Problem Set – Week 3 – Class 2 Solutions

ICGN104 Mathematics and Its Contemporary Applications

1. Let $\log 2 = a$, $\log 3 = b$, and $\log 5 = c$. Express the indicated logarithm in terms of a, b, and c.

   (a) $\log 30$

   Solution: We have $\log 30 = \log(2 \cdot 3 \cdot 5) = \log 2 + \log 3 + \log 5 = a + b + c$.

   (b) $\log \frac{2}{3}$

   Solution: We have $\log \frac{2}{3} = \log 2 - \log 3 = a - b$.

   (c) $\log_2 3$

   Solution: We have $\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = \frac{\log 3}{\log 2} = \frac{b}{a}$.

2. Let $\log 2 = a$, $\log 3 = b$, and $\log 5 = c$. Express the indicated logarithm in terms of a, b, and c.

   (a) $\log 1024$

   Solution: We have $\log 1024 = \log 2^{10} = 10 \log 2 = 10a$.

   (b) $\log \frac{5}{2}$

   Solution: We have $\log \frac{5}{2} = \log 5 - \log 2 = c - a$.

   (c) $\log_3 5$

   Solution: We have $\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = \frac{\log 5}{\log 3} = \frac{c}{b}$.

3. Determine the value of the expression without the use of a calculator.

   (a) $\log_{11}(11\sqrt[3]{11})^7$

   Solution: Since $11\sqrt[3]{11} = 11 \cdot 11^{\frac{1}{3}} = 11^{1+\frac{1}{3}} = 11^{\frac{4}{3}}$, we have

   $$\log_{11}(11\sqrt[3]{11})^7 = \log_{11}(11^{\frac{4}{3}})^7 = \log_{11}(11^{\frac{28}{3}}) = \frac{28}{3}\log_{11} 11 = \frac{28}{3}.$$

(b) $\log 0.0000001$

Solution: We have $\log 0.0000001 = \log(10^{-7}) = -7$.

(c) $\ln \dfrac{1}{\sqrt{e}}$

Solution: We have $\ln \dfrac{1}{\sqrt{e}} = \ln(e^{-\frac{1}{2}}) = -\dfrac{1}{2}\ln e = -\dfrac{1}{2}$.

4. Determine the value of the expression without the use of a calculator.

(a) $\log_7 7^{48}$

Solution: We have $\log_7 7^{48} = 48 \log_7 7 = 48$.

(b) $10^{\log 3.4}$

Solution: We have $10^{\log 3.4} = 10^{\log_{10} 3.4} = 3.4$.

(c) $\log \dfrac{1}{10} + \ln e^3$

Solution: We have $\log \dfrac{1}{10} + \ln e^3 = \log(10^{-1}) + \ln e^3 = -1 + 3 = 2$.

5. Write the expression in terms of $\ln x, \ln(x+1)$, and $\ln(x+2)$.

(a) $\ln(x(x+1)^2)$

Solution: We have $\ln(x(x+1)^2) = \ln x + \ln(x+1)^2 = \ln x + 2\ln(x+1)$.

(b) $\ln \dfrac{\sqrt{x}}{(x+1)^2(x+2)^3}$

Solution: We have

$$\begin{aligned}
\ln \frac{\sqrt{x}}{(x+1)^2(x+2)^3} &= \ln(x^{\frac{1}{2}}) - \ln\left[(x+1)^2(x+2)^3\right] \\
&= \frac{1}{2}\ln x - \left[\ln(x+1)^2 + \ln(x+2)^3\right] \\
&= \frac{1}{2}\ln x - \left[2\ln(x+1) + 3\ln(x+2)\right] \\
&= \frac{1}{2}\ln x - 2\ln(x+1) - 3\ln(x+2).
\end{aligned}$$

(c) $\ln\left(\dfrac{1}{x+2}\sqrt[5]{\dfrac{x^2}{x+1}}\right)$

Solution: We have

$$\ln\left(\frac{1}{x+2}\sqrt[5]{\frac{x^2}{x+1}}\right) = \ln\left(\frac{1}{x+2}\left(\frac{x^2}{x+1}\right)^{\frac{1}{5}}\right) = \ln\frac{x^{\frac{2}{5}}}{(x+2)(x+1)^{\frac{1}{5}}}$$
$$= \ln x^{\frac{2}{5}} - \ln\left((x+2)(x+1)^{\frac{1}{5}}\right)$$
$$= \frac{2}{5}\ln x - \ln(x+2) - \frac{1}{5}\ln(x+1).$$

6. Write the expression in terms of $\ln x, \ln(x+1)$, and $\ln(x+2)$.
   (a) $\ln(x(x+1))^3$

   Solution: We have $\ln(x(x+1))^3 = 3\ln(x(x+1)) = 3(\ln x + \ln(x+1))$.

   (b) $\ln\dfrac{x^2(x+1)}{x+2}$

   Solution: We have

$$\ln\frac{x^2(x+1)}{x+2} = \ln(x^2(x+1)) - \ln(x+2) = \ln x^2 + \ln(x+1) - \ln(x+2)$$
$$= 2\ln x + \ln(x+1) - \ln(x+2).$$

   (c) $\ln\sqrt[4]{\dfrac{x^2(x+2)^3}{(x+1)^5}}$

   Solution: We have

$$\ln\sqrt[4]{\frac{x^2(x+2)^3}{(x+1)^5}} = \frac{1}{4}\ln\frac{x^2(x+2)^3}{(x+1)^5} = \frac{1}{4}(\ln x^2 + \ln(x+2)^3 - \ln(x+1)^5)$$
$$= \frac{1}{2}\ln x + \frac{3}{4}\ln(x+2) - \frac{5}{4}\ln(x+1).$$

7. Express each of the given forms as a single logarithm.
   (a) $\log 6 + \log 4$

   Solution: We have $\log 6 + \log 4 = \log(6\cdot 4) = \log 24$.

   (b) $2 + 10\log 1.05$

   Solution: We have $2 + 10\log 1.05 = \log 100 + \log(1.05)^{10} = \log\left(100(1.05)^{10}\right)$.

8. Express each of the given forms as a single logarithm.
   (a) $\log_3 10 - \log_3 5$

Solution: We have $\log_3 10 - \log_3 5 = \log_3 \dfrac{10}{5} = \log_3 2$.

(b) $\dfrac{1}{2}(\log 215 + 8\log 6 - 3\log 169)$

Solution: We have

$$\begin{aligned}\frac{1}{2}(\log 215 + 8\log 6 - 3\log 169) &= \frac{1}{2}(\log 215 + \log 6^8 - \log 169^3 = \frac{1}{2}\log\frac{215(6^8)}{169^3}\\ &= \log\sqrt{\frac{215(6^8)}{169^3}}.\end{aligned}$$

9. Determine the values of the expressions without using a calculator.

(a) $e^{4\ln 3 - 3\ln 4}$

Solution: We have $e^{4\ln 3-3\ln 4} = e^{\ln 3^4 - \ln 4^3} = e^{\ln\left(\frac{3^4}{4^3}\right)} = \dfrac{3^4}{4^3} = \dfrac{81}{64}$.

(b) $\log_3(\ln(\sqrt{7+e^3}+\sqrt{7}) + \ln(\sqrt{7+e^3}-\sqrt{7}))$

Solution: We have

$$\begin{aligned}\ln(\sqrt{7+e^3}+\sqrt{7}) + \ln(\sqrt{7+e^3}-\sqrt{7}) &= \ln\left[(\sqrt{7+e^3}+\sqrt{7})(\sqrt{7+e^3}-\sqrt{7})\right]\\ &= \ln(7+e^3-7) = \ln e^3 = 3.\end{aligned}$$

Hence $\log_3(\ln(\sqrt{7+e^3}+\sqrt{7}) + \ln(\sqrt{7+e^3}-\sqrt{7})) = \log_3 3 = 1$.

10. Determine the values of the expressions without using a calculator.

(a) $\log_6 54 - \log_6 9$

Solution: We have $\log_6 54 - \log_6 9 = \log_6 \dfrac{54}{9} = \log_6 6 = 1$.

(b) $\log_3\sqrt{3} - \log_2\sqrt[3]{2} - \log_5\sqrt[4]{5}$

Solution: We have

$$\begin{aligned}\log_3\sqrt{3} - \log_2\sqrt[3]{2} - \log_5\sqrt[4]{5} &= \frac{1}{2}\log_3 3 - \frac{1}{3}\log_2 2 - \frac{1}{4}\log_5 5\\ &= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = -\frac{1}{12}.\end{aligned}$$

11. Solve the equation.

(a) $4^{\log_4 x + \log_4 2} = 3$

Solution: The equation is equivalent to $4^{\log_4(2x)} = 3$ which implies that $2x = 3$ or $x = \dfrac{3}{2}$.

(b) $e^{3\ln x} = 8$

Solution: The equation is equivalent to $e^{\ln x^3} = 8$ which implies that $x^3 = 8$ or $x = 2$.

12. Solve the equation.

(a) $e^{\ln(2x)} = 5$

Solution: The equation implies that $2x = 5$ or $x = \frac{5}{2}$.

(b) $10^{\log(x^2+2x)} = 3$

Solution: The equation implies that $x^2+2x = 3$ or $x^2+2x-3 = (x+3)(x-1) = 0$ or $x = -3$ or $x = 1$. We check the answer and see that both are the solutions to the equation.

13. Write each expression in terms of natural logarithms.

(a) $\log_2(2x+1)$

Solution: From the change of base formula, we have

$$\log_2(2x+1) = \frac{\log_e(2x+1)}{\log_e 2} = \frac{\ln(2x+1)}{\ln 2}.$$

(b) $\log_3(x^2+2x+2)$

Solution: From the change of base formula, we have

$$\log_3(x^2+2x+2) = \frac{\log_e(x^2+2x+2)}{\log_e 3} = \frac{\ln(x^2+2x+2)}{\ln 3}.$$

14. Write each expression in terms of natural logarithms.

(a) $\log_3(x^2+1)$

Solution: From the change of base formula, we have

$$\log_3(x^2+1) = \frac{\log_e(x^2+1)}{\log_e 3} = \frac{\ln(x^2+1)}{\ln 3}.$$

(b) $\log_7(x^2+1)$

Solution: From the change of base formula, we have

$$\log_7(x^2+1) = \frac{\log_e(x^2+1)}{\log_e 7} = \frac{\ln(x^2+1)}{\ln 7}.$$

15. Find x. Round your answer to three decimal places.

(a) $\log(5x+1) = \log(4x+2)$

Solution: This implies $5x+1 = 4x+2$ or $\boxed{x=1}$.

(b) $2(10)^x + (10)^{x+1} = 4$

Solution: We have $2(10)^x + (10)^x(10) = 4$ or $(10^x)(12) = 4$ or $10^x = \frac{1}{3}$. Hence $\boxed{x = \log\frac{1}{3} \approx -0.477}$.

(c) $\log_2(5x+1) = 4 - \log_2(3x-2)$

Solution: We have $\log_2(5x+1) + \log_2(3x-2) = 4$ or $\log_2[(5x+1)(3x-2)] = 4$ or $(5x+1)(3x-2) = 16$ or $15x^2 - 7x - 18 = 0$. We find $x \approx 1.353$ or $x \approx -0.887$. However, $x \approx 1.353$ is the only value that satisfies the original equation. Hence the solution is $\boxed{x \approx 1.353}$.

16. Find x. Round your answer to three decimal places.

(a) $\log_2 x + 3\log_2 2 = \log_2 \frac{2}{x}$

Solution: We have $\log_2 x + \log_2 2^3 = \log_2 \frac{2}{x}$ or $\log_2(8x) = \log_2 \frac{2}{x}$ which implies that $8x = \frac{2}{x}$ or $x^2 = \frac{1}{4}$. We find $x = \frac{1}{2}$ or $x = -\frac{1}{2}$. However, $x = \frac{1}{2}$ is the only value that satisfies the original equation. Hence the solution is $\boxed{x = \frac{1}{2}}$.

(b) $(4)5^{3-x} - 7 = 2$

Solution: We have $5^{3-x} = \frac{9}{4}$ or $3 - x = \log_5 \frac{9}{4}$ or $x = 3 - \log_5 \frac{9}{4}$ or $\boxed{x \approx 2.496}$.

(c) $\log_2\left(\frac{2}{x}\right) = 3 + \log_2 x$

Solution: We have $\log_2 2 - \log_2 x = 3 + \log_2 x$ or $1 - 2\log_2 x = 3$ or $\log_2 x = -1$, so that $\boxed{x = 2^{-1} = \frac{1}{2}}$.

17. In Springfield the population P grows at the rate of 3% per year. The equation $P = 1,500,000(1.03)^t$ gives the population t years after 2009. Find the value of t for which the population will be 2,000,000. Give your answer to the nearest tenth.

Solution: We want to solve the equation $2,000,000 = 1,500,000(1.03)^t$. This gives $t = \frac{\ln\frac{4}{3}}{\ln 1.03} \approx 9.7$ years.

18. The demand equation for a consumer product is $q = 80 - 2^p$. Solve for p and express your answer in terms of common logarithms. Evaluate p to two decimal places when $q = 60$.

Solution: From $q = 80 - 2^p$, we have $2^p = 80 - q$ or $p = \log_2(80 - q)$. When $q = 60$, this gives $p = \log_2(80 - 60) = \log_2 20 \approx 4.32$.

19. The equation $A = P(1.105)^t$ gives the value A at the end of t years of an investment of P dollars compounded annually at an annual interest rate of 10.5%. How many years will it take for an investment to double? Give your answer to the nearest year.

Solution: The investment doubles when $A = 2P$. We have $2P = P(1.105)^t$ or $2 = (1.105)^t$. Solving for t gives $\ln 2 = \ln(1.105)^t$ or $\ln 2 = t \ln 1.105$ or $t = \dfrac{\ln 2}{\ln 1.105} \approx 7$.

20. Suppose that the daily output of units of a new product on the tth day of a production run is given by

$$q = 100(1 - e^{-0.1t}).$$

Such an equation is called a *learning equation* and indicates that as time progresses, output per day will increase. This may be due to a gain in a worker's proficiency at his or her job. Determine, to the nearest complete unit, the output on (a) the first day and (b) the tenth day after the start of a production run. (c) After how many days will a daily production run of 80 units be reached? Give your answer to the nearest day.

Solution:

(a) If $t = 1$, then $q = 100(1 - e^{-0.1}) \approx 10$.

(b) If $t = 10$, then $q = 100(1 - e^{-1}) \approx 63$.

(c) We solve the equation $80 = 100(1 - e^{-0.1t})$ or $\dfrac{4}{5} = 1 - e^{-0.1t}$ or $e^{-0.1t} = \dfrac{1}{5}$ or $-0.1t = \ln\dfrac{1}{5} = -\ln 5$ or $t = \dfrac{\ln 5}{0.1} \approx 16$.