Isomorphisms
Math 130 Linear Algebra
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Frequently in mathematics we look at two algebraic structures A and B of the same kind and want to compare them. For instance, we might think they're really the same thing, but they have different names for their elements. That leads to the concept of isomorphism $f : A \stackrel{\simeq}{\to} B$, and we'll talk about that first. Other times we'll know they're not the same thing, but there is a relation between them, and that will lead to the next concept, homomorphism, $f : A \to B$. We'll then look as some special homomorphisms such as monomorphisms. When we have a homomorphism $f : A \to A$, we'll call it an endomorphism, and when an isomorphism $f : A \stackrel{\simeq}{\to} A$, we'll call it an automorphism. We'll take each of these variants in turn.

Injections, surjections, and bijections of functions between sets. These are words that describe certain functions $f : A \to B$ from one set to another.

An *injection*, also called a *one-to-one function* is a function that maps distinct elements to distinct elements, that is, if $x \neq y$, then $f(x) \neq f(y)$. Equivalently, if $f(x) = f(y)$ then $x = y$. If A is a subset of B, then there is a natural injection $\iota : A \to B$, called the *inclusion function*, defined by $\iota(x) = x$.

A *surjection*, also called an *onto function* is one that includes all of B in its image, that is, if $y \in B$, then there is an $x \in A$ such that $f(x) = y$.

A *bijection*, also called a *one-to-one correspondence*, is a function that is simultaneously injective and bijective. Another way to describe a bijection $f : A \to B$ is to say that there is an inverse function $g : B \to A$ so that the composition $g \circ f : A \to A$ is the identity function on A while $f \circ g : B \to B$ is the identity function on B. The usual notation for the function inverse to f is f^{-1}.

If f and g are inverse to each other, that is, if g is the inverse of f, $g = f^{-1}$, then f is the inverse of g, $f = g^{-1}$. Thus, $(f^{-1})^{-1} = f$.

An important property of bijections is that you can convert equations involving f to equations involving f^{-1}:

$$f(x) = y \text{ if and only if } x = f^{-1}(y)$$

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Isomorphisms of algebraic structures. There are lots of different kinds of algebraic structures. We've already studied two of them, namely, fields and vector spaces.

We'll say two algebraic structures A and B are isomorphic if they have exactly the same structure, but their elements may be different. For instance, let A be the vector space $\mathbf{R}[x]$ of polynomials in the variable x, and let B be the vector space $\mathbf{R}[y]$ of polynomials in y. They're both just polynomials in one variable, it's just that the choice of variable is different in the two rings.

We're studying vector spaces, so we need a precise definition of isomorphism for them.

Definition 1 (Isomorphism of vector spaces)**.** Two vector spaces V and W over the same field F are *isomorphic* if there is a bijection $T : V \to W$ which preserves addition and scalar multiplication, that is, for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, and all scalars $c \in F$,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(c\mathbf{v}) = cT(\mathbf{v}).$$

The correspondence T is called an *isomorphism of vector spaces*.

When $T : V \to W$ is an isomorphism we'll write $T : V \stackrel{\simeq}{\to} W$ if we want to emphasize that it is an isomorphism. When V and W are isomorphic, but the specific isomorphism is not named, we'll just write $V \cong W$.

Of course, the identity function $I_V : V \stackrel{\simeq}{\to} V$ is an isomorphism.

After we introduce linear transformations (which is what homomorphisms of vector spaces are called), we'll have another way to describe isomorphisms.

You can prove various properties of vector space isomorphisms from this definition.

Since the structure of vector spaces is defined in terms of addition and scalar multiplication, if T preserves them, it will preserve structure defined in terms of them. For instance, T preserves $\mathbf{0}$, negation, subtraction, and linear transformations.

Theorem 2. If $T : V \stackrel{\simeq}{\to} W$ is an isomorphism of vector spaces, then its inverse $T^{-1} : W \stackrel{\simeq}{\to} V$ is also an isomorphism.

Proof. Since T is a bijection, T^{-1} exists as a function $W \to V$. We have to show T^{-1} preserves addition and scalar multiplication.

First, we'll do addition. Let $\mathbf{w}$ and $\mathbf{x}$ be elements of W. We have to show that

$$T^{-1}(\mathbf{w}+\mathbf{x}) = T^{-1}(\mathbf{w}) + T^{-1}(\mathbf{x}).$$

We'll show that by simplifying it to logically equivalent statements until we reach one which we know is true. Since T and T^{-1} are inverse functions, that equation holds if and only if

$$\mathbf{w}+\mathbf{x} = T(T^{-1}(\mathbf{w}) + T^{-1}(\mathbf{x})).$$

Since T is an isomorphism, we can rewrite that as

$$\mathbf{w}+\mathbf{x} = T(T^{-1}(\mathbf{w})) + T(T^{-1}(\mathbf{x}))$$

which simplifies to $\mathbf{w}+\mathbf{x} = \mathbf{w}+\mathbf{x}$ which is true.

Scalar multiplication is left to you. Show $T^{-1}(c\mathbf{w}) = cT^{-1}(\mathbf{w})$. Q.E.D.

We'll omit the proof of the next theorem.

Theorem 3. If $S : V \stackrel{\simeq}{\to} W$ and $T : W \stackrel{\simeq}{\to} X$ are both isomorphisms of vector spaces, then so is their composition $(T \circ S) : V \stackrel{\simeq}{\to} X$.

Example 4. Consider P_3, the vector space of polynomials over $\mathbf{R}$ of degree 3 or less. Define $T : P_3 \stackrel{\simeq}{\to} \mathbf{R}^4$ by $T(a_1x^3 + a_2x^2 + a_3x + a_4) = (a_1, a_2, a_3, a_4)$. It just associates to a polynomial its 4-tuple of coefficients starting with the coefficient of x^3 and going down in degree. This T preserves addition and scalar multiplication, it is one-to-one, and it is onto. (Those statements are easy to verify.)

This is not the only isomorphism $P_3 \stackrel{\simeq}{\to} \mathbf{R}^4$. A cubic polynomial is determined by its value at any four points. The association $f(x)$ to the 4-tuple $(f(1), f(2), f(3), f(4))$ is also an isomorphism.

Theorem 5. If $T : V \to W$ is an isomorphism, then T carries linearly independent sets to linearly independent sets, spanning sets to spanning sets, and bases to bases.

Proof. For the first statement, let S be a set of linearly independent vectors in V. We'll show that its image $T(S)$ is a set of linearly independent vectors in W. If $\mathbf{0}$ were a nontrivial linear combination of vectors in $T(S)$, then an application of T^{-1} would yield a nontrivial linear combination of vectors in S, but there is none since S is independent. Therefore, $T(S)$ is linear independent.

For the second statement, let $\mathbf{w}$ be any vector in W, then $T^{-1}(\mathbf{w})$ is a linear combination of vectors in V. Apply T to that linear combination to see that $\mathbf{w}$ is a linear combination of vectors in W.

Since T carries both independent and spanning sets from V to W, it carries bases to bases. Q.E.D.

More generally, any property of vector spaces defined in terms of the structure of vector spaces (addition and scalar multiplication) is preserved by isomorphisms.

Coordinates with respect to a basis determine an isomorphism. One of the main uses of a basis $\beta = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ for a vector space V over a field is to impose coordinates on V. Each vector $\mathbf{v}$ in V is a unique linear combination of of the basis vectors

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n.$$

The coefficients are used as coordinates for $\mathbf{v}$ with the respect to the basis β

$$[\mathbf{v}]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Let's denote the function that assigns these coordinates ϕ_β.

Theorem 6. The correspondence $\mathbf{v}$ to $[\mathbf{v}]_\beta$ is an isomorphism $\phi_\beta : V \overset{\simeq}{\to} F^n$.

To prove that theorem, you'll need to note that this is a bijection, prove that $[\mathbf{u}+\mathbf{v}]_\beta = [\mathbf{u}]_\beta + [\mathbf{v}]_\beta$, and prove that $[c\mathbf{v}]_\beta = c[\mathbf{v}]_\beta$.

Since the correspondence ϕ_β is an isomorphism, it means we can work with coordinates with respect to a basis β of V just like ordinary coordinates.

Corollary 7. Two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

Proof. If they're isomorphic, then there's an isomorphism T from one to the other, and it carries a basis of the first to a basis of the second. Therefore they have the same dimension.

On the other hand, if they have the same dimension n, then they're each isomorphic to F^n, and therefore they're isomorphic to each other. Q.E.D.

Linear transformations. Next we'll look at linear transformations of vector spaces.

Whereas isomorphisms are bijections that preserve the algebraic structure, homomorphisms are simply functions that preserve the algebraic structure. In the case of vector spaces, the term linear transformation is used in preference to homomorphism.

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