CHAPTER 10

The Mathematics of Finance

This chapter presents several topics in the mathematics of finance, including compound and simple interest, annuities, and amortization. Computations are carried out in the traditional way, with formulas, and with technology.

10.1 Interest

Compound and Simple Interest

When you deposit money into a savings account, the bank pays you a fee for the use of your money. This fee is called **interest** and is determined by the amount deposited, the duration of the deposit, and the interest rate. The amount deposited is called the **principal** or **present value**, and the amount to which the principal grows (after the addition of interest) is called the **future value** or **balance**.

The entries in a hypothetical bank statement are shown in Table 1. Note the following facts about this statement:

1. The principal is \$100.00. The future value after 1 year is \$104.06.
2. Interest is being paid four times per year (or, in financial language, *quarterly*).
3. Each quarter, the amount of the interest is 1% of the previous balance. That is, \$1.00 is 1% of \$100.00, \$1.01 is 1% of \$101.00, and so on. Since $4 \times 1\%$ is 4%, we say that the money is earning 4% *annual interest compounded quarterly*.

Table 1

Date	Deposits	Withdrawals	Interest	Balance
1/1/16	\$100.00			\$100.00
4/1/16			\$1.00	101.00
7/1/16			1.01	102.01
10/1/16			1.02	103.03
1/1/17			1.03	104.06

As in the statement shown in Table 1, interest rates are usually stated as *annual interest rates*, with the interest to be **compounded** (i.e., computed) a certain number of times per year. Some common frequencies for compounding are listed in Table 2.

Table 2

Number of Interest Periods Per Year	Length of Each Interest Period	Interest Compounded
1	One year	Annually
2	Six months	Semiannually
4	Three months	Quarterly
12	One month	Monthly
52	One week	Weekly
365	One day	Daily

Of special importance is the *interest rate per period*, denoted i, which is calculated by dividing the annual interest rate by the number of interest periods per year. For example, in our statement in Table 1, the annual interest rate is 4%, the interest is compounded quarterly, and the interest rate per period is $4\%/4 = \frac{.04}{4} = .01$.

DEFINITION If interest is compounded m times per year and the annual interest rate is r, then the **interest rate per period** is

$$i = \frac{r}{m}.$$

EXAMPLE 1 **Determining Interest Rate Per Period** Determine the interest rate per period for each of the following annual interest rates.

(a) 3% interest compounded semiannually

(b) 2.4% interest compounded monthly

SOLUTION **(a)** The annual interest rate is 3%, and the number of interest periods is 2. Therefore,

$$i = \frac{3\%}{2} = \frac{.03}{2} = .015.$$

(b) The annual interest rate is 2.4%, and the number of interest periods is 12. Therefore,

$$i = \frac{2.4\%}{12} = \frac{.024}{12} = .002.$$

>> Now Try Exercise 1

Consider a savings account in which the interest rate per period is i. Then the interest earned during a period is i times the previous balance. That is, at the end of an interest period, the new balance, B_{new}, is computed by adding this interest to the previous balance, B_{previous}. Therefore,

$$B_{\text{new}} = 1 \cdot B_{\text{previous}} + i \cdot B_{\text{previous}}$$
$$B_{\text{new}} = (1 + i) \cdot B_{\text{previous}}. \quad (1)$$

Formula (1) says that the balances for successive interest periods are computed by multiplying the previous balance by $(1 + i)$.

EXAMPLE 2 **Computing Interest and Balances** Compute the balance for the first two interest periods for a deposit of \$1000 at 2% interest compounded semiannually.

SOLUTION Here, the initial balance is \$1000 and $i = 1\% = .01$. Let B_1 be the balance at the end of the first interest period and B_2 be the balance at the end of the second interest period. By formula (1),

$$B_1 = (1 + .01)1000 = 1.01 \cdot 1000 = 1010.$$

Similarly, applying formula (1) again, we get

$$B_2 = 1.01 \cdot B_1 = 1.01 \cdot 1010 = 1020.10.$$

Therefore, the balance is \$1010 after the first interest period and \$1020.10 after the second interest period. **>> Now Try Exercises 37(a), (b)**

A simple formula for the balance after any number of interest periods can be derived from formula (1) as follows:

Principal (present value)	P
Balance after 1 interest period	$(1 + i)P$
Balance after 2 interest periods	$(1 + i) \cdot (1 + i)P$ or $(1 + i)^2P$
Balance after 3 interest periods	$(1 + i) \cdot (1 + i)^2P$ or $(1 + i)^3P$
Balance after 4 interest periods	$(1 + i)^4P$
⋮	⋮
Balance after n interest periods	$(1 + i)^nP$.

Future Value Formula for Compound Interest The future value F after n interest periods is

$$F = (1 + i)^nP, \tag{2}$$

where i is the interest rate per period in decimal form, and P is the principal (or present value).

EXAMPLE 3 **Computing Future Values** Apply formula (2) to the savings account statement discussed at the beginning of this section, and calculate the future value after **(a)** 1 year and **(b)** 5 years.

SOLUTION

(a) $F = (1 + i)^nP$ — Future value formula for compound interest

$= (1.01)^4 \cdot 100$ — $n = 1 \cdot 4 = 4$, $i = \frac{.04}{4} = .01$, $P = 100$

$= \$104.06$ — Calculate. Round to nearest cent.

(b) $F = (1 + i)^nP$ — Future value formula for compound interest

$= (1.01)^{20} \cdot 100$ — $n = 5 \cdot 4 = 20$, $i = \frac{.04}{4} = .01$, $P = 100$

$= \$122.02$ — Calculate. Round to nearest cent. **>> Now Try Exercise 13**

Table 3 shows the effects of interest rates (compounded quarterly) on the future value of \$100.

Table 3

Principal = $100.00		
	Future Value	
Interest Rate	5 Years	10 Years
1%	$105.12	$110.50
2%	$110.49	$122.08
3%	$116.12	$134.83
4%	$122.02	$148.89
5%	$128.20	$164.36
6%	$134.69	$181.40
7%	$141.48	$200.16
8%	$148.59	$220.80
9%	$156.05	$243.52
10%	$163.86	$268.51

EXAMPLE 4 **Computing a Present Value** How much money must be deposited now in order to have $1000 after 5 years if interest is paid at a 4% annual interest rate compounded quarterly?

SOLUTION As in Example 3(b), we have $i = .01$ and $n = 20$. However, now we are given F and are asked to solve for P.

$$F = (1 + i)^n P \quad \text{Future value formula for compound interest}$$

$$1000 = (1.01)^{20} P \quad F = 1000,\ i = \tfrac{.04}{4} = .01,\ n = 5 \cdot 4 = 20$$

$$P = \frac{1000}{(1.01)^{20}} \quad \text{Divide both sides by } (1.01)^{20}\text{. Rewrite.}$$

$$P = 819.54 \quad \text{Calculate. Round to two decimal places.}$$

We say that $819.54 is the present value of $1000, 5 years from now, at 4% interest compounded quarterly. The concept of "time value of money" says that, at an interest rate of 4% compounded quarterly, $1000 in 5 years is equivalent to $819.54 now.

» Now Try Exercise 21

Compound interest problems involve the four variables P, i, n, and F. Given the values of any three of the variables, we can find the value of the fourth. As we have seen, the formula used to find the value of F is

$$F = (1 + i)^n P.$$

Solving this formula for P gives the present value formula for compound interest.

Present Value Formula for Compound Interest The present value P of F dollars to be received n interest periods in the future is

$$P = \frac{F}{(1 + i)^n},$$

where i is the interest rate per period in decimal form.

EXAMPLE 5 **Computing a Present Value** Determine the present value of a \$10,000 payment to be received on January 1, 2027, if it is now May 1, 2018, and money can be invested at 3% interest compounded monthly.

SOLUTION Here, $n = 104$ (the number of months between the two given dates).

$$P = \frac{F}{(1+i)^n} \qquad \text{Present value formula for compound interest}$$

$$= \frac{10{,}000}{(1.0025)^{104}} \qquad F = 10{,}000,\ i = \tfrac{.03}{12} = .0025,\ n = 104$$

$$= 7713.02 \qquad \text{Calculate. Round to two decimal places.}$$

Therefore, \$7713.02 invested on May 1, 2018, will grow to \$10,000 by January 1, 2027.

>> Now Try Exercise 19

The interest that we have been discussing so far is the most prevalent type of interest and is known as **compound interest**. There is another type of interest, called **simple interest**, which is used in some financial circumstances.

Interest rates for simple interest are given as an annual interest rate r. Interest is earned *only* on the principal P, and the interest is rP for each year. Therefore, at the end of the year, the new balance, B_{new} is computed by adding this interest to the previous balance, B_{previous}. Therefore,

$$B_{\text{new}} = B_{\text{previous}} + rP$$

This formula says that the balances for successive years are computed by adding rP to the previous balance. Therefore, the interest earned in n years is nrP. So the future value F after n years is the original amount plus the interest earned. That is,

$$F = P + nrP = 1 \cdot P + nrP = (1 + nr)P.$$

Future Value Formula for Simple Interest The future value F after n years is

$$F = (1 + nr)P,$$

where r is the interest rate per year and P is the principal (or present value).

EXAMPLE 6 **Computing a Balance with Simple Interest** Calculate the future value after 4 years if \$1000 is invested at 2% simple interest.

SOLUTION

$$F = (1 + nr)P \qquad \text{Future value formula for simple interest}$$

$$= [1 + 4(.02)]1000 \qquad n = 4,\ r = .02,\ P = 1000$$

$$= (1.08)1000 \qquad \text{Multiply and add.}$$

$$= 1080 \qquad \text{Calculate.}$$

Therefore, the future value is \$1080.00. **>> Now Try Exercise 41**

In Example 6, had the money been invested at 2% compound interest with annual compounding, then the future value would have been \$1082.43. Money invested at simple interest is earning interest only on the principal amount. However, with compound interest, after the first interest period, the interest is also earning interest.

Effective Rate of Interest

The annual rate of interest is also known as the **nominal rate** or the **stated rate**. Its true worth depends on the number of compounding periods. The nominal rate does not help you decide, for instance, whether a savings account paying 3.65% interest compounded quarterly is better than a savings account paying 3.6% interest compounded monthly. The *effective rate of interest* provides a standardized way of comparing investments.

DEFINITION The **effective rate of interest** is the simple interest rate that yields the same amount of interest after one year as the compounded annual (nominal) rate of interest. The effective rate is also known as the *annual percentage yield* (APY) or the *true interest rate*.

Suppose that the annual rate of interest r is compounded m times per year. Then, with compound interest, P dollars will grow to $P(1+i)^m$ in one year, where $i = r/m$. With simple interest r_{eff}, the balance after one year will be $P(1+r_{eff})$. Equating the two balances,

$$P(1+r_{eff}) = P(1+i)^m$$

$$1 + r_{eff} = (1+i)^m \quad \text{Divide both sides by } P.$$

$$r_{eff} = (1+i)^m - 1. \quad \text{Subtract 1 from both sides.} \tag{3}$$

Effective Rate of Interest Formula If interest is compounded m times per year, then

$$r_{eff} = (1+i)^m - 1,$$

where i is the interest rate per period in decimal form.

EXAMPLE 7 **Compare Two Interest Rates** Calculate and compare the effective rate of interest for savings accounts paying

(a) a nominal rate of 3.65% compounded quarterly.
(b) a nominal rate of 3.6% compounded monthly.

SOLUTION

(a)

$$r_{eff} = (1+i)^m - 1 \quad \text{Effective rate of interest formula}$$

$$= (1.009125)^4 - 1 \quad m = 4,\ i = \tfrac{r}{m} = \tfrac{.0365}{4} = .009125$$

$$\approx .0370 \quad \text{Calculate. Round to four decimal places.}$$

(b)

$$r_{eff} = (1+i)^m - 1 \quad \text{Effective rate of interest formula}$$

$$= (1.003)^{12} - 1 \quad m = 12,\ i = \tfrac{r}{m} = \tfrac{.036}{12} = .003$$

$$\approx .0366 \quad \text{Calculate. Round to four decimal places.}$$

Therefore, the first savings account is better than the second. » ***Now Try Exercise 55***

Let us summarize the key formulas developed so far.

Compound Interest

Future value: $F = (1+i)^n P$

Present value: $P = \dfrac{F}{(1+i)^n}$

Effective rate: $r_{eff} = (1+i)^m - 1$

Here, i is the interest rate per period, n is the total number of interest periods, and m is the number of times interest is compounded per year.

Simple Interest

Future value: $F = (1 + nr)P$,

Here, r is the annual interest rate and n is the number of years.

INCORPORATING TECHNOLOGY

Display Balances Graphing calculators can easily display successive balances in a savings account. Consider the situation from Table 1, in which \$100 is deposited at 4% interest compounded quarterly. In Fig. 1, successive balances are displayed in the home screen. In Fig. 2, successive balances are graphed, and in Fig. 3, they are displayed in a table.

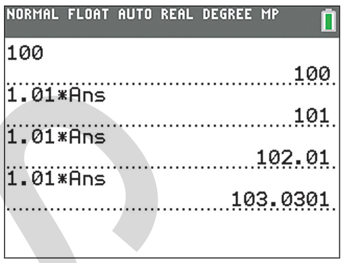

Figure 1

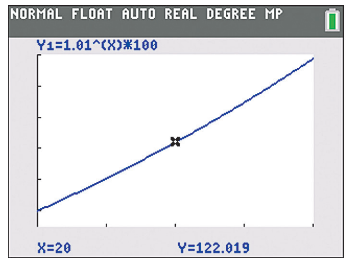

Figure 2

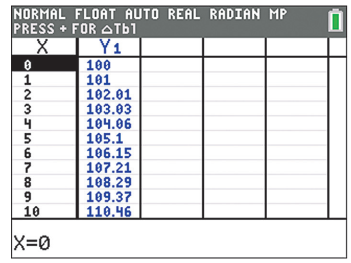

Figure 3

Display Successive Balances on the Home Screen Successive balances can be determined with the relation $B_{\text{new}} = (1 + i)B_{\text{previous}}$. In Fig. 1, after the principal (**100**) is entered, the last displayed value (**100**) is assigned to **Ans**. The instruction **1.01*Ans** generates the next balance (**101**) and assigns it to **Ans**. Each subsequent press of ENTER generates another balance.

Display Balances in a Graph In order to obtain the graph in Fig. 2, $\mathbf{Y_1}$ was set to $\mathbf{1.01^X{*}100}$ in the **Y=** editor—that is, $(1 + i)^X \cdot P$. Here, X, instead of n, is used to represent the number of interest periods. Then the window was set to [0, 40] *by* [90, 150], with an Xscl= 10 and a Yscl= 10. Then, we pressed GRAPH to display the function and pressed TRACE to obtain the trace cursor. *Note:* After $\mathbf{Y_1}$ has been defined, values of $\mathbf{Y_1}$ can be displayed on the home screen by entering expressions such as $\mathbf{Y_1(20)}$.

Display Balances in a Table In order to obtain the table in Fig. 3, $\mathbf{Y_1}$ was set to $\mathbf{1.01^X{*}100}$ in the **Y=** editor. Then, 2nd [TBLSET] was pressed to bring up the TABLE SETUP screen. **TblStart** was set to 0, and **ΔTbl** was set to 1. Finally, 2nd [TABLE] was pressed to bring up the table.

Financial Functions The Finance menu contains the functions Eff and Nom that are used to calculate effective and nominal interest rates. See Appendix B for details.

TVM Solver The details for using this financial calculation tool are presented in Appendix B. In this chapter, TVM Solver screens will be displayed in order to confirm answers to many of the examples. The eight lines of the TVM Solver screen hold the following information.

- **N:** Number of interest periods.
- **I%:** Annual interest rate given as a percent, such as 6 or 4.5. In this chapter, its value is $100r$.
- **PV:** Present value or principal.
- **PMT:** Periodic payment. It has value 0 in Section 10.1, but will have nonzero values in the rest of the chapter.
- **FV:** Future value.
- **P/Y:** Number of payments per year. For our purposes, it is the same as C/Y.
- **C/Y:** Number of times interest is compounded per year. That is, 1, 2, 4, 12, 52, or 365.
- **PMT:** When payments are paid—the *end* or *beginning* of each interest period. For our purposes, it will always be set to END.

A small black square appears to the left of the computed value.

In Figs. 4, 5, and 6, TVM Solver shows the solutions to Examples 3(b), 4, and 5. *Note:* The values for PV are negative, since they represent money paid to the bank.

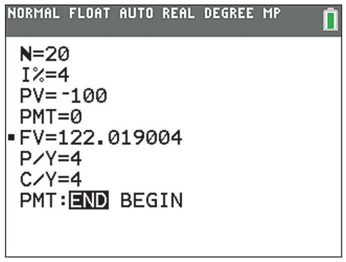

Figure 4 Example 3(b)

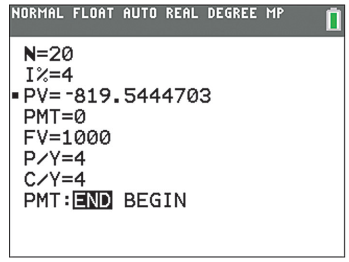

Figure 5 Example 4

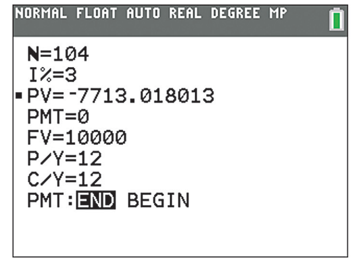

Figure 6 Example 5

Excel Spreadsheet The section "Using Excel's Financial Functions" of Appendix C shows how the functions FV, PV, and EFFECT are used to calculate future values, present values, and effective rates of interest.

WolframAlpha The formulas presented in this section can be evaluated directly with Wolfram|Alpha. In addition, the answers to examples from this section can be obtained with the following instructions:

Example 3(b): **compound interest PV=$100,i=1%,n=20,annual**
Example 4: **compound interest FV=$1000,i=1%,n=20,annual**
Example 6: **simple interest PV=$1000,i=2%,n=4**
Example 7(a): **EffectiveInterestRate with nominal=3.65%,n=4**

Suppose $100 deposited into a savings account grows to $200 after 36 interest periods. Then the compound interest rate per period is given by **interest rate PV=$100,FV=$200,n=36**, and the simple interest rate per period is given by **simple interest rate PV=$100,FV=$200,n=36**.

Also, a table similar to the one in Fig. 3 can be generated with an instruction such as
Table[100*(1.01)^x,{x,0,10,1}].

Check Your Understanding 10.1

Solutions can be found following the section exercises.

1. Calculate the present value of $1000 to be received 10 years in the future at 6% interest compounded annually.
2. Calculate the future value after 2 years of $1 at 26% interest compounded weekly.
3. Calculate the future amount of $2000 after 6 months if invested at 6% simple interest.

EXERCISES 10.1

In Exercises 1–6, give the values of i and n under the given conditions.

1. 3% interest compounded monthly for 2 years
2. 2% interest compounded quarterly for 5 years
3. 2.2% interest compounded semiannually for 20 years
4. 6% interest compounded annually for 3 years
5. 4.5% interest compounded monthly from January 1, 2013, to July 1, 2016
6. 2.4% interest compounded quarterly from January 1, 2013, to October 1, 2016

In Exercises 7–12, give the values of i, n, P, and F.

7. $500 invested at 2.8% interest compounded annually grows to $558.40 in 4 years.
8. $800 invested on January 1, 2011, at 1.8% interest compounded monthly, grows to $957.64 by January 1, 2021.
9. $7174.85 is deposited on January 1, 2011. The balance on July 1, 2020 is $9000, and the interest is 2.4% compounded semiannually.
10. The amount of money that must be deposited now at 2.6% interest compounded weekly in order to have $7500 in 1 year is $7307.56.
11. $3000 deposited at 6% interest compounded monthly will grow to $18,067.73 in 30 years.
12. In 1626, Peter Minuit, the first director-general of New Netherlands province, purchased Manhattan Island for trinkets and cloth valued at about $24. Had this money been invested at 8% interest compounded quarterly, it would have amounted to $677,454,102,888,106 by 2017.

In Exercises 13–38, solve each problem.

13. **Future Value** Calculate the future value of $1000 after 2 years if deposited at 2.1% interest compounded monthly.
14. **Future Value** Calculate the future value of $1000 after 1 year if deposited at 2.19% interest compounded daily.
15. **Future Value** Six thousand dollars is deposited in a savings account at 2.7% interest compounded monthly. Find the balance after 3 years and the amount of interest earned during that time.

16. **Future Value** Two thousand dollars is deposited in a savings account at 4% interest compounded semiannually. Find the balance after 7 years and the amount of interest earned during that time.

17. **Future Value** If you had invested $10,000 on January 1, 2010, at 4% interest compounded quarterly, how much would you have had on January 1, 2016?

18. **Future Value** Ms. Garcia has just invested $100,000 at 2.5% interest compounded annually. How much money will she have in 20 years?

19. **Present Value** Calculate the present value of $100,000 payable in 25 years at 2.4% interest compounded monthly.

20. **Present Value** Calculate the present value of $10,000 payable in 5 years at 2.4% interest compounded semiannually.

21. **Savings Account** Mr. Smith wishes to purchase a $10,000 sailboat upon his retirement in 3 years. He has just won the state lottery and would like to set aside enough cash in a savings account paying 3.4% interest compounded quarterly to buy the boat upon retirement. How much should he deposit?

22. **Investment** In order to have $10,000 on his 25th birthday, how much would a person who just turned 21 have to invest if the money will earn 1.5% interest compounded monthly?

23. **Comparing Payouts** Is it more profitable to receive $1400 now or $1700 in 9 years? Assume that money can earn 2.5% interest compounded annually.

24. **Comparing Payouts** Is it more profitable to receive $7000 now or $10,000 in 9 years? Assume that money can earn 4% interest compounded quarterly.

25. **Interest Rates** In 1999, the NASDAQ Composite Index grew at a rate of 62.2% compounded weekly. Is this rate better or worse than 85% compounded annually?

26. **Interest Rates** Would you rather earn 3% interest compounded annually or 2.92% interest compounded daily?

27. **Interest** If $1000 is deposited into a savings account earning 1.6% interest compounded quarterly, how much interest is earned during the first quarter year? Second quarter year? Third quarter year?

28. **Interest** If $2000 is deposited into a savings account earning 1.2% interest compounded quarterly, how much interest is earned during the first quarter year? Second quarter year? Third quarter year?

29. **Savings Account** If $2000 is deposited into a savings account at 3% interest compounded quarterly, how much interest is earned during the first 2 years? During the third year?

30. **Savings Account** If $5000 is deposited into a savings account at 1.8% interest compounded monthly, how much interest is earned during the first year? During the second year?

31. **Interest** If $1000 is deposited for 5 years in a savings account earning 2.6% interest compounded quarterly, how much interest is earned during the fifth year?

32. **Interest** If $1000 is deposited for 6 years in a savings account earning 3.6% interest compounded monthly, how much interest is earned during the sixth year?

33. **Savings Account** How much money must you deposit at 4% interest compounded quarterly in order to earn $406.04 interest in 1 year?

34. **Savings Account** How much money must you deposit at 2.1% interest compounded monthly in order to earn $347.58 interest in 3 years?

35. **Interest Rate** Clara would like to have $2100 in 7 years to give her granddaughter as a 21st birthday present. She has $1500 to invest in a 7-year certificate of deposit (CD). What rate of interest compounded annually must the CD earn?

36. **Interest Rate** Juan would like to have $2300 in 3 years in order to buy a motorcycle. He has $2000 to invest in a 3-year certificate of deposit (CD). What rate of interest compounded annually must the CD earn?

37. **Savings Account** Consider the following savings account statement:

	1/1/17	2/1/17	3/1/17
Deposit	$10,000.00		
Withdrawal			
Interest		$20.00	$20.04
Balance	$10,000.00	$10,020.00	$10,040.04

(a) What interest rate is this bank paying?
(b) Give the interest and balance on 4/1/17.
(c) Give the interest and balance on 1/1/19.

38. **Savings Account** Consider the following savings account statement:

	1/1/17	4/1/17	7/1/17
Deposit	$10,000.00		
Withdrawal			
Interest		$40.00	$40.16
Balance	$10,000.00	$10,040.00	$10,080.16

(a) What interest rate is this bank paying?
(b) Give the interest and balance on 10/1/17.
(c) Give the interest and balance on 1/1/19.

Exercises 39–52 concern simple interest.

39. **Simple Interest** Determine r, n, P, and F for each of the following situations:
(a) $500 invested at 1.5% simple interest grows to $503.75 in 6 months.
(b) In order to have $525 after 2 years at 2.5% simple interest, $500 must be invested.

40. **Simple Interest** Determine r, n, P, and F for each of the following situations:
(a) At 3% simple interest, $1000 deposited on January 1, 2017, was worth $1015 on June 1, 2017.
(b) At 4% simple interest, in order to have $3600 in 5 years, $3000 must be deposited now.

41. **Future Value** Calculate the future value after 3 years if $1000 is deposited at 1.2% simple interest.

42. **Future Value** Calculate the future value after 18 months if $2000 is deposited at 2% simple interest.

43. **Present Value** Find the present value of $3000 in 10 years at 2% simple interest.

44. **Present Value** Find the present value of $2000 in 8 years at 3.5% simple interest.

45. **Interest Rate** Determine the (simple) interest rate at which \$980 grows to \$1000 in 6 months.

46. **Interest Rate** At what (simple) interest rate will \$1000 grow to \$1200 in 5 years?

47. **Time Period** How many years are required for \$500 to grow to \$800 at 1.5% simple interest?

48. **Time Period** How many years are required for \$1000 to grow to \$1240 at 2.4% simple interest?

49. **Time Interval** Determine the amount of time required for money to double at 2% simple interest.

50. **Interest Rate** Derive the formula for the (simple) interest rate r at which P dollars grow to F dollars in n years. That is, express r in terms of P, F, and n.

51. **Present Value** Derive the formula for the present value P of F dollars in n years at simple interest rate r.

52. **Time Interval** Derive the formula for the number of years n required for P dollars to grow to F dollars at simple interest rate r.

53. **Future Value** Compute the future value after 1 year for \$100 invested at 4% interest compounded quarterly. What simple interest rate will yield the same amount in 1 year?

54. **Future Value** Compute the future value after 1 year for \$100 invested at 2.7% interest compounded monthly. What simple interest rate will yield the same amount in 1 year?

Effective Rate of Interest In Exercises 55–58, calculate the effective rate of interest corresponding to the given nominal rate.

55. 4% interest compounded semiannually

56. 8.45% interest compounded weekly

57. 4.4% interest compounded monthly

58. 7.95% interest compounded quarterly

59. **Savings Account** On January 1, 2014, a deposit was made into a savings account paying interest compounded quarterly. The balance on January 1, 2017 was \$10,000, and the balance on April 1, 2017 was \$10,100. How large was the deposit?

60. **Savings Account** During the 1990s, a deposit was made into a savings account paying 4% interest compounded quarterly. On January 1, 2017, the balance was \$2020. What was the balance on October 1, 2016?

61. **Future Value** Suppose that a principal of \$100 is deposited for 5 years in a savings account at 6% interest compounded semiannually. Which of items (a)–(d) can be used to fill in the blank in the following statement? (*Note:* Before computing, use your intuition to guess the correct answer.)
    If the _______ is doubled, then the future value will double.
    (a) principal
    (b) interest rate
    (c) number of interest periods per year
    (d) number of years

62. **Future Value** Suppose that a principal of \$100 is deposited for 5 years in a savings account at 6% simple interest. Which of items (a)–(c) can be used to fill in the blank in the following statement? (*Note:* Before computing, use your intuition to guess the correct answer.)
    If the _______ is doubled, then the future value will double.
    (a) principal
    (b) interest rate
    (c) number of years

63. **Doubling Time** If a \$1000 investment at compound interest doubles every 6 years, how long will it take the investment to grow to \$8000?

64. **Doubling Time** (True or False) An investment growing at the rate of 12% compounded monthly will double (that is, increase by 100%) in about 70 months. Therefore, the investment should increase by 50% in about 35 months.

65. **Future Value** If the value of an investment grows at the rate of 4% compounded annually for 10 years, then it grows about _______ over the 10-year period.
    (a) 25% **(b)** 40% **(c)** 44% **(d)** 48%

66. **Future Value** If your stock portfolio gained 20% in 2014 and 30% in 2015, then it gained ________ over the two-year period.
    (a) 50% **(b)** 56% **(c)** 60% **(d)** 100%

67. **Comparing Investments** The same amount of money was invested in each of two different investments on January 1, 2015. Investment A increased by 2.5% in 2015, 3% in 2016, and 8.4% in 2017. Investment B increased by the same amount, r%, in each of the 3 years and was worth the same as investment A at the end of the 3-year period. Determine the value of r.

68. **Future Value** If you increase the compound interest rate for an investment by 20%, will the future value increase by 20%?

TECHNOLOGY EXERCISES

In Exercises 69–74, give the settings or statements to determine the solution with TVM Solver, Excel, or Wolfram|Alpha.

69. Exercise 13
70. Exercise 14
71. Exercise 19
72. Exercise 20
73. Exercise 35
74. Exercise 36

75. **Savings Account** One thousand dollars is deposited into a savings account at 2.7% interest compounded annually. How many years are required for the balance to reach \$1946.53? After how many years will the balance exceed \$2500?

76. **Savings Account** Ten thousand dollars is deposited into a savings account at 1.8% interest compounded monthly. How many months are required for the balance to reach \$10,665.74? After how many months will the balance exceed \$11,000?

77. **Investment** Tom invests \$500,000 at 1.9% interest compounded annually. When will Tom be a millionaire?

78. **Savings Account** How many years are required for \$100 to double if deposited at 2.2% interest compounded quarterly?

79. **Comparing Investments** Consider the following two interest options for an investment of \$1000: (A) 4% simple interest, (B) 3% interest compounded annually. After how many years will option B outperform option A?

80. **Comparing Investments** Consider the following two interest options for an investment of \$1000: (A) 4% simple interest, (B) 4% interest compounded daily. After how many years will option B outperform option A by at least \$100?

Solutions to Check Your Understanding 10.1

1. Here, we are given the value in the future F and are asked to find the present value P. Interest compounded annually has just one interest period per year.

$$P = \frac{F}{(1+i)^n} \quad \text{Present value formula for compound interest}$$
$$= \frac{1000}{(1.06)^{10}} \quad F = 1000,\ i = \tfrac{.06}{1} = .06,\ n = 10 \cdot 1 = 10$$
$$= \$558.39 \quad \text{Calculate. Round to nearest cent.}$$

2. Here, we are given the present value, $P = 1$, and are asked to find the value F at a future time. Interest compounded weekly has 52 interest periods each year, so $n = 2 \times 52 = 104$ and $i = 26\%/52 = (1/2)\% = .005$. Using the future value formula, we have

$$F = (1+i)^n P = (1.005)^{104} \cdot 1 = \$1.68.$$

3. In simple interest problems, time should be expressed in terms of years. Therefore, 6 months is 1/2 of a year.

$$F = (1 + nr)P \quad \text{Future value formula for simple interest}$$
$$= [1 + \tfrac{1}{2}(.06)]2000 \quad n = \tfrac{1}{2},\ r = .06,\ P = 2000$$
$$= (1.03)2000 \quad \text{Multiply and add.}$$
$$= \$2060 \quad \text{Calculate.}$$

10.2 Annuities

An **annuity** is a sequence of equal payments made at regular intervals of time. Here are two illustrations.

1. As the proud parent of a newborn daughter, you decide to save for her college education by depositing \$200 at the end of each month into a savings account paying 2.7% interest compounded monthly. Eighteen years from now, after you make the last of 216 payments, the account will contain \$55,547.79.
2. Having just won the state lottery, you decide not to work for the next 5 years. You want to deposit enough money into the bank so that you can withdraw \$5000 at the end of each month for 60 months. If the bank pays 2.1% interest compounded monthly, you must deposit \$284,551.01.

The periodic payments in the foregoing financial transactions are called **rent**. The amount of a rent payment is denoted by the letter R. Thus, in the preceding examples, we have $R = \$200$ and $R = \$5000$, respectively.

In illustration 1, you make equal payments to a bank in order to generate a large sum of money in the future. This sum, namely \$55,547.79, is called the **future value of the annuity**. Since the amount of money in the savings account increases each time a payment is made, this type of annuity is an **increasing annuity.**

In illustration 2, the bank will make equal payments to you in order to pay back (with interest) the sum of money that you currently deposit. The value of the current deposit, namely, \$284,551.01, is called the **present value of the annuity**. Since the amount of money in the savings account decreases each time a payment is made, this type of annuity is a **decreasing annuity**.

Increasing Annuities

DEFINITION The **future value of an increasing annuity** of n equal payments is the value of the annuity after the nth payment.

Consider an annuity consisting of n payments of R dollars, each deposited into an account paying compound interest at the rate of i per interest period, and suppose that each payment is made at the end of an interest period. Let us derive a formula for the future value of the annuity—that is, a formula for the balance of the account immediately after the last payment.

Each payment accumulates interest for a different number of interest periods, so let us calculate the balance in the account as the sum of n future values, one corresponding to each payment (Table 1). Denote the future value of the annuity by F.

Table 1 Increasing Annuity

Payment	Amount	Number of Interest Periods on Deposit	Future Value of Payment
1	R	$n-1$	$(1+i)^{n-1}R$
2	R	$n-2$	$(1+i)^{n-2}R$
⋮	⋮	⋮	⋮
$n-2$	R	2	$(1+i)^2R$
$n-1$	R	1	$(1+i)R$
n	R	0	R

Then F is the sum of the numbers in the right-hand column:

$$F = R + (1+i)R + (1+i)^2R + (1+i)^3R + \cdots + (1+i)^{n-1}R$$

A compact expression for F can be obtained by multiplying both sides of this equation by $(1+i)$ and then subtracting the original equation from the new equation:

$$\begin{aligned}(1+i)F &= \phantom{R+{}}(1+i)R + (1+i)^2R + (1+i)^3R + \cdots + (1+i)^{n-1}R + (1+i)^nR\\ F &= R + (1+i)R + (1+i)^2R + (1+i)^3R + \cdots + (1+i)^{n-1}R\\ \hline iF &= (1+i)^nR - R\end{aligned}$$

The last equation can be written $iF = [(1+i)^n - 1]\cdot R$. Dividing both sides by i yields

$$F = \frac{(1+i)^n - 1}{i}\cdot R.$$

Solving for R yields

$$R = \frac{i}{(1+i)^n - 1}\cdot F.$$

Formulas for an Increasing Annuity The future value F and the rent R of an increasing annuity of n payments with interest compounded at the rate i per interest period are related by the formulas

$$F = \frac{(1+i)^n - 1}{i}\cdot R \quad \text{and} \quad R = \frac{i}{(1+i)^n - 1}\cdot F.$$

EXAMPLE 1 **Calculating the Future Value of an Increasing Annuity** Calculate the future value of an increasing annuity of \$100 per month for 5 years at 3% interest compounded monthly.

SOLUTION

$$F = \frac{(1+i)^n - 1}{i}\cdot R \qquad \text{Formula for future value}$$

$$= \frac{1.0025^{60} - 1}{.0025}\cdot 100 \qquad i = \frac{.03}{12} = .0025,\ n = 5\cdot 12 = 60,\ R = 100$$

$$= 6464.67 \qquad \text{Calculate. Round to two decimal places.}$$

The future value is \$6464.67.

» Now Try Exercise 5

We can derive a formula that illustrates exactly how the future value of an increasing annuity changes from period to period. Each new balance can be computed from the previous balance with the formula

$$B_{\text{new}} = (1 + i)B_{\text{previous}} + R.$$

That is, the new balance equals the growth of the previous balance due to interest, plus the amount paid. So, for instance, if B_1 is the balance after the first payment is made, B_2 is the balance after the second payment is made, and so on, then

$$\begin{aligned} B_1 &= R \\ B_2 &= (1 + i)B_1 + R \\ B_3 &= (1 + i)B_2 + R \\ &\vdots \end{aligned}$$

Successive balances are computed by multiplying the previous balance by $(1 + i)$ and adding R.

Successive Balances If R dollars is deposited into an increasing annuity at the end of each interest period with interest rate i per period, then

$$B_{\text{new}} = (1 + i)B_{\text{previous}} + R \tag{1}$$

can be used to calculate each new balance from the previous balance.

EXAMPLE 2 **Calculating the Future Value of an Increasing Annuity** Consider the annuity of Example 1. Determine the future value after 61 months.

SOLUTION From Example 1, $i = .0025$, the balance after 60 months is \$6464.67, and $R = 100$. Therefore, by (1)

$$\begin{aligned} [\text{balance after 61 months}] &= (1 + .0025)[\text{balance after 60 months}] + 100 \\ &= 1.0025(6464.67) + 100 \\ &= 6580.83. \end{aligned}$$

Therefore, the annuity will have accumulated to \$6580.83 after 61 months. «

EXAMPLE 3 **Determining the Rent for an Increasing Annuity** Ms. Adams would like to buy a \$300,000 airplane when she retires in 8 years. How much should she deposit at the end of each half-year into an account paying 4% interest compounded semiannually so that she will have enough money to purchase the airplane?

SOLUTION

$$R = \frac{i}{(1 + i)^n - 1} \cdot F \qquad \text{Formula for rent}$$

$$= \frac{.02}{1.02^{16} - 1} \cdot 300{,}000 \qquad i = \frac{.04}{2} = .02,\ n = 8 \cdot 2 = 16,\ F = 300{,}000$$

$$= 16{,}095.04 \qquad \text{Calculate. Round to two decimal places.}$$

She should deposit \$16,095.04 at the end of each half-year period. «

Decreasing Annuities

DEFINITION The **present value of a decreasing annuity** is the amount of money necessary to finance the sequence of annuity payments. That is, the present value of the annuity is the amount you would need to deposit in order to provide the desired sequence of annuity payments and leave a balance of zero at the end of the term.

To find a formula for the present value P of an annuity, consider an annuity consisting of n payments of R dollars made at the end of each interest period, with interest compounded at a rate i per interest period.

Situation 1: Suppose that the P dollars were just left in the account and that the annuity payments were not withdrawn. At the end of the n interest periods, there would be $(1 + i)^n P$ dollars in the account.

Situation 2: Suppose that the payments are withdrawn but are immediately redeposited into another account having the same rate of interest. At the end of the n interest periods, there would be $\frac{(1+i)^n-1}{i} \cdot R$ dollars in the new account.

In both of these situations, P dollars is deposited and it, together with all of the interest generated, is earning income at the same interest rate for the same amount of time. Therefore, the final amounts of money in the accounts should be the same. That is,

$$(1 + i)^n P = \frac{(1 + i)^n - 1}{i} \cdot R$$

$$P = (1 + i)^{-n} \cdot \frac{(1 + i)^n - 1}{i} \cdot R \quad \text{Multiply both sides by } (1 + i)^{-n}.$$

$$= \frac{1 - (1 + i)^{-n}}{i} \cdot R. \quad \text{Multiply numerator of fraction by } (1 + i)^{-n}.$$

Solving for R yields

$$R = \frac{i}{1 - (1 + i)^{-n}} \cdot P.$$

Formulas for a Decreasing Annuity The present value P and the rent R of a decreasing annuity of n payments with interest compounded at the rate i per interest period are related by the formulas

$$P = \frac{1 - (1 + i)^{-n}}{i} \cdot R \quad \text{and} \quad R = \frac{i}{1 - (1 + i)^{-n}} \cdot P.$$

EXAMPLE 4 **Determining the Present Value for a Decreasing Annuity** How much money must you deposit now at 6% interest compounded quarterly in order to be able to withdraw \$3000 at the end of each quarter year for 2 years?

SOLUTION We are asked to calculate the present value of the sequence of payments.

$$P = \frac{1 - (1 + i)^{-n}}{i} \cdot R \quad \text{Formula for present value}$$

$$= \frac{1 - 1.015^{-8}}{.015} \cdot 3000 \quad i = \frac{.06}{4} = .015,\ n = 2 \cdot 4 = 8,\ R = 3000$$

$$= 22{,}457.78 \quad \text{Calculate. Round to two decimal places.}$$

The present value is \$22,457.78.

>> Now Try Exercise 25

EXAMPLE 5 **Determining the Rent for a Decreasing Annuity** If you deposit \$10,000 into a fund paying 6% interest compounded monthly, how much can you withdraw at the end of each month for 1 year?

SOLUTION We are asked to calculate the rent for the sequence of payments.

$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \qquad \text{Formula for rent}$$

$$= \frac{.005}{1-1.005^{-12}} \cdot 10{,}000 \qquad i = \frac{.06}{12} = .005,\ n = 1 \cdot 12 = 12,\ P = 10{,}000$$

$$= 860.66 \qquad \text{Calculate. Round to two decimal places.}$$

The rent is $860.66.

» Now Try Exercise 11

NOTE In this section, we have considered only annuities with payments made at the end of each interest period. Such annuities are called *ordinary annuities*. Annuities that have payments at the beginning of the interest period are called *annuities due*. Annuities whose payment period is different from the interest period are called *general annuities*. «

Two key formulas in this section are as follows:

1. Increasing Annuity: $F = \frac{(1+i)^n - 1}{i} \cdot R$
2. Decreasing Annuity: $P = \frac{1-(1+i)^{-n}}{i} \cdot R$

The following tips help us recall these formulas:

- Both formulas have i in the denominator and R on the right side.
- Since an increasing annuity builds up a future value, formula 1 involves F.
- Since a decreasing annuity depletes a present value, formula 2 involves P.
- Associate *increasing* with *positive* and *decreasing* with *negative*. Formula 1 has a positive exponent, and formula 2 has a negative exponent.

NOTE The following two formulas, obtained by solving formulas 1 and 2 for R, are frequently used in annuity computations.

3. Increasing Annuity: $R = \frac{i}{(1+i)^n - 1} \cdot F$
4. Decreasing Annuity: $R = \frac{i}{1-(1+i)^{-n}} \cdot P$ «

EXAMPLE 6 **Determining the Time for a Future Value to Exceed a Specified Amount** Consider the situation in which $100 is deposited monthly at 6% interest compounded monthly. Use technology to determine when the balance will exceed $10,000.

SOLUTION There are three ways to answer this question with a graphing calculator—with a graph, with a table, and with TVM Solver.

1. (Graph) Set $\mathbf{Y_2=10000}$, and find its intersection with the graph of

$$\mathbf{Y_1} = \frac{\mathbf{1.005^X - 1}}{\mathbf{.005}} * \mathbf{100}.$$

See Fig. 1. (The window was set to [0, 120] *by* [−4000, 17000] with an X-scale of 10 and a Y-scale of 1000.)
2. (Table) Scroll down the table for $\mathbf{Y_1}$ until the balance exceeds 10,000. See Fig. 2.
3. (TVM Solver) See Fig. 3. *Note:* The value for PMT is negative, since it represents money paid to the bank.

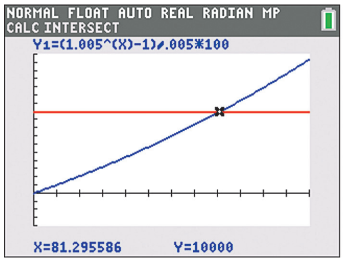

Figure 1

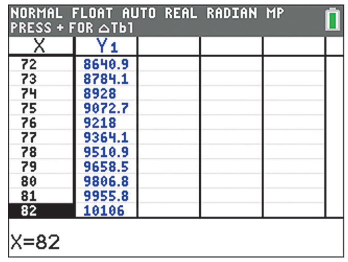

Figure 2

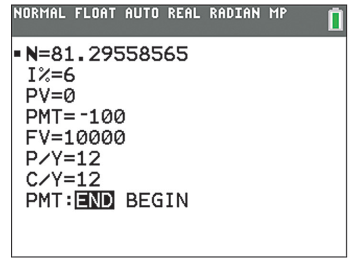

Figure 3

Entering =**NPER(0.005,−100,0,10000)** into a cell produces 81.29558565. *Note:* The payment is negative, since it represents money paid to the bank.

WolframAlpha The instruction **annuityperiods,i = .5%,pmt = $100,FV = $10000** produces "number of periods is 81.3."

In each case, we see that the balance will exceed $10,000 after 82 months—that is, after 6 years and 10 months. «

INCORPORATING TECHNOLOGY

Display Successive Balances Graphing calculators can easily display successive balances in an annuity. (We limit our discussion here to increasing annuities. The corresponding analysis for decreasing annuities is similar.) Consider the situation in which $100 is deposited at the end of each month at 6% interest compounded monthly. In Fig. 4, successive balances are displayed in the home screen. In Fig. 5, successive balances are graphed, and in Fig. 6, they are displayed in a table.

These figures were obtained with the same processes as Figs. 1–3 in Section 10.1. Successive balances were determined with the relation $B_{\text{new}} = (1 + i)B_{\text{previous}} + R$. The function to be graphed was specified as

$$\mathtt{Y}_1 = \frac{1.005^{\mathtt{X}} - 1}{.005} * 100.$$

The window was set to [0, 120] *by* [−4000, 17000] with an X-scale of 10 and a Y-scale of 1000. In the TABLE SETUP menu, both **TblStart** and **ΔTbl** were set to 1.

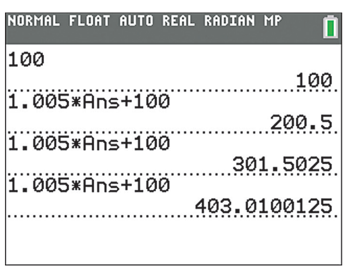

Figure 4

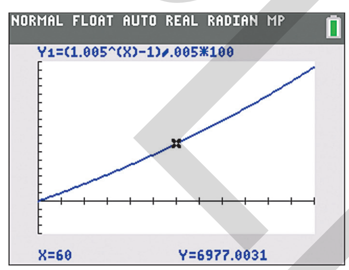

Figure 5

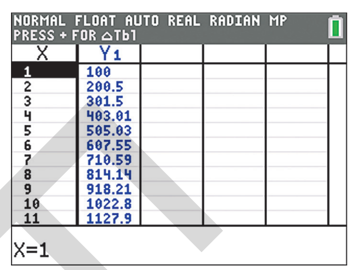

Figure 6

TVM Solver In Figs. 7, 8, and 9, TVM Solver is used to solve Examples 1, 3, and 4. PMT, the monthly payment, is the value we have been calling *rent*. The value of PV for an increasing annuity is 0 since initially there is no money in the bank. The value for PV for a decreasing annuity is negative, since it represents money paid to the bank.

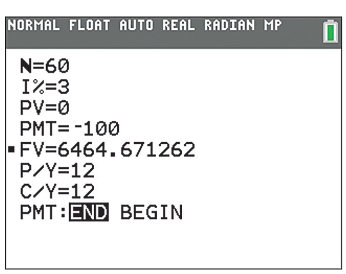

Figure 7 Example 1

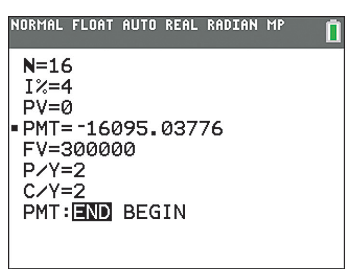

Figure 8 Example 3

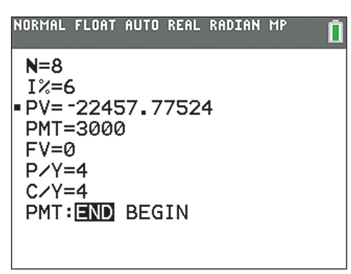

Figure 9 Example 4

Excel Spreadsheet The "Using Excel's Financial Functions" section in Appendix C shows how the functions FV, PV, PMT, and NPER are used to calculate future values (F), present values (P), monthly payments (R), and number of payments (n) for annuities.

WolframAlpha A graph and the first 10 balances for an increasing annuity can be generated with an instruction of the form **B(n) = (1 + i)B(n−1) + R,B(0) = 0**. For instance, the graph and table in Figs. 5 and 6 can be produced with the instruction **B(n) = 1.005B(n−1) + 100,B(0) = 0**. The corresponding instruction for a decreasing annuity is **B(n) = (1 + i)B(n−1)−R,B(0) = P**.

The formulas presented in this section can be evaluated directly. In addition, the answers to the examples from this section can be obtained with the instructions that follow. (*Note:* The answers will sometimes be rounded.)

Example 1: **annuity FV,i = .25%,n = 60,pmt = $100**
Example 2: **annuity FV,i = .25%,n = 61,pmt = $100**
Example 3: **annuity pmt,FV = $300000,i = 2%,n = 16**
Example 4: **annuity PV,i = 1.5%,n = 8,pmt = $3000**
Example 5: **annuity pmt,PV = $10000,i = .5%,n = 12**

Check Your Understanding 10.2

Solutions can be found following the section exercises.

Decide whether or not each of the given annuities is an ordinary annuity—that is, the type of annuity considered in this section. If so, identify n, i, and R, and calculate the present value or the future value, whichever is appropriate.

1. You make a deposit at 9% interest compounded monthly into a fund that pays you $1 at the end of each month for 5 years.
2. At the end of each week for 2 years, you deposit $10 into a savings account earning 6% interest compounded monthly.
3. At the end of each month for 2 years, you deposit $10 into a savings account earning 6% interest compounded monthly.

EXERCISES 10.2

Exercises 1 and 2 describe increasing annuities. Specify i, n, R, and F.

1. If, at the end of each month, $100 is deposited into a savings account paying 2.1% interest compounded monthly, the balance after 10 years will be $13,340.09.
2. Mr. Smith is saving to buy a $200,000 yacht in January 2020. Since January 2010, he has been depositing $8231.34 at the end of each half year into a fund paying 4% interest compounded semiannually.

Exercises 3 and 4 describe decreasing annuities. Specify i, n, R, and P.

3. In order to receive $2000 at the end of each quarter year beginning in 2015 until the end of 2019, Ms. Williams deposited $36,642.08 into an investment paying 3.4% interest compounded quarterly.
4. A retiree deposits $185,288.07 into an investment paying 2.7% interest compounded monthly and withdraws $1000 at the end of each month for 20 years.

In Exercises 5 and 6, calculate the future value of the increasing annuity.

5. At the end of each half year, for 5 years, $1500 is deposited into an investment paying 2.6% interest compounded semiannually.
6. At the end of each quarter year, for 6 years, $1200 is deposited into an investment paying 3.4% interest compounded quarterly.

In Exercises 7 and 8, calculate the rent of the increasing annuity.

7. A deposit is made at the end of each month into a savings account paying 1.8% interest compounded monthly. The balance after 1 year is $1681.83.
8. Money is deposited at the end of each week into an investment paying 2.6% interest compounded weekly. The balance after 3 years is $16,382.52.

In Exercises 9 and 10, calculate the present value of the decreasing annuity.

9. At the end of each month, for two years, $3000 will be withdrawn from a savings account paying 1.5% interest compounded monthly.
10. At the end of each month, for one year, $1000 will be withdrawn from a savings account paying 1.2% interest compounded monthly.

In Exercises 11 and 12, calculate the rent of the decreasing annuity.

11. A withdrawal is made at the end of each quarter year for 3 years from a savings account paying 1.6% interest compounded quarterly. The account initially contained $47,336.25.
12. A withdrawal is made at the end of each half year for 5 years from a savings account paying 1.8% interest compounded semiannually. The account initially contained $57,781.39.

13. **Savings Account** Ethan deposits $500 into a savings account at the end of every month for 4 years at 3% interest compounded monthly.
    - **(a)** Find the balance at the end of 4 years.
    - **(b)** How much money did Ethan deposit into the account?
    - **(c)** How much interest did Ethan earn during the 4 years?
    - **(d)** Prepare a table showing the balance and interest for the first 3 months.

14. **Savings Account** Emma deposits $2000 into a savings account at the end of every quarter year for 5 years at 3% interest compounded quarterly.
    - **(a)** Find the balance at the end of 5 years.
    - **(b)** How much money did Emma deposit into the account?
    - **(c)** How much interest did Emma earn during the 5 years?
    - **(d)** Prepare a table showing the balance and interest for the first 3 quarters.

15. **Savings Account** A person deposits $10,000 into a savings account at 2.2% interest compounded quarterly and then withdraws $1000 at the end of each quarter year. Prepare a table showing the balance and interest for the first 3 quarters.

16. **Savings Account** A person deposits $5000 into a savings account at 2.4% interest compounded monthly and then withdraws $300 at the end of each month. Prepare a table showing the balance and interest for the first 4 months.

In Exercises 17–20, determine the amount of interest earned by the specified annuity.

17. The increasing annuity in Exercise 1
18. The increasing annuity in Exercise 2
19. The decreasing annuity in Exercise 3
20. The decreasing annuity in Exercise 4

21. **Comparing Payouts** Is it more profitable to receive $1000 at the end of each month for 10 years or to receive a lump sum of $140,000 at the end of 10 years? Assume that money can earn 3% interest compounded monthly.

22. **Comparing Payouts** Is it more profitable to receive a lump sum of $9,000 at the end of 3 years or to receive $750 at the end of each quarter-year for 3 years? Assume that money can earn 2.2% interest compounded quarterly.

23. **Comparing Bonus Plans** When Bridget takes a new job, she is offered the choice of a $2300 bonus now or an extra $200 at the end of each month for the next year. Assume money can earn an interest rate of 3.3% compounded monthly.
    - **(a)** What is the future value of payments of $200 at the end of each month for 12 months?
    - **(b)** Which option should Bridget choose?

24. **Comparing Lottery Payouts** A lottery winner is given two options:

    Option I: Receive a $20 million lump-sum payment.

    Option II: Receive 25 equal annual payments totaling $60 million, with the first payment occurring immediately.

    If money can earn 6% interest compounded annually during that long period, which option is better? *Hint:* With each option, calculate the amount of money earned at the end of 24 years if all of the funds are to be deposited into a savings account as soon as they are received.

25. **College Allowance** During Jack's first year at college, his father had been sending him $200 per month for incidental expenses. For his sophomore year, his father decided instead to make a deposit into a savings account on August 1 and have his son withdraw $200 on the first of each month from September 1 to May 1. If the bank pays 1.8% interest compounded monthly, how much should Jack's father deposit?

26. **Magazine Subscription** Suppose that a magazine subscription costs $45 per year and that you receive a magazine at the end of each month. At an interest rate of 2.1% compounded monthly, how much are you actually paying for each issue?

27. **Savings Account** Suppose that $1000 was deposited on January 1, 1989, into a savings account paying 8% interest compounded quarterly, and an additional $100 was deposited into the account at the end of each quarter year. How much would have been in the account on January 1, 2000?

28. **Savings Account** Suppose that you opened a savings account on January 1, 2014 and made a deposit of $100. In 2016, you began depositing $10 into the account at the end of each month. If the bank pays 2.7% interest compounded monthly, how much money will be in the account on January 1, 2020?

29. **Savings Account** Ms. Jones deposited $100 at the end of each month for 10 years into a savings account paying 2.1% interest compounded monthly. However, she deposited an additional $1000 at the end of the seventh year. How much money was in the account at the end of the 10th year?

30. **Savings Account** Redo Exercise 29 for the situation in which Ms. Jones withdrew $1000 at the end of the seventh year instead of depositing it.

31. **Savings Account** How much money must you deposit into a savings account at the end of each year at 2% interest compounded annually in order to earn $3400 interest during a 10-year period? (*Hint:* The future value of the annuity will be $10R + \$3400$.)

32. **Savings Account** How much money must you deposit into a savings account at the end of each quarter at 4% interest compounded quarterly in order to earn $403.80 interest during a 5-year period? (*Hint:* The future value of the annuity will be $20R + \$403.80$.)

33. **Annuity** Suppose that you deposit $600 every 6 months for 5 years into an annuity at 4% interest compounded semiannually. Which of items (a)–(d) can be used to fill in the blank in the statement that follows? (*Note:* Before computing, use your intuition to guess the correct answer.)

    If the _________ doubles, then the amount accumulated will double.
    - **(a)** rent
    - **(b)** interest rate
    - **(c)** number of interest periods per year
    - **(d)** number of years

34. **Total Rent** Suppose that you deposit $100,000 into an annuity at 4% interest compounded semiannually and withdraw an equal amount at the end of each interest period so that the account is depleted after 10 years. Which of items (a)–(d) can be used to fill in the blank in the statement that follows? (*Note:* Before computing, use your intuition to guess the correct answer.)

If the ________ doubles, then the total amount withdrawn each year will double.
(a) amount deposited
(b) interest rate
(c) number of interest periods per year
(d) number of years

35. **Municipal Bond** A municipal bond pays 4% interest compounded semiannually on its face value of \$5000. The interest is paid at the end of every half-year period. Fifteen years from now, the face value of \$5000 will be returned. How much should you pay for the bond?

36. **Present Value** What is the present value of a loan that pays \$800 at the end of every quarter for 4 years and pays an additional \$10,000 at the end of the 4 years? Assume an interest rate of 3% compounded quarterly.

37. **Business Loan** A business loan for \$200,000 carries an interest rate of 9% compounded monthly. Suppose that the business pays only interest for the first 5 years and then repays the loan amount plus interest in equal monthly installments for the next 5 years.
(a) How much money will be paid each month during the first 5 years?
(b) Calculate the monthly payments during the second 5-year period in order to pay off the \$200,000 still owed.

38. **Lottery Payoff** A lottery winner is to receive \$1000 a month for the next 5 years. How much is this sequence of payments worth today if interest rates are 1.8% compounded monthly? How is the difference between this amount and the \$60,000 paid out beneficial to the agency running the lottery?

A *sinking fund* is an increasing annuity set up by a corporation or government to repay a large debt at some future date. Exercises 39–44 concern such annuities.

39. **Sinking Fund** A city has a debt of \$1,000,000 due in 15 years. How much money must it deposit at the end of each half year into a sinking fund at 4% interest compounded semiannually in order to pay off the debt?

40. **Sinking Fund** A corporation wishes to deposit money into a sinking fund at the end of each half year in order to repay \$50 million in bonds in 10 years. It can expect to receive a 6% (compounded semiannually) return on its deposits to the sinking fund. How much should the deposits be?

41. **Sinking Fund** The Federal National Mortgage Association (Fannie Mae) puts \$30 million at the end of each month into a sinking fund paying 4.8% interest compounded monthly. The sinking fund is to be used to repay bonds that mature 15 years from the creation of the fund. How large is the face amount of the bonds, assuming that the sinking fund will exactly pay it off?

42. **Saving for an Upgrade** A corporation sets up a sinking fund to replace some aging machinery. It deposits \$100,000 into the fund at the end of each month for 10 years. The annuity earns 12% interest compounded monthly. The equipment originally cost \$13 million. However, the cost of the equipment is rising 6% each year. Will the annuity be adequate to replace the equipment? If not, how much additional money is needed?

43. **Saving for a New Headquarters** A corporation borrows \$5 million to erect a new headquarters. The financing is arranged by the use of bonds, to be repaid in 15 years. How much should the corporation deposit into a sinking fund at the end of each quarter if the fund earns 2.6% interest compounded quarterly?

44. **Saving for a New Warehouse** A corporation sets up a sinking fund to replace an aging warehouse. The cost of the warehouse today would be \$8 million. However, the corporation plans to replace the warehouse in 5 years. It estimates that the cost of the warehouse will increase by 5% annually. The sinking fund will earn 3.6% interest compounded monthly. What should be the monthly payment to the fund?

A *perpetuity* is similar to a decreasing annuity, except that the payments continue forever. Exercises 45 and 46 concern such annuities.

45. **Scholarship** A grateful alumnus decides to donate a permanent scholarship of \$12,000 per year. How much money should be deposited in the bank at 5% interest compounded annually in order to be able to supply the \$12,000 for the scholarship at the end of each year so that the amount of money in the account remains constant?

46. **Deposit** Show that establishing a perpetuity paying R dollars at the end of each interest period requires a deposit of R/i dollars, where i is the interest rate per interest period.

A *deferred annuity* is a type of decreasing annuity whose term is to start at some future date. Exercises 47–50 concern such annuities.

47. **College Fund** On her 10th birthday, Emma inherits \$20,000, which is to be used for her college education. The money is deposited into a trust fund at 3% interest compounded annually that will pay her R dollars on her 18th, 19th, 20th, and 21st birthdays.
(a) How much money will be in the trust fund on Emma's 17th birthday?
(b) Find the value of R. *Note:* R is the rent on a decreasing annuity with a duration of 4 years.

48. **College Fund** Refer to Exercise 47. Find the size of the inheritance that would result in \$10,000 per year during the college years (ages 18–21, inclusive).

49. **Trust Fund** On December 1, 2014, a philanthropist set up a permanent trust fund to buy Christmas presents for needy children. The fund will provide \$90,000 each year beginning on December 1, 2024. How much must have been set aside in 2014 if the money earns 3% interest compounded annually?

50. **Rent** Show that the rent paid by a deferred annuity of n payments that are deferred by m interest periods is given by the formula

$$R = \frac{i(1+i)^{n+m}}{(1+i)^n - 1} \cdot P.$$

TECHNOLOGY EXERCISES

In Exercises 51–58, give the settings or statements to determine the solution with TVM Solver, Excel, or Wolfram|Alpha.

51. Exercise 5
52. Exercise 6
53. Exercise 7
54. Exercise 8
55. Exercise 9
56. Exercise 10
57. Exercise 11
58. Exercise 12

59. **Time Interval** A person deposits \$1000 at the end of each year into an annuity earning 5% interest compounded annually.

How many years are required for the balance to reach \$30,539? After how many years will the balance exceed \$50,000?

60. **Time Interval** A person deposits \$800 at the end of each quarter-year into an annuity earning 2.2% interest compounded quarterly. How many quarters are required for the balance to reach \$17,754? After how many quarters will the balance exceed \$20,000?

61. **Bicycle Fund** Jane has taken a part-time job to save for a \$503 racing bike. If she puts \$15 each week into a savings account paying 5.2% interest compounded weekly, when will she be able to buy the bike?

62. **Home Repairs Fund** Bob needs \$3064 to have some repairs done on his house. He decides to deposit \$150 at the end of each month into an annuity earning 2.7% interest compounded monthly. When will he be able to do the repairs?

63. **Comparing Expense Ratios** Some stock funds charge an annual fee, called the *expense ratio,* that usually ranges from about .3% to about 1.5% of the amount of money in the account at the end of the year. For instance, if the expense ratio is .5% and you have \$10,000 in the account at the end of the year, then you would be charged $.005 \cdot 10000 = \$50$ in fees. Suppose that you put \$5000 at the beginning of each year into an IRA and that the funds are invested in a stock fund earning 6% per year, before expenses. How much more will you have at the end of 10 years if the stock fund has an expense ratio of .3%, as opposed to 1.5%?

Solutions to Check Your Understanding 10.2

1. An ordinary decreasing annuity with $n = 60$, $i = (3/4)\% = .0075$, and $R = 1$. You will make a deposit now, in the present, and then withdraw money each month. The amount of this deposit is the present value of the annuity.

$$\begin{aligned} P &= \frac{1-(1+i)^{-n}}{i} \cdot R \\ &= \frac{1-1.0075^{-60}}{.0075} \cdot 1 \\ &= \$48.17 \end{aligned}$$

Note: For this transaction, the future value of the annuity needn't be computed. At the end of 5 years, the fund will have a balance of 0.

2. Not an ordinary annuity, since the payment period (1 week) is different from the interest period (1 month).

3. An ordinary increasing annuity with $n = 24$, $i = .5\% = .005$, and $R = \$10$. There is no money in the account now, in the present. However, in 2 years, in the future, money will have accumulated. So for this annuity, only the future value is of concern.

$$\begin{aligned} F &= \frac{(1+i)^{n}-1}{i} \cdot R \\ &= \frac{1.005^{24}-1}{.005} \cdot 10 \\ &= \$254.32 \end{aligned}$$

10.3 Amortization of Loans

In this section, we analyze the mathematics of paying off loans. The loans that we shall consider will be repaid in a sequence of equal payments at regular time intervals, with the payment intervals coinciding with the interest periods. The process of paying off such a loan is called **amortization**. In order to obtain a feeling for the amortization process, let us consider a particular case, the amortization of a \$563 loan to buy a high-definition TV. Suppose that this loan charges interest at a 12% rate with interest compounded monthly on the unpaid balance and that the monthly payments are \$116 for 5 months. The repayment process is summarized in Table 1.

Table 1 Payments on a Loan

Payment Number	Amount	Interest	Applied to Principal	Unpaid Balance
1	\$116	\$5.63	\$110.37	\$452.63
2	116	4.53	111.47	341.16
3	116	3.41	112.59	228.57
4	116	2.29	113.71	114.85
5	116	1.15	114.85	0.00

Note the following facts about the financial transactions:

1. Payments are made at the end of each month. The payments have been carefully calculated to pay off the debt, with interest, in the specified time interval.
2. Since $i = 1\%$, the interest to be paid each month is 1% of the unpaid balance at the end of the previous month. That is, 5.63 is 1% of 563, 4.53 is 1% of 452.63, and so on.

3. Although we write just one check each month for \$116, we regard part of the check as being applied to payment of that month's interest. The remainder (namely, 116 − [interest]) is regarded as being applied to repayment of the principal amount.
4. The unpaid balance at the end of each month is the previous unpaid balance minus the portion of the payment applied to the principal. A loan can be paid off early by paying the current unpaid balance.

The four factors that describe the amortization process just described are as follows:

the principal	\$563
the interest rate	12% compounded monthly
the term	5 months
the monthly payment	\$116

The important fact to recognize is that the sequence of payments in the preceding amortization constitutes a decreasing annuity for the bank. That is, when thinking of the loan as a decreasing annuity, the roles of the bank and the person are reversed. The person receives a sum of money and returns it in a sequence of equal payments (including interest) to the bank. Therefore, the mathematical tools developed in Section 10.2 suffice to analyze the amortization. In particular, we can determine the monthly payment or the principal once the other three factors are specified.

EXAMPLE 1 **Calculating Values Associated with a Loan** Suppose that a loan has an interest rate of 12% compounded monthly and a term of 5 months.

(a) Given that the principal is \$563, calculate the monthly payment.
(b) Given that the monthly payment is \$116, calculate the principal.
(c) Given that the monthly payment is \$116, calculate the unpaid balance after 3 months.

SOLUTION The sequence of payments constitutes a decreasing annuity with the monthly payments as rent and the principal as the present value.

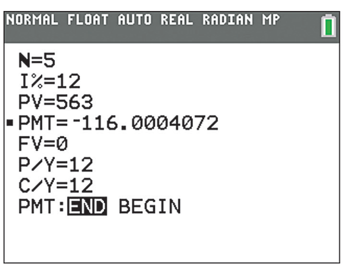

Figure 1 The value for PMT is negative since it denotes money paid to the bank.

(a)
$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \quad \text{Formula for rent}$$
$$= \frac{.01}{1-1.01^{-5}} \cdot 563 \quad i = \frac{.12}{12} = .01,\ n = 5,\ P = 563$$
$$= 116.00 \quad \text{Calculate. Round to two decimal places.}$$

The rent is \$116.00. (The TVM Solver screen in Fig. 1 confirms this result.)

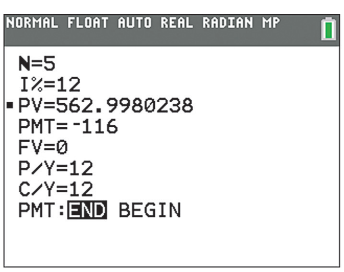

Figure 2 The value for PMT is negative since it denotes money paid to the bank.

(b)
$$P = \frac{1-(1+i)^{-n}}{i} \cdot R \quad \text{Formula for present value (that is, the principal)}$$
$$= \frac{1-1.01^{-5}}{.01} \cdot 116 \quad i = \frac{.12}{12} = .01,\ n = 5,\ R = 116$$
$$= 563.00 \quad \text{Calculate. Round to two decimal places.}$$

The principal is \$563.00. (The TVM Solver screen in Fig. 2 confirms this result.)

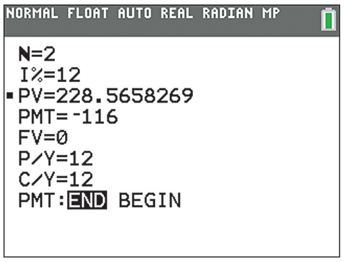

Figure 3 The value for PMT is negative since it denotes money paid to the bank.

(c) The unpaid balance is most easily calculated by regarding it as the amount necessary to retire the debt. Therefore, it must be sufficient to generate the sequence of two remaining payments. That is, the unpaid balance is the present value of a decreasing annuity of two payments of \$116.

$$P = \frac{1-(1+i)^{-n}}{i} \cdot R \quad \text{Formula for present value (that is, the unpaid balance)}$$

$$= \frac{1-1.01^{-2}}{.01} \cdot 116 \quad i = \frac{.12}{12} = .01, n = 2, R = 116$$

$$= 228.57 \quad \text{Calculate. Round to two decimal places.}$$

The unpaid balance after 3 months is \$228.57. (The TVM Solver screen in Fig. 3 confirms this result.)

» Now Try Exercises 1 and 5

A **mortgage** is a long-term loan used to purchase real estate. The real estate is used as collateral to guarantee that the loan will be repaid.

EXAMPLE 2

Calculating Values Associated with a Mortgage On December 31, 1996, a house was purchased with the buyer taking out a 30-year, \$112,475 mortgage at 9% interest compounded monthly. The mortgage payments are made at the end of each month. (*Note:* Computations will be made under the assumption that the loan was not refinanced.)

(a) Calculate the amount of the monthly payment.

(b) Calculate the unpaid balance of the loan on December 31, 2022, just after the 312th payment.

(c) How much interest will be paid during the month of January 2023?

(d) How much of the principal will be paid off during the year 2022?

(e) How much interest will be paid during the year 2022?

(f) What is the total amount of interest paid during the 30 years?

SOLUTION

(a)

$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \quad \text{Formula for rent (that is, the monthly payment)}$$

$$= \frac{.0075}{1-1.0075^{-360}} \cdot 112{,}475 \quad i = \frac{.09}{12} = .0075, n = 360, P = 112{,}475$$

$$= 905.00 \quad \text{Calculate. Round to two decimal places.}$$

The monthly payment is \$905.00. (The TVM Solver screen in Fig. 4 confirms this result.)

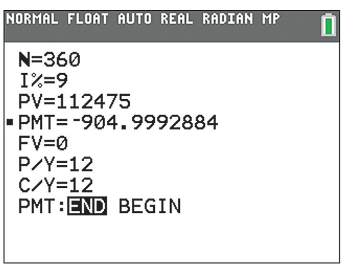

Figure 4 The value for PMT is negative since it denotes money paid to the bank.

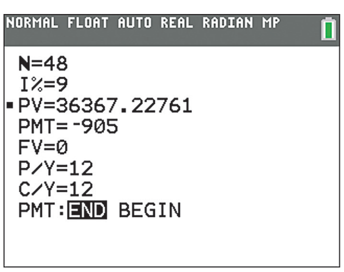

Figure 5 The value for PMT is negative since it denotes money paid to the bank.

(b) The remaining payments constitute a decreasing annuity of 48 payments.

$$[\text{unpaid balance}] = \frac{1-(1+i)^{-n}}{i} \cdot R \quad \text{Formula for present value (that is, the unpaid balance)}$$

$$= \frac{1-1.0075^{-48}}{.0075} \cdot 905 \quad i = \frac{.09}{12} = .0075, n = 48, R = 905$$

$$= 36{,}367.23 \quad \text{Calculate. Round to two decimal places.}$$

The unpaid balance after the 312th payment is \$36,367.23. (The TVM Solver screen in Fig. 5 confirms this result.)

(c) The interest paid during 1 month is i, the interest rate per month, times the unpaid balance at the end of the preceding month. Therefore,

$$[\text{interest for January 2023}] = \tfrac{3}{4}\% \text{ of } \$36{,}367.23$$

$$= .0075 \cdot 36{,}367.23$$

$$= \$272.75.$$

(d) Since the portions of the monthly payments applied to repay the principal have the effect of reducing the unpaid balance, this question may be answered by calculating how much the unpaid balance will be reduced during 2022. Reasoning as in part (b), we determine that the unpaid balance on December 31, 2021 (just after the 300th payment), is equal to \$43,596.90. Therefore,

$$\begin{aligned}&[\text{amount of principal repaid in 2022}]\\&\quad= [\text{unpaid balance Dec. 31, 2021}] - [\text{unpaid balance Dec. 31, 2022}]\\&\quad= \$43{,}596.90 - \$36{,}367.23\\&\quad= \$7229.67.\end{aligned}$$

(e) During the year 2022, the total amount paid is $12 \times 905 = \$10{,}860$. But by part (d), \$7229.67 is applied to repayment of principal, the remainder being applied to interest.

$$\begin{aligned}[\text{interest in 2022}] &= [\text{total amount paid}] - [\text{principal repaid in 2022}]\\&= \$10{,}860 - \$7229.67\\&= \$3630.33\end{aligned}$$

(f) The total amount of money paid during the 30 years is

$$360 \cdot 905 = \$325{,}800.$$

Since the principal is \$112,475, the rest of the total amount paid is interest. That is,

$$\begin{aligned}[\text{total amount of interest paid}] &= 325{,}800 - 112{,}475\\&= \$213{,}325.\end{aligned}$$

» Now Try Exercise 17

In the early years of a mortgage, most of each monthly payment is applied to interest. For the mortgage described in Example 2, the interest portion will exceed the principal portion until the 23rd year. Figure 6 shows how the monthly payment is apportioned between interest and repayment of principal. Figure 7 shows the decrease in the balance for the duration of the mortgage.

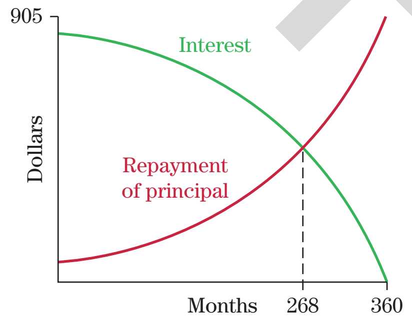

Figure 6

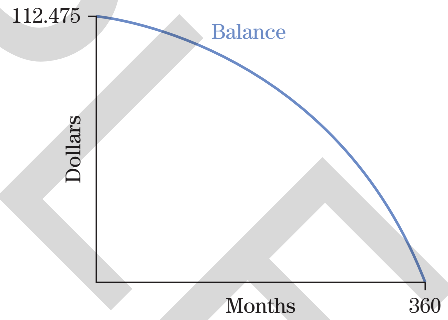

Figure 7

Each new (unpaid) balance can be computed from the previous balance with the formula

$$B_{\text{new}} = (1 + i)B_{\text{previous}} - R.$$

That is, the new balance equals the growth of the previous balance due to interest, minus the amount paid. Successive balances are computed by multiplying the previous balance by $(1 + i)$ and subtracting R.

Successive Balances If P dollars is borrowed at interest rate i per period and R dollars is paid back at the end of each interest period, then the formula

$$B_{\text{new}} = (1 + i)B_{\text{previous}} - R \tag{1}$$

can be used to calculate each new balance, B_{new}, from the previous balance, B_{previous}.

The Excel spreadsheet in Table 2 calculates the payment per period and shows the successive balances for the annuity of Example 2. The entry in cell B4 was set to **=-PMT(B1,B2,B3)**. The form of the payment function is **PMT(*rate*,*nper*,*pv*)**. The value of the payment function is a negative number, since the money is paid to the bank. B10 was set to B3, and B11 was set to **=(1+$B$1)*B10−$B$4**. The remainder of the B column was created by selecting the cell B11 and dragging its fill handle. C11 was set to **=$B$1*B10**, and D11 was set to **=$B$4−C11**. The remainder of the C and D columns was created by dragging fill handles.

Table 2 Amortization Table

	A	B	C	D
1	Interest rate	.75%		
2	Number of periods	360		
3	Principal	$112,475.00		
4	Payment per period	$905.00		
5				
6		Amortization Table		
7				
8		Unpaid		Applied to
9	Payment number	balance	Interest	principal
10	0	$112,475.00		
11	1	$112,413.56	$843.56	$61.44
12	2	$112,351.67	$843.10	$61.90
13	3	$112,289.30	$842.64	$62.36
14	4	$112,226.47	$842.17	$62.83
15	5	$112,163.17	$841.70	$63.30
16	6	$112,099.40	$841.22	$63.78
17	7	$112,035.14	$840.75	$64.25
18	8	$111,970.41	$840.26	$64.74
19	9	$111,905.19	$839.78	$65.22
20	10	$111,839.48	$839.29	$65.71
21	11	$111,773.27	$838.80	$66.20
22	12	$111,706.57	$838.30	$66.70

	A	B	C	D
359	349	$9,521.20	$77.61	$827.38
360	350	$8,687.61	$71.41	$833.59
361	351	$7,847.77	$65.16	$839.84
362	352	$7,001.63	$58.86	$846.14
363	353	$6,149.14	$52.51	$852.49
364	354	$5,290.26	$46.12	$858.88
365	355	$4,424.94	$39.68	$865.32
366	356	$3,553.13	$33.19	$871.81
367	357	$2,674.78	$26.65	$878.35
368	358	$1,789.84	$20.06	$884.94
369	359	$898.26	$13.42	$891.58
370	360	$0.00	$6.74	$898.26
371	Payment number	Unpaid	Interest	Applied to
372		balance		principal

EXAMPLE 3 **Calculating the Unpaid Balance on a Mortgage** Refer to Example 2. Compute the unpaid balance of the loan on January 31, 2023, just after the 313th payment.

SOLUTION By formula (1),

$$\begin{aligned} B_{\text{Jan}} &= (1 + i)B_{\text{Dec}} - R \\ &= (1.0075) \cdot 36{,}367.23 - 905 \\ &= \$35{,}734.98. \end{aligned}$$

«

Sometimes, amortized loans stipulate a **balloon payment** at the end of the term. For instance, you might pay $200 at the end of each quarter year for 3 years and an additional $1000 at the end of the third year. The $1000 is a balloon payment. Balloon payments are more common in commercial real estate than in residential real estate.

EXAMPLE 4 **Analyzing the Apportionment of Mortgage Payments** Consider the situation from Example 2 in which $112,475 is borrowed at 9% interest compounded monthly and repaid with 360 monthly payments of $905. Use technology to determine when the repayment of the principal portion of the payment will surpass the interest portion.

SOLUTION The question can be answered with a graph. Define Y_1, Y_2, and Y_3 as follows:

$$Y_1 = \frac{1 - 1.0075^{X-360}}{.0075} * 905 \quad \text{Unpaid balance}$$

$$Y_2 = .0075 * Y_1(X - 1) \quad \text{Interest on the previous balance}$$

$$Y_3 = 905 - Y_2 \quad \text{Repayment of principal}$$

Then, set the window as in Fig. 8, graph Y_2 and Y_3, and find their point of intersection as shown in Fig. 9.

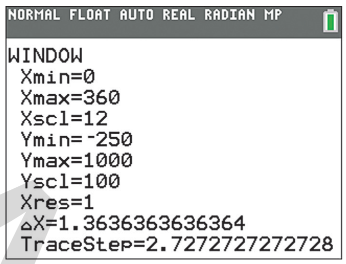

Figure 8

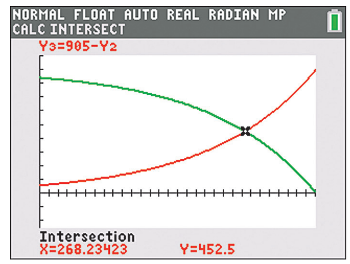

Figure 9

WolframAlpha The instruction

solve 905(1-1.0075^(x-361)) = 905-905(1-1.0075^(x-361))

yields 268.234.

In each case, we see that principal repayment will exceed interest after 269 months—that is, after 22 years and 5 months.

» *Now Try Exercise 51*

EXAMPLE 5 **Determining How Much You Can Afford to Borrow** How much money can you borrow at 8% interest compounded quarterly if you agree to pay $200 at the end of each quarter year for 3 years and, in addition, a balloon payment of $1000 at the end of the third year?

SOLUTION Here, you are borrowing in the present and repaying in the future. The amount of the loan will be the present value of *all* future payments. The future payments consist of an annuity and a lump-sum payment. Let us calculate the present values of each of these separately.

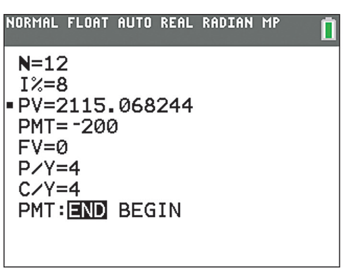

Figure 10

$$[\text{present value of annuity}] = \frac{1 - (1 + i)^{-n}}{i} \cdot R \quad \text{Formula for present value}$$

$$= \frac{1 - 1.02^{-12}}{.02} \cdot 200 \quad i = \frac{.08}{4} = .02,\ n = 3 \cdot 4 = 12,\ R = 200$$

$$= 2115.07 \quad \text{Calculate. Round to two decimal places.}$$

The present value of the annuity is $2115.07. (The TVM Solver screen in Fig. 10 confirms this result.)

$$[\text{present value of balloon payment}] = \frac{F}{(1 + i)^n} \quad \text{Formula for present value}$$

$$= \frac{1000}{1.02^{12}} \quad F = 1000,\ i = \frac{.08}{4} = .02,\ n = 3 \cdot 4 = 12$$

$$= 788.49 \quad \text{Calculate. Round to two decimal places.}$$

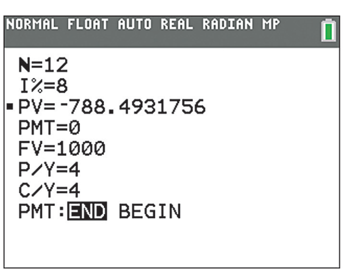

Figure 11

The present value of the balloon payment is $788.49. (The TVM Solver screen in Fig. 11 confirms this result.)

Therefore, the amount that you can borrow is

$$\$2115.07 + \$788.49 = \$2903.56.$$

«

INCORPORATING TECHNOLOGY

Display Balances Graphing calculators can easily display successive balances for a loan. Consider the situation from Example 2 in which $112,475 is borrowed at 9% interest compounded monthly and repaid with 360 monthly payments of $905. In Fig. 12, successive balances are displayed in the home screen. In Fig. 13, successive balances are graphed, and in Fig. 14, they are displayed as part of an amortization table. (The $\mathbf{Y_2}$ column gives the interest portion of the monthly payment.)

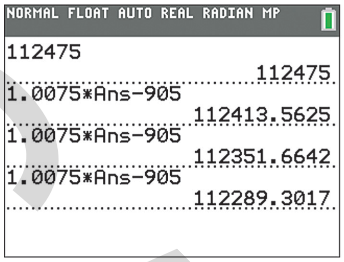

Figure 12

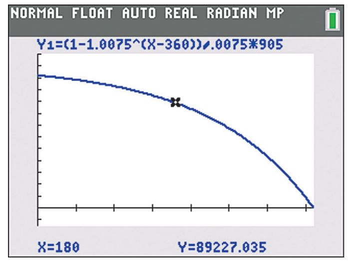

Figure 13

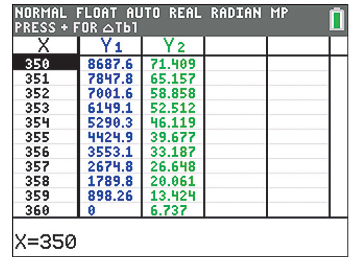

Figure 14

Figures 12–14 were obtained with the same processes as Figures 1–3 in Section 10.1. Successive balances were determined with the relation

$$B_{\text{new}} = (1 + i)B_{\text{previous}} - R.$$

The unpaid balance function to be graphed was specified as

$$\mathbf{Y_1} = \frac{\mathbf{1 - 1.0075^{X-360}}}{\mathbf{.0075}} * \mathbf{905}.$$

The window was set to [0, 360] *by* [−15000, 130000], with an X-scale of 50 and a Y-scale of 10000. In the TABLE SETUP menu, **TblStart** was set to **350**. The second function was defined as $\mathbf{Y_2 = .0075*Y_1(X-1)}$—that is, the interest on the previous balance. The up-arrow key can be used to generate earlier balances and interest payments.

Excel Spreadsheet The section "Using Excel's Financial Functions" of Appendix C shows how the functions IPMT and PPMT are used to calculate the interest and principal reduction portions of a specific payment—that is, the values in columns C and D of Table 2. The annuity functions FV, PV, PMT, and NPER also apply to mortgages.

WolframAlpha Since a mortgage is a decreasing annuity with the borrower serving as the bank, questions about mortgages can be answered with the annuity instructions presented in Section 10.2. For instance, the following instructions give the solutions to Example 1:

(a) annuity pmt,PV = $563,i = 1%,n = 5

(b) annuity PV,pmt = $116,i = 1%,n = 5

(c) annuity PV,pmt = $116,i = 1%,n = 2

The instruction

mortgage $112475,30 years,9%

generates an amortization table for the mortgage of Example 2. The table gives the monthly payment and displays values at the end of each year. The instruction also produces graphs similar to the ones in Figs. 6 and 7.

Check Your Understanding 10.3

Solutions can be found following the section exercises.

1. The word "amortization" comes from the French "à mort," meaning "at the point of death." Justify the word.
2. **Mortgage** Determine the *y*-coordinate of the intersection point in Fig. 6 on page 452.
3. Explain why only present values and not future values arise in amortization problems.

EXERCISES 10.3

1. **Loan Payment** A car loan of $10,000 is to be repaid with quarterly payments for 5 years at 6.4% interest compounded quarterly. Calculate the quarterly payment.
2. **Loan Payment** A loan of $5000 is to be repaid with quarterly payments for 2 years at 5.6% interest compounded quarterly. Calculate the quarterly payment.
3. **Loan Payment** A loan of $4000 is to be repaid with semiannual payments for 3 years at 5.2% interest compounded semiannually. Calculate the semiannual payment.
4. **Loan Payment** A loan of $3000 is to be repaid with semiannual payments for 4 years at 4.8% interest compounded semiannually. Calculate the semiannual payment.
5. **Loan Amount** The weekly payment on a 2-year loan at 7.8% compounded weekly is $23.59. Calculate the amount of the loan.
6. **Loan Amount** The quarterly payment on a 5-year loan at 6.8% compounded quarterly is $235.82. Calculate the amount of the loan.
7. **Mortgage Payment** Find the monthly payment on a $100,000, 25-year mortgage at 5.4% interest compounded monthly.
8. **Mortgage Payment** Find the monthly payment on a $250,000, 30-year mortgage at 4.8% interest compounded monthly.
9. **Mortgage Amount** Find the amount of a 30-year mortgage at 4.5% interest compounded monthly where the monthly payment is $724.56.
10. **Mortgage Amount** Find the amount of a 25-year mortgage at 4.2% interest compounded monthly where the monthly payment is $1121.00.
11. **Mortgage Balance** A 30-year mortgage at 4.2% interest compounded monthly with a monthly payment of $1019.35 has an unpaid balance of $10,000 after 350 months. Find the unpaid balance after 351 months.
12. **Mortgage Balance** A 25-year mortgage at 4.5% interest compounded monthly with a monthly payment of $258.96 has an unpaid balance of $5,000 after 280 months. Find the unpaid balance after 281 months.
13. **Loan Interest** A loan with a weekly payment of $100 has an unpaid balance of $2000 after 5 weeks and an unpaid balance of $1903 after 6 weeks. If interest is compounded weekly, find the interest rate.
14. **Loan Interest** A loan with a quarterly payment of $1500 has an unpaid balance of $10,000 after 30 quarters and an unpaid balance of $8670 after 31 quarters. If interest is compounded quarterly, find the interest rate.
15. **Amortization Schedule** Write out a complete amortization schedule (as in Table 1 at the beginning of this section) for the amortization of a $830 loan with monthly payments of $210.10 at 6% interest compounded monthly for 4 months.
16. **Amortization Schedule** Write out a complete schedule (as in Table 1) for the amortization of a $945 loan with payments of $173.29 every 6 months at 5.6% interest compounded semiannually for 3 years.
17. **Mortgage** Consider a $204,700, 30-year mortgage at interest rate 4.8%, compounded monthly, with a $1073.99 monthly payment.
    - **(a)** How much interest is paid the first month?
    - **(b)** How much of the first month's payment is applied to paying off the principal?
    - **(c)** What is the unpaid balance after 1 month?
18. **Mortgage** James buys a house for $370,000. He puts $70,000 down and then finances the rest at 6.3% interest compounded monthly for 25 years.
    - **(a)** Find his monthly payment.
    - **(b)** Find the total amount he pays for the house.
    - **(c)** Find the total amount of interest he pays.
19. **Car Loan** Susie takes out a car loan for $9480 for a term of 3 years at 6% interest compounded monthly.
    - **(a)** Find her monthly payment.
    - **(b)** Find the total amount she pays for the car.
    - **(c)** Find the total amount of interest she pays.
    - **(d)** Find the amount she still owes after 1 year.
    - **(e)** Find the amount she still owes after 2 years.
    - **(f)** Find the total interest she pays in year 2.
    - **(g)** Prepare the amortization schedule for the first 4 months.
20. **Loan** Consider a $21,281.27 loan for 7 years at 8% interest compounded quarterly with a payment of $1000 per quarter-year.
    - **(a)** Compute the unpaid balance after 5 years.
    - **(b)** How much interest is paid during the fifth year?
    - **(c)** How much principal is repaid in the first payment?
    - **(d)** What is the total amount of interest paid on the loan?
    - **(e)** Prepare the amortization schedule for the first 4 months.
21. **Comparing Financing Options** In a recent year, Toyota was offering the choice of a .9% loan for 60 months or $500 cash back on the purchase of an $18,000 Toyota Corolla.
    - **(a)** If you take the .9% loan offer, how much will your monthly payment be?
    - **(b)** If you take the $500 cash-back offer and can borrow money from your local bank at 6% interest compounded monthly for 5 years, how much will your monthly payment be?
    - **(c)** Which of the two offers is more favorable for you?
22. **Comparing Financing Options** In a recent year, Ford was offering the choice of a 1.9% loan for 36 months or $1000 cash back on the purchase of a $28,000 Taurus.
    - **(a)** If you take the 1.9% loan offer, how much will your monthly payment be?
    - **(b)** If you take the $1000 cash-back offer and can borrow money from your local bank at 6% interest compounded monthly for 3 years, how much will your monthly payment be?
    - **(c)** Which of the two offers is more favorable for you?
23. **Comparing Financing Options** A bank makes the following two loan offers to its credit card customers:

    Option I: Pay 0% interest for 5 months and then 9% interest compounded monthly for the remainder of the duration of the loan.

    Option II: Pay 6% interest compounded monthly for the duration of the loan.

    With either option, there will be a minimum payment of $100 due each month. Suppose that you would like to borrow $5500 and pay it back in a year and a half. For which option do you pay the least amount to the bank?

24. Buy Now or Later? According to an article in the *New York Times* on August 9, 2008, economists were predicting that the average interest rate for a 30-year mortgage would increase from 6.7% to 7.1% during the next year and home prices would decline by 9% during that same period. In August 2008, the Johnson family was considering purchasing a $400,000 home. They intended to make a 20% down payment and finance the rest of the cost. What would their monthly payment have been if they bought the house in August 2008? A comparable house in August 2009?

25. Balloon Payment A loan is to be amortized over an 8-year term at 6.4% interest compounded semiannually, with payments of $905.33 every 6 months and a balloon payment of $5,000 at the end of the term. Calculate the amount of the loan.

26. Balloon Payment A loan of $127,000.50 is to be amortized over a 5-year term at 5.4% interest compounded monthly, with monthly payments and a $10,000 balloon payment at the end of the term. Calculate the monthly payment.

27. Car Loan A car is purchased for $6287.10, with $2000 down and a loan to be repaid at $100 a month for 3 years, followed by a balloon payment. If the interest rate is 6% compounded monthly, how large will the balloon payment be?

28. Cash Flow for a Rental Property You are considering the purchase of a condominium to use as a rental property. You estimate that you can rent the condominium for $1500 per month and that taxes, insurance, and maintenance costs will run about $300 per month. If interest rates are 4.8% compounded monthly, how large of a 25-year mortgage can you assume and still have the rental income cover all of the monthly expenses?

29. Repayment of Principal If you take out a 30-year mortgage at 6.8% interest compounded monthly, what percentage of the principal will be paid off after 15 years?

30. Repayment of Principal If you take out a 20-year mortgage at 6.4% interest compounded monthly, what percentage of the principal will be paid off after 10 years?

31. Terminating a Mortgage In 2006, Emma purchased a house and took out a 25-year, $50,000 mortgage at 6% interest compounded monthly. In 2016, she sold the house for $150,000. How much money did she have left after she paid the bank the unpaid balance on the mortgage?

32. Refinancing a Mortgage A real estate speculator purchases a tract of land for $1 million and assumes a 25-year mortgage at 4.2% interest compounded monthly.

(a) What is his monthly payment?

(b) Suppose that at the end of 5 years the mortgage is changed to a 10-year term for the remaining balance. What is the new monthly payment?

(c) Suppose that after 5 more years, the mortgage is required to be repaid in full. How much will then be due?

33. Loan Payment The total interest paid on a 3-year loan at 6% interest compounded monthly is $1,085.16. Determine the monthly payment for the loan. (*Hint:* Use the fact that the loan amount equals $36 \cdot R -$ [total interest].)

34. Loan Payment The total interest paid on a 5-year loan at 8% interest compounded quarterly is $2,833.80. Determine the quarterly payment for the loan. (*Hint:* Use the fact that the loan amount equals $20 \cdot R -$ [total interest].)

35. Total of Loan Payment Suppose that you borrow $10,000 at 4% interest compounded semiannually and pay off the loan in 7 years. Which of items (a)–(d) can be used to fill in the blank in the statement that follows? (*Note:* Before computing, use your intuition to guess the correct answer.)

If the ____________ is doubled, then the total amount paid will double.

(a) amount of the loan

(b) interest rate

(c) number of interest periods per year

(d) term of the loan

36. Loan Amount Suppose that you borrow money at 4% interest compounded semiannually and pay off the loan in payments of $1000 per interest period for 5 years. Which of items (a)–(d) can be used to fill in the blank in the statement that follows? (*Note:* Before computing, use your intuition to guess the correct answer.)

If the ____________ is doubled, then the amount that can be borrowed will double.

(a) amount paid per interest period

(b) interest rate

(c) number of interest periods per year

(d) term of the loan

37. Let $B_n =$ balance of a loan after n payments, $I_n =$ the interest portion of the nth payment, and $Q_n =$ the portion of the nth payment applied to the principal. Equation (1) states that $B_n = (1 + i)B_{n-1} - R$.

(a) Use the fact that $I_n + Q_n = R$ and $I_n = iB_{n-1}$ to show that

$$B_{n-1} = \frac{R - Q_n}{i}.$$

(b) Use equation (1) and the results that

$$B_{n-1} = \frac{R - Q_n}{i} \quad \text{and} \quad B_n = \frac{R - Q_{n+1}}{i}$$

from part (a) to derive the formula $Q_{n+1} = (1 + i)Q_n$.

(c) State in your own words the formula obtained in part (b).

(d) Suppose that, for the 10th monthly payment on a loan at 1% interest per month, $100 was applied to the principal. How much of the 11th and 12th payments will be applied to the principal?

38. Let Q_n and I_n be as defined in Exercise 37.

(a) Use the result in part (b) of Exercise 37 and the facts that $Q_{n+1} = R - I_{n+1}$ and $Q_n = R - I_n$ to show that $I_{n+1} = (1 + i)I_n - iR$.

(b) State in your own words the formula obtained in part (a).

(c) Suppose that, for a loan with a monthly payment of $400 and an interest rate of 1% per month, $100 of the 9th monthly payment goes toward paying off interest. How much of the 10th and 11th payments will be used to pay off interest?

TECHNOLOGY EXERCISES

In Exercises 39–46, give settings or statements to determine the solution with TVM Solver, Excel, or Wolfram|Alpha.

39. Exercise 1

40. Exercise 4

41. Exercise 5

42. Exercise 6

43. Exercise 7

44. Exercise 8

45. Exercise 9

46. Exercise 10

47. **Financing a Computer** Bill buys a top-of-the-line computer for \$4193.97 and pays off the loan (at 4.8% interest compounded monthly) by paying \$100 at the end of each month. After how many months will the loan be paid off?

48. **Financing Medical Equipment** Alice borrows \$20,000 to buy some medical equipment and pays off the loan (at 7% interest compounded annually) by paying \$4195.92 at the end of each year. After how many years will the loan be paid off?

49. **Debt Reduction** A loan of \$10,000 at 9% interest compounded monthly is repaid in 80 months with monthly payments of \$166.68. After how many months will the loan be one-quarter paid off? One-half? Three-quarters?

50. **Debt Reduction** A loan of \$4000 at 6% interest compounded monthly is repaid in 8 years with monthly payments of \$52.57. After how many months will the loan be one-quarter paid off? One-half? Three-quarters?

51. **Debt Reduction** A 25-year mortgage of \$124,188.57 at 8.5% interest compounded monthly has monthly payments of \$1000. After how many months will at least 75% of the monthly payment go toward debt reduction?

52. **Debt Reduction** A 30-year mortgage of \$118,135.40 at 8.4% interest compounded monthly has monthly payments of \$900. After how many months will the amount applied to the reduction of the debt be more than twice the amount applied to interest?

Solutions to Check Your Understanding 10.3

1. A portion of each payment is applied to reducing the debt, and by the end of the term the debt is totally annihilated.

2. 452.5. Since the *amount of interest* equals the *repayment of principal* at the point of intersection and the two numbers add up to 905, they must each have the value 905/2; that is, 452.5

3. The debt is formed when the creditor gives you a lump sum of money now, in the present. The lump sum of money is gradually repaid by you, with interest, thereby generating the decreasing annuity. At the end of the term, in the future, the loan is totally paid off, so there is no more debt. That is, the future value is always zero!

NOTE You are actually functioning like a savings bank, since you are paying the interest. Think of the creditor as depositing the lump sum with you and then making regular withdrawals until the balance is 0. «

10.4 Personal Financial Decisions

You will be faced with several financial decisions shortly after graduating from college. Financial advisers suggest that you open an individual retirement account as soon as you start earning an income. You will probably buy a car and some major appliances with consumer loans. You will consider buying a condominium or a house and will need a mortgage. This section will help you answer the following questions:

- How important is it to open an individual retirement account as soon as possible? What kind of IRA should you open?
- What is the difference between the finance charges on a consumer loan that employs the add-on method and one that employs the compounding method of Section 10.3?
- How do discount points work, and when are they a good choice?
- How do you choose between two mortgages if they have different interest rates and up-front costs?
- What are interest-only and adjustable-rate mortgages and when are they a good choice?

Individual Retirement Accounts

Money earned in an ordinary savings account is subject to federal, state, and local income taxes. However, a special type of savings account, called an **individual retirement account** (IRA), provides a shelter from these taxes. IRAs are highly touted by financial planners and are available to any individual whose income does not exceed a certain limit. The maximum annual contribution for persons under 50 in 2016 was about \$5500. (However, the amount contributed must not exceed the person's earned income.) Typically, money in an IRA cannot be withdrawn without penalty until the person reaches $59\frac{1}{2}$ years of age. However, funds may be withdrawn earlier under certain circumstances.

The two main types of IRAs are known as a **traditional IRA** and a **Roth IRA**. Contributions to a traditional IRA are tax deductible, but all withdrawals are taxed. Interest earned is not taxed until it is withdrawn. In contrast, contributions to a Roth IRA are not tax deductible, but withdrawals are not taxed. Therefore, interest earned is never taxed.

EXAMPLE 1

Calculating Values Associated with a Traditional IRA Suppose that, for the year 2016, you deposit \$5000 of earned income into a traditional IRA, you earn an annual interest rate of 6% compounded annually, and you are in a 30% marginal tax bracket for the duration of the account. (That is, for each additional dollar you earn, you pay an additional \$.30 in taxes. Likewise, for each dollar you can deduct, you save \$.30 in taxes.) We recognize that interest rates and tax brackets are subject to change over a long period of time, but some assumptions must be made in order to evaluate the investment.

(a) How much income tax on earnings will you save for the year 2016?

(b) Assuming that no additional deposits are made, how much money will be in the account after 48 years?

(c) Suppose that you withdraw all of the money in the IRA after 48 years (assuming that you are older than $59\frac{1}{2}$). How much will you have left after you pay the taxes on the money?

SOLUTION

(a)
$$\begin{aligned}[\text{income tax saved}] &= [\text{tax bracket}] \cdot [\text{amount}] \\ &= .30 \cdot 5000 \\ &= 1500\end{aligned}$$

Therefore, \$1500 in income taxes will be saved. You will have the full \$5000 of earned income to deposit into your traditional IRA.

(b)
$$\begin{aligned}[\text{balance after 48 years}] &= P(1+i)^n \\ &= 5000(1.06)^{48} \\ &= 81{,}969.36\end{aligned}$$

Therefore, after 48 years, the balance in the IRA will be \$81,969.36.

(c) When you withdraw the money, 30% of the money will be paid in taxes. Hence, 70% will be left for you.

$$\begin{aligned}[\text{amount after taxes}] &= .70(81{,}969.36) \\ &= \$57{,}378.55\end{aligned}$$

«

EXAMPLE 2

Comparing a Savings Account with a Traditional IRA Rework Example 1 with the money going into an ordinary savings account.

SOLUTION

(a) None. With regard to a savings account, there are no savings on income tax, since you pay the taxes as money is earned on the account. Hence,

$$\begin{aligned}[\text{earnings after income tax}] &= [1 - \text{tax bracket}] \cdot [\text{amount}] \\ &= .70 \cdot 5000 \\ &= 3500.\end{aligned}$$

Therefore, \$3500 will be deposited into the savings account.

(b) The interest earned each year will be taxed during that year. Therefore, each year, 30% of the interest will be paid in taxes, and hence, only 70% will be kept. Since 70% of 6% is 4.2%, you will effectively earn just 4.2% interest per year. Hence, the future value of your investment is as follows:

$$[\text{balance after 48 years}] = P(1+i)^n = 3500(1.042)^{48} = \$25{,}218.46$$

(c) \$25,218.46, since the interest was taxed in the year that it was earned. «

Comparing the results from Examples 1 and 2, we see why financial planners encourage people to set up IRAs. The balance upon withdrawal is \$32,160.09 greater with the IRA. The amount of money in the traditional IRA is more than 2.25 times the amount of money in an ordinary savings account.

EXAMPLE 3

Calculating the Future Value of a Roth IRA Consider the following variation of Example 1. Suppose that you earned \$5000, paid your taxes, and then deposited the remainder into a Roth IRA. How much money would you have if you closed out the account after 48 years?

SOLUTION

$$\begin{aligned}[\text{earnings after income tax}] &= [1 - \text{tax bracket}] \cdot [\text{amount}] \\ &= \quad .70 \quad \cdot \quad 5000 \\ &= \$3500\end{aligned}$$

Therefore, \$3500 will be deposited into the Roth IRA.

$$[\text{balance after 48 years}] = P(1 + i)^n = 3500(1.06)^{48} = \$57{,}378.55$$

Therefore, after 48 years, the balance will be \$57,378.55, the same amount found in part (c) of Example 1. Since you do *not* pay taxes on the interest earned, you will have all of this money available to you upon withdrawal. «

The result of Example 3 suggests the following general conjecture, which is easily verified:

> **Equivalence of Traditional and Roth IRAs** With the assumption that your tax bracket does not change, the net earnings upon withdrawal from contributing P dollars to a traditional IRA account is the same amount as would result from paying taxes on the P dollars but then contributing the remaining money to a Roth IRA.

Verification of the Equivalence Conjecture Suppose that P dollars is deposited into a traditional IRA at interest rate r compounded annually for n years. Also, suppose that your marginal tax bracket is k. (In Example 1, $k = .30$.) In general, if taxes are paid on A dollars, then the amount remaining after taxes is

$$A - k \cdot A = A \cdot (1 - k).$$

With a traditional IRA, the future value (or balance) after n years is $[P(1 + r)^n]$ and the amount left after taxes are paid is $[P(1 + r)^n] \cdot (1 - k)$.

With a Roth IRA, taxes are paid initially on the P dollars, so the amount deposited is $[P \cdot (1 - k)]$. The future value (or balance) after n years is obtained by multiplying the amount deposited by $(1 + r)^n$. That is, the balance is $[P \cdot (1 - k)] \cdot (1 + r)^n$.

Each of the preceding amounts is the product of the same three numbers; only the order is different. Therefore, the two investments have the same future value. «

Financial planners encourage clients not only to establish IRAs, but also to begin making contributions at an early age. The following example illustrates the advantage of beginning as soon as possible:

EXAMPLE 4

The Value of Starting an IRA Early Earl and Larry each begin full-time jobs in January 2017 and plan to retire in January 2065 after working for 48 years. Assume that any money that they deposit into IRAs earns 6% interest compounded annually.

(a) Suppose that Earl opens a traditional IRA account immediately and deposits \$5000 into his account at the end of each year for 12 years. After that, he makes no further deposits and just lets the money earn interest. How much money will Earl have in his account when he retires in January 2065?

(b) Suppose that Larry waits 12 years before opening his traditional IRA and then deposits \$5000 into the account at the end of each year until he retires. How much money will Larry have in his account when he retires in January 2065?
(c) Who paid more money into his IRA?
(d) Who had more money in his account upon retirement?

SOLUTION **(a)** Earl's contributions consist of an increasing annuity with $R = 5000$, $i = .06$, and $n = 12$. The future value (or balance) in the account on January 1, 2029 will be

$$F = \frac{(1+i)^n - 1}{i} \cdot R = \frac{1.06^{12} - 1}{.06} \cdot 5000 = \$84{,}349.71.$$

This money then earns interest compounded annually for 36 years. Using the formula $F = P(1+i)^n$, it grows to

$$84{,}349.71 \cdot (1.06)^{36} = \$687{,}218.34.$$

(b) Larry's contributions consist of an increasing annuity with $R = 5000$, $i = .06$, and $n = 36$. The balance in the account on January 1, 2065 will be

$$F = \frac{(1+i)^n - 1}{i} \cdot R = \frac{1.06^{36} - 1}{.06} \cdot 5000 = \$595{,}604.33.$$

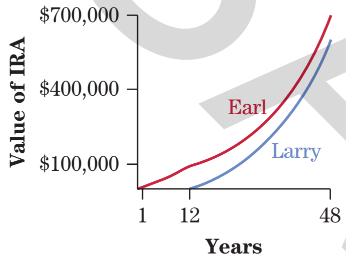

Figure 1

(c) Larry pays in $36 \cdot 5000 = \$180{,}000$ versus the $12 \cdot 5000 = \$60{,}000$ paid in by Earl. That is, Larry pays in three times as much money as Earl does.

(d) Earl has \$91,614.01 more than Larry. Figure 1 shows the growths of the accounts.

» Now Try Exercise 9

Consumer Loans and the Truth-in-Lending Act

Consumer loans are paid off with a sequence of monthly payments in much the same way as mortgages. However, the most widely used method for determining finance charges on consumer loans is the **add-on method**, which results in an understated interest rate. When an add-on method interest rate is stated as r per year, the term of the loan is t years, and the amount of the loan is P, then the total interest to be paid is $P \cdot r \cdot t$. The monthly payment is then determined by dividing the sum of the principal and the interest by the total number of months. For instance, suppose that you take out a two-year loan of \$1000 at a stated annual interest rate of 6%.

$$\begin{aligned}[\text{monthly payment}] &= \frac{1000 + 1000 \cdot .06 \cdot 2}{12 \cdot 2} \\ &= \frac{1000(1 + .06 \cdot 2)}{24} \\ &= \$46.67\end{aligned}$$

The general formula for the monthly payment R is

$$R = \frac{P(1+rt)}{12t}.$$

This formula can be solved for r to obtain

$$r = \frac{\frac{12Rt - P}{t}}{P} = \frac{12Rt - P}{Pt}.$$

Since $12Rt - P$ is the amount of interest paid for the duration of the loan, $\frac{12Rt - P}{t}$ is the amount of interest paid per year and $\frac{12Rt - P}{Pt}$ is the annual percent of the principal that consists of the interest, namely, r.

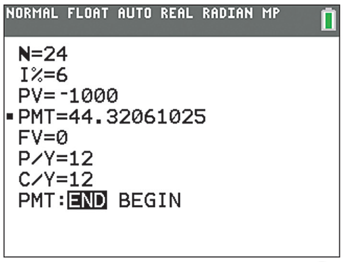

Figure 2

If this loan were amortized as in Section 10.3, the interest rate would be $i = .06/12 = .005$. The monthly payment would be

$$[\text{monthly payment}] = \frac{i}{1-(1+i)^{-n}} \cdot P \qquad \text{Formula for rent (that is, the monthly payment)}$$

$$= \frac{.005}{1-1.005^{-24}} \cdot 1000 \qquad i = \frac{.06}{12} = .005,\ n = 2 \cdot 12 = 24,\ P = 1000$$

$$= 44.32 \qquad \text{Calculate. Round to two decimal places.}$$

The monthly payment is \$44.32. (The TVM Solver screen in Fig. 2 confirms this result.)

This means that the add-on method results in the borrower being charged an additional $46.67 - 44.32 = 2.35$ dollars in interest per month. The problem with the add-on method is that each month, the borrower is being charged interest on the entire principal even though, after the first month, part of the principal has already been repaid.

In 1968, Congress passed the federal Truth-in-Lending Act as a part of the Consumer Protection Act. The law's stated goal is

> to assure a meaningful disclosure of credit terms so that the consumer will be able to compare more readily the various credit terms available . . . .

The Truth-in-Lending Act requires a lender to disclose what is known as the **annual percentage rate** (or **APR**) for any advertised loan. For the preceding loan, the monthly payment of \$46.67 corresponds to an interest rate of 11.134% when the payment is calculated as in Section 10.3. Therefore, the APR is 11.134%.

EXAMPLE 5 **Calculating the Monthly Payment for a Loan with the Add-on Method** A one-year loan for \$5000 is advertised at 8% by the add-on method. What is the monthly payment?

SOLUTION The interest on the loan is $.08 \cdot (5000) = \$400$. The total amount (principal plus interest) of \$5400 is to be paid in 12 monthly payments of $5400/12 = \$450$.

» Now Try Exercise 11

EXAMPLE 6 **Determining the Interest Rate on a Consumer Loan** Suppose that a consumer loan of \$100,000 for 10 years has an APR of 8% compounded monthly and a monthly payment of \$1213.28. What interest rate would be stated with the add-on method?

SOLUTION The total amount paid by the borrower is $120 \cdot 1213.28 = \$145{,}593.60$. Therefore, the interest paid on the loan is $145{,}593.60 - 100{,}000 = \$45{,}593.60$. The interest paid per year is $45{,}593.60/10 = \$4559.36$, which is about 4.56% of \$100,000. Therefore, with the add-on method, the interest rate would be given as 4.56%.

» Now Try Exercise 15

The computation of the APR is easily accomplished with a spreadsheet or graphing calculator. For instance, with Excel, the APR of 11.134% discussed previously is calculated with the RATE function as 12*RATE(24,−46.67,1000,0). This same value is easily obtained on a TI-84 Plus calculator with TVM Solver. To determine the APR on an arbitrary graphing calculator, graphically solve the equation $1000 = \frac{1-(1+X)^{-24}}{X} \cdot 46.67$ for X, and then multiply by 12.

Mortgages with Discount Points

Home mortgages have a stated annual interest rate, called the **contract rate**. However, some loans also carry **points** or **discount points** to reduce the contract rate. Each point requires that you pay up-front additional interest equal to 1% of the stated loan amount. For instance, for a \$200,000 mortgage with 3 discount points you must pay \$6000 immediately. This has the effect of reducing the loan to \$194,000, yet requiring the same monthly payment as a \$200,000 mortgage having no points.

EXAMPLE 7 **Discount Points and Monthly Payments for Mortgages** Compute the monthly payment for each of the following 30-year mortgages:

Mortgage A: \$200,000 at 6% interest compounded monthly, one point

Mortgage B: \$198,000 at 6.094% interest compounded monthly, no points

SOLUTION Mortgage A:

$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \qquad \text{Formula for rent (that is, the monthly payment)}$$

$$= \frac{.005}{1-1.005^{-360}} \cdot 200{,}000 \qquad i = \frac{.06}{12} = .005,\ n = 30 \cdot 12 = 360,\ P = 200{,}000$$

$$= 1199.10 \qquad \text{Calculate. Round to two decimal places.}$$

The monthly payment is \$1199.10. (The TVM Solver screen in Fig. 3 confirms this result.)

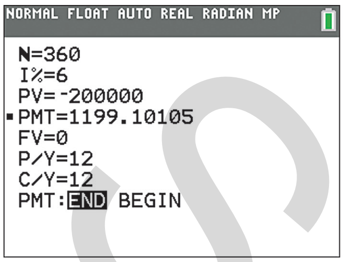

Figure 3

Mortgage B:

$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \qquad \text{Formula for rent (that is, the monthly payment)}$$

$$= \frac{\frac{.06094}{12}}{1-\left(1+\frac{.06094}{12}\right)^{-360}} \cdot 198{,}000 \qquad i = \frac{.06094}{12},\ n = 30 \cdot 12 = 360,\ P = 198{,}000$$

$$= 1199.10 \qquad \text{Calculate. Round to two decimal places.}$$

The monthly payment is \$1199.10. (The TVM Solver screen in Fig. 4 confirms this result.) «

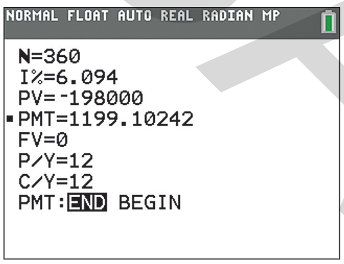

Figure 4

The two monthly payments in Example 7 are the same. With mortgage A, the borrower is essentially borrowing \$198,000 (since \$2000 is paid back to the lender immediately) and then making the same monthly payments as on a \$198,000 loan at 6.094%. The Truth-in-Lending Act requires the lender of mortgage A to specify that the loan has an APR of 6.094%. Purchasing discount points has the effect of changing the APR, which can be calculated by following the steps shown next.

APR The following steps calculate the APR for a mortgage loan having a term of n months:

1. Calculate the monthly payment (call it R) on the stated loan amount.
2. Subtract the up-front costs (such as points) from the stated loan amount. Denote the result by P.
3. Find the interest rate that produces the monthly payment R for a loan of P dollars to be repaid in n months.

The interest rate discussed in step 3 can be calculated in Excel with the RATE function as

$$12{*}\text{RATE}(n, -R, P, 0).$$

To find the interest rate in step 3 with a TI-84 Plus calculator, use TVM Solver. On a graphing calculator or with Wolfram|Alpha, find the intersection of the graphs of $Y_1 = P$ and $Y_2 = \frac{1-(1+X)^{-n}}{X} \cdot R$, and multiply the X-coordinate by 12.

EXAMPLE 8 **Calculating the APR for a Mortgage with Discount Points** Use the three steps just given to calculate the APR for mortgage A of Example 7.

TECHNOLOGY SOLUTION

1. From Example 7, the monthly payment is \$1199.10.
2. $P = 200{,}000 - 2000 = \$198{,}000$, since the up-front costs consist of one discount point.

3. The interest rate can be calculated in Excel with the RATE function. The value of

$$12*\text{RATE}(360, -1199.10, 198000, 0)$$

is .06093981.

TVM Solver gives an interest rate of 6.093981074%. See Fig. 5.

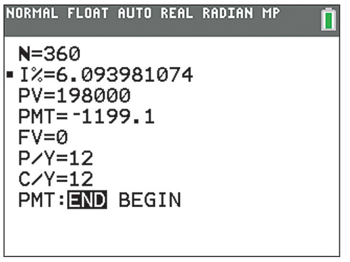

Figure 5

Therefore, the APR for mortgage A is 6.094%. **» Now Try Exercise 27**

The APR allows you to compare loans that are kept for their full terms. However, the typical mortgage is refinanced or terminated after approximately five years—that is, 60 months. Mortgage analysts often rely on the **effective mortgage rate** that takes into account the length of time the loan will be held and the unpaid balance at that time. For the \$200,000 mortgage A discussed in Example 7, the monthly payment is \$1199.10, and we show in Example 9 that the unpaid balance after 60 months is \$186,108.55. With a 5-year lifetime, the effective mortgage rate is the interest rate for which a decreasing annuity with a beginning balance of \$198,000 (\$200,000 loan − \$2000 cost of the point) and monthly payment of \$1199.10 will decline to \$186,108.55 after 60 months. A graphing calculator or a spreadsheet can be used to determine that the effective rate of interest for mortgage A is 6.24%. If terminated after 5 years, mortgage A is equivalent to a mortgage of \$200,000 at 6.24% interest compounded monthly and having no discount points. That is, with the assumption that mortgage A will be held for only 5 years, the interest rate is effectively 6.24%.

EXAMPLE 9 **Calculating Values Associated with Mortgages** Consider mortgage A of Example 7. The mortgage had an up-front payment of \$2000, an interest rate of 6% compounded monthly, and a monthly payment of \$1199.10. Let mortgage C be a 360-month loan of \$200,000 at 6.24% compounded monthly, with no points.

(a) Find the future value of \$2000 after 60 months at 6.24% compounded monthly.
(b) Find the future value of an increasing annuity consisting of 60 monthly payments of \$1199.10 with an interest rate of 6.24%.
(c) Find the unpaid balance on mortgage A after 60 months.
(d) Find the monthly payment for mortgage C.
(e) Find the future value of an increasing annuity consisting of 60 monthly payments of the amount found in part (d), with an interest rate of 6.24%.
(f) Find the unpaid balance on mortgage C after 60 months.
(g) Compare the sum of the three amounts found in parts (a), (b), and (c) with the sum of the two amounts found in parts (e) and (f).

SOLUTION **(a)**

$$\begin{aligned} F &= (1+i)^n P && \text{Formula for future value} \\ &= (1.0052)^{60}\cdot 2000 && n = 60,\ i = \frac{.0624}{12} = .0052,\ P = 2000 \\ &= 2730.10 && \text{Calculate. Round to two decimal places.} \end{aligned}$$

The future value of \$2000 after 60 months is \$2730.10. (The TVM Solver screen in Fig. 6 confirms this result.)

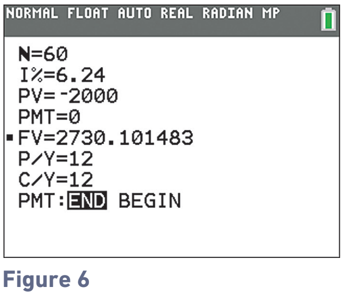

Figure 6

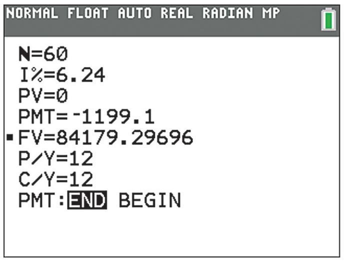

Figure 7

(b) $F = \frac{(1+i)^n - 1}{i} \cdot R$ Formula for future value of an increasing annuity

$= \frac{1.0052^{60} - 1}{.0052} \cdot 1199.10$ $i = \frac{.0624}{12} = .0052,\ n = 60,\ R = 1199.10$

$= 84{,}179.30$ Calculate. Round to two decimal places.

The future value after 60 months is \$84,179.30. (The TVM Solver screen in Fig. 7 confirms this result.)

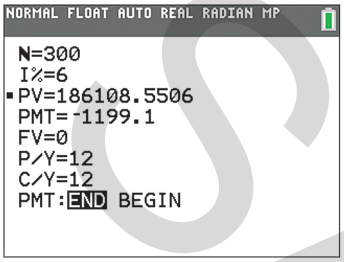

Figure 8

(c) After 5 years, there are $360 - 60 = 300$ months remaining. The unpaid balance is calculated as the present value of a sequence of $n = 300$ payments. Therefore,

$[\text{unpaid balance}] = \frac{1-(1+i)^{-n}}{i} \cdot R$ Formula for present value

$= \frac{1 - 1.005^{-300}}{.005} \cdot 1199.10$ $i = \frac{.06}{12} = .005,\ R = 1199.10$

$= 186{,}108.55$ Calculate. Round to two decimal places.

The unpaid balance after 60 months is \$186,108.55. (The TVM solver screen in Fig. 8 confirms this result.)

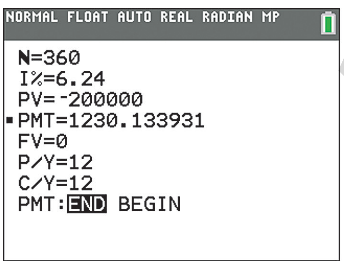

Figure 9

(d) $R = \frac{i}{1-(1+i)^{-n}} \cdot P$ Formula for rent (that is, monthly payment)

$= \frac{.0052}{1 - 1.0052^{-360}} \cdot 200{,}000$ $i = \frac{.0624}{12} = .0052,\ n = 360,\ P = 200{,}000$

$= 1230.13$ Calculate. Round to two decimal places.

The monthly payment is \$1230.13. (The TVM Solver screen in Fig. 9 confirms this result.)

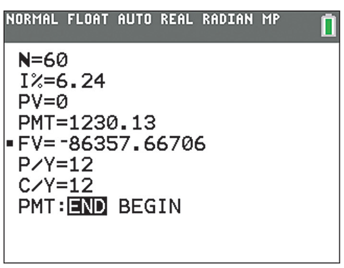

Figure 10

(e) $F = \frac{(1+i)^n - 1}{i} \cdot R$ Formula for future value of an increasing annuity

$= \frac{1.0052^{60} - 1}{.0052} \cdot 1230.13$ $i = \frac{.0624}{12} = .0052,\ n = 60,\ R = 1230.13$

$= 86{,}357.67$ Calculate. Round to two decimal places.

The future value after 60 months is \$86,357.67. (The TVM Solver screen in Fig. 10 confirms this result.)

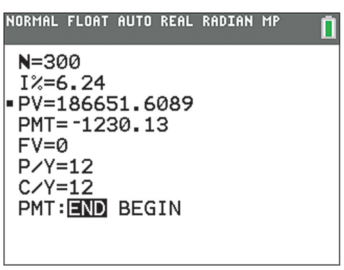

Figure 11

(f) Mortgage C has $360 - 60 = 300$ months to go.

$[\text{unpaid balance}] = \frac{1-(1+i)^{-n}}{i} \cdot R$ Formula for present value

$= \frac{1 - 1.0052^{-300}}{.0052} \cdot 1230.13$ $i = \frac{.0624}{12} = .0052,\ n = 300,\ R = 1230.13$

$= 186{,}651.61$ Calculate. Round to two decimal places.

The unpaid balance after 60 months is \$186,651.61. (The TMV Solver screen in Fig. 11 confirms this result.)

(g) $[\text{sum of amounts found for mortgage } A] = 2730.10 + 84{,}179.30 + 186{,}108.55 = \$273{,}017.95$

$[\text{sum of amounts found for mortgage } C] = 86{,}357.67 + 186{,}651.61 = \$273{,}009.28$

The two sums are very close. This demonstrates that the two loans are essentially equivalent after 60 months. (The difference of \$8.67 is due to round-off error.) «

Effective Mortgage Rate The following steps calculate the effective mortgage rate for a mortgage loan expected to be held for m months:

1. Calculate the monthly payment (call it R) on the stated loan amount.
2. Subtract the up-front costs (such as points) from the stated loan amount. Denote the result by P.
3. Determine the unpaid balance on the stated loan amount after m months. Denote the result by B.
4. Find the interest rate that causes a decreasing annuity with a beginning balance of P dollars and monthly payment R to decline to B dollars after m months.

The interest rate discussed in step 4 can be calculated in Excel with the RATE function as

$$12*\text{RATE}(m, -R, P, -B).$$

To calculate the interest rate with TVM Solver, set N $= m$, PV $= P$, PMT $= -R$, FV $= -B$, P/Y $= 12$ and solve for I%. The interest rate also can be found on a calculator by graphically solving the equation

$$R - iB = (R - iP)\cdot(1 + i)^m$$

for i and then multiplying the value of i by 12. The preceding equation says that the amount of the monthly payment applied to principal in the $(m + 1)$st payment is equal to the future value of the amount applied to principal in the first payment. There are several equations that involve m, R, P, and i. We chose this equation because it is one of the simplest.

EXAMPLE 10 **Calculating Effective Mortgage Rates** Use the four steps just given to calculate the effective mortgage rate for mortgage A of Example 7, assuming that the mortgage will be held for 5 years.

SOLUTION

1. The monthly payment $R = \$1199.10$ was calculated in Example 7.
2. $P = 200{,}000 - 2000 = 198{,}000$, since the up-front fee consists only of the one discount point.
3. The unpaid balance after 60 months, $B = \$186{,}108.55$, was computed in part (c) of Example 9.
4. The interest rate can be calculated in Excel with the RATE function. The value of

$$12*\text{RATE}(60, -1199.10, 198000, -186108.55)$$

is .0624069.

TVM Solver gives an interest rate of 6.240691672%, as shown in Fig. 12. With a graphing calculator or Wolfram|Alpha, the interest rate is found by graphically solving

$$1199.10 - X(186108.55) = (1199.10 - X(198000))(1 + X)^{60}$$

for X and then multiplying the answer by 12. The value obtained is .06240696. Therefore, the effective mortgage rate for mortgage A is $\approx$ 6.24%, which is the rate of mortgage C in Example 9. » *Now Try Exercise 31*

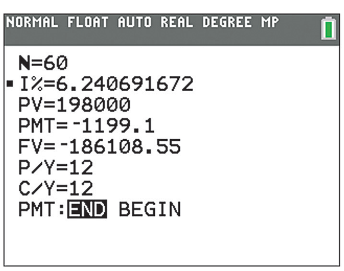

Figure 12

NOTE Mortgage analysts have a rule of thumb for calculating the effective mortgage rate for a mortgage with points. For mortgages that will be held for 4 to 6 years, each discount point adds about ¼% to the stated interest rate. For instance, for the mortgage in Example 10, the rule of thumb predicts that the effective mortgage rate should be ¼% higher than the stated rate for a mortgage having one discount point. This is very close to the actual increase, .24%. Table 1 extends the rule of thumb to other lifetimes. «

Table 1 Rule of Thumb for the Significance of Discount Points

Lifetime	Difference between Effective Mortgage Rate and Stated Interest Rate, Per Discount Point
1 year	1 percentage point
2 years	$\frac{1}{2}$ of a percentage point
3 years	$\frac{1}{3}$ of a percentage point
4 to 6 years	$\frac{1}{4}$ of a percentage point
7 to 9 years	$\frac{1}{6}$ of a percentage point
10 to 12 years	$\frac{1}{7}$ of a percentage point
More than 12 years	$\frac{1}{8}$ of a percentage point

Example 11 shows another way to take the expected lifetime of a loan into account when deciding between two mortgages. If you intend to either sell the house or refinance the loan within a few years, then you should avoid a loan carrying excess points, since the large amount of money paid up front will not be recovered. The longer you keep the mortgage, the less expensive points become.

EXAMPLE 11 **Using Expected Lifetime to Compare Mortgages** Suppose that a lender gives you a choice between the following two 30-year mortgages of $100,000:

Mortgage A: 6.1% interest compounded monthly, two points, monthly payment of $605.99

Mortgage B: 6.6% interest compounded monthly, no points, monthly payment of $638.66

The best choice of mortgage depends on the length of time that you expect to hold the mortgage. Mortgage B is the better choice if you plan to terminate or refinance the mortgage fairly early. Assuming that you can invest money at 3.6% interest compounded monthly, determine the length of time that you must retain the mortgage in order for mortgage A to be the better choice.

SOLUTION The difference in the monthly payments is $638.66 - 605.99 = \$32.67$. Although mortgage A saves $32.67 per month, it carries two points and therefore requires the payment of an additional $2000 up front. With mortgage B, the $2000 could be used to generate an income stream of $32.67 per month for a certain number of months, n, by investing it in an annuity at 3.6% compounded monthly.

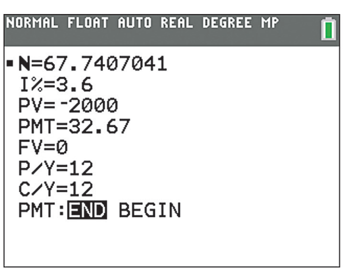

Figure 13

The number of months, n, can be calculated in Excel with the NPER function. The value of NPER(.003, 32.67, −2000, 0) is approximately 67.74. (*Note:* $.003 = .036/12$.)

TVM Solver gives a duration of 67.7407041 months, as shown in Fig. 13. The number of months, n, also can be found graphically on a calculator by solving

$$P = \frac{1-(1+i)^{-n}}{i} \cdot R$$

for n. This equation can be shown to be equivalent to

$$\left(\frac{1}{1+i}\right)^n = 1 - \frac{iP}{R}.$$

(See Exercise 40 for details.) With the settings $i = .003$, $P = 2000$, and $R = 32.67$, the equation has solution $n = 67.740704$.

Therefore, if you plan to hold the loan for 68 months (about $5\frac{2}{3}$ years) or more, you should choose mortgage A. **>> Now Try Exercise 53**

Interest-Only and Adjustable-Rate Mortgages

With an **interest-only mortgage**, the monthly payment for a certain number of years (usually, five or ten years) consists only of interest payments. That is, there is no amount applied to paying off the principal. At the end of the interest-only period, the monthly payment is determined by the number of months remaining. We will assume that the interest rate for the second period is the same as the interest rate for the first period. (With some interest-only mortgages, the interest rate is reset to conform to prevailing interest rates at that time.)

EXAMPLE 12 **Calculating Monthly Payments for an Interest-Only Mortgage** Consider a 30-year mortgage of \$400,000 at 6.6% interest compounded monthly, where the loan is interest-only for 10 years.

(a) What is the monthly payment during the first 10 years?

(b) What is the monthly payment during the last 20 years?

(c) Compare the monthly payment to the monthly payment of \$2554.64 for a conventional 30-year mortgage at 6.6% interest compounded monthly.

SOLUTION Here, $P = 400{,}000$ and $i = \frac{.066}{12} = .0055$.

(a) During the first 10 years, the monthly payment is

$$\begin{aligned} i \cdot P &= .0055(400{,}000) \\ &= \$2200. \end{aligned}$$

(b) At the beginning of the 11th year, the principal is still $P = 400{,}000$, and there are $20 \cdot 12 = 240$ months left to pay off the loan.

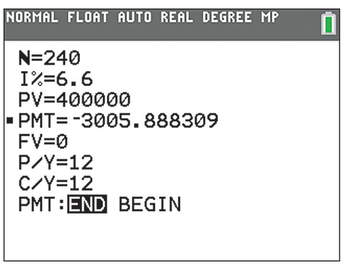

Figure 14

$$[\text{monthly payment}] = \frac{i}{1-(1+i)^{-n}} \cdot P$$

Formula for rent (that is, monthly payment)

$$= \frac{.0055}{1-1.0055^{-240}} \cdot 400{,}000$$

$i = \frac{.066}{12} = .0055$, $n = 240$, $P = 400{,}000$

$$= 3005.89$$

Calculate. Round to two decimal places.

Therefore, beginning in the 11th year, the monthly payment is \$3005.89. (The TVM Solver screen in Fig. 14 confirms this result.)

(c) Since $2554.64 - 2200 = 354.64$, the borrower's payment is \$354.64 a month less than a conventional loan for the first 10 years. However, since $3005.89 - 2554.64 = 451.25$, the borrower pays an additional \$451.25 per month for the last 20 years.

» Now Try Exercise 41

Some disadvantages of interest-only mortgages are as follows:

- Interest-only mortgages are riskier for lenders than fixed-rate mortgages and therefore tend to have slightly higher interest rates.
- When the higher monthly payment kicks in, the borrower might have difficulty making the payments.

Some advantages of interest-only mortgages are as follows:

- Borrowers with a modest current income who are reasonably certain that their income will increase in the future can afford to buy a more expensive house.
- In the early years of the mortgage, the savings can be contributed to an IRA that otherwise might not have been affordable.

Another type of mortgage in which monthly payments change is the **adjustable-rate mortgage** (ARM). With an ARM, the interest rate changes periodically as determined by a measure of current interest rates, called an **index**. The most common index in the United States, called the 1-year CMT (constant-maturity Treasury), is based on the yield of one-year Treasury bills. (During the past 10 years, the CMT index has ranged from about .10% to about 5.22%.) The interest rate on the ARM is reset by adding a fixed percent called the **margin** to the index percent. For instance, if the current value of the CMT index is 3% and the margin is 2.7%, then the adjusted interest rate for the mortgage would be 5.7%. The margin usually remains fixed for the duration of the loan.

Of the many types of ARMs, we will consider the so-called 5/1 ARM. With this type of mortgage, the interest rate is fixed for the first 5 years, and then is readjusted each year depending on the value of an index.

EXAMPLE 13 **Calculating Values Associated with an ARM** Consider a $200,000 5/1 ARM that has a 2.7% margin, is based on the CMT index, and has a 30-year maturity. Suppose that the interest rate is initially 5.7% and the value of the CMT index is 4.5% 5 years later when the rate adjusts. Assume that all interest rates use monthly compounding.

(a) Calculate the monthly payment for the first 5 years.
(b) Calculate the unpaid balance at the end of the first 5 years.
(c) Calculate the monthly payment for the 6th year.

SOLUTION **(a)** The monthly payment for the first 5 years is calculated as follows:

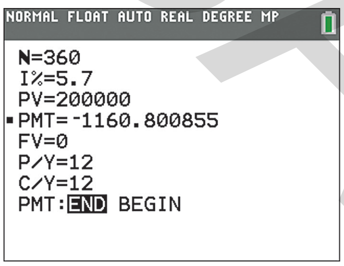

Figure 15

$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \qquad \text{Formula for rent (that is, monthly payment)}$$

$$= \frac{.00475}{1-1.00475^{-360}} \cdot 200{,}000 \qquad i = \frac{.057}{12} = .00475,\ n = 360,\ P = 200{,}000$$

$$= 1160.80 \qquad \text{Calculate. Round to two decimal places.}$$

The monthly payment for the first 5 years is $1160.80. (The TVM Solver screen in Fig. 15 confirms this result.)

(b) After 5 years, there are 360 − 60 = 300 months remaining. The unpaid balance is calculated as the present value of a sequence of 300 payments of $1160.80 each.

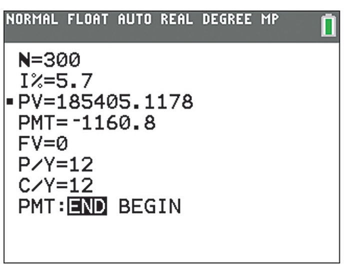

Figure 16

$$[\text{unpaid balance}] = \frac{1-(1+i)^{-n}}{i} \cdot R \qquad \text{Formula for present value}$$

$$= \frac{1-1.00475^{-300}}{.00475} \cdot 1160.80 \qquad i = \frac{.057}{12} = .00475,\ n = 300,\ R = 1160.80$$

$$= 185{,}405.12 \qquad \text{Calculate. Round to two decimal places.}$$

The balance after 5 years is $185,405.12. (The TVM Solver screen in Fig. 16 confirms this result.)

(c) Here, $P = \$185{,}405.12$ and $n = 300$. The annual rate of interest for the 6th year will be 4.5% + 2.7% = 7.2%. The new monthly payment is calculated as before.

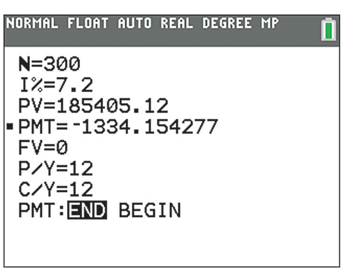

Figure 17

$$R = \frac{i}{1-(1+i)^{-n}} \cdot P \qquad \text{Formula for rent (that is, monthly payment)}$$

$$= \frac{.006}{1-1.006^{-300}} \cdot 185{,}405.12 \qquad i = \frac{.072}{12} = .006,\ n = 300,\ P = 185{,}405.12$$

$$= 1334.15 \qquad \text{Calculate. Round to two decimal places.}$$

Therefore, the monthly payment for the 6th year is $1334.15. (The TVM Solver screen in Fig. 17 confirms this results.)

>> Now Try Exercise 43

Caps There are several types of caps that can be specified for ARMs. **Periodic adjustment caps** limit the amount that the interest rate can adjust from one adjustment period to the next. **Lifetime caps** limit the interest-rate increase over the lifetime of the loan. **Payment caps** limit the amount that the monthly payment may increase from one adjustment to the next. (Periodic adjustment caps and payment caps usually do not apply to the first adjustment.) We will focus on payment caps.

Although payment caps limit the increase to the monthly payment, they can cause a slowdown in the repayment of the principal. The payment cap can even cause the unpaid balance to rise, a phenomenon called **negative amortization**.

EXAMPLE 14

Calculating Values Associated with an ARM Consider the mortgage discussed in Example 13, and assume that it carries a payment cap of 8%. (That is, the amount of the monthly payment in the seventh year can be at most 8% higher than the monthly payment for the sixth year.) Also, assume that the CMT index is 6.9% at the beginning of the seventh year.

(a) Calculate the unpaid balance of the loan at the beginning of the seventh year—that is, at the beginning of the 73rd month.
(b) Calculate the monthly payment for the seventh year, without using the payment cap.
(c) Calculate the monthly payment for the seventh year, using the payment cap.
(d) How much interest is due for the 73rd month?
(e) Determine the unpaid balance at the end of the 73rd month.

SOLUTION

(a) $[\text{unpaid balance}] = \frac{1-(1+i)^{-n}}{i} \cdot R$ — Formula for present value

$= \frac{1-1.006^{-288}}{.006} \cdot 1334.15$ — $i = \frac{.072}{12} = .006$, $n = 360 - 72 = 288$, $R = 1334.15$

$= 182{,}654.27$ — Calculate. Round to two decimal places.

The balance after 5 years is \$182,654.27. (The TVM Solver screen in Fig. 18 confirms this result.)

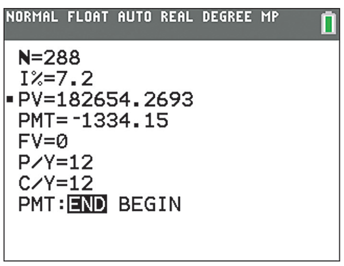

Figure 18

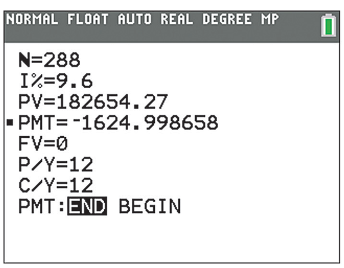

Figure 19

(b) The annual rate of interest for the seventh year will be 6.9% + 2.7% = 9.6%.

$[\text{new monthly payment}] = \frac{i}{1-(1+i)^{-n}} \cdot P$ — Formula for rent

$= \frac{.008}{1-1.008^{-288}} \cdot 182{,}654.27$ — $i = \frac{.096}{12} = .008$, $n = 288$, $P = 182{,}654.27$

$= 1625.00$ — Calculate. Round to two decimal places.

The new monthly payment for the seventh year (without using the payment cap) is \$1625.00. (The TVM Solver screen in Fig. 19 confirms this result.)

(c) Without the payment cap, the percentage increase in the monthly payment from the sixth year to the seventh year would be

$$\frac{1625.00 - 1334.15}{1334.15} \approx .2180 = 21.8\%.$$

Since 21.8% is greater than the payment cap of 8%, the payment cap kicks in to mandate that the monthly payment for the seventh year be

$$1.08 \cdot 1334.15 = \$1440.88.$$

(d) The interest due for the 73rd month is

$$.008 \cdot 182{,}654.27 = \$1461.23.$$

(e) The interest due is greater than the monthly payment by

$$1461.23 - 1440.88 = \$20.35.$$

Therefore, negative amortization occurs. The \$20.35 is added to the previous unpaid balance. The unpaid balance at the end of the 73rd month is

$$182{,}654.27 + 20.35 = \$182{,}674.62.$$

The increased unpaid balance will be taken into account when the next adjustment of the monthly payment is calculated. **» Now Try Exercise 45**

Here are a few words of caution about ARMs:

- Usually, monthly payments go up when interest rates rise and go down when interest rates fall—but not always.
- Some lenders offer ARMs with an artificially low interest rate, called a *discounted interest rate* or *teaser rate*. Such loans can produce payment shock when the interest rate adjusts.
- If you want to pay off your ARM early to avoid higher monthly payments, you might have to pay a penalty.

Check Your Understanding 10.4

Solutions can be found following the section exercises.

1. Rework Example 11 under the assumption that the \$2000 will not earn any interest.
2. Consider the two loans discussed in Example 7. Calculate the unpaid balance of each loan after 1 month.
3. Which is the better deal for a 30-year mortgage: (a) 6% interest plus three discount points (6.286% APR), or (b) 6.3% interest plus one discount point (6.396% APR)?

EXERCISES 10.4

In Exercises 1 and 2, fill in the blank with the word "free" or "deferred."

1. Interest earned in a traditional IRA is tax ______.
2. Interest earned in a Roth IRA is tax _______.
3. **Traditional IRA** Carlos is 60 years old, is in the 45% marginal tax bracket, and has \$300,000 in his traditional IRA. How much money will he have after taxes if he withdraws all of the money from the account?
4. **Roth IRA** Rework Exercise 3 for a Roth IRA.
5. **Traditional IRA** If you are 18 years old, deposit \$5000 each year into a traditional IRA for 52 years, at 6% interest compounded annually, and retire at age 70, how much money will be in the account upon retirement?
6. **Comparing IRAs** Rework Examples 1 and 3 for the case where the marginal tax bracket is 30% when the money is contributed but is only 25% when the balance is withdrawn. Which type of IRA is most advantageous?
7. **Roth IRA** Rework Exercise 5 for a Roth IRA.
8. **Comparing IRAs** Rework Examples 1 and 3 for the case where the marginal tax bracket is 30% when the money is contributed but rises to 35% when the balance is withdrawn. Which type of IRA is most advantageous?
9. **Value of Starting an IRA Early** Redo Example 4 where Earl and Larry are each in a 40% marginal tax bracket, have Roth IRAs, and contribute the remainder of \$5000 after taxes are deducted.
10. **Advantage of Prompt IRA Contributions** The contribution into an IRA for a particular year can be made any time from January 1 of that year to April 15 of the following year. Suppose that Enid and Lucy both set up traditional IRA accounts on January 1 of 2017 and each contributes \$5000 into her account for 10 years at 6% interest compounded annually. Assume that Enid makes her contributions as soon as possible and Lucy makes her contributions 1 year later. Calculate the balances in the two accounts at the time Lucy makes her final contribution.

In Exercises 11–14, use the add-on method to determine the monthly payment.

11. \$4000 loan at 10% stated interest rate for 1 year
12. \$6000 loan at 7% stated interest rate for 1 year
13. \$3000 loan at 9% stated interest rate for 3 years
14. \$10,000 loan at 8% stated interest rate for 2 years

In Exercises 15–18, give the add-on interest rate.

15. A \$2000 loan for 1 year at 5% APR with a monthly payment of \$171.21
16. A \$5000 loan for 1 year at 6% APR with a monthly payment of \$430.33
17. A \$20,000 loan for 3 years at 6% APR with a monthly payment of \$608.44
18. A \$10,000 loan for 2 years at 7% APR with a monthly payment of \$447.73
19. **Discount Method** Another method used by lenders to determine the monthly payment for a loan is the *discount method*. In this case, the stated interest rate is called the *discount rate* and the formula for determining the total amount paid is

$$[\text{total payment}] = \frac{[\text{loan amount}]}{1 - rt},$$

where r is the discount rate and t is the term of the loan in years. The monthly payment is then [total payment]/$12t$.
(a) Calculate the monthly payment for a 2-year loan of \$880 with a discount rate of 6%.
(b) If the add-on method were used instead, would the monthly payment be greater than or less than the amount found in part (a)?

20. True or False For any mortgage with discount points, the APR will be greater than the stated interest rate.

21. True or False For any mortgage that will be terminated before its full term, the effective mortgage rate will be greater than the stated rate.

22. True or False If a mortgage is expected to be held for its entire term, the effective mortgage rate will be the same as the APR.

23. True or False The longer a loan with discount points is expected to be held, the greater its effective mortgage rate will be.

24. True or False Increasing the discount points for a mortgage increases its APR and its effective mortgage rate.

25. True or False Changing the amount of a loan while keeping the stated interest rate, the term, and the up-front costs the same will have no effect on the APR.

26. True or False Refer to Example 10. If the stated interest rate were 9% instead of 6%, then the effective mortgage rate would be 9.24%. That is, the interest rate would increase by the same amount as before, .24%.

APR In multiple-choice Exercises 27–30, assume that the loan amount is \$250,000 and confirm your answer by calculating monthly payments. (*Note:* The two monthly payments will not be identical, due to round-off error.)

27. The APR for a 25-year mortgage at 9% interest compounded monthly and with two discount points is
(a) 8.75%. **(b)** 9.05%. **(c)** 12%. **(d)** 9.25%.

28. The APR for a 30-year mortgage at 5.9% interest compounded monthly and having one discount point is
(a) 9%. **(b)** 6%. **(c)** 5.7%.

29. The APR for a 20-year mortgage at 5.5% interest compounded monthly and with four discount points is
(a) 6%. **(b)** 5%. **(c)** 6.5%.

30. The APR for a 25-year mortgage at 9% interest compounded monthly and having four discount points is
(a) 8.5%. **(b)** 9.5%. **(c)** 11%.

Effective Mortgage Rate In multiple-choice Exercises 31–34, assume that the loan amount is \$100,000 and confirm your answer by calculating sums as in Example 9.

31. The effective mortgage rate for a 30-year mortgage at 5.71% interest compounded monthly with one discount point that is expected to be kept for 4 years is
(a) 6%. **(b)** 5.54%. **(c)** 6.56%. **(d)** 7%.

(*Hint:* For a 5.71%, 30-year mortgage, $R = \$581.03$ and the unpaid balance after 4 years is \$94,341.50.)

32. The effective mortgage rate for a 15-year mortgage at 5.481% interest compounded monthly, with three discount points, that is expected to be kept for 10 years is
(a) 5%. **(b)** 6%. **(c)** 6.5%. **(d)** 7%.

(*Hint:* For a 5.481%, 15-year mortgage, $R = \$816.08$ and the unpaid balance after 10 years is \$42,743.72.)

33. The effective mortgage rate for a 30-year mortgage at 6% interest compounded monthly, with three discount points, that is expected to be kept for 7 years is
(a) 5.5%. **(b)** 6.2%. **(c)** 6.56%. **(d)** 9%.

34. The effective mortgage rate for a 30-year mortgage at 9% interest compounded monthly, with two discount points, that is expected to be kept for 7 years is
(a) 9%. **(b)** 8.5%. **(c)** 9.4%. **(d)** 10%.

In Exercises 35 and 36, give the two Excel formulas that can be used to obtain the APR and the effective mortgage rate for the mortgage. When calculating the effective mortgage rate, assume that the mortgage will be held for 10 years.

35. A 20-year mortgage of \$80,000 carrying a stated interest rate of 6% and three points

36. A 15-year mortgage of \$180,000 carrying a stated interest rate of 9% and two points

In Exercises 37 and 38, give the two equations that can be solved for i with a graphing calculator to obtain the monthly interest rates corresponding to the APR and the effective mortgage rate. Assume that the loans will be held for 5 years.

37. A 15-year mortgage of \$120,000 carrying a stated interest rate of 9% and one point

38. A 20-year mortgage of \$150,000 carrying a stated interest rate of 6% and three points

39. Loan Quandary You decide to buy a \$1250 entertainment system for your apartment. The salesman asks you to pay 20% down and is willing to finance the remaining \$1000 with a 2-year loan having an APR of 5% and a monthly payment of \$43.87. (The total of your payments on the loan will be \$1052.88.) Suppose that you actually have \$1000 in your savings account (after the down payment) and can invest it at 4% compounded monthly. You must decide whether or not to take the loan. The salesman tells you that you will come out ahead if you take the loan, since \$1000 invested at 4% interest compounded monthly grows to \$1083.14, which is greater than the total payments on the loan. This doesn't make sense to you since you would be borrowing money at 5% and only earning 4% on the money you invest. What is wrong with the salesman's reasoning? (*Note:* Ignore taxes.)

40. Perform algebraic manipulations on the equation $P = \frac{1-(1+i)^{-n}}{i} \cdot R$ to obtain $\left(\frac{1}{1+i}\right)^n = 1 - \frac{iP}{R}$. *Hint:* Multiply the equation by $\frac{i}{R}$, replace $(1+i)^{-n}$ by $\left(\frac{1}{1+i}\right)^n$, and then solve for $\left(\frac{1}{1+i}\right)^n$.

41. Interest-Only Mortgage Consider a 15-year mortgage of \$200,000 at 6.9% interest compounded monthly, where the loan is interest-only for 5 years. What is the monthly payment during the first 5 years? Last 10 years?

42. Interest-Only Mortgage Consider a 15-year mortgage of \$300,000 at 7.2% interest compounded monthly, where the loan is interest-only for 5 years. What is the monthly payment during the first 5 years? Last 10 years?

43. Adjustable-Rate Mortgage Consider a 25-year \$250,000 5/1 ARM having a 2.5% margin and based on the CMT index. Suppose that the interest rate is initially 6% and the value of the CMT index is 4.4% five years later. Assume that all interest rates use monthly compounding.

(a) Calculate the monthly payment for the first 5 years.
(b) Calculate the unpaid balance at the end of the first 5 years.
(c) Calculate the monthly payment for the sixth year.

44. **Adjustable-Rate Mortgage** Consider a 15-year $300,000 5/1 ARM having a 3% margin and based on the CMT index. Suppose that the interest rate is initially 6.3% and the value of the CMT index is 3% 5 years later. Assume that all interest rates use monthly compounding.
(a) Calculate the monthly payment for the first 5 years.
(b) Calculate the unpaid balance at the end of the first 5 years.
(c) Calculate the monthly payment for the sixth year.

45. **Adjustable-Rate Mortgage** Consider the mortgage discussed in Exercise 43, and assume that it carries a payment cap of 7%. Also, assume that the CMT index is 7.7% at the beginning of the seventh year.
(a) Calculate the unpaid balance of the loan at the beginning of the seventh year.
(b) Calculate the monthly payment for the seventh year, without using the payment cap.
(c) Calculate the monthly payment for the seventh year, using the payment cap.
(d) How much interest is due for the 73rd month?
(e) Determine the unpaid balance at the end of the 73rd month.

46. **Adjustable-Rate Mortgage** Consider Example 14. What is the unpaid balance at the end of the 74th month?

TECHNOLOGY EXERCISES

47. Find the APR on a $10,000 loan with an add-on method interest rate of 6% compounded monthly for 3 years.

48. Find the APR on a $15,000 loan with an add-on method interest rate of 8% compounded monthly for 2 years.

49. Find the APR on a 25-year mortgage of $300,000 carrying a stated interest rate of 6.5% compounded monthly and having three points.

50. Find the APR on a 30-year mortgage of $250,000 carrying a stated interest rate of 6% compounded monthly and having two points.

51. Find the effective mortgage rate for a 25-year $140,000 mortgage at 6.5% interest compounded monthly, with three discount points. Assume that the mortgage will be held for 7 years.

52. Find the effective mortgage rate for a 30-year $250,000 mortgage at 6% interest compounded monthly, with two discount points, that is expected to be kept for 6 years.

53. **Comparing Mortgages** Suppose that a lender gives you a choice between the following two 25-year mortgages of $175,000:

Mortgage A: 6.3% interest compounded monthly, with three points, monthly payment of $1159.84

Mortgage B: 6.5% interest compounded monthly, with two points, monthly payment of $1181.61

Assuming that you can invest money at 2.4% interest compounded monthly, determine the length of time that you must retain the mortgage in order for mortgage A to be the better choice.

54. **Comparing Mortgages** Suppose that a lender gives you a choice between the following two 15-year mortgages of $250,000:

Mortgage A: 6.5% interest compounded monthly, with one point, monthly payment of $2177.77

Mortgage B: 6.1% interest compounded monthly, with two points, monthly payment of $2123.17

Assuming that you can invest money at 3.6% interest compounded monthly, determine the length of time that you must retain the mortgage in order for mortgage B to be the better choice.

55. **Comparing Mortgages** A lender gives you a choice between the following two 30-year mortgages of $200,000:

Mortgage A: 6.65% interest compounded monthly, with one point, monthly payment of $1283.93

Mortgage B: 6.8% interest compounded monthly, with no points, monthly payment of $1303.85

Assuming that you can invest money at 4.8% interest compounded monthly, determine the length of time that you must retain the mortgage in order for mortgage A to be the better choice.

56. **Comparing Mortgages** A lender gives you a choice between the following two 30-year mortgages of $235,000:

Mortgage A: 6.9% interest compounded monthly, with one point, monthly payment of $1547.71

Mortgage B: 6.5% interest compounded monthly, with three points, monthly payment of $1,485.36

Assuming that you can invest money at 3.3% interest compounded monthly, determine the length of time that you must retain the mortgage in order for mortgage B to be the better choice.

Solutions to Check Your Understanding 10.4

1. In this case, we merely have to solve $n \cdot (32.67) = 2000$.

$$n = \frac{2000}{32.67} = 61.218$$

Therefore, take the additional points only if you expect to hold onto the loan for more than 61 months (that is, a little more than 5 years).

NOTE This computation is much simpler than the one in Example 11 and can be used to get a rough estimate of the proper number of months. «

2. Successive balances can be calculated with the formula $B_{\text{new}} = (1 + i)B_{\text{previous}} - R$. Mortgage A was a $200,000 loan at 6% interest with a monthly payment of $1199.10.

$$[\text{balance after 1 month}] = (1.005) \cdot 200{,}000 - 1199.10 = \$199{,}800.90$$

Mortgage B was a $198,000 loan at 6.094% interest with a monthly payment of $1199.10. The monthly interest $i = .06094/12 = .0050783333$.

$$[\text{balance after 1 month}] = (1.0050783333) \cdot 198{,}000 - 1199.10 = \$197{,}806.41$$

The balance for mortgage A is \$1994.49 higher. The balance will remain higher until the final payment is made.

3. It depends on how long you intend to keep the mortgage. If you expect to hold the mortgage for 30 years, then mortgage A is clearly superior because it has a lower APR. On the other hand, if you expect to terminate the mortgage after two years, then Table 1 on page 467 says that the effective rates are approximately 7.5% $\left[3 \cdot \left(\frac{1}{2}\%\right) + 6\%\right]$ and 6.8% $\left[1 \cdot \left(\frac{1}{2}\%\right) + 6.3\%\right]$, respectively, obviously favoring mortgage B.

10.5 A Unifying Equation

In this chapter we discuss a number of topics from the mathematics of finance: compound interest, simple interest, mortgages, and annuities. As we shall see, all such financial transactions can be described by a single type of equation, called a *difference equation.* Furthermore, the same type of difference equation can be used to model many other phenomena, such as the spread of information, radioactive decay, and population growth, to mention just a few.

NOTE If you have not read the first three sections of this chapter, or would like a refresher of the financial terms discussed there, read the appendix at the end of this section. «

In the first three sections of this chapter, we encountered the following four formulas describing the change in the balance B at the end of each interest period. (Here, r is the stated annual interest rate, i is the interest rate per interest period, P is the principal, and R is the payment made at the end of each interest period.)

$$\begin{aligned} \text{Compound interest:} \quad & B_{\text{new}} = (1 + i)B_{\text{previous}} \\ \text{Simple interest:} \quad & B_{\text{new}} = B_{\text{previous}} + rP \\ \text{Increasing annuity:} \quad & B_{\text{new}} = (1 + i)B_{\text{previous}} + R \\ \text{Decreasing annuity:} \quad & B_{\text{new}} = (1 + i)B_{\text{previous}} - R \end{aligned}$$

Each formula has the form

$$B_{\text{new}} = a \cdot B_{\text{previous}} + b, \tag{1}$$

where a and b are numbers. For compound interest, $a = (1 + i)$, $b = 0$; for simple interest, $a = 1$, $b = rP$; for an increasing annuity, $a = (1 + i)$, $b = R$; and for a decreasing annuity, $a = (1 + i)$, $b = -R$. *Note:* A loan is actually a decreasing annuity with the borrower paying the interest.

If we let y_n represent the balance after n interest periods, then formula (1) can be written as

$$y_n = a \cdot y_{n-1} + b. \tag{2}$$

Formula (2) can be used to generate successive balances once the beginning balance, y_0, is given. The combination of formula (2) and the value of y_0 is called a **difference equation**. The number y_0 is called the **initial value** of the difference equation. (Specifically, the difference equation is called a **linear difference equation with an initial value**.) Table 1 summarizes the financial difference equations.

Table 1 Summary of Financial Difference Equations

Compound interest	$y_n = (1 + i) \cdot y_{n-1}$	y_0 = initial amount deposited
Simple interest	$y_n = y_{n-1} + ry_0$	y_0 = initial amount deposited
Increasing annuity	$y_n = (1 + i) \cdot y_{n-1} + R$	y_0 = initial amount deposited
Decreasing annuity (or loan)	$y_n = (1 + i) \cdot y_{n-1} - R$	y_0 = initial amount deposited (or borrowed)

EXAMPLE 1 **Generate Values of a Difference Equation** Determine the first five values generated by the difference equation $y_n = .2y_{n-1} + 4.8$, $y_0 = 1$.

SOLUTION Beginning with the value of y_0, formula (2) can be used to successively obtain y_1, y_2, y_3, and so on. The first five values are as follows:

$$y_0 = 1$$
$$y_1 = .2y_0 + 4.8 = .2(1) + 4.8 = 5$$
$$y_2 = .2y_1 + 4.8 = .2(5) + 4.8 = 5.8$$
$$y_3 = .2y_2 + 4.8 = .2(5.8) + 4.8 = 5.96$$
$$y_4 = .2y_3 + 4.8 = .2(5.96) + 4.8 = 5.992$$

>> Now Try Exercise 15(a)

Not only can we generate as many values as we like for a difference equation, there are formulas that directly give any specific value. Such a formula is called a **solution** of the difference equation. The two formulas below are derived at the end of the section.

Solution of a Difference Equation The solution of the difference equation $y_n = a \cdot y_{n-1} + b$, y_0 is

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \qquad \text{when } a \neq 1$$

$$y_n = y_0 + bn \qquad \text{when } a = 1$$

EXAMPLE 2 **Solve a Difference Equation** Determine the solution of the difference equation in Example 1 and use it to calculate y_3.

SOLUTION The difference equation is $y_n = .2y_{n-1} + 4.8$, $y_0 = 1$. Here $a = .2$ and $b = 4.8$. Therefore,

$$\frac{b}{1-a} = \frac{4.8}{1-.2} = \frac{4.8}{.8} = 6$$

and

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \qquad \text{General solution}$$
$$= 6 + (1-6)(.2)^n \qquad \frac{b}{1-a} = 6,\ y_0 = 1,\ a = .2$$
$$= 6 + (-5)(.2)^n \qquad \text{Subtract.}$$
$$= 6 - 5(.2)^n \qquad \text{Rewrite.}$$

Therefore, the solution of the difference equation is $y_n = 6 - 5(.2)^n$.

When $n = 3$,

$$y_3 = 6 - 5(.2)^3 = 6 - 5(.008) = 5.96$$

Notice that the value of y_3 agrees with the value found in Example 1.

>> Now Try Exercise 15(b)

EXAMPLE 3 **Financial Difference Equations** Determine the difference equation for each of the following financial transactions where y_n is the balance after n interest periods.

(a) \$1000 is deposited into a savings account earning 4% interest compounded quarterly.
(b) \$1000 is deposited into a savings account earning 3% annual simple interest.

(c) At the end of each week, \$100 is deposited into a savings account earning 2.6% interest compounded weekly.
(d) A \$304,956.29 mortgage has an interest rate of 4.8% compounded monthly and a monthly payment of \$1600.

SOLUTION

(a) $y_n = (1 + i)y_{n-1}$, y_0 given — Difference equation for compound interest

$y_n = (1 + .01)y_{n-1}$, $y_0 = 1000$ — $i = \frac{.04}{4} = .01$, $y_0 = 1000$

$y_n = (1.01)y_{n-1}$, $y_0 = 1000$ — Add.

(b) $y_n = y_{n-1} + ry_0$, y_0 given — Difference equation for simple interest

$y_n = y_{n-1} + .03 \cdot 1000$, $y_0 = 1000$ — $r = .03$, $y_0 = 1000$

$y_n = y_{n-1} + 30$, $y_0 = 1000$ — Multiply.

(c) $y_n = (1 + i)y_{n-1} + R$, y_0 given — Difference equation for increasing annuity

$y_n = (1 + .0005)y_{n-1} + 100$, $y_0 = 0$ — $i = \frac{.026}{52} = .0005$, $R = 100$

$y_n = 1.0005y_{n-1} + 100$, $y_0 = 0$ — Add.

(d) $y_n = (1 + i)y_{n-1} - R$, y_0 given — Difference equation for decreasing annuity

$y_n = (1 + .004)y_{n-1} - 1600$, $y_0 = 304{,}956.29$ — $i = \frac{.048}{12} = .004$, $R = 1600$, $y_0 = 304{,}956.29$

$y_n = 1.004y_{n-1} - 1600$, $y_0 = 304{,}956.29$ — Add.

›› Now Try Exercises 1, 3, 5, 9

The financial formulas presented in the first part of this chapter can be derived from difference equations.

EXAMPLE 4 **Future Value of an Increasing Annuity** Use a difference equation to derive the formula for the future value of an increasing annuity in which R dollars is deposited into a savings account at the end of each interest period and the interest rate per period is i.

SOLUTION The difference equation for an increasing annuity is $y_n = (1 + i)y_{n-1} + R$, $y_0 = 0$. Now find the solution to this equation.

$$\frac{b}{1-a} = \frac{R}{1-(1+i)} = \frac{R}{-i} = -\frac{R}{i}$$

Therefore,

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \qquad \text{General solution}$$

$$= -\frac{R}{i} + \left[0 - \left(-\frac{R}{i}\right)\right](1+i)^n \qquad \text{Replace } \frac{b}{1-a} \text{ with } -\frac{R}{i}, \text{ and } a \text{ with } (1+i).$$

$$= -\frac{R}{i} + \frac{R}{i}(1+i)^n \qquad 0 - \left(-\frac{R}{i}\right) = \frac{R}{i}$$

$$= \frac{R}{i}(1+i)^n - \frac{R}{i}\cdot 1 \qquad \text{Reorder terms. Replace } \frac{R}{i} \text{ with } \frac{R}{i}\cdot 1.$$

$$= \frac{R}{i}[(1+i)^n - 1] \qquad \text{Factor out } \frac{R}{i}.$$

$$= \frac{(1+i)^n - 1}{i}\cdot R \qquad \text{Rewrite.}$$

›› Now Try Exercise 27

EXAMPLE 5 **Balance in a Mortgage** The amortization table in Section 10.3 shows that for a \$112,475 mortgage at 9% interest compounded monthly with a monthly payment of \$905, the balance after 12 months is \$111,706.57. Use a difference equation to derive a formula for the balances of the mortgage and use the formula to calculate the balance after 12 months.

SOLUTION Let y_n be the balance after n months.

$$y_n = (1+i)y_{n-1} - R,\ y_0 = \text{amount borrowed} \qquad \text{Difference equation for a loan}$$
$$= (1.0075)y_{n-1} - 905,\ y_0 = 112{,}475 \qquad i = \tfrac{.09}{12} = .0075,\ R = 905,\ y_0 = 112{,}475$$

Therefore,

$$\frac{b}{1-a} = \frac{-905}{1-(1.0075)} = \frac{-905}{-.0075} = \frac{905}{.0075} \qquad \text{Calculate } \frac{b}{1-a} \text{ with } a = 1.0075,\ b = -905.$$

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \qquad \text{General solution}$$

$$= \frac{905}{.0075} + \left(112{,}475 - \frac{905}{.0075}\right)(1.0075)^n \qquad \text{Substitute values for } \frac{b}{1-a},\ y_0, \text{ and } a.$$

The balance after 12 months is

$$y_{12} = \frac{905}{.0075} + \left(112{,}475 - \frac{905}{.0075}\right)(1.0075)^{12} \qquad \text{Substitute 12 for } n.$$

$$= \$111{,}706.57 \qquad \text{Calculate. Round to two decimal places.}$$

» Now Try Exercise 33

NOTE The solution to Example 5 also can be obtained by solving the general difference equation to obtain $y_n = \frac{R}{i} + \left(y_0 - \frac{R}{i}\right)(1+i)^n$, and then substituting the given values. «

EXAMPLE 6 **Loan Payment** A car loan of \$10,000 is to be repaid with quarterly payments for 5 years at 6.4% interest compounded quarterly. Calculate the quarterly payment R.

SOLUTION Let y_n be the balance after n quarters. The balance owed on the loan after n quarterly payments satisfies the difference equation

$$y_n = (1.016)y_{n-1} - R,\ y_0 = 10{,}000 \qquad 1 + i = 1 + \tfrac{.064}{4} = 1 + .016 = 1.016$$

Since

$$\frac{b}{1-a} = \frac{-R}{1-1.016} = \frac{-R}{-.016} = \frac{R}{.016}, \qquad \text{Calculate } \frac{b}{1-a} \text{ with } a = 1.016,\ b = -R.$$

the solution of the difference equation is

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \qquad \text{General solution}$$

$$y_n = \frac{R}{.016} + \left(10{,}000 - \frac{R}{.016}\right)(1.016)^n \qquad \frac{b}{1-a} = \frac{R}{.016},\ y_0 = 10{,}000,\ a = 1.016$$

After 20 quarters, the loan will be paid off (that is, $y_{20} = 0$) and the equation becomes

$$\frac{R}{.016} + \left(10{,}000 - \frac{R}{.016}\right)(1.016)^{20} = 0 \qquad n = 20$$

Now solve this equation for R.

$$\frac{R}{.016} + 10{,}000(1.016)^{20} - \left(\frac{R}{.016}\right)(1.016)^{20} = 0 \quad \text{Multiply values in parentheses by } (1.016)^{20}.$$

$$\frac{R}{.016}[1 - (1.016)^{20}] + 10{,}000(1.016)^{20} = 0 \quad \text{Subtract like terms and factor out } \frac{R}{.016}.$$

$$\frac{R}{.016}[1 - (1.016)^{20}] = -10{,}000(1.016)^{20} \quad \text{Subtract } 10{,}000(1.016)^{20}.$$

$$\frac{R}{.016} = \frac{-10{,}000(1.016)^{20}}{1 - (1.016)^{20}} \quad \text{Divide by } [1 - (1.016)^{20}].$$

$$R = .016\left(\frac{-10{,}000(1.016)^{20}}{1 - (1.016)^{20}}\right) \quad \text{Multiply by .016.}$$

$$R = 588.22 \quad \text{Calculate. Round to two decimal places.}$$

Therefore, the quarterly payment is $588.22.

›› Now Try Exercise 37

Nonfinancial Applications

The difference equation developed in this section can be used to model topics in many fields—such as demographics, economics, biology, psychology, sociology, and physics.

EXAMPLE 7 **Population Growth** Suppose the population of a certain country is currently 6 million. The growth of this population attributable to an excess of births over deaths is 2% per year. Further, the country is experiencing immigration at the rate of 40,000 people per year. Let y_n denote the population (in millions) of the country after n years.

(a) Determine a difference equation for y_n.

(b) Solve the difference equation in part (a) and use it to calculate the population of the country after 35 years.

SOLUTION **(a)** $y_0 = 6$. The growth of the population in year n due to an excess of births over deaths is $.02y_{n-1}$. There are .04 (million) immigrants each year. Therefore,

$$y_n = y_{n-1} + .02y_{n-1} + .04$$

$$\begin{bmatrix}\text{new}\\ \text{population}\end{bmatrix} = \begin{bmatrix}\text{previous}\\ \text{population}\end{bmatrix} + \begin{bmatrix}\text{natural}\\ \text{increase}\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{immigrants}\end{bmatrix}$$

Thus, the difference equation for the population is

$$y_n = 1.02y_{n-1} + .04,\ y_0 = 6.$$

(b)

$$\frac{b}{1-a} = \frac{.04}{1-(1.02)} = \frac{.04}{-.02} = -2 \quad b = .04,\ a = 1.02$$

Therefore, the solution of the difference equation is

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \quad \text{General solution}$$

$$= -2 + [6 - (-2)](1.02)^n \quad \frac{b}{1-a} = -2,\ y_0 = 6,\ a = 1.02$$

$$= -2 + 8(1.02)^n. \quad \text{Subtract.}$$

The population after 35 years will be

$$y_{35} = -2 + 8(1.02)^{35} \quad \text{Substitute 35 for } n.$$

$$= 14. \quad \text{Calculate. Round to a whole number.}$$

That is, after 35 years the population will be about 14 million.

›› Now Try Exercise 41

EXAMPLE 8

Supply and Demand This year's level of production and price for most agricultural products affects the level of production and price for the next year. Suppose that the current crop of soybeans in a certain country is 80 million bushels. Let q_n denote the quantity of soybeans (in millions of bushels) grown n years from now, and let p_n denote the market price (in dollars per bushel) in n years. Suppose that experience has shown that q_n and p_n are related by the following equations:

$$p_n = 20 - .1q_n \qquad q_n = 5p_{n-1} - 10.$$

(a) Determine a difference equation for q_n.

(b) Solve the difference equation in part (a) and use it to calculate the number of bushels of soybeans produced after 4 years.

SOLUTION

(a)

$$\begin{aligned} q_0 &= 80 && \text{Initial value} \\ q_n &= 5p_{n-1} - 10 && \text{Given equation} \\ &= 5(20 - .1q_{n-1}) - 10 && \text{Substitute } (20 - .1q_{n-1}) \text{ for } p_{n-1}. \\ &= 100 - .5q_{n-1} - 10 && \text{Multiply.} \\ &= -.5q_{n-1} + 90 && \text{Combine numbers.} \end{aligned}$$

Therefore, the difference equation is $q_n = -.5q_{n-1} + 90$, $q_0 = 80$.

(b) Since $a = -.5$ and $b = 90$,

$$\frac{b}{1-a} = \frac{90}{1-(-.5)} = \frac{90}{1.5} = 60.$$

Therefore, the solution of the difference equation is

$$\begin{aligned} q_n &= \frac{b}{1-a} + \left(q_0 - \frac{b}{1-a}\right)a^n && \text{General solution} \\ &= 60 + (80 - 60)(-.5)^n && \frac{b}{1-a} = 60,\ q_0 = 80,\ a = -.5 \\ &= 60 + 20(-.5)^n. && \text{Subtract.} \\ q_4 &= 60 + 20(-.5)^4 && \text{Quantity produced after 4 years} \\ &= 61.25 && \text{Calculate.} \end{aligned}$$

That is, after 4 years, 61.25 million bushels of soybeans will be produced. «

INCORPORATING TECHNOLOGY

Display Successive Values Graphing calculators can easily generate successive values of a difference equation. Consider the difference equation $y_n = .2y_{n-1} + 4.8$, $y_0 = 1$, of Examples 1 and 2. In Fig. 1, successive values are displayed in the home screen, and in Fig. 2, they are displayed in a table.

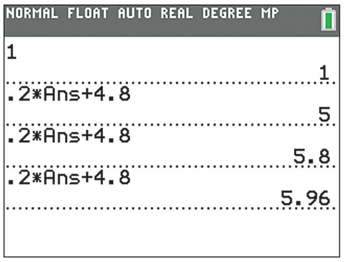

Figure 1

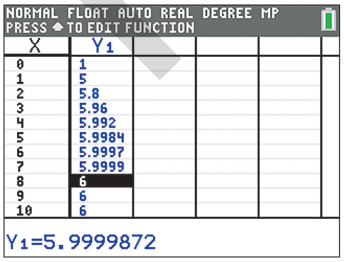

Figure 2

Display Successive Values on the Home Screen In Fig. 1, after the initial value (1) is entered, the last displayed value (1) is assigned to **Ans**. The instruction **.2*Ans+4.8** generates the next value (5) and assigns it to **Ans**. Each subsequent press of ENTER generates another value.

Display Successive Values in a Table To obtain the table in Fig. 2, set **Y₁=6−5*.2^x** in the **Y=** editor. Then press [2nd] [TBLSET] to bring up the TABLE SETUP screen. Set **TblStart** to 0, and set **ΔTbl** to 1. Finally, press [2nd] [TABLE] to bring up the table. The down-arrow key can be used to generate further values.

The table in Fig. 3 is easily created in an Excel spreadsheet. The following steps create the two columns of the table.

1. To enter the numbers 0 through 6 in the first column, enter 0 into cell A2, enter 1 into cell A3, select the two cells, and drag the fill handle down to A8.
2. Type 1 into cell B2, and press [ENTER].
3. Type =.2*B2+4.8 into cell B3, and press [ENTER].
4. Click on cell B3, and drag its fill handle down to B8.

	A	B
1	n	y_n
2	0	1
3	1	5
4	2	5.8
5	3	5.96
6	4	5.992
7	5	5.9984
8	6	5.99968

Figure 3

n	$y(n)$
0	1
1	5
2	5.8
3	5.96
4	5.992
5	5.9984
6	5.99968

Figure 4

WolframAlpha The instruction **y(n) = .2y(n−1) + 4.8, y(0) = 1** generates a table of values for the difference equation. See Fig. 4.

Derivation of the Solution to the Difference Equation for $a \neq 1$ From the difference equation $y_n = ay_{n-1} + b$, we get

$$y_1 = ay_0 + b$$
$$y_2 = ay_1 + b = a(ay_0 + b) + b = a^2y_0 + ab + b$$
$$y_3 = ay_2 + b = a(a^2y_0 + ab + b) + b = a^3y_0 + a^2b + ab + b$$
$$y_4 = ay_3 + b = a(a^3y_0 + a^2b + ab + b) + b = a^4y_0 + a^3b + a^2b + ab + b$$

The pattern that clearly develops is

$$y_n = a^ny_0 + a^{n-1}b + a^{n-2}b + \cdots + a^2b + ab + b. \tag{3}$$

Multiply both sides of (3) by a, and then subtract the new equation from (3). Notice that many terms drop out.

$$\begin{aligned} y_n &= a^ny_0 + a^{n-1}b + a^{n-2}b + \cdots + a^2b + ab + b \\ ay_n &= a^{n+1}y_0 + a^nb + a^{n-1}b + a^{n-2}b + \cdots + a^2b + ab \\ \hline y_n - ay_n &= a^ny_0 - a^{n+1}y_0 - a^nb + b \end{aligned}$$

The last equation can be written as

$$(1 - a)y_n = (1 - a)a^ny_0 - ba^n + b.$$

Now, divide both sides of the equation by $(1 - a)$. *Note:* Since $a \neq 1$, $1 - a \neq 0$.

$$\begin{aligned} y_n &= y_0a^n - \frac{b}{1-a}\cdot a^n + \frac{b}{1-a} \\ &= \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \end{aligned}$$

«

Derivation of the Solution to the Difference Equation for $a = 1$ The preceding reasoning also gives the solution in the case $a = 1$—namely, formula (3) holds for any value of a. In particular, for $a = 1$, formula (3) reads

$$\begin{aligned} y_n &= a^n y_0 + a^{n-1}b + a^{n-2}b + \cdots + a^2b + ab + b \\ &= 1^n y_0 + 1^{n-1}b + 1^{n-2}b + \cdots + 1^2b + 1b + b \\ &= y_0 + b + b + \cdots + b + b + b \\ &= y_0 + bn \end{aligned}$$

«

APPENDIX Financial Terms

Saving Accounts

Money in a savings account grows with either compound interest or simple interest. The money initially deposited is called the **principal** (denoted P) and the money in the account at a specified time in the future is called the **future value** (denoted F) of the money.

With **compound interest**, the year is divided into interest periods. The interest rate per period is

$$i = \frac{r}{m} = \frac{\text{annual rate of interest}}{\text{number of times compounded during year}}.$$

The rate r is expressed as a percentage or a decimal (usually between 0 and 1). The most common values of m are 1 (annual compounding), 2 (semiannual compounding), 4 (quarterly compounding), and 12 (monthly compounding). The interest earned during each interest period is $i \cdot B_{\text{previous}}$ where B_{previous} is the balance at the end of the previous interest period. Therefore, the new balance at the end of each interest period is given by the formula

$$B_{\text{new}} = B_{\text{previous}} + i \cdot B_{\text{previous}} = (1 + i)B_{\text{previous}}.$$

With **simple interest,** the interest earned each year is rP. Therefore

$$B_{\text{new}} = B_{\text{previous}} + rP.$$

Annuities

An **increasing annuity** is a bank account into which a sequence of equal deposits are made at the end of regular time periods. The successive balances in the account increase due to the compound interest earned and the deposits. We will assume that a deposit of R dollars is made at the end of each interest period and i is the interest per period. Therefore, the new balance at the end of each interest period is given by the formula

$$B_{\text{new}} = \underbrace{B_{\text{previous}}}_{\text{balance at beginning of interest period}} + \underbrace{i \cdot B_{\text{previous}}}_{\text{interest earned for the interest period}} + \underbrace{R}_{\text{amount deposited}} = (1 + i)B_{\text{previous}} + R.$$

A **decreasing annuity** is a bank account into which an amount of money is deposited and then a sequence of equal withdrawals are made at the end of regular time periods. We will assume that the withdrawal of R dollars is made at the end of each interest period, and that the amount withdrawn is greater than the compound interest earned during the interest period. The successive balances in the account decrease. The new balance at the end of each interest period is given by the formula

$$B_{\text{new}} = \underbrace{B_{\text{previous}}}_{\text{balance at beginning of interest period}} + \underbrace{i \cdot B_{\text{previous}}}_{\text{interest earned for the interest period}} - \underbrace{R}_{\text{amount withdrawn}} = (1 + i)B_{\text{previous}} - R.$$

A **mortgage** is a long-term loan that is used to purchase real estate and is paid back (with interest) by a sequence of equal payments made at the end of each interest period. We can think of it as a decreasing annuity where the borrower functions as the bank.

Check Your Understanding 10.5

Solutions can be found following the section exercises.

Mortgage The difference equation for the monthly balance of a mortgage is

$$y_n = (1.004)y_{n-1} - 1015,$$
$$y_0 = 177{,}138.81$$

and the solution of the difference equation is

$$y_n = 253{,}750 - 76{,}611.19(1.004)^n.$$

1. What are the amount, the annual rate of interest, and the monthly payment of the mortgage?
2. Is the duration of the mortgage 15 years, 25 years, or 30 years?

EXERCISES 10.5

In Exercises 1–10, give a difference equation for y_n, the balance after n interest periods.

1. **Compound Interest** \$1000 is deposited into a savings account paying 4% interest compounded quarterly.
2. **Compound Interest** \$1250 is deposited into a savings account paying 3% interest compounded monthly.
3. **Simple Interest** \$2500 is deposited into a savings account paying 2.5% simple interest.
4. **Simple Interest** \$500 is deposited into a savings account paying 2.75% simple interest.
5. **Mortgage** A \$204,700 mortgage with interest rate 4.8% compounded monthly is repaid with monthly payments of \$1073.99.
6. **Mortgage** A \$143,000 mortgage with interest rate 4.5% compounded monthly with monthly payments of \$724.56.
7. **Decreasing Annuity** \$25,000 is deposited into a savings account paying 3.9% interest compounded weekly and \$100 is withdrawn from the account at the end of each week.
8. **Loan** A car loan of \$9480 with interest rate 6% compounded monthly is repaid in 3 years with monthly payments of \$288.40.
9. **Increasing Annuity** \$2500 is deposited into a savings account paying 2.8% interest compounded semiannually and \$1000 is added to the account at the end of each half year.
10. **Increasing Annuity** \$1000 is deposited into a savings account paying 5.2% interest compounded weekly and \$25 is added to the account at the end of each week.

In Exercises 11 and 12, answer the questions.

11. **Increasing Annuity** The difference equation for the quarterly balance in an increasing annuity with interest compounded quarterly and money added at the end of each quarter year is
$$y_n = 1.008y_{n-1} + 1196,\ y_0 = 37{,}780$$
and the solution of the difference equation is
$$y_n = -149{,}500 + 187{,}280(1.008)^n.$$
   (a) What are the amount initially deposited into the annuity, the annual rate of interest, and the quarterly amount deposited into the annuity?
   (b) Will the value of the annuity reach \$100,000 after 7 years, 8 years, or 9 years?
12. **Decreasing Annuity** The difference equation for the weekly balance in a decreasing annuity with interest compounded weekly and money withdrawn at the end of each week is
$$y_n = 1.0005y_{n-1} - 200,\ y_0 = 24{,}792$$
and the solution of the difference equation is
$$y_n = 400{,}000 - 375{,}208(1.0005)^n.$$
   (a) What are the amount initially deposited into the annuity, the annual rate of interest, and the weekly amount withdrawn from the annuity?
   (b) Will the annuity be depleted after 128 weeks or 138 weeks?

In Exercises 13–18, **(a)** determine the first five values generated by the difference equation, and **(b)** find the solution of the difference equation.

13. $y_n = y_{n-1} + 5,\ y_0 = 1$
14. $y_n = y_{n-1} - 2,\ y_0 = 50$
15. $y_n = .4y_{n-1} + 3,\ y_0 = 7$
16. $y_n = 3y_{n-1} - 12,\ y_0 = 10$
17. $y_n = -5y_{n-1},\ y_0 = 2$
18. $y_n = -.7y_{n-1} + 3.4,\ y_0 = 3$

In Exercises 19–38, use difference equations to answer the question.

19. **Compound Interest** Calculate the future value of \$1000 after 2 years if deposited at 2.1% interest compounded monthly.
20. **Compound Interest** Calculate the future value of \$3000 after 4 years if deposited at 2.1% interest compounded monthly.
21. **Compound Interest** Calculate the present value of \$40,100 payable in 3 years at 4.4% interest compounded quarterly.
22. **Compound Interest** Calculate the present value of \$101,850 payable in 6 years at 4% interest compounded quarterly.
23. **Simple Interest** Calculate the future value after 3 years if \$1000 is deposited at 4.5% simple interest.
24. **Simple Interest** Calculate the future value after 5 years if \$4000 is deposited at 3% simple interest.
25. **Simple Interest** Calculate the present value of \$2000 in 10 years at 2.5% simple interest.
26. **Simple Interest** Calculate the present value of \$1000 in 7 years at 4% simple interest.
27. **Increasing Annuity** Seventeen thousand dollars is deposited into a savings account at 5% interest compounded semiannually and \$1500 is deposited at the end of each half year. How much money will be in the account after 5 years?

28. **Increasing Annuity** Suppose P dollars is deposited into a savings account at 3.2% interest compounded quarterly and then \$500 is added to the account at the end of each quarter year. For what value of P will the account contain \$41,000 after 6 years?

29. **Increasing Annuity** Suppose you deposit P dollars into a savings account at 4.5% interest compounded quarterly and then add \$1500 to the account at the end of each quarter. For what value of P will the account contain \$73,000 after 4 years?

30. **Increasing Annuity** Thirty-one thousand dollars is deposited into a savings account paying 3.9% interest compounded semiannually and \$1000 is deposited at the end of each half year. How much money is in the account after 5 years?

31. **Increasing Annuity** Suppose you deposit \$10,500 into a savings account paying 3.75% interest compounded annually. How much money should you deposit into the account at the end of each year in order to have \$20,000 in the account at the end of 7 years?

32. **Increasing Annuity** Suppose you deposit \$12,700 into a savings account paying 3.9% interest compounded weekly. How much money should you deposit into the account at the end of each week in order to have \$25,000 in the account at the end of 6 years?

33. **Decreasing Annuity** Forty-three thousand dollars is deposited into a savings account paying 3.2% interest compounded quarterly and \$700 is withdrawn at the end of each quarter year. How much money is in the account after 5 years?

34. **Mortgage** Calculate the monthly payment for a 30-year mortgage for \$300,062 at 4.5% interest compounded monthly.

35. **Mortgage** Calculate the monthly payment for a 25-year mortgage for \$300,080 at 5.1% interest compounded monthly.

36. **Decreasing Annuity** Thirty-one thousand dollars is deposited into a savings account paying 3% interest compounded quarterly and \$1000 is withdrawn at the end of each quarter. How much money is in the account after 4 years?

37. **Loan** Suppose you borrow \$2710 at 5% interest compounded quarterly. How much money should you pay back each quarter in order to pay back the loan in 3 years?

38. **Loan** Suppose you borrow \$49,000 at 6.8% interest compounded quarterly. How much money should you pay back each quarter in order to pay back the loan in 15 years?

39. **Present Value of a Decreasing Annuity** Use a difference equation to derive the formula for the present value of a decreasing annuity in which R dollars is withdrawn from a savings account at the end of each interest period and where the interest rate per period is i.

40. **Parachuting** A parachutist opens her parachute after reaching a speed of 100 feet per second. Suppose that y_n, her speed n seconds after opening the parachute, satisfies the difference equation $y_n = .1y_{n-1} + 14.4$, $y_0 = 100$.
   (a) Solve the difference equation
   (b) Her speed will get closer and closer to what speed?

41. **Population Dynamics** A small city with current population 50,000 is experiencing a departure of 600 people each year. Assume that each year the increase in population due to natural causes is 1% of the population at the start of that year.
   (a) Find the difference equation for y_n, the population after n years.
   (b) Solve the difference equation from part (a).
   (c) Use the solution from part (b) to calculate the population after 10 years.

42. **Elevation and Atmospheric Pressure** The atmospheric pressure at sea level is 14.7 pounds per square inch. Suppose that, at any elevation, an increase of 1 mile results in a decrease of 20% of the atmospheric pressure at that elevation.
   (a) Find the difference equation for y_n, the atmospheric pressure at elevation n miles.
   (b) Solve the difference equation from part (a).
   (c) Use the solution from part (b) to calculate the atmospheric pressure at 12 miles above sea level.

43. **Spread of Information** A sociological study was made to examine the process by which doctors decide to adopt a new drug. Certain doctors who had little interaction with other physicians were called *isolated*. Out of 100 isolated doctors, the number who adopted the new drug at the end of each month was 8% of those who had not yet adopted the drug at the beginning of the month. (*Source:* James S. Coleman, Elihu Katz, and Herbert Menzel, "The Diffusion of an Innovation Among Physicians.")
   (a) Find a difference equation for y_n, the number of isolated physicians adopting the drug after n months.
   (b) Solve the difference equation from part (a).
   (c) Use the solution from part (b) to calculate the number of isolated doctors who had adopted the drug after 11 months.

44. **Solute Concentration** A cell is put into a fluid containing an 8 milligram/liter concentration of a solute. (This concentration stays constant throughout.) Initially, the concentration of the solute in the cell is 3 milligrams/liter. The solute passes through the cell membrane at such a rate that each minute the increase in concentration in the cell is 40% of the difference between the outside concentration and the inside concentration.
   (a) Find the difference equation for y_n, the concentration of the solute in the cell after n minutes.
   (b) Solve the difference equation from part (a).
   (c) Use the solution from part (b) to calculate the concentration of the solute in the cell after 7 minutes.

45. **Learning Curve** Psychologists have found that, in certain learning situations in which there is a maximum amount that can be learned, the additional amount learned each minute is 30% of the amount yet to be learned at the beginning of that minute. Let 12 units of information be the maximum amount that can be learned.
   (a) Find the difference equation for y_n, the amount learned after n minutes.
   (b) Solve the difference equation from part (a).
   (c) Use the solution from part (b) to calculate the amount learned after 6 minutes.

46. **Genetics** Consider two genes A and a in a population, where A is a dominant gene and a is a recessive gene controlling the same genetic trait. (That is, A and a belong to the same locus.) Suppose that, initially, 80% of the genes are A and 20% are a. Suppose that, in each generation, .003% of gene A mutate to gene a.

(a) Find the difference equation for y_n, the percentage of gene a after n generations. [*Note:* The percentage of gene $A = 100$ – (the percentage of gene a).]
(b) Solve the difference equation from part (a).
(c) Use the solution from part (b) to calculate the percentage of gene a after 1800 generations.

47. Thermodynamics When a cold object is placed in a warm room, each minute its increase in temperature is 20% of the difference between the room temperature and the temperature of the object at the beginning of the minute. Suppose that the room temperature is 70°F and the initial temperature of the object is 40°F.
(a) Find the difference equation for y_n, the temperature of the object after n minutes.
(b) Solve the difference equation from part (a).
(c) Use the solution from part (b) to calculate the temperature of the object after 5 minutes.

48. Electricity Usage Suppose that the annual amount of electricity used in the United States will increase at a rate of 1.8% each year and that, this year, 3.2 trillion kilowatt-hours are being used.
(a) Find the difference equation for y_n, the number of trillion kilowatt-hours to be used during the year that is n years from now.
(b) Solve the difference equation from part (a).
(c) Use the solution from part (b) to calculate the number of trillion kilowatt-hours that will be used during the year that is 7 years from now.

49. Probability Suppose a coin has probability p of landing on heads. What is the probability of obtaining heads an even number of times in n tosses of the coin? *Note:* Let y_n be the probability of obtaining heads an even number of times in n tosses. Then $y_0 = 1$ since, if you make no tosses, you get zero heads, and zero is an even number. The tree diagram in Fig. 4 displays the probabilities associated with n tosses for $n \geq 1$.

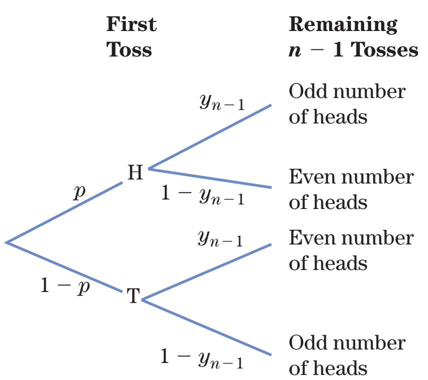

Figure 4 Tree diagram for probability problem with H = heads, T = tails

(a) Use the tree diagram to obtain a difference equation for y_n.
(b) Solve the difference equation in part (a).

50. Drug Absorption The popular sleep-aid drug Zaleplon has a half-life of one hour. That is, the amount of the drug present in the body is halved each hour. Suppose 10 mg of Zaleplon is taken at bedtime.
(a) Find the difference equation for y_n, the amount of the drug present in the body after n hours.
(b) Solve the difference equation from part (a).
(c) Use the solution from part (b) to calculate the amount of the drug present in the body after 8 hours.

TECHNOLOGY EXERCISES

51. Compound Interest From 1984 to 2014, the value of Berkshire Hathaway stock grew at a rate of about 18% compounded annually. Give the difference equation for an investment of \$1000 earning 18% compounded annually, and determine the number of years required for the investment to more than triple.

52. Compound Interest A loan of \$5614 at 6% interest compounded monthly is paid off with monthly payments of \$818.12. Give the difference equation for the monthly balances and determine the number of months until the loan is repaid.

53. Thermodynamics A steel rod of temperature 600°F is immersed in a large vat of water at temperature 75°F. At the end of each minute, its decrease in temperature is 30% of the difference between the water temperature and the temperature of the rod at the beginning of the minute.
(a) Give the difference equation for y_n, the temperature of the rod after n minutes.
(b) What is the temperature after 2 minutes?
(c) When will the temperature drop below 80°F?

54. Population Dynamics The birth rate in a certain city is 3.5% per year, and the death rate is 2% per year. Also, there is a net movement of population out of the city at a steady rate of 300 people per year. The population in 2016 was 5 million.
(a) Give the difference equation for y_n, the population after n years.
(b) Estimate the 2021 population.
(c) When will the population exceed 6 million?
(d) When will the population have doubled to 10 million?

55. Spread of Information Consider the sociological study of Exercise 43. Out of a group of *nonisolated* doctors, let y_n be the percent who adopted the new drug after n months. Then y_n satisfies the difference equation

$$y_n = .0025y_{n-1}(500 - y_{n-1}),\ y_0 = 3.$$

That is, initially 3% had adopted the drug.
(a) What percent had adopted the drug after 1 year?
(b) When had over half of the doctors adopted the drug?
(c) When had over 99% adopted the drug?

56. Caffeine Absorption After caffeine is absorbed into the body, 13% is eliminated from the body each hour. Assume that a person drinks an 8-oz cup of brewed coffee containing 130 mg of caffeine, and the caffeine is absorbed immediately into the body.
(a) Find the difference equation for y_n, the amount of caffeine in the body after n hours.
(b) After how many hours will 65 mg (one-half the original amount) remain in the body?
(c) How much caffeine will be in the body 24 hours after the person drank the coffee?

57. Caffeine Absorption Refer to Exercise 56. Suppose that the person drinks a cup of coffee at 7 a.m. and then drinks a cup of coffee at the end of each hour until 7 a.m. the next day.
(a) Find the difference equation for y_n, the amount of caffeine in the body after n hours.
(b) How much caffeine will be in the body at the end of the 24 hours?

Solutions to Check Your Understanding 10.5

1. The general form of the difference equation for the balances in a mortgage is

$$y_n = (1 + i)y_{n-1} - R,$$
$$y_0 = \text{amount borrowed}$$

Therefore, the amount of the mortgage is \$177,138.81. The monthly payment is the value of R, namely \$1015. Using the fact that $1 + i = 1.004 = 1 + .004$, we see that the monthly rate of interest (in decimal form) is .004. Therefore, the annual rate of interest is $12 \cdot .004 = .048$ or 4.8%.

2. We are given that the duration of the mortgage is either 15 years (that is, 180 months), 25 years (that is, 300 months), or 30 years (that is, 360 months). To determine which duration is correct, we should set $n = 180$, 300, and 360 in the solution of the difference equation and see which one gives a balance of 0. Rounded to two decimal places, $y_{180} = 96{,}583.31$, $y_{300} = 0.00$, and $y_{360} = -68{,}675.08$. Therefore, the mortgage will be paid off after 25 years.

CHAPTER 10 Summary

KEY TERMS AND CONCEPTS	EXAMPLES
10.1 Interest	
Interest is money paid for the use of money.	
Compound interest is calculated each period on the original principal and all interest accumulated during past periods. When P dollars, the **principal**, is invested at annual interest rate r compounded m times per year, then the **balance** after n interest periods is $P(1 + i)^n$, where $i = r/m$ is the **interest per period**.	Find the balance after 10 years when \$1000 is deposited at interest rate 4% compounded quarterly. *Answer:* $P = 1000$, $i = \frac{.04}{4} = .01$, and $n = 4 \cdot 10 = 40$. Therefore, the balance is $1000(1.01)^{40} = \$1488.86$.
Present value is the value today of an amount of money in the future. The present value P of F dollars n interest periods in the future is $$P = \frac{F}{(1 + i)^n}.$$	Find the present value of \$1000 to be received five years from now at interest rate 3% compounded monthly. *Answer:* $F = 1000$, $i = \frac{.03}{12} = .0025$, and $n = 12 \cdot 5 = 60$. Therefore, the present value is $\frac{1000}{1.0025^{60}} = \$860.87$.
The **effective rate** is the simple interest rate that produces the same amount of interest as the compound interest rate. $r_{\text{eff}} = (1 + i)^m - 1$, where interest is compounded m times per year.	Find the effective rate of 3% interest compounded quarterly. *Answer:* $i = \frac{.03}{4} = .0075$, and $m = 4$. Therefore, $r_{\text{eff}} = 1.0075^4 - 1 = .030339 \approx 3.034\%$.
Simple interest is calculated on the original principal only. When P dollars is invested at a simple interest rate r for n years, the balance is $(1 + nr)P$.	Find the balance after 10 years when \$1000 is deposited at simple interest rate 4%. *Answer:* $P = 1000$, $n = 10$, and $r = .04$. Therefore, the balance is $(1 + 10 \cdot .04)1000 = (1.4)1000 = \$1400$.
10.2 Annuities	
An **increasing (decreasing) annuity** is a sequence of equal deposits (withdrawals) made at the ends of regular time periods.	
Successive **balances in an increasing annuity** are calculated with the formula $$B_{\text{new}} = (1 + i)B_{\text{previous}} + R,$$ where i is the interest rate per time period and R is the amount of the regular deposit. The **future value of an increasing annuity** is calculated with the formula $$F = \frac{(1 + i)^n - 1}{i} \cdot R.$$	If \$100 is deposited into an account at the end of each month paying an interest rate of 3% compounded monthly ($i = .0025$), then successive monthly balances can be calculated with the formula $$B_{\text{new}} = (1.0025)B_{\text{previous}} + 100.$$ The balance after 5 years (60 months) is $$\frac{1.0025^{60} - 1}{.0025} \cdot 100 = \$6464.67.$$

KEY TERMS AND CONCEPTS	EXAMPLES
Successive **balances in a decreasing annuity** are calculated with the formula $B_{\text{new}} = (1+i)B_{\text{previous}} - R,$ where i is interest per time period and R is the amount of the regular withdrawal. The **present value of a decreasing annuity** is calculated with the formula $P = \frac{1-(1+i)^{-n}}{i} \cdot R.$	If \$100 is withdrawn from an account at the end of each month paying an interest rate of 3% compounded monthly, then successive monthly balances can be calculated with the formula $B_{\text{new}} = (1.0025)B_{\text{previous}} - 100.$ Find the present value of a decreasing annuity from which \$100 will be withdrawn at the end of each month for 5 years (60 months). *Answer:* $R = 100$, $n = 60$, and $i = .0025$. Therefore, the present value is $\frac{1-1.0025^{-60}}{.0025} \cdot 100 = \$5565.24.$
10.3 Amortization of Loans	
Amortization is a method of retiring a loan (such as a mortgage). It consists of a steady stream of equal payments at regular periods that pays down the loan with interest. The formula $P = \frac{1-(1+i)^{-n}}{i} \cdot R$ relates the principal P to the regular payment R.	Find the amount of a 30-year mortgage at 4.6% compounded monthly with a monthly payment of \$1025. Here, $R = 1025$, $i = \frac{.046}{12} = .0038333333$, and $n = 12 \cdot 30 = 360$. Therefore, the amount of the mortgage is $\frac{1-1.0038333333^{-360}}{.003833333} \cdot 1025 \approx \$200{,}000.$
10.4 Personal Financial Decisions	
An **individual retirement account** (IRA) is an increasing annuity in which the annual interest earned is either tax free (Roth IRA) or tax-deferred (traditional IRA). Contributions are tax deductible only with a traditional IRA.	If \$5000 is deposited annually into a traditional IRA at 6% interest compounded annually, the balance after 36 years will be \$595,604.33.
When the **add-on method** is used to calculate finance charges on a consumer loan, the interest paid each month is a fixed percentage of the principal.	With the add-on method, the monthly payment on a two-year loan of \$1000 at a stated annual interest rate of 6% is \$46.67. (The payment would be \$44.32 if calculated as in Section 10.3.)
A **discount point** accompanying a mortgage loan requires the borrower to pay additional interest up front equal to 1% of the amount borrowed.	A \$200,000 mortgage with three discount points requires an up-front payment of \$6000.
The **APR** of a mortgage is determined by calculating the interest rate corresponding to the loan amount after deducting certain up-front costs.	The APR for a mortgage of \$200,000 with one point at 6% interest compounded monthly is 6.094%.
The **effective mortgage rate** is the interest rate corresponding to an annuity that decreases the loan amount (after the deduction of certain up-front costs) to the balance after the expected lifetime of the loan.	With the assumption that the aforementioned mortgage will be held for 5 years, the effective mortgage rate is 6.24%.
The **interest-only mortgage** and the **adjustable-rate mortgage** (ARM) are alternatives to the standard fixed-rate mortgage. The interest rate of an ARM is reset periodically, depending on a floating interest rate determined by an index.	A 30-year mortgage of \$400,000 at 6.6% interest compounded monthly that is interest-only for 10 years has a monthly payment of \$2200 for the first 10 years and \$3005.89 thereafter.
10.5 A Unifying Equation	
The **difference equation** $y_n = ay_{n-1} + b$ with y_0 given and $a \neq 1$ has the **solution** $y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n.$	Solve $y_n = .2y_{n-1} + 4.8$, $y_0 = 1$. Since $\frac{b}{1-a} = \frac{4.8}{1-.2} = \frac{4.8}{.8} = 6,$ $y_n = 6 + (1-6)(.2)^n = 6 - 5(.2)^n$
The difference equation $y_n = y_{n-1} + b$ with y_0 given has the **solution** $y_n = y_0 + bn.$	The solution of $y_n = y_{n-1} + 2$, $y_0 = 3$ is $y_n = 3 + 2n.$

KEY TERMS AND CONCEPTS	EXAMPLES
The **interest formulas** $y_n = y_0 + (iy_0)n$ simple interest; $y_n = y_0(1+i)^n$ compound interest; each give the balance after n interest periods when y_0 dollars is deposited at an interest rate i per interest period.	If \$100 is deposited at 2% interest compounded annually, the balance after 10 years will be $y_{10} = 100 + 2 \cdot 10 = \$120.00$ simple interest; $y_{10} = 100(1.02)^{10} = \$121.90$ compound interest
Successive balances in an **increasing annuity** can be calculated with the difference equation $y_n = (1+i)y_{n-1} + R,\ y_0 = 0$ where i is the interest rate per period and R is the periodic deposit.	If \$100 is deposited into an account at the end of each month paying an interest rate of 3% compounded monthly, then successive monthly balances can be calculated with the difference equation $y_n = (1.0025)y_{n-1} + 100,\ y_0 = 0.$
Successive balances in a **decreasing annuity** (or a loan) can be calculated with the difference equation $y_n = (1+i)y_{n-1} - R,\ y_0 = \text{initial balance}$ where i is the interest rate per period and R is the periodic payment.	A 30-year mortgage of \$200,000 at 4.6% compounded monthly has a monthly payment of \$1025. Successive balances are calculated with the difference equation $y_n = (1.003833)y_{n-1} - 1025,\ y_0 = 200{,}000.$
Phenomena in physics, biology, sociology, and economics can be described by difference equations.	Radioactive decay, growth in a bacteria culture, spread of information, and supply and demand can be modeled with difference equations.

CHAPTER 10 Fundamental Concept Check Exercises

1. What is meant by *principal*?
2. What is the difference between compound interest and simple interest?
3. What is meant by the *balance* in a savings account? *Future value*?
4. How is the interest rate per period determined from the annual interest rate?
5. Explain how compound interest works.
6. Explain how simple interest works.
7. What is meant by the *present value* of a sum of money to be received in the future?
8. Explain the difference between the *nominal* and *effective* rates for compound interest.
9. What is an annuity?
10. What is meant by the *future value* of an annuity? *Present value*? *Rent*?
11. Describe the two types of annuities discussed in this chapter. In each case, identify the present and future values.
12. Give the formula for computing a new balance from a previous balance for each type of annuity.
13. Give a formula relating F and R in an increasing annuity.
14. Give a formula relating P and R in a decreasing annuity.
15. What are the components of an amortization table of a loan?
16. Give the formula for computing a new balance from a previous balance for a loan.
17. What is a balloon payment?
18. Explain how *traditional* and *Roth IRAs* work.
19. How are finance charges on a consumer loan calculated with the *add-on method*?
20. What are *discount points*?
21. What is the difference between the effective mortgage rate of a mortgage and the APR?
22. What is an interest-only mortgage?
23. What is an adjustable-rate mortgage?
24. Explain how a sequence of numbers is generated by a difference equation of the form $y_n = ay_{n-1} + b$, y_0 given.
25. What is meant by an *initial value for a difference equation*?
26. Give the solution of the difference equation $y_n = ay_{n-1} + b$, y_0 given, with $a \neq 1$. With $a = 1$.

CHAPTER 10 Review Exercises

1. **Future Valve** If \$100 earns 3% interest compounded annually, find the future value after 10 years.
2. **Saving for Retirement** Mr. West wishes to purchase a condominium for \$240,000 cash upon his retirement 10 years from now. How much should he deposit at the end of each month into an annuity paying 2.7% interest compounded monthly in order to accumulate the required savings?
3. **Mortgage Considerations** The income of a typical family in a certain city is currently \$39,216 per year. Family finance experts recommend that mortgage payments not exceed 25%

of a family's income. Assuming a current mortgage interest rate of 4.2% compounded monthly for a 30-year mortgage with monthly payments, how large a mortgage can the typical family in that city afford?

4. **Future Value** Calculate the future value of $50 after a year if it is deposited at 2.19% compounded daily.

5. **Comparing Payouts** Which is a better investment: 3% compounded annually or 2.92% compounded daily?

6. **Bond Fund** Ms. Smith deposits $200 at the end of each month into a bond fund yielding 3% interest compounded monthly. How much are her holdings worth after 5 years?

7. **Nonstandard Mortgage** A real estate investor takes out a $200,000 mortgage subject to the following terms: For the first 5 years, the payments will be the same as the monthly payments on a 15-year mortgage at 4.5% interest compounded monthly. The unpaid balance will then be payable in full.
   (a) What are the monthly payments for the first 5 years?
   (b) What balance will be owed after 5 years?

8. **College Expenses** College expenses at a private college currently average $35,000 per year. It is estimated that these expenses are increasing at the rate of ½% per month. What is the estimated cost of a year of college 10 years from now?

9. **Present Value** What is the present value of $50,000 in 10 years at 3% interest compounded monthly?

10. **Investment Value** An investment will pay $10,000 in 2 years and then $5000 1 year later. If the current market interest rate is 2.7% compounded monthly, what should a rational person be willing to pay for the investment?

11. **Car Loan** A woman purchases a car for $12,000. She pays $3,000 as a down payment and finances the remaining amount at 6% interest compounded monthly for 4 years. What is her monthly car payment?

12. **Nonstandard Loan** A businessman buys a $100,000 piece of manufacturing equipment on the following terms: Interest will be charged at a rate of 4% compounded semiannually, but no payments will be made until 2 years after purchase. Starting at that time, equal semiannual payments will be made for 5 years. Determine the semiannual payment.

13. **Retirement Account** A retired person has set aside a fund of $105,003.50 for his retirement. This fund is in a bank account paying 2.4% interest compounded monthly. How much can he draw out of the account at the end of each month so that there is a balance of $30,000 at the end of 15 years? (*Hint:* First compute the present value of the $30,000.)

14. **Balloon Payment** A business loan of $500,000 is to be paid off in monthly payments for 10 years with a $100,000 balloon payment at the end of the 10th year. The interest rate on the loan is 6% compounded monthly. Calculate the monthly payment.

15. **Savings Plan** Ms. Jones saved $100 per month for 30 years at 6% interest compounded monthly. How much were her accumulated savings worth?

16. **Loan** An apartment building is currently generating an income of $2000 per month. Its owners are considering a 10-year loan at 4.5% interest compounded monthly in order to pay for repairs. How large a loan can the income of the apartment house support?

17. **Comparing Investments** Investment *A* generates $1000 at the end of each year for 10 years. Investment *B* generates $5000 at the end of the fifth year and $5000 at the end of the tenth year. Assume a market rate of interest of 2.5% compounded annually. Which is the better investment?

18. **Bond Value** A 5-year bond has a face value of $1000 and is currently selling for $800. The bond pays $5 interest at the end of each month and, in addition, will repay the $1000 face value at the end of the fifth year. The market rate of interest is currently 4.8% compounded monthly. Is the bond a bargain? Why or why not?

19. **Effective Rate** Calculate the effective rate for 2.2% interest compounded semiannually.

20. **Effective Rate** Calculate the effective rate for 2.7% interest compounded monthly.

21. **Annuities with Extra Deposit** A person makes an initial deposit of $10,000 into a savings account and then deposits $1000 at the end of each quarter year for 15 years. If the interest rate is 2.2% compounded quarterly, how much money will be in the account after 15 years?

22. **Car Loan** A $10,000 car loan at 6% interest compounded monthly is to be repaid with 36 equal monthly payments. Write out an amortization schedule for the first 6 months of the loan.

23. **Savings Plan** A person pays $200 at the end of each month for 10 years into a fund paying .15% interest per month compounded monthly. At the end of the 10th year, the payments cease, but the balance continues to earn interest. What is the value of the fund at the end of the 20th year?

24. **Savings Fund** A savings fund currently contains $300,000. It is decided to pay out this amount with 1.8% interest compounded monthly over a 5-year period. What are the monthly payments?

25. **Mortgage Payment** What is the monthly payment on a $150,000, 30-year mortgage at 4.8% interest compounded monthly?

26. **Traditional IRA** Elisa, age 60, is currently in the 30% tax bracket and has $30,000 in a traditional IRA that earns 6% interest compounded annually. She anticipates being in the 35% tax bracket 5 years from now.
   (a) How much money will Elisa have after paying taxes if she withdraws her money now?
   (b) How much will Elisa have after paying taxes if she waits 5 years and then withdraws the money from the account?

27. **Roth IRA** Rework Exercise 26 for a Roth IRA.

28. **Consumer Loan** A consumer loan of $5000 for 2 years has an APR of 9% compounded monthly and a monthly payment of $228.42. What interest rate would be stated with the add-on method?

29. **Comparing Loans** Spike is considering purchasing a new computer and has decided to take one of two consumer loans:

   Consumer Loan *A*: 1 year, $3000, 10% APR, monthly payment of $263.75
   Consumer Loan *B*: 1 year, $3000, 6% add-on interest

   Which loan should Spike take and why?

30. **Mortgage with Points** Consider a 15-year mortgage of $90,000 at 6% interest compounded monthly with two discount points

and a monthly payment of $759.47. The APR for the mortgage is obtained by solving $P = \frac{1 - (1 + i)^{-n}}{i} \cdot R$ for i and then multiplying by 12. What are the values of P, n, and R?

31. **Mortgage with Points** Consider a 20-year mortgage of $100,000 at 6% interest compounded monthly with three discount points and a monthly payment of $716.43. Assume that the loan is expected to be held for 6 years, at the end of which time the unpaid balance will be $81,298.44. The effective mortgage rate is obtained in Excel by evaluating 12*RATE(m, $-R$, P, $-B$). What are the values of m, R, P, and B?

32. **Comparing Mortgages** Suppose that a lender gives you a choice between the following two 25-year mortgages of $200,000:

    Mortgage A**:** 6.5% interest compounded monthly, two points, monthly payment of $1350.41
    Mortgage B**:** 7% interest compounded monthly, one point, monthly payment of $1413.56

    Assume that you can invest money at 3.5% compounded monthly. The length of time that you must retain the mortgage in order for mortgage A to be the better choice is obtained in Excel by evaluating NPER(i, R, $-P$). What are the values of i, R, and P?

33. **Mortgage with Points** A 20-year mortgage of $250,000 at 5.75% interest compounded monthly, with two discount points, has a monthly payment of $1755.21. Show that the APR for this mortgage is 6%.

34. **Mortgage with Points** A 30-year mortgage of $100,000 at 5.5% interest compounded monthly, with three discount points, has a monthly payment of $567.79. Assume that the loan is expected to be terminated after 8 years, at which time the unpaid balance will be $86,837.98. Show that the effective mortgage rate is 6%.

35. **Interest-Only Mortgage** Consider a 25-year mortgage of $380,000 at 6.9% interest compounded monthly, where the loan is interest-only for 10 years. What is the monthly payment during the first 10 years? Last 15 years?

36. **Adjustable-Rate Mortgage** Consider a 25-year $220,000 5/1 ARM with a 2.8% margin and which is based on the CMT index. Suppose the value of the CMT index is 3.5% when the loan is initiated and is 4.55% 5 years later. Assume that all interest rates use monthly compounding.
    (a) Calculate the monthly payment for the first 5 years.
    (b) Calculate the unpaid balance at the end of the first 5 years.
    (c) Calculate the monthly payment for the sixth year.

37. Consider the difference equation $y_n = -3y_{n-1} + 8$, $y_0 = 1$.
    (a) Generate y_1, y_2, y_3 from the difference equation.
    (b) Solve the difference equation.
    (c) Use the solution in part (b) to obtain y_4.

38. Consider the difference equation $y_n = y_{n-1} - \frac{3}{2}$, $y_0 = 10$.
    (a) Generate y_1, y_2, y_3 from the difference equation.
    (b) Solve the difference equation.
    (c) Use the solution in part (b) to obtain y_6.

In Exercises 39–43, use difference equations to answer the question.

39. **Account Balance** How much money would you have to deposit into a savings account initially at 3.05% interest compounded quarterly in order to have $2474 after 7 years?

40. **Account Balance** How much money would you have in the bank after 2 years if you deposited $1000 at 5.2% interest compounded weekly?

41. **Annuity** How much money must be deposited at the end of each week into an annuity at 2.6% interest compounded weekly in order to have $36,000 after 21 years?

42. **Mortgage Payments** Find the monthly payment on a $33,100 20-year mortgage at 6% interest compounded monthly.

43. **State Legislature** Suppose that 100 people were just elected to a certain state legislature and that after each term, 8% of those still remaining from this original group will either retire or not be reelected. Let y_n be the number of legislators from the original group of 100 who are still serving after n terms. Find a difference equation for y_n.

Conceptual Exercises

44. If you decrease the interest rate for an investment by 10%, will the future value decrease by 10%?

45. If interest is compounded semiannually, will the effective rate be higher or lower than the nominal rate?

46. If you increase the number of months in which to pay off a loan, will you increase or decrease the monthly payment? The total amount of interest paid? Give an example to justify your answers.

47. Consider a decreasing annuity. If the amount withdrawn each month increases by 5%, will the duration decrease by 5%? Give an example to justify your answer.

48. Give an intuitive explanation for why successive payments for a mortgage contribute steadily more toward repayment of the principal.

CHAPTER 10 PROJECT

Two Items of Interest

Item 1: Successive Interest Computations

1. Suppose that a $100,000 investment grows 3% during the first year and 4% during the second year. By what percent will it have grown after the 2-year period? (*Note:* The answer is not 7%.)

2. Rework Exercise 1 for the case in which the investment earns 4% during the first year and 3% during the second year. Is the answer to Exercise 2 greater than, less than, or equal to the answer to Exercise 1?

3. Consider an annuity in which \$100,000 is invested and \$10,000 is withdrawn at the end of each year. Suppose that the interest rate is 3% during the first year and 4% during the second year. What is the balance at the end of the second year?

4. Rework Exercise 3, where the interest rate is 4% during the first year and 3% during the second year. Is the balance at the end of the second year greater than, less than, or equal to the answer to Exercise 3? Explain why your answer to this question makes sense.

5. Show that, if an investment of P dollars declines by 4% during a year, the balance at the end of the year is $P \cdot (1 - .04)$—that is, $P \cdot (.96)$.

6. Show that, if an investment of P dollars declines by r% during a year, the balance at the end of the year is $P \cdot (1 - r/100)$. We say that the investment earned $-r$%.

7. Show that, if an investment of P dollars earns r% one year and s% the following year, then the balance after the 2-year period is $P(1 + r/100)(1 + s/100)$. Conclude that the order of the two numbers r and s does not affect the balance after 2 years. (*Note:* The numbers r and s can be positive or negative.)

8. Which of the following two statements is true?

(a) Suppose that an investment of P dollars earns 4% one year and loses 4% the next year. Then the value of the investment at the end of the 2-year period will be P dollars.

(b) Suppose that investment A earns 4% one year and then loses 3% the next year, and investment B loses 3% one year and then gains 4% the next year. Then the balances of the two investments will be the same after the 2-year period.

Item 2: Rule of 72

9. Show that, if \$1000 is invested at 8% interest compounded annually, then it will double in about 9 years. (*Note:* $9 = 72/8$.)

10. Show that, if \$1000 is invested for 6 years, then it will approximately double in that time if it appreciates at 12% per year. (*Note:* $12 = 72/6$.)

> **Rule of 72** If money is invested at r% interest compounded annually, then it will double in about $72/r$ years. Alternatively, if money is invested for n years, then it will double during that time if it appreciates by about $(72/n)$% per year.

Table 1

m	Number Associated with Increasing m-Fold
2	72
3	114
4	144
5	167
6	186

11. Conclude from the Rule of 72 that, for an interest rate of r%, $(1 + r/100)^{72/r} \approx 2$. Also, conclude that for a number of years, n, $(1 + .72/n)^n \approx 2$.

We needn't restrict ourselves to doubling. For instance, the *Rule of 114* says that, if money is invested at r% interest, then it will triple in about $114/r$ years. Table 1 gives the numbers associated with several different multiples. Let us denote the number associated with m by N_m. That is, $N_2 = 72$ and $N_3 = 114$.

12. Notice that $N_6 = N_2 + N_3$. Explain why this makes sense.

13. Explain why $N_{n \cdot p} = N_n + N_p$ for any positive numbers n and p.

14. Find N_{10}, and then use that number to estimate the amount of time required for money to increase tenfold if invested at 8% interest compounded annually. Check your answer by raising (1.08) to that power.

15. Use the values of N_2 and N_3 to find $N_{1.5}$, the number associated with money increasing by one-half. Estimate the amount of time required for \$1000 to grow to \$1500 when invested at 7% interest compounded annually.