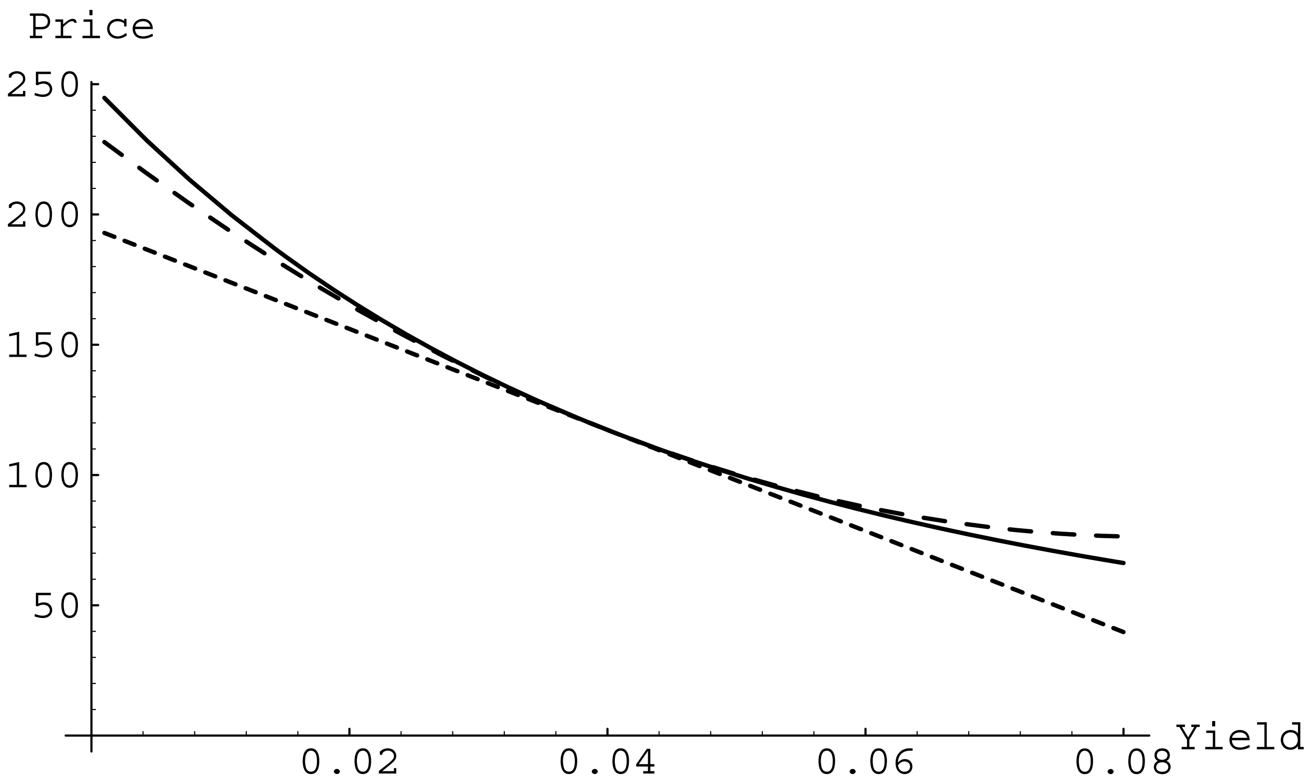
Convexity

- Convexity is defined as

$$\textsf{convexity} \text{ (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.$$

- The convexity of a level-coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).
- So between two bonds with the same price and duration, the one with a higher convexity is more valuable.[a]

[a] Do you spot a problem here (Christensen and Sørensen (1994))?


Price
250
200
150
100
50
Yield
0.02
0.04
0.06
0.08

Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

$$\textsf{convexity (in years)} = \frac{\textsf{convexity (in periods)}}{k^2}$$

when there are k periods per annum.

Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- For larger yield changes, use

$$\begin{aligned}\frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\,\frac{1}{P}\,\Delta y + \frac{1}{2}\,\frac{\partial^2 P}{\partial y^2}\,\frac{1}{P}\,(\Delta y)^2 \\ &= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.\end{aligned}$$

- Recall the figure on p. 98.

The Practices

- Convexity is usually expressed in percentage terms — call it $C_{\%}$ — for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by

$$-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$$

  when the yield increases instantaneously by $\Delta r\%$.

  – Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

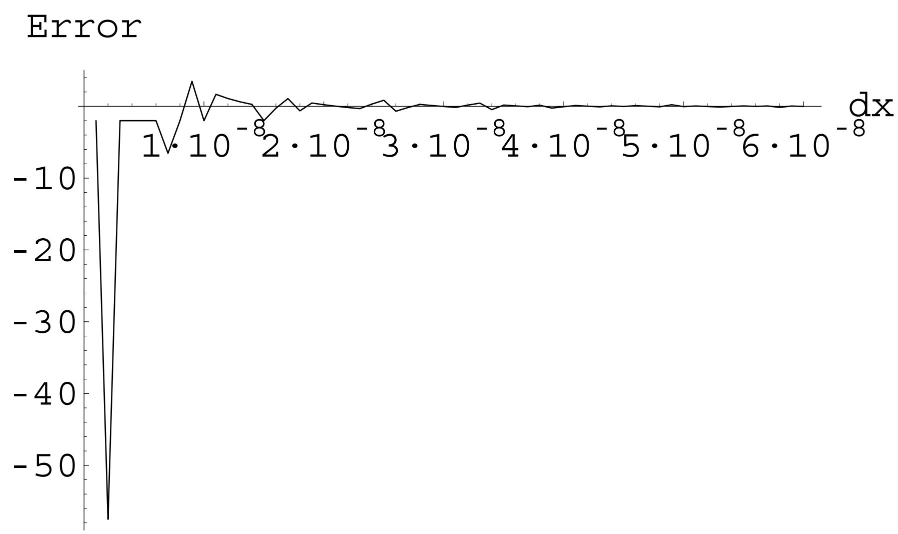
- $C_{\%}$ equals convexity divided by 100 (prove it!).

Effective Convexity

- The effective convexity is defined as
$$\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},$$
  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.
- Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.
- Numerically, choosing the right Δy is a delicate matter.

Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2$ and 2:



- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 777ff).

Interest Rates and Bond Prices: Which Determines Which?[a]

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

---

[a] Contributed by Mr. Wang, Cheng (`R01741064`) on March 5, 2014.

Term Structure of Interest Rates

Why is it that the interest of money is lower,
when money is plentiful?
— Samuel Johnson (1709–1784)

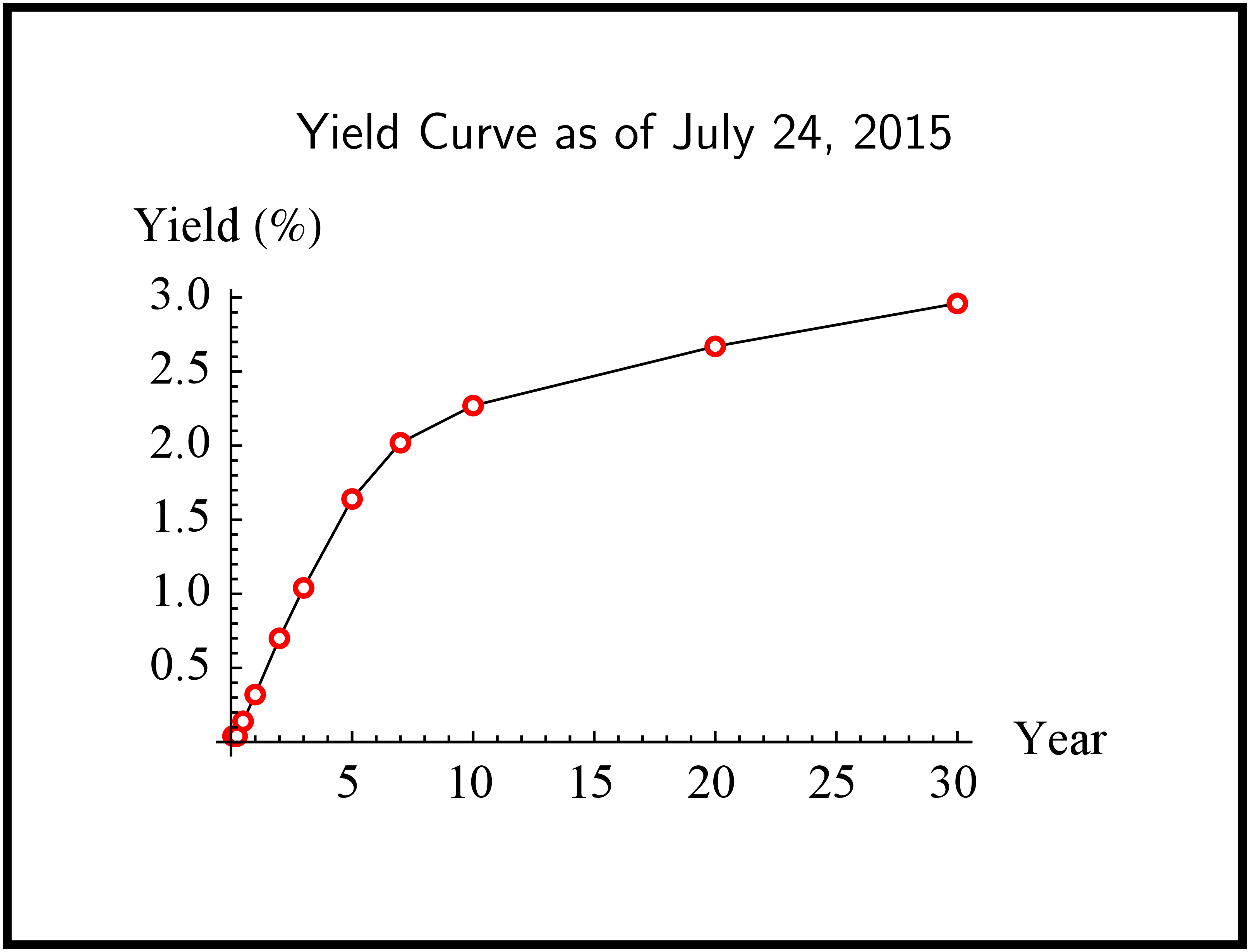
If you have money, don't lend it at interest.
Rather, give [it] to someone
from whom you won't get it back.
— Thomas Gospel 95

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.

Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots the yields to maturity of coupon bonds against maturity.
- A par yield curve is constructed from bonds trading near par.


Yield Curve as of July 24, 2015
Yield (%)
3.0
2.5
2.0
1.5
1.0
0.5
Year
5
10
15
20
25
30

Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is by definition

$$[\, 1 + S(i) \,]^{-i}.$$

  - It is the price of an i-period zero-coupon bond.[a]
- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity:

$$S(1), S(2), \ldots, S(n).$$

---

[a] Recall Eq. (9) on p. 58.

Problems with the PV Formula

- In the bond price formula (3) on p. 32,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield y.

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

$$\sum_{i=1}^{n_1} \frac{C}{(1+y_1)^i} + \frac{F}{(1+y_1)^{n_1}},$$

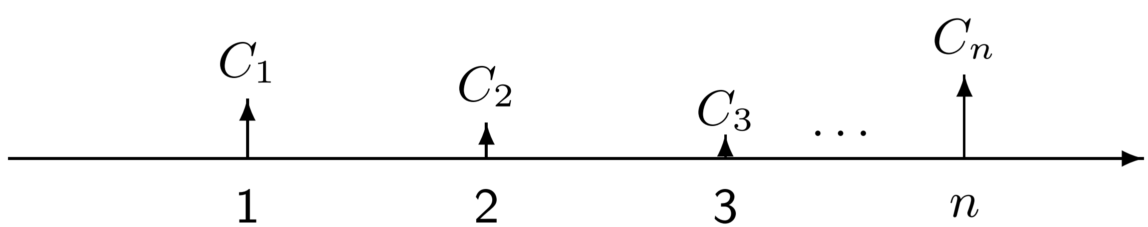
$$\sum_{i=1}^{n_2} \frac{C}{(1+y_2)^i} + \frac{F}{(1+y_2)^{n_2}}.$$

Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with *different* rates.
- But shouldn't they be discounted at the *same* rate?

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.



Spot Rate Discount Methodology (concluded)

- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1+S(i)\,]^{i}} + \frac{F}{[\,1+S(n)\,]^{n}}. \tag{16}$$

- This pricing method incorporates information from the term structure.
- It discounts each cash flow at the corresponding spot rate.

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  - Above, $d(i) \equiv [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.
  - $d(i)$ is the PV of one dollar i periods from now.
  - This formula, now just a definition, will be justified on p. 202.

- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1+S(1)} + \frac{C+100}{[\,1+S(2)\,]^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price P of the n-period coupon bond and

$$S(1), S(2), \ldots, S(n-1).$$

- Then $S(n)$ can be computed from Eq. (16) on p. 115, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}.$$

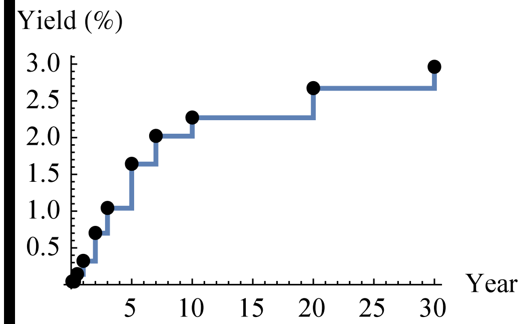
- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.

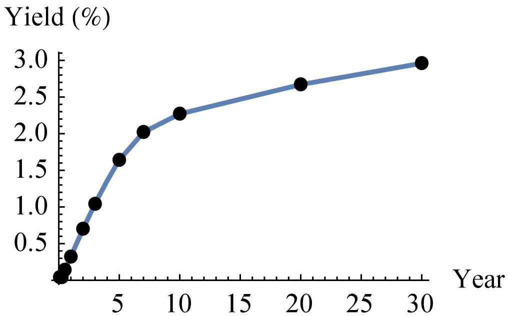
Some Problems

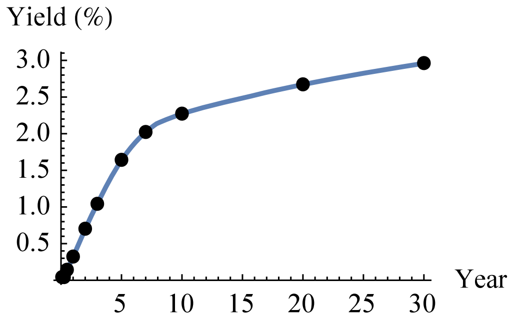
- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.[a]

[a] Any economic justifications?

Which One?


Yield (%)
3.0
2.5
2.0
1.5
1.0
0.5
5
10
15
20
25
30
Year


Yield (%)
3.0
2.5
2.0
1.5
1.0
0.5
5
10
15
20
25
30
Year


Yield (%)
3.0
2.5
2.0
1.5
1.0
0.5
5
10
15
20
25
30
Year

Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.

Static Spread

- Were the risky bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$

- But as risk must be compensated, in reality $P < P^*$.
- The static spread is the amount s by which the spot rate curve has to shift *in parallel* to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

- Unlike the yield spread, the static spread incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

[a] See a counterexample in the text.